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CONVERGENCE RATE OF EMPIRICAL ESTIMATES IN STOCHASTIC PROGRAMMING

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Abstract. It is well known that many practical optimization problems with random elements lead from the mathematical point of view to deterministic optimization problems depending on the random elements through probability laws only. Further, it is also well known that these probability laws are known very seldom. Consequently, statistical estimates of the unknown probability measure, if they exist, must be employed to obtain some estimates of the optimal value and the optimal solution, at least.

If the theoretical distribution function is completely unknown then an empirical distribution usually substitutes it [2, 3, 9, 17, 31]. The great attention has been already paid to the studying of statistical properties of such arised empirical estimates, in the literature. We can remember here the works [4, 5, 6, 10, 13, 16, 32], for example. The aim of this paper is to discuss the convergence rate. For this we shall employed the assertions of the papers [10, 11, 13].

Key words: stochastic programming, problem with penalty, deterministic equivalent, random sequence fulfilling Φ-mixing condition.

1. Introduction. Let $(\Omega, \mathcal{S}, P)$ be probability space, $\xi = \xi(\omega) = [\xi_1(\omega, \ldots, \xi_s(\omega)]$ be an s-dimensional random vector defined on $(\Omega, \mathcal{S}, P)$, $F(z)$ be the distribution function of the random vector $\xi(\omega)$, $Z \subset E_s$ denote the support of the probability measure corresponding to the distribution function $F(z)$, $g_i(x, z)$, $i = 0, 1, 2, \ldots, \ell$ be real valued continuous functions defined on $E_n \times E_s$, $X \subset E_n$ be a nonempty set (E_n, $n \geqslant 1$ denotes an n-dimensional Euclidean space).

The general optimization problem with random element can be introduced as the problem to find

$$\min \{ g_0(x,\xi(\omega)) \mid x \in X : \; g_i(x,\xi(\omega)) \leqslant 0, \; i = 1,2,\ldots,\ell \}. \qquad (1)$$

It happens rather often that the solution x must be found without the realization knowledge of the random vector $\xi(\omega)$. Evidently, it is necessary, first to determine the decision rule, in such situation. It means to set to the original stochastic optimization problem (1) some deterministic equivalent. Two well known types of the deterministic equivalents can be introduced as the following deterministic problems.

I. To find

$$\inf \{ E\bar{g}(x,\xi(\omega)) \mid x \in X \}.$$

Stochastic programming problems with penalty and two-stage stochastic programming problems belong to this class of optimization problems.

II. To find

$$\inf \{ Eg(x,\xi(\omega)) \mid x \in X :: P\{\omega : \; g_i(x,\xi(\omega)) \leqslant 0, \; i = 1,2,\ldots,\ell\} \geqslant \alpha \}.$$

This deterministic equivalent is known as the chance constrained stochastic programming problem, from the literature.

$\alpha \in \langle 0,1 \rangle$ is a parameter, $\bar{g}(x,z)$, $g(x,z)$ some real valued continuous functions defined on $E_n \times E_s$, E denotes the operator of the mathematical expectation.

REMARKS:

1. The choice of the functions $\bar{g}(.,.)$, $g(.,.)$ depends on the character of the original stochastic problem.
2. It can generally happen that some above mentioned symbols are not meaningful. However, this situation cannot appear under the assumptions considered in this paper.
3. It is easy to see that I is a special case of II, from the mathematical point of view. We consider these problems separately, according to the historical development and their specific properties.

If we define the sets $Z(x)$, $X(\alpha)$, $\alpha \in E_1$, $x \in E_n$ by the prescription

$$X(\alpha) = \begin{cases} \{x \in X : P[Z(x)] \geqslant \alpha\} & \text{for } \alpha \in \langle 0, 1 \rangle, \\ X(1) & \text{for } \alpha > 1, \\ X(0) & \text{for } \alpha < 0, \end{cases} \tag{2}$$
$$Z(x) = \{z \in E_S : g_i(x, z) \leqslant 0, \ i = 1, 2, \ldots, \ell\},$$

then we can rewrite the deterministic equivalent II in the form: to find

$$\inf_{X(\alpha)} Eg(x, \xi(\omega)). \tag{3}$$

If, further, $\xi^k(\omega) = \xi^k = [\xi_1^k(\omega), \ldots, \xi_s^k(\omega)]$ is a sequence of random vectors with the common distribution function $F(z)$, the functions $U_i(z, \omega) = U_i(z)$, $F_N(z, \omega) = F_N(z)$, $z = (z_1, \ldots, z_s) \in E_s$, $\omega \in \Omega$, $k = 1, 2, \ldots, N$, $N = 1, 2, \ldots$ are defined by

$$\begin{aligned} U_k(z, \omega) &= 1 \iff \xi_j^k(\omega) < z_j \quad \text{for all} \quad j = 1, 2, \ldots, s, \\ &= 0 \iff \xi_j^k(\omega) \geqslant z_j \quad \text{for at least one} \quad j \in \{1, 2, \ldots, s\}, \end{aligned}$$

$$F_N(z, \omega) = \left(\frac{1}{N}\right) \sum_{k=1}^{N} U_k(z, \omega).$$

E_N and P_N, $N = 1, 2, \ldots$ denote the operator of mathematical expectation and probability measure corresponding to the distribution function $F_N(\cdot)$, the set mapping $X_N(\alpha) = X_N(\alpha, \omega)$, $N = 1, 2, \ldots$, $\alpha \in \langle 0, 1 \rangle$, $\omega \in \Omega$ is defined by the prescription

$$X_N(\alpha) = X_N(\alpha, \omega) = \{x \in X : P_N[Z(x)] \geqslant \alpha\},$$

then under general conditions $\inf_{x \in X} E_N \bar{g}(x, \xi(\omega))$ estimates the value $\inf_{x \in X} E\bar{g}(x, \xi(\omega))$ in the case of the deterministic equivalent I. In the case of the deterministic equivalent II the theoretical value $\inf_{X(\alpha)} Eg(x, \xi(\omega))$ can be estimated by the value $\inf_{X_N(\alpha)} E_N g(x, \xi(\omega))$.

The statistical behaviour of the just introduced estimates has been studied in the literature many times. We can remember here the works [2, 3, 6, 9, 12, 32], where the conditions are given under which these estimates are consistent. Further, the rate convergence was studied in [10, 11, 22, 29, 30]. In detail, mostly the upper bound for the expressions

$$P\Big\{\omega : \Big|\inf_X E_N \bar{g}(x,\xi(\omega)) - \inf_X E\bar{g}(x,\xi(\omega))\Big| > t\Big\},$$

$$P\Big\{\omega : \Big|\inf_{X_N(\alpha)} E_N g(x,\xi(\omega)) - \inf_{X(\alpha)} Eg(x,\xi(\omega))\Big| > t\Big\},$$

$t > 0$, were presented there.

Further, the asymptotic distributions of the random value $\sqrt{N}(\bar{x}_N(\omega) - \bar{x})$ was studied in [4, 5, 16, 17, 32],

$$\bar{x} = \arg\min_X E\bar{g}(x,\xi(\omega)), \tag{4}$$

$$\bar{x}_N(\omega) \doteq \bar{x}_N = \arg\min_X E_N \bar{g}(x,\xi(\omega)).$$

In this paper we shall try to present some tighter results for the convergence rate. In detail, we shall determine $\beta > 0$ such that

$$P\Big\{\omega : N^\beta\Big|\inf_X E_N \bar{g}(x,\xi(\omega)) - \inf_X E\bar{g}(x,\xi(\omega))\Big| > t\Big\} \to_{(N\to\infty)} 0, \tag{5}$$

$$P\Big\{\omega : N^\beta\Big|\inf_{X_N(\alpha)} E_N g(x,\xi(\omega)) - \inf_{X(\alpha)} Eg(x,\xi(\omega))\Big| > t\Big\} \to_{(N\to\infty)} 0 \tag{6}$$

for every $t > 0$.

Further, if $\bar{g}(\cdot,z)$, $g(\cdot,z)$ are uniformly strongly convex with parameter $\rho > 0$ functions of $x \in X$ (for the definition of strongly convex functions see Definition 3), then we have also $\beta > 0$ such that

$$P\{\omega : N^\beta\|\bar{x}_N(\omega) - \bar{x}\|^2 > t\} \to_{(N\to\infty)} 0 \tag{7}$$

and

$$P\{\omega : N^\beta\|x_N(\alpha,\omega) - x(\alpha)\|^2 > t\} \to_{(N\to\infty)} 0, \tag{8}$$

$\bar{x}_N(\omega)$, $\bar{x}$ fulfil the relation (4) and $x_N(\alpha,\omega) = x_N(\alpha)$, $x(\alpha)$ are defined by

$$x_N(\alpha,\omega) = x_N(\alpha) = \arg\min_{X_N(\alpha)} E_N g(x,\xi(\omega)), \tag{9}$$

$$x(\alpha) = \arg\min_{X(\alpha)} Eg(x,\xi(\omega)).$$

We shall employ the works [10, 11, 13] to obtain the above mentioned results. Consequently, some types of stochastic dependent random samples will be included into our investigation. However, we restrict our consideration to the case when

$$g_i(x,z) = f_i(x) - z_i, \qquad i = 1,2,\ldots,\ell,$$

in the case of the deterministic equivalent II. $f_i(x)$, $i = 1,2,\ldots,\ell$ are some real valued continuous functions defined on E_n.

Evidently, it holds $\ell = s$ in the case.

2. Some auxiliary definitions. In this part we shall remember some definitions. The Hausdorff distance between two subsets in E_n is defined by the following way:

DEFINITION 1. *If X', $X'' \subset E_n$, $n \geqslant 1$ are two non-empty sets then the Hausdorff distance of these sets is defined by*

$$\Delta_n[X', X''] = \max\left[\delta_n(X', X''), \delta_n(X'', X')\right],$$
$$\delta_n[X', X''] = \sup_{x' \in X'} \inf_{x'' \in X''} \|x' - x''\|.$$

$\|\cdot\|$ denotes the Euclidean norm in E_n.

(We usually omit the subscripts in the symbols Δ_n, δ_n.)

Let $\{\eta^k(\omega)\}_{k=-\infty}^{\infty}$ be an s-dimensional strongly stationary random sequence defined on $(\Omega, \mathcal{S}, P)$, $\mathcal{B}(-\infty, a)$ be the σ-algebra given by $\ldots, \eta^{a-1}(\omega), \eta^a(\omega)$, $\mathcal{B}(b, +\infty)$ be the σ-algebra given by $\eta^b(\omega)$, $\eta^{b+1}(\omega), \ldots$: ($a$, b are integers).

If $\mathcal{N}$ denotes the set of all natural numbers, $\phi(\cdot)$ is a nonnegative real valued function defined on $\mathcal{N}$ then we can define the ϕ-mixing random sequence by the following definition from [1].

DEFINITION 2. *We say that the strongly stationary random sequence $\{\eta^k(\omega)\}_{k=-\infty}^{\infty}$ fulfils the condition of ϕ-mixing if*

$$|P(A_1 \cap A_2) - P(A_1)P(A_2)| \leqslant \phi(N)P(A_1)$$

for $A_1 \in \mathcal{B}(-\infty, m)$, $A_2 \in \mathcal{B}(m+N, +\infty)$, $-\infty < m < +\infty$, $N \geqslant 1$, m is an integral number.

Further, we shall remember one definition of convex analysis.

DEFINITION 3. *Let $\bar{h}(x)$ be a real valued function defined on a convex set $\mathcal{K} \subset E_n$. $\bar{h}(x)$ is a strongly convex function with the parameter $\rho > 0$ if*

$$\bar{h}(\lambda x_1 + (1-\lambda)x_2) \leqslant \lambda \bar{h}(x_1) + (1-\lambda)\bar{h}(x_2) - \lambda(1-\lambda)\rho\|x_1 - x_2\|^2$$

for every $x_1,\ x_2 \in \mathcal{K},\ \lambda \in \langle 0, 1\rangle$.

REMARK. The assumptions under which a function is a strongly convex one with a parameter $\rho > 0$ are introduced in [20], for example.

It is known that the class of logarithmic concave probability measures is very important for change constrained stochastic programming problems [8, 18, 19]. If $\mathcal{B}_s$ denotes the Borel σ-algebra of the subsets of E_s then we can already remember the corresponding definition [18].

DEFINITION 4. *A probability measure $P(\cdot)$ defined on the $\mathcal{B}_s$ is logarithmic concave if for every $A,\ B \in \mathcal{B}_s$ and for every $\lambda \in \langle 0, 1\rangle$ the following inequality*

$$P[\lambda A + (1-\lambda)B] \geqslant [P(A)]^{\lambda}[P(B)]^{1-\lambda}$$

holds. ($\lambda A + (1-\lambda)B$ means Minkowski addition of sets.)

If $D \subset E_n$ is a bounded set then there exist $d'_j,\ d''_j \in E_1,\ j = 1, 2, \ldots, n$ and natural numbers $m_j = m_j(D, d),\ j = 1, 2, \ldots, n$ for $d > 0,\ d \in E_1,\ d < \min_j(d''_j - d'_j)$ such that

$$\begin{aligned} d'_j(D) = d'_j &= \inf\{x_j : x = (x_1, \ldots, x_n) \in D\}, \qquad (10)\\ d''_j(D) = d''_j &= \sup\{x_j : x = (x_1, \ldots, x_n) \in D\}, \\ D_j \frac{\sqrt{n}}{d} \leqslant m_j &< D_j \frac{\sqrt{n}}{d} + 1, \qquad D_j = d''_j - d'_j. \end{aligned}$$

Further, we can define $x_{j_1}, \ldots, x_{jm_j},\ j = 1, 2, \ldots, n$ such that

$$\begin{aligned} &d'_j = x_{j_1},\ x_{j_r} = x_{j_{r-1}} + \left(\frac{d}{\sqrt{n}}\right),\ r = 2, \ldots, m_j, \\ &x_{jm_j - 1} < d''_j,\ x_{jm_j} \geqslant d''_j,\ j = 1, 2, \ldots, n, \end{aligned}$$

and the system S as the following

$$S = S(D,d) = \{x = [x_1, \dots, x_n] : x_i \in [x_{ij_1}, \dots, x_{im_i}],\ i = 1, 2, \dots, n\}.$$

It holds that

$$\inf_{x' \in S} \|x - x'\| \leqslant d \qquad \text{for all} \qquad x \in D,$$

$$\inf_{x' \in S} \|x - x'\| \leqslant d \qquad \text{for all} \qquad x \in \prod_{j=1}^{n} \langle d'_j, d''_j \rangle$$

and

$$m = \prod_{j=1}^{n} m_j, \tag{11}$$

where we denote by $m = m[D, d]$ the number of the elements of the system S.

At the end of this part we denote the surroundings of a non-empty set $X \subset E_n$ by the symbol $X[\delta]$. It means that

$$X[\delta] = \{x \in E_n : x = x_1 + x_2,\ x_1 \in X,\ x_2 \in \mathcal{B}(\delta)\},$$

where $\mathcal{B}(\delta)$ is the δ-surroundings of zero in E_n.

3. Main results. The aim of this section is to present the values $\beta > 0$ for which the relations (5), (6), (7), (8) hold. More precisely, we shall try to obtain the results for the case when either $\{\xi^k(\omega)\}_{k=1}^{\infty}$ is a sequence of independent random vectors or when $\{\xi^k(\omega)\}_{k=-\infty}^{\infty}$ fulfils the ϕ-mixing condition.

Our investigation will be devided into two parts. We shall deal separately with the deterministic equivalent I and the deterministic equivalent II. The reason for this is practical especially in the mathematical technic of proof. Moreover, the interval for β will be rather greater in the deterministic equivalent I. In detail, this interval does not depend on the dimension of the random vector $\xi(\omega)$. Further, the results achieved for this simpler case will be employed to obtain the results for the deterministic equivalent II. However, we start our investigation by very simple case of deterministic equivalent I. By this access it will be seen the dependence of interval for β on the complexity of the corresponding problems.

a) Deterministic equivalent I. We shall consider a special, simple case, at the beginning. So first, let us assume that there exist a natural number s_1 and continuous functions $g_i^*(z)$, $h_i^*(x)$, $i = 1, 2, \ldots, s_1$, defined on $E_s \times E_n$ such that

$$\bar{g}(x, z) = \sum_{i=1}^{s_1} g_i^*(z) h_i^*(x). \tag{12}$$

We shall see that in this very special case (however from the mathematical point of view enough important) the terminal result will be the same in the both considered cases.

Theorem 1. *Let X be a compact set, the function $\bar{g}(x, z)$ fulfils the relation (12), where $g_i^*(z)$ and $h_i^*(x)$, $i = 1, 2, \ldots, s_1$, are continuous, real valued, bounded functions defined on $Z \times X$. If either*

1. $\{\xi^k(\omega)\}_{k=1}^{\infty}$ *is a sequence of independent random vectors*

or

2. $\{\xi^k(\omega)\}_{k=-\infty}^{\infty}$ *is a strongly stationary random sequence fulfilling the ϕ-mixing condition for which there exists*

$$\overline{\lim_{N \to \infty}} \frac{1}{N} \sum_{k=1}^{N-1} (N - k)\phi(k) < +\infty,$$

then

$$P\Big\{\omega : \ N^{\beta} \Big| \inf_{x \in X} E_N \bar{g}(x, \xi(\omega)) - \inf_{x \in X} E\bar{g}(x, \xi(\omega)) \Big| > t\Big\} \to_{(N \to \infty)} 0$$

for $t > 0$, $\beta \in (0, \frac{1}{2})$.

If moreover

3. *X is a convex set,*
4. *$h_i^*(x)$ $i = 1, 2, \ldots, s_1$, are convex functions on X,*
5. *there exists $i \in \{1, 2, \ldots, s_1\}$ such that $h_i^*(x)$ is a strongly convex with a parameter $\rho > 0$ function on X,*

then also

$$P\{\omega : \ N^{\beta} \| \bar{x}_N(\omega) - \bar{x} \|^2 > t\} \to_{(N \to \infty)} 0$$

for $t > 0$, $\beta \in (0, 1/2)$.

(Remark: $\bar{x}_N(\omega)$, $\bar{x}$, $N = 1, 2, \ldots$ are defined by the relation (4).)

Proof. The assertion of Theorem 1 follows immediately from the next Theorem. It is sufficient to employe there the substitution $t := (t/(N^{\beta}))$ and generally known limit properties of the corresponding functions.

Theorem 2. *Let X be a a compact set and the function $\bar{g}(x, z)$ fulfil the relation* (12) *where $g_i^*(z)$ and $h_i^*(x)$, $i = 1, 2, \ldots, s_1$, are continuous real valued functions defined on $Z \times X$. Let further K_1, M_1 be real valued constants such that*

$$|g_i^*(z)| \leqslant K_1, \quad |h_i^*(x)| \leqslant M_1, \quad x \in X, \ z \in Z, \ i = 1, 2, \ldots, s_1.$$

If

1. $\{\xi^k(\omega)\}_{k=1}^{\infty}$ *is a sequence of independent random vectors, then*

$$\begin{aligned} &P\Big\{\omega : \Big|\inf_{x \in X} E_N \bar{g}(x, \xi(\omega)) - \inf_{x \in X} E\bar{g}(x, \xi(\omega))\Big| > t\Big\} \\ &\leqslant 2s_1 \cdot \exp\Big\{-\frac{Nt^2}{2M_1^2 K_1^2 s_1^2}\Big\} \quad \text{for } N = 1, 2, \ldots, \ t > 0, \end{aligned} \tag{13}$$

2. $\{\xi^k(\omega)\}_{k=-\infty}^{\infty}$ *is a strongly stationary random sequence fulfilling the ϕ-mixing condition then*

$$\begin{aligned} &P\Big\{\omega : \Big|\inf_{x \in X} E_N \bar{g}(x, \xi(\omega)) - \inf_{x \in X} E\bar{g}(x, \xi(\omega))\Big| > t\Big\} \\ &\leqslant s_1^3 \frac{4K_1^2 M_1^2}{t^2 N^2}\Big[N + \sum_{k=1}^{N-1}(N - k)\phi(k)\Big], \end{aligned} \tag{14}$$

$N = 1, 2, \ldots, \ t > 0.$

If moreover

3. *X is a convex set,*
4. *$h_i^*(x)$ $i = 1, 2, \ldots, s_1$, are convex functions on X,*
5. *there exists $i \in \{1, 2, \ldots, s_1\}$ such that $h_i^*(x)$ is a strongly convex with a parameter $\rho > 0$ function,*

then also

in the case 1

$$P\left\{\omega : \|\bar{x}_N(\omega) - \bar{x}\|^2 > \frac{2t}{\rho}\right\} \leqslant 4s_1 \exp\left\{-\frac{Nt^2}{2M_1^2K_1^2s_1^2}\right\}, \tag{15}$$
for $N = 1, 2, \dots,\ t > 0$,

and in the case 2

$$P\left\{\omega : \|\bar{x}_N(\omega) - \bar{x}\|^2 > \frac{2t}{\rho}\right\} \leqslant 2s_1^3 \frac{4K_1^2M_1^2}{t^2N^2}\left[N + \sum_{k=1}^{N-1}(N-k)\phi(k)\right], \tag{16}$$
for $N = 1, 2, \dots,\ t > 0$.

The proof of Theorem 2 will be given in the Appendix.

It follows from the assertion of Theorem 1 that the interval for β is the same in the both considered cases. According to well known results of [4, 5, 16, 32] we can recognize that it is not possible this interval to expand. However, Theorem 1 deals only with the very special form of the function $\bar{g}(x, z)$. Further, we shall consider rather general case.

Theorem 3. *Let*

1. *$X \subset E_n$ be a compact set,*
2. *$\bar{g}(x, z)$ be a continuous, bounded function on $X[d] \times Z$ for some $d > 0$,*
3. *$\bar{g}(x, z)$ be for every $z \in Z$ a Lipschitz function of $x \in X[d]$ with the Lipschitz constant L not depending on $z \in Z$.*

If

4. *either*
   a) *$\{\xi^k(\omega)\}_{k=1}^{\infty}$ is a sequence of independent random vectors and simultaneously $0 < \beta < 1/2$,*

   or

   b) *$\{\xi^k(\omega)\}_{k=-\infty}^{\infty}$ is a strongly stationary random sequence fulfilling the ϕ-mixing condition for which*

   $$\overline{\lim_{N \to \infty}} \frac{1}{N} \sum_{k=1}^{N-1}(N-k)\phi(k) < +\infty,$$

   and simultaneously $0 < \beta < 1/(n+2)$, then

$$P\Big\{\omega : N^{\beta}\Big|\inf_{x\in X} E_N\bar{g}(x,\xi(\omega)) - \inf_{x\in X} E\bar{g}(x,\xi(\omega))\Big| > t\Big\} \to_{(N\to\infty)} 0,$$

for $t > 0$.

If moreover

5. *X is a convex set,*
6. *$\bar{g}(x,z)$ is for every $z \in Z$ a strongly convex with a parameter $\rho > 0$ function of $x \in X[d]$,*

then also

$$P\{\omega :\ N^{\beta}\|\bar{x}_N(\omega) - \bar{x}\|^2 > t\} \to_{(N\to\infty)} 0,$$

for $t > 0$.

To prove the assertion of Theorem 3 we shall employ some former results [10, 11]. However, we introduce them in a modified form.

Theorem 4. *If the assumptions* 1, 2, 3 *of Theorem* 3 *are fulfilled and if $\{\xi^k(\omega)\}_{k=1}^{\infty}$ is a sequence of independent random vectors then*

$$P\Big\{\omega :\ |E_N\bar{g}(x,\xi(\omega)) - E\bar{g}(x,\xi(\omega))| > tL \text{ for at least one } x \in X[d]\Big\}$$
$$\leqslant 2m[X,(\tfrac{t}{3})]\exp\Big\{-\frac{Nt^2}{18M^2}\Big\},$$

and simultaneously (17)

$$P\Big\{\omega :\ \Big|\inf_{X} E_N\bar{g}(x,\xi(\omega)) - \inf_{x\in X} E\bar{g}(x,\xi(\omega))\Big| > tL\Big\}$$
$$\leqslant 2m[X,(\tfrac{t}{3})]\exp\Big\{-\frac{Nt^2}{18M^2}\Big\},$$

for $0 < t < 3d$ and a constant M fulfilling the inequality $|\bar{g}(x,z)| \leqslant M$, $x \in X[d]$, $z \in Z$.

If moreover the assumptions 5, 6 *of Theorem* 3 *are satisfied, then also*

$$P\Big\{\omega :\ \|\bar{x}_N(\omega) - \bar{x}\|^2 \geqslant tL\cdot\Big(\frac{2}{\rho}\Big)\Big\} \leqslant 4m[X,(\tfrac{t}{3})]\exp\Big\{-\frac{Nt^2}{18M^2}\Big\}. \quad (18)$$

Theorem 5. *If the assumptions* 1, 2, 3 *of Theorem* 3 *are fulfilled and if* $\{\xi^k(\omega)\}_{k=-\infty}^{\infty}$ *is a strongly stationary random sequence fulfilling the* ϕ*-mixing conditions, then*

$$P\Big\{\omega : \Big|\inf_{x\in X} E_N\bar{g}(x,\xi(\omega)) - \inf_{x\in X} E\bar{g}(x,\xi(\omega))\Big| > tL\Big\}$$
$$\leqslant m[X,(\tfrac{t}{3})]\,\frac{36M^2}{t^2N^2}\Big[N + \sum_{k=1}^{N-1}(N-k)\phi(k)\Big],$$

and simultaneously (19)

$$P\{\omega : |E_N\bar{g}(x,\xi(\omega)) - E\bar{g}(x,\xi(\omega))| > tL \text{ for at least one } x\in X[d]\}$$
$$\leqslant m[X,(\tfrac{t}{3})]\Big[N + \sum_{k=1}^{N-1}(N-k)\phi(k)\Big]\cdot\frac{36M^2}{t^2N^2},$$

for $0 < t < 3d$ *and a constant* M *fulfilling the inequality* $|\bar{g}(x,z)| \leqslant M$, $x\in X[d]$, $z\in Z$.

If moreover the assumptions 5, 6 *of Theorem* 3 *are satisfied, then also*

$$P\Big\{\omega : \|\bar{x}_N(\omega) - \bar{x}\|^2 \geqslant \frac{2Lt}{\rho}\Big\}$$
$$\leqslant 2m[X,(\tfrac{t}{3})]\,\frac{36M^2}{t^2N^2}\Big[N + \sum_{k=1}^{N-1}(N-k)\phi(k)\Big]. \tag{20}$$

The assertions given by the relations (17) and (19) follow immediately from the results of the papers [10, 11]. (Theorem 2 and its proof in [10] and also Theorem 2 and the corresponding proof in [11]). It remains to prove the relation (18), (20). However this results will be proved in the Appendix. Here we shall prove the assertion of Theorem 3 only.

Proof of Theorem 3. To prove the assertion of Theorem 3 we shall first find an upper estimate of the number $m[X,d]$. For this we shall verify the validity of the auxiliary assertion.

Lemma 1. *If* $X \subset E_n$ *is a nonempty, bounded set,* $d > 0$, *then*

$$m[X,d] \leqslant \Big[\frac{\operatorname{rad} X\cdot\sqrt{n}}{d} + 1\Big]^n,$$

where rad $X = \sup_{x,x' \in X} \|x - x'\|$.

Proof. Since it follows from definition of the number $m[X, d]$ that

$$m[X,d] \leqslant \prod_{j=1}^{n} \left[\frac{D_j\sqrt{n}}{d} + 1\right],$$

we can see that the assertion of Lemma 1 holds.

Consequently, as X is a compact set, it follows from Lemma 1 that

$$m[X, (\tfrac{t}{3})] \leqslant \sum_{k=0}^{n} \binom{n}{k} \left[\frac{\operatorname{rad} X}{t} 3\sqrt{n}\right]^k,$$

for $t > 0$.

Further, if we substitute (in Theorems 4, 5) $t := t/(LN^{\beta})$ then there exists a natural number N_0 and a constant $K = K(n, X)$ such that

$$LN^{\beta}\frac{\operatorname{rad} X}{t}\sqrt{n} > 1 \qquad \text{for} \quad N > N_0,$$

and simultaneously

$$m\left[X, \frac{t}{3LN^{\beta}}\right] \leqslant K(n, X)\frac{N^{n\beta}}{t^n}. \tag{21}$$

Now already the validity of the assertion of Theorem 3 follows from the last inequality, Theorems 4 and 5 and well known limit properties of the corresponding functions.

We have finished the part of the section 3 corresponding to the deterministic equivalent I. Comparing these results with the ones of the works [4, 6] (where the problems of asymptotic distribution of $\sqrt{N}(\bar{x}_N(\omega) - \bar{x})$ is discussed), we can believe that it is impossible to expand the interval for β at least in the case of independent random samples.

Further, it follows from Theorem 1 that the same interval can be achieved in special cases for dependent samples too. Of course, it was done only for the case when the optimalized function satisfy the relation (12) and the random sample satisfy the ϕ-mixing condition.

The chance constrained stochastic programming problems will be considered in the next part of this section. We shall see that the interval for β will be smaller everywhere, it means in the case of independent random samples too. This is evidently draw by new ineccuraccy arising by the set $X(\alpha)$ approximation.

b) Deterministic equivalent II. We have already mentioned that employing the results of the works [12, 13] we have to restrict the original stochastic problem in the case of the deterministic equivalent II. In detail, we have to deal with the original stochastic problems in which the random element appears on the constraints right-hand side only. So we shall assume that there exist real valued functions $f_i(x)$, $i = 1, 2, \ldots, \ell$ defined on E_n such that

$$g_i(x, z) = f_i(x) - z_i, \qquad i = 1, 2, \ldots, \ell.$$

Consequently $\ell = s$. Of course, the original optimalized function may depend on the random element as well. Now, we shall repeat the definition of some symbols in this case.

Let $f_i(x)$, $i = 1, 2, \ldots, \ell$ be real valued, continuous functions defined on E_n, $n \geqslant 1$,

$$\begin{aligned}
X &= E_n^+, \\
Z &\subset E_\ell^+, \\
Z(x) &= \{z \in E_\ell^+ : \; z = (z_1, \ldots, z_\ell), \; f_i(x) \leqslant z_i, \; i = 1, 2, \ldots, \ell\}, \\
X(\alpha) &= \{x \in E_n^+ : \; P[Z(x)] \geqslant \alpha\} && \text{for } \alpha \in \langle 0, 1 \rangle, \qquad (22) \\
&\equiv X(1) && \text{for } \alpha > 1, \\
&\equiv X(0) && \text{for } \alpha < 0, \\
X_N(\alpha) &= \{x \in E_n^+ : \; P_N[Z(x)] \geqslant \alpha\} && \text{for } \alpha \in \langle 0, 1 \rangle,
\end{aligned}$$

where $P[Z(x)] = P\{\omega : \; \xi(\omega) \in Z(x)\}$; $P_N[\,] = P_N\{\cdot, \omega\}$ is the empirical probability measure corresponding to the distribution function $F_N(\cdot)$, $E_n^+ = \{x \in E_n : \; x = (x_1, \ldots, x_n), \; x_i \geqslant 0, \; i = 1, 2, \ldots, n\}$.

Let further

$$\begin{aligned}
x(\beta) &= \arg \min_{X(\beta)} Eg(x, \xi(\omega)), \qquad (23) \\
x_N(\beta, \omega) &= \arg \min_{X_N(\beta)} E_N g(x, \xi(\omega)) \quad \text{for } \beta \in (0, 1).
\end{aligned}$$

If $\delta > 0$, $\alpha \in (0,1)$ are arbitrary chosen but fix in the sequel we set the following assumptions.

i) $f_i(x)$, $i = 1,2,\dots,\ell$ are real valued continuous functions on E_n^+ such that

a) $f_i(0) = 0$, $i = 1,2,\dots,\ell$, $0 \in E_n$,

b) there exist a constant γ_1 such that

$$f_i(x) - f_i(x') \geqslant \gamma_1 \sum_{j=1}^{n} (x_j - x'_j),$$

for every $x = (x_1,\dots,x_n)$, $x' = (x'_1,\dots,x'_n)$, $x \geqslant x'$ componentwise, $i = 1,2,\dots,\ell$, $x, x' \in E_n^+$,

c) there exists a constant $\gamma_2 > 0$ such that

$$|f_i(x) - f_i(x')| \leqslant \gamma_2 \|x - x'\|,$$

for $i = 1,2,\dots,\ell$, $x, x' \in X(\alpha, 2\delta)$, $x < x'$ componentwise, $X(\alpha,\delta)$ is defined by the following relation

$$X(\alpha,\delta) = \{x = x_1 + x_2 : \; x_1 \in X(\alpha), \; x_2 \in B(\delta)\},$$

where $B(\delta)$ denotes the δ-surrounding of $0 \in E_n$.

ii) $\xi(\omega)$ fulfils the conditions:

a) the probability measure of the random vector $\xi(\omega)$ is absolutely continuous with respect to the Lebesgue measure in E_ℓ. Let us denote by $h(z)$ the probability density corresponding to the distribution function $F(z)$ of the random vector $\xi(\omega)$;

b) there exist real numbers c_j, $j = 1,2,\dots,\ell$ such that $c_j > 0$ and $Z = \prod_{j=1}^{\ell} \langle 0, c_j \rangle$ (it means $P\{\omega : \; \xi(\omega) \in \prod_{j=1}^{\ell} \langle 0, c_j \rangle\} = 1$);

c) there exist ϑ_1, ϑ_2 such that

$$0 < \vartheta_1 \leqslant h(z) \leqslant \vartheta_2 \quad \text{for every } z \in \prod_{j=1}^{\ell} \langle 0, c_j \rangle.$$

iii)

a) $g(x,z)$ is a bounded function on $E_n^+ \times Z$,

b) $g(x,z)$ is for every $z \in Z$ a Lipschitz function of $x \in X(\alpha,\delta)$ with Lipschitz constant L not depending on $z \in Z$,

iv) at least one of the following assumptions is satisfied:

a) $\{\xi^k(\omega)\}_{k=1}^{\infty}$ is a sequence of independent random vectors and simultaneously $0 < \beta < 1/(2\ell)$,

b) $\{\xi^k(\omega)\}_{k=-\infty}^{\infty}$ is a strongly stationary random sequence fulfilling the ϕ-mixing condition for which

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{k=1}^{N-1} (N-k)\phi(k) < +\infty$$

and simultaneously $0 < \beta < 1/((n+2)\ell)$.

v) $g(x, z)$ is for every $z \in Z$, strongly convex with parameter $\rho > 0$ function on $X(\alpha, \delta)$,

vi) $f_i(x)$, $i = 1, 2, \dots, \ell$ are convex function on E_n^+,

vii) the probability measure corresponding to $F(z)$ is logarithmic concave.

Now we can already introduce the main result of this part of the paper.

Theorem 6. *Let* $X = E_n^+$ *and the assumptions* i, ii, iii, iv *be fulfilled for given arbitrary* $\alpha \in (0,1)$, $\delta > 0$. *If* $t > 0$ *then*

$$P\Big\{\omega : N^{\beta}\Big| \inf_{x\in X(\alpha)} Eg(x, \xi(\omega)) - \inf_{x\in X_N(\alpha)} E_N g(x, \xi(\omega))\Big| > t\Big\} \to_{(N\to\infty)} 0.$$

If moreover the assumptions v, vi, vii *are fulfilled, then also*

$$P\{\omega : N^{\beta}\|x_N(\alpha,\omega) - x(\alpha)\|^2 > t\} \to_{(N\to\infty)} 0.$$

To prove the assertion of Theorem 6 we shall employ the former results [13]. However we shall have to employ them in modified forms, again.

Theorem 7. *Let* $\alpha \in (0,1)$, $\delta > 0$, $t > 0$, $t_0 = 4(\sqrt{n}/\gamma_1))\sqrt[\ell]{2t/(\vartheta_1)}$. *Let, further, the assumptions* i, ii, iii *be fulfilled. If* $d < \min(\delta, t/6)$, $(\sqrt{n}/(\gamma_1))\sqrt[\ell]{2t/(\vartheta_1)} < \delta$, $\vartheta_2\gamma_2 d \sum_{i=1}^{\ell} \prod_{v\neq i} c_v < t/6$, M *is a constant for which* $|g(x,z)| \leqslant M$, $x \in X(\alpha, 2\delta)$, $z \in Z$ *and if* $\{\xi^k(\omega)\}_{k=1}^{\infty}$ *is a sequence of independent random vectors, then*

$$\begin{aligned} P\Big\{\omega : \Big|\inf_{x\in X(\alpha)} Eg(x,\xi(\omega)) - \inf_{x\in X_N(\alpha)} E_N g(x,\xi(\omega))\Big| > t_0 L\Big\} \\ \leqslant 2m[X(\alpha,2\delta), d]\exp\{-Nt^2/18\} \\ + 2m[X(\alpha,2\delta), d]\exp\{-Nt_0^2 L^2/4\cdot 18M^2\}. \end{aligned} \tag{24}$$

If moreover the assumptions v, vi, vii *are fulfilled, then also*

$$P\Big\{\omega : \|x_N(\alpha,\omega) - x(\alpha)\|^2 \geqslant 4\frac{t_0 L}{\rho}\Big\} \leqslant 6m[X(\alpha,2\delta),d]\exp\{-Nt^2/18\} + 4m[X(\alpha,2\delta),d]\exp\{-Nt_0^2L^2/4\cdot 18M^2\}. \tag{25}$$

Theorem 8. *Let* $\alpha \in (0,1)$, $\delta > 0$, $t > 0$, $t_0 = 4(\sqrt{n}/(\gamma_1))\sqrt[\ell]{2t/(\vartheta_1)}$. *Let, further, the assumptions* i, ii, iii *be fulfilled. If* $d < \min(\delta, t/6)$, $(\sqrt{n}/(\gamma_1))\sqrt[\ell]{2t/(\vartheta_1)} < \delta$, $\vartheta_2\gamma_2^\ell d\sum_{i=1}^{\ell}\prod_{v\neq i} c_v < t/6$, M *is a constant for which* $|g(x,z)| \leqslant M$, $x \in X(\alpha,2\delta)$, $z \in Z$ *and if* $\{\xi^k(\omega)\}_{k=-\infty}^{\infty}$ *is a random sequence fulfilling a* ϕ*-mixing condition, then*

$$P\Big\{\omega : \Big|\inf_{X(\alpha)} Eg(x,\xi(\omega)) - \inf_{X_N(\alpha)} E_N g(x,\xi(\omega))\Big| > t_0 L\Big\} \leqslant m[X(\alpha,2\delta),d]\Big[N + \sum_{k=1}^{N-1}(N-k)\phi(k)\Big]\Big[\frac{36\cdot 4}{t^2N^2} + \frac{4\cdot 36M^2}{L^2t_0^2N^2}\Big]. \tag{26}$$

If moreover the assumptions v, vi, vii *are fulfilled, then also*

$$P\Big\{\omega : \|x_N(\alpha,\omega) - x(\alpha)\|^2 \geqslant 4\frac{t_0 L}{\rho}\Big\} \leqslant 3m[X(\alpha,2\delta),d]\Big[N + \sum_{k=1}^{N-1}(N-k)\phi(k)\Big]\cdot\frac{36\cdot 4}{t^2N^2} + 2m[X(\alpha,2\delta),d]\Big[N + \sum_{k=1}^{N-1}(N-k)\phi(k)\Big]\cdot\frac{4\cdot 36M^2}{L^2t_0^2N^2}. \tag{27}$$

The assertions given by the relations (24) and (26) follows immediately from the results of the paper [13]. It remains to prove the relation (25), (27). The proof of this results will be given in the Appendix. Here we shall prove the assertion of Theorem 6, only.

Proof of Theorem 6. To prove the assertion of Theorem 6 we shall have first to determine an upper estimate of the number $m[X(\alpha,2\delta),d]$ for $d = t/(7N^\beta)$. However, employing the results of Lemma 1 and further following the corresponding part of Theorem 3 proof we obtain that

$$m[X(\alpha,2\delta),d] \leqslant \tilde{k}(n, X(\alpha))\frac{N^{n\beta}}{t^n},$$

for enough large N and some constant $\bar{k}(n, X(\alpha))$.

Now already, the validity of the assertion of Theorem 6 follows from the last inequality, Theorems 7 and 8, the substitution $t_0 = t/(LN^{\beta})$ and well known limit properties of the corresponding functions.

4. Appendix. The aim of this section is to give a proof of the introduced but unverified results. First, we prove some auxiliary assertions.

Lemma 2. *Let $K \subset E_n$ be a non-empty, compact convex set. Let, further, $\bar{h}(x)$ be a strongly convex with a parameter $\rho > 0$, continuous, real valued function defined on K. If $x_0 \in K$ is determined by the relation*

$$x_0 = \arg\min_{x \in K} \bar{h}(x),$$

then

$$\|x - x_0\|^2 \leqslant \left(\frac{2}{\rho}\right)[\bar{h}(x) - \bar{h}(x_0)],$$

for every $x \in K$.

Proof. We refer to the paper [14], for this proof. There namely, the proof of the corresponding assertion for concave functions is presented. Besides this, the assertion of Lemma 2 has already been introduced in [28], too.

If we denote by the symbol B_s the Borel σ-algebra in E_s then we can remember one well known inequality from the probability theory.

Lemma 3. *If*

1. *$\kappa(z)$ is a measurable (according to B_s) function defined on E_s such that there exists a constant $\overline{M}$ fulfilling the inequality $|\kappa(z)| \leqslant \overline{M}$ for all $z \in E_s$,*
2. *$\{\xi^k(\omega)\}_{k=1}^{\infty}$ is a sequence of independent random vectors,*

then

$$P\{\omega : \; E_N\kappa(\xi(\omega)) - E\kappa(\xi(\omega)) > t\} \leqslant \exp\Big\{-\frac{Nt^2}{2\overline{M}^2}\Big\},$$

for every $t \in E_1$, $t > 0$.

Proof. The inequality introduced in Lemma 3 has been first proved in [7].

Further, we present one result for ϕ-mixing random sequences.

Lemma 4. *If the assumption 1 of Lemma 3 is fulfilled and if $\{\xi^k(\omega)\}_{k=-\infty}^{\infty}$ is a strongly stationary random sequence fulfilling the ϕ-mixing condition, then*

$$P\{\omega : \; |E_N\kappa(\xi(\omega)) - E\kappa(\xi(\omega))| > t\} \leqslant \frac{2\overline{M}^2}{t^2N^2}\Big[N + \sum_{k=1}^{N-1}(N-k)\phi(k)\Big].$$

Proof. First, it follows from Lemma 2, Chapter 4 of [1]

$$|E[\kappa(\xi^k(\omega)) - E\kappa(\xi(\omega))][\kappa(\xi^r(\omega)) - E\kappa(\xi(\omega))]| \leqslant 2\overline{M}^2\phi(|r-k|) \quad (28)$$

for $r \neq k$, $r, k = \ldots -2, -1, 0, 1, 2, \ldots$.

Since it follows from Chebyshev's inequality that

$$P\{\omega : \; |E_N\kappa(\xi(\omega)) - E\kappa(\xi(\omega))| > t\} \leqslant \frac{1}{t^2N^2}E\Big|\sum_{k=1}^{N}[\kappa(\xi^i(\omega) - E\kappa(\xi(\omega))]\Big|^2$$

and since

$$E[\kappa(\xi^i(\omega) - E\kappa(\xi(\omega))] = 0 \quad \text{for every } i = \ldots, -2, -1, 0, 1, 2, \ldots,$$

we obtain the assertion of Lemma 4 immediately from the relation (28).

Now, we can already present the proof of Theorem 2.

Proof of Theorem 2. Let $t > 0$ be arbitrary given. Since it follows from the relation (12) that

$$|E_N\bar{g}(x, \xi(\omega)) - E\bar{g}(x, \xi(\omega))| \leqslant \sum_{i=1}^{s_1} M_1|E_Ng_i^*(\xi(\omega)) - Eg_i^*(\xi(\omega))|,$$

we can obtain successfully further

$$
\begin{aligned}
P\Big\{\omega : \Big|\inf_{x\in X} E_N\bar{g}(x,\xi(\omega)) - \inf_{x\in X} E\bar{g}(x,\xi(\omega))\Big| > t\Big\} \\
\leqslant P\{\omega : |E_N\bar{g}(x,\xi(\omega)) - E\bar{g}(x,\xi(\omega))| > t \text{ for at least one } x \in X\} \\
\leqslant \sum_{i=1}^{s_1} P\Big\{\omega : |E_N g_i^*(\xi(\omega)) - E g_i^*(\xi(\omega))| > \frac{t}{M_1 s_1}\Big\}.
\end{aligned}
\tag{29}
$$

Employing now Lemma 3 and Lemma 4 we obtain immediately the validity of the assertion (13) and (14).

It remains to prove the validity of the relations (15), (16). However, evidently if the assumptions 3, 4, 5 of Theorem 2 are satisfied then $E\bar{g}(x,\xi(\omega))$ is a strongly convex with the parameter ρ function. So, according to Lemma 2 it is

$$\|\bar{x}_N(\omega) - \bar{x}\|^2 \leqslant \Big(\frac{2}{\rho}\Big)\big|E\bar{g}(\bar{x}_N(\omega),\xi(\omega)) - E\bar{g}(\bar{x},\xi(\omega))\big|,$$

for all $\omega \in \Omega$, $N = 1,2,\dots$ and $E\bar{g}(\bar{x}_N(\omega),\xi(\omega)) = \big[E\bar{g}(x,\xi(\omega))\big]_{x=\bar{x}_N(\omega)}$.

Employing the triangular inequality we get

$$
\begin{aligned}
\|\bar{x}_N(\omega) - \bar{x}\|^2 \leqslant \Big(\frac{2}{\rho}\Big)\{&|E\bar{g}(\bar{x}_N(\omega),\xi(\omega)) - E_N\bar{g}(\bar{x}_N(\omega),\xi(\omega))| \\
&+ |E_N\bar{g}(\bar{x}_N(\omega),\xi(\omega)) - E\bar{g}(\bar{x},\xi(\omega))|\},
\end{aligned}
$$

for all $\omega \in \Omega$, $N = 1,2,\dots$ and so also

$$
\begin{aligned}
&P\{\omega : \|\bar{x}_N(\omega) - \bar{x}\|^2 > t\cdot\Big(\frac{2}{\rho}\Big)\} \\
&\quad\leqslant P\{\omega : |E\bar{g}(x,\xi(\omega)) - E_N\bar{g}(x,\xi(\omega))| > \Big(\frac{t}{2}\Big) \text{ for at least one } x \in X\} \\
&\quad+ P\Big\{\omega : \Big|\inf_{x\in X} E_N\bar{g}(x,\xi(\omega)) - \inf_{x\in X} E\bar{g}(x,\xi(\omega))\Big| > \Big(\frac{t}{2}\Big)\Big\}.
\end{aligned}
$$

Now already we obtain the validity of the relations (15), (16) on the basis of the inequalities given by the relations (13), (14), (29), Lemma 3 and Lemma 4. By this we have finished the proof of Theorem 2.

Theorems 4 and 5 generalize the results of Theorem 2 to rather great class of the optimalized functions. Of course, the achieved upper bound is higher. Namely, there appears the factor $m(X,(t/3))$

in the relations (17), (18), (19), (20). We shall present here the proof of the new part of the assertion of Theorem 4.

Proof of Theorem 4. The assertion given by the relations (17) follows immediately from the results of the paper [10] (Theorem 2 and its proof). So it remains to prove the assertion given by the relation (18). Since $E\bar{g}(x, \xi(\omega))$ is a strongly convex with the parameter $\rho > 0$ function on $X[d]$ we can apply the idea of the second part proof of Theorem 2 to get successfully

$$\begin{aligned} \|\bar{x}_N(\omega) - \bar{x}\|^2 &\leqslant \Big(\frac{2}{\rho}\Big) \big|E\bar{g}(\bar{x}_N(\omega), \xi(\omega)) - E\bar{g}(\bar{x}, \xi(\omega))\big| \\ &\leqslant \Big(\frac{2}{\rho}\Big) \big\{ \big|E\bar{g}(\bar{x}_N(\omega), \xi(\omega)) - E_N\bar{g}(\bar{x}_N(\omega), \xi(\omega))\big| \\ &\quad + \big|E_N\bar{g}(\bar{x}_N(\omega), \xi(\omega)) - E\bar{g}(\bar{x}, \xi(\omega))\big| \big\}, \end{aligned}$$

for all $\omega \in \Omega$, $N = 1, 2, \ldots$ and $E\bar{g}(\bar{x}_N(\omega), \xi(\omega)) = \big[E\bar{g}(x, \xi(\omega))\big]_{x=\bar{x}_N(\omega)}$.

However, now we can already see that the validity of (18) follows immediately from the last inequalities system and from the relations (17).

Proof of Theorem 5. As the proof of Theorem 5 is very similar to the proof of Theorem 4, we omit it. It is necessary to employ there the results of the paper [11] (Theorem 2 and the corresponding proof) and Lemma 4 instead of the results of the paper [10] and Lemma 3.

We have finished the proof of the assertions corresponding to the deterministic equivalent I. Now we shall deal with the assertions belonging to the deterministic equivalent II.

Proof of Theorem 7. The proof of the relation (24) is given in [13]. Since this proof is rather complicated and long we shall not repeat it here. However we shall verify carefully the validity of the relation (25). For this let $t > 0$ fulfil the assumptions of Theorem 7. We define the set Ω_t by the relation

$$\Omega_t = \{\omega \in \Omega : \ X_N(\alpha) \subset X(\alpha - t)\}.$$

The following auxiliary assertion follows from [13].

Lemma 5. *Let $\alpha \in (0,1)$, $\delta > 0$. If the assumptions* i, ii *are fulfilled and if $\{\xi^k(\omega)\}_{k=1}^{\infty}$ is a sequence of independent random vectors then for $t > 0$, $d > 0$ such that $d < \delta$,*

$$\frac{\sqrt{n}}{\lambda_1}\sqrt{\frac{2t}{\vartheta_1}} < \delta, \qquad \vartheta_2\lambda_2 d\sum_{i=1}^{\ell}\prod_{v\neq i} c_v < \frac{t}{3},$$

then

$$P\{\omega:\ X(\alpha+t)\subset X_N(\alpha)\subset X(\alpha-t)\} \geqslant 1-2m[X(\alpha,2\delta),d]\exp\{-Nt^2/18\}.$$

Proof. It is proved in [13] (Lemma 5) that under our assumptions

$$P\{\omega:\ X(\alpha+t)\subset X_N(\alpha)\subset X(\alpha-t)\} \geqslant 1-\sum_{x^v\in S(X(\alpha,2\delta),d)} P\{\omega:\ |P_N(Z(x^v)-P(Z(x^v)|>\frac{t}{3}\}.$$

However now already the assertion of Lemma 5 follows immediately from the last inequality, Lemma 3 and the definition of the number $\overline{m}(X(\alpha,2\delta),d)$.

We can continue in the proof of Theorem 7.

According to Lemma 5 we obtain that

$$P\{\omega:\ \omega\in\Omega-\Omega_t\}\leqslant 2m[X(\alpha,2\delta),d]\exp\{-Nt^2/18\},$$

and so also

$$\begin{aligned} P\Big\{\omega:\ ||x_N(\alpha,\omega)-x(\alpha)||^2\geqslant 4\frac{t_0L}{\rho}\Big\} &\leqslant 2m[X(\alpha,2\delta),d]\exp\{-Nt^2/18\} \\ &\quad + P\Big\{\Big[\omega:\ ||x_N(\alpha,\omega)-x(\alpha)||^2\geqslant 4\frac{t_0L}{\rho}\Big]\cap\Omega_1\Big\}. \end{aligned} \tag{30}$$

Since

$$||x_N(\alpha,\omega)-x(\alpha)||^2\leqslant 2\{||x_N(\alpha,\omega)-x(\alpha-t)||^2+||x(\alpha)-x(\alpha-t)||^2\}$$

and since $Eg(x, \xi(\omega))$ is a strongly convex with the parameter $\rho > 0$ function and $X(\alpha)$ is a convex set [19], we obtain for $\omega \in \Omega_t$ successively

$$\begin{aligned} \|x_N(\alpha,\omega) - x(\alpha)\|^2 \leqslant &(\frac{4}{\rho})\{|Eg(x_N(\alpha,\omega),\xi(\omega)) - Eg(x(\alpha - t),\xi(\omega))| \\ &+ |Eg(x(\alpha),\xi(\omega)) - Eg(x(\alpha - t),\xi(\omega))|\} \\ \leqslant &(\frac{4}{\rho})\{|Eg(x_N(\alpha,\omega),\xi(\omega)) - E_N g(x_N(\alpha - t),\xi(\omega))| \\ &+ |E_N g(x_N(\alpha,\omega),\xi(\omega)) - Eg(x(\alpha)\xi(\omega))| \\ &+ |Eg(x(\alpha),\xi(\omega)) - Eg(x(\alpha - t),\xi(\omega))| \\ &+ |Eg(x(\alpha),\xi(\omega)) - Eg(x(\alpha - t),\xi(\omega))|\}, \end{aligned} \tag{31}$$

where $Eg(x_N(\alpha,\omega),\ \xi(\omega)) = [Eg(x,\xi(\omega))]_{x=x_N(\alpha,\omega)}$.

The next auxiliary assertion was proved in [13] too.

Lemma 6. *Let* $\alpha \in (0,1)$. *If the assumptions* i, ii *are fulfilled then for* $t > 0$ *the inequality*

$$\Delta[X(\alpha), X(\alpha - t)] < \frac{\sqrt{n}}{\lambda_1}\sqrt[\ell]{(\frac{2t}{\vartheta_1})}$$

holds.

However according to this assertion and to the relations (24), (31) it is easy to see that the relation (25) will be proved if we verify the relation

$$\begin{aligned} P\Big\{\omega \in \Omega_t :\ &|Eg(x_N(\alpha,\omega),\xi(\omega)) - E_N g(x_N(\alpha,\omega),\xi(\omega))| \\ &+ |E_N g(x_N(\alpha,\omega),\xi(\omega)) - Eg(x(\alpha),\xi(\omega))| \geqslant (\frac{t_0 L}{2})\Big\} \\ \leqslant &4m[X(\alpha,2\delta),d]\exp[-Nt^2/18] \\ &+ 4m[X(\alpha,2\delta),d]\exp\{-Nt_0^2L^2/4\cdot 18M^2\}. \end{aligned} \tag{32}$$

However as

$$\begin{aligned} P\Big\{\omega \in \Omega_t :\ &|Eg(x_N(\alpha,\omega),\xi(\omega)) - E_N g(x_N(\alpha,\omega),\xi(\omega))| \geqslant (\frac{t_0 L}{4})\Big\} \\ \leqslant P\Big\{\omega :\ &|Eg(x_N(\alpha,\omega),\xi(\omega)) - E_N g(x_N(\alpha,\omega),\xi(\omega))| \geqslant (\frac{t_0 L}{4}) \\ &\text{for at least one } x \in X(\alpha,2\delta)\Big\}, \end{aligned}$$

and as $X(\alpha, 2\delta)$ is a compact set we see that the validity of the relation (32) follows immediately by Theorem 4 and the inequality (24).

Theorem 7 deal with the case of independent random samples. The ϕ-mixing case is considered in Theorem 8. Since the proof of Theorem 8 is very similar to the proof of Theorem 7 we omit it. We remember here only that instead of the results of Theorem 4 in this case the results of Theorem 5 is employed.

REMARK. A proof of measurability of the random vectors $\bar{x}_N(\omega)$, $x_N(\alpha, \omega)$ is omited in this paper. But it follows from the paper [30].

5. Conclusion. The presented paper have dealt with convergence rate of the empirical estimates in stochastic programming problems. Former results on this topic are improved.

It is seen that the interval for β fulfiling the relations (5), (6), (7), (8) are greater in simplest case, of course. Especially this interval is rather smaller in the case of deterministic equivalent II. This reality is evidently caused by new inaccuracy that arised by the approximation of the constraints set $X(\alpha)$. We can recognize this following the proof of the corresponding results and proofs introduced in [13]. However this question will not be discussed more in this paper.

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