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Comparison of H-Logical Norm with some t-Norms

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Abstract. The paper presents the comparison of fuzzy conclusions derived from the use of t-norms with the fuzzy conclusions derived from the use of H-norm. The idea was to examine the application of H-logical norm to fuzzy reasoning, which is not monotonious while it is known that t-norms used in fuzzy reasoning have the characteristic of monotonicity. The comparison of fuzzy conclusions was performed by the use of coherence measure. The research was carried out on a relatively small number of examples (nine) and it was shown that the fuzzy conclusions derived from the use of H-norm were the closest to the fuzzy conclusions derived from T_L-norm.

Key words: fuzzy reasoning, fuzzy conclusion, t-norms, H-norm, coherence measure, comparison of fuzzy conclusions.

1. Introduction

Working on the theory of probabilistic metric spaces and using Menger (1942) as a source, Schweitzer and Sklar (1958; 1960) introduced a class of real binary operations which they named triangular norms (t-norms). Today, there are different families of t-norms which are used in fuzzy logic, neuro-fuzzy systems, games theory, information theory, and in other fields related to the measure theory.

In the fuzzy sets theory, t-norms have a significant place since they provide models for intersection and union operations on fuzzy sets. The laws of commutativity, associativity and monotonicity as well as the boundary conditions are satisfied by t-norms in the interval [0, 1]. There are a great number of t-norms known today, and the best known ones are applied in this paper (Pedrycz and Gomide, 1998):

1) $T_M(x,y) = \min(x,y)$;
2) $T_P(x,y) = xy$;
3) $T_L(x,y) = \max(0, x+y-1)$;

and their dual t-co-norms:

1) $S_M(x,y) = \min(x,y)$;
2) $S_P(x,y) = x+y-xy$;
3) $S_L(x,y) = \min(1, x+y)$.

The basic idea of the paper is to compare fuzzy conclusions derived by applying t-norms with fuzzy conclusions derived by applying H-norm. H-norm was introduced by Professor Petar Hotomski and it is described in Section 2. It is a rational Boolean norm [0, 1], which satisfies all laws of Boolean logic, even those which t-norms do not satisfy. The H-norm possesses the properties of commutativity, associativity and the boundary conditions, the same as the t-norms. However, H-norm is not monotonous, so it is interesting to compare the fuzzy conclusions obtained by applying this norm to the fuzzy conclusions obtained by applying t-norms.

In Section 3 the cutting method is described, according to (Jager, 1995), which gives a mechanism for deriving fuzzy conclusions from fuzzy rule base. There are nine examples of fuzzy rule base, which vary in number and complexity of fuzzy rules.

The comparison of fuzzy conclusions, obtained by the cutting method for the H-norm and the t-norms, is performed by using coherence measure which is based on the maximum distance between the elements of the two fuzzy sets. A short review of the characteristics and methods of construction of coherence measure is given in Section 4.

In Section 5, the comparison between fuzzy conclusions derived by the use of the H-norm with the fuzzy conclusions derived by the use of t-norms is described, as well as the order between average measures of similarity for every two norms. The average measure of similarity for every norm with other norms is also given.

The paper ends with the conclusion where the obtained results are analyzed.

2. *H*-Norm

The definition and the characteristics of H-norm appears according to (Hotomski, 1975; Hotomski and Radojević, 2002).

DEFINITION 1. Let $f(z_1, z_2, \ldots, z_k) = 2^{k-1}z_1 + 2^{k-2}z_2 + \cdots + 2^0 z_k$, $z_i \in \{0, 1\} = L_2$, $*$ be one of the operators AND, OR and $x = f(a_1, a_2, \ldots, a_k)$, $y = f(b_1, b_2, \ldots, b_k)$, $x, y \in L_2^k$.

$$x*y =^{\text{def}}= f\,(a_1 * b_1, a_2 * b_2, \ldots, a_k * b_k)\,,$$
$$\neg\, x =^{\text{def}}= f\,(\neg\, a_1, \neg\, a_2, \ldots, \neg\, a_k)\,. \tag{1}$$

For these operations L_2^k is regular Boolean algebra with the first element 0 and the last $(2^k - 1)$.

For each propositional formula, $F(p_1, p_2, \ldots, p_n)$, a term $F(k_1, k_2, \ldots, k_n)$ is introduced in which $k_i = I_n/(I_{n-i} + 2)$, $I_n = 2^s - 1$, $s = 2^n$, $i = 1, 2, \ldots, n$ and it is shown that the following applies:

$$F(p_1, p_2, \ldots, p_n) \text{ is tautology if and only if } F(k_1, k_2, \ldots, k_n) = I_n.$$

Additionally, the following holds for the operations on $\{0, 1, 2, \ldots, I_n\}$:

$$\neg\, x = I_n - x; \quad x \vee y = x + y - x \wedge y = x + \neg\, x \wedge y; \quad x \Rightarrow y = \neg\, x + x \wedge y;$$
$$x \Leftrightarrow y = \neg\, x \wedge \neg\, y + x \wedge y = \neg\,(x + y) + 2(x \wedge y),$$

where $+$, $-$ are ordinary arithmetic operations.

Dividing every element of L_2^s, $s = 2^n$ by I_n leads to the set $\{0, 1/I_n, 2/I_n, \ldots, 1\}$ which contains $I_n + 1$ element. By increasing the value of n, sufficiently fine segmentation of [0, 1] may be obtained.

Definition and characteristics of H-norm

Let $\{0, 1/I_n, 2/I_n, \ldots, (I_n - 1)/I_n, 1\}$ be a segmentation of the interval [0, 1], where $I_n = 2^s - 1$, $s = 2^n$. Let $x, y \in [0, 1]$ such that $x, y \in \{0, 1/I_n, 2/I_n, \ldots, (I_n - 1)/I_n, 1\}$. Then:

$$phc_n(x, y) \overset{\text{def}}{=} (xI_n \wedge yI_n)/I_n, \tag{2}$$

where $\wedge$ is AND operator on $\{0, 1, 2, \ldots, I_n\}$ determined by Definition 1.

Lemma 1. *If there exists n, so that $x, y \in \{0, 1/I_n, 2/I_n, \ldots, (I_n - 1)/I_n, 1\}$, then $phc_n(x, y) = phc_{n+1}(x, y)$, i.e., phc is independent of n.*

Lemma 2. *There exists a rational number $x \in [0, 1]$, for which there is no segmentation, so that xI_n is a natural number.*

Consequence

For each segmentation of the interval [0, 1] determined by $\{0,\ 1/I_n,\ 2/I_n, \ldots,$ $(I_n - 1)/I_n, 1\}$ and $x \in [0, 1]$ it is true that $k/I_n \leqslant x \leqslant (k + 1)/I_n$, i.e., $k \leqslant x \cdot I_n \leqslant (k + 1)$, $k \in \{0, 1, 2, \ldots, I_n - 1\}$.

The function round $(x \cdot I_n) = k_1$ performs rounding of the arguments respecting the evenness rule, so k_1 is a natural number k or $k + 1$. For $xI_n = k + 0.5$, the value of k_1 is equal either to k (if k is an even number) or to $k + 1$ (if k is an odd number).

Since $x_1 = k_1/I_n = \text{round}(x \cdot I_n)/I_n = (x \cdot I_n + R)/I_n = x + R/I_n$, $|R| \leqslant 0.5$, $|\ x_1 - x| \leqslant 1/(2I_n)$, the upper error margin when substituting x by x_1 is $1/(2I_n)$. We name $1/(2I_n)$ *sensitivity coefficient* on an integer k. By proper selection of n, the sensitivity coefficient may be decreased to the required value. For $n = 4$, I_4 is equal to 65535, so that the sensitivity coefficient becomes 0.000008, the value that completely satisfies practical requirements since it provides the accuracy up to five decimals for $x \in [0, 1]$.

Now, $phc(x, y)$ can be defined for any $x, y \in [0, 1]$.

DEFINITION 2. H-norm and H-co-norm are defined by the following:

$$phc(x, y) \overset{\text{def}}{=} (\text{round}\,(xI_n) \wedge \text{round}\,(yI_n))\,/I_n, \tag{3}$$

where $\wedge$ is AND operator on the set $\{0, 1, 2, \ldots, I_n\}$, $I_n = 2^s - 1$, $s = 2^n$ defined by (1), while round $(xI_n) = x_1 I_n = k_1 \in \{0, 1, 2, \ldots, I_n\}$ provides the substitution of argument x by x_1 which is accurate up to $1/(2I_n)$.

$$phd(x, y) \stackrel{\text{def}}{=} (\text{round}\,(xI_n) \vee \text{round}\,(yI_n))\,/I_n, \tag{4}$$

where $\vee$ is OR operator on the set $\{0, 1, 2, \ldots, I_n\}$, $I_n = 2^s - 1$, $s = 2^n$ defined by (1).

H-norm $phc(x, y)$ and H-co-norm $phd(x, y)$ define operators AND and OR, whereby all the Boolean properties of these operators are preserved.

The following properties are directly verified for $x \approx x_1$ and $y \approx y_1$ within the limits constituted by the sensitivity coefficient $1/(2I_n)$:

1) $phd(x, y) = x + y - phc(x, y)$,
2) $phc(x, x) = x$,
3) $phc(0, y) = 0$,
4) $phc(1, y) = y$,
5) $phc(x, y)$ is not monotone,
6) $phc(x, y)$ is extremely sensitive to the change of x, y, i.e., a small modification of arguments significantly changes the value of phc (because of the property of $\wedge$ on the set $\{0, 1, 2, \ldots, I_n\}$).

Other logical operations on [0, 1] can be analogously defined.

Negation is defined as:

$$\neg\, x \stackrel{\text{def}}{=} \neg\, \text{round}\,(xI_n)/I_n = \neg\,(x_1 I_n)/I_n = (I_n - x_1 I_n)/I_n = 1 - x_1.$$

Within the limits of the sensitivity coefficient, for $x \approx x_1$ it is true that $\neg\, x = 1 - x$; $phc(x, \neg\, x) = 0$; $phd(x, \neg\, x) = 1$. The Boolean H-norm is the natural generalization of the logical conjunction of the set $\{0, 1\}$ to the set $\{0, 1, 2, \ldots, I_n\}$, $I_n = 2^s$, $s = 2^n$, i.e., to [0, 1]. The monotonisity is absent here, but the characteristics of AND operator are kept in the regular Boolean algebra. Fig. 1 presents a graphic description of H-norm.

3. Deriving Fuzzy Conclusions

The fuzzy propositions are the basic elements of fuzzy logic and fuzzy reasoning. Logic junctions AND and OR are used to combine fuzzy propositions. These logic junctions are respectively interpreted as t-norms and t-co-norms. There is no general rule for choosing t-norm and t-co-norm in the process of fuzzy reasoning, since the choice depends on the domain of application. T_M-norm, proposed by Zadeh, enables getting the same information by combining the two identical fuzzy propositions, which will not be the case with application of other t-norms. If the fuzzy propositions are not the same but joined and if they affect each other, other t-norms should be applied.

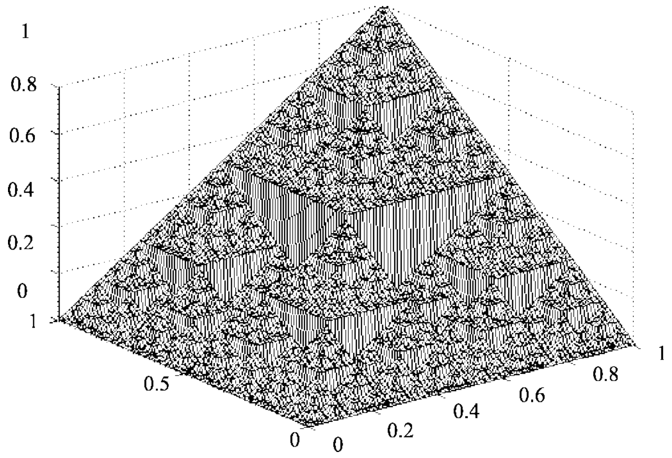

Fig. 1. Visualization of H-norm (Hotomski, 2002).

The fuzzy rules are "*if-then*" statements, the premises and consequences of which consist of fuzzy propositions. Dubois and Prade (1991) made a summary of different types of fuzzy implications. One of them, the T-implication, is discussed in this paper. T-implication is an implication interpreted as a t-norm, and is usually used in fuzzy control (Jager, 1995).

The cutting method is used in the paper to derive fuzzy conclusion for the given fuzzy rule or the fuzzy rule base.

The cutting method is one of the methods of deriving fuzzy conclusion for the given fuzzy rule, or for the given fuzzy rule base (Jager, 1995):

Let the given base of the Nr fuzzy rules and let r_k be the rule:

$$r_k : \text{ If } x_1 \text{ is } A_{1,k} \text{ AND } x_2 \text{ is } A_{2,k}, \text{ then } y \text{ is } B_k,$$

and let the initial facts be A1' and A2'.

By the cutting method, the degree of matching $\alpha_{i,k}$ of the initial fact A'_i and the fuzzy set $A_{i,k}$ is defined as:

$$\alpha_{i,k} = hgt(Ai' \cap A_{i,k}),$$

where the intersection of fuzzy sets is interpreted by t-norm. For the given example, the degree of matching α_k of the fuzzy rule r_k with the initial facts is calculated by application of the t-norm to the degrees of matching $\alpha_{i,k}$. If in the fuzzy rule r_k, the fuzzy propositions are joined with the junction OR, then the degree of matching α_k is calculated by application of t-co-norm to the degrees of matching $\alpha_{i,k}$.

Deriving the fuzzy conclusion B'_k for fuzzy rule r_k is defined as:

$$B'_k = T(\alpha_k, B_k),$$

where T is t-norm.

The final fuzzy conclusion B is derived by aggregation operator for the fuzzy rule base. The combination of fuzzy rule base in a single fuzzy relation is called aggregation. The aggregation operator is interpreted by disjunction, which is most frequently represented by max operator (as in this paper), but it is generally represented by t-co-norm.

The local inference is used in the paper in order to derive fuzzy conclusion from the fuzzy rule base. Apparently, the inference of each individual fuzzy rule is done first and after that the aggregation of the derived fuzzy conclusions into one final conclusion. By global inference the fuzzy rule base, presented by fuzzy relations, is first transformed into one fuzzy relation by use of aggregation operator. If T-implications are used, there is no difference in the results obtained by local and global inference, because disjunction is used as aggregation operator. If fuzzy implications based on classic implications are used, differences in results occur. Local inference gives fuzzy conclusions which carry less information than the fuzzy conclusions derived by global inference.

Deriving fuzzy conclusions from the fuzzy rule base by the use of H-norm and the above-mentioned t-norms has been performed for nine examples (Nikolic, 2002). In the first example, the fuzzy rule base is simple (one initial fact and one fuzzy rule), and in each following example the fuzzy rule base is more complex. Table 1 reviews these nine examples. The fuzzy sets A_i and B_i are defined on the set $X = \{x_1, x_2, x_3\}$ and are given in the third column of Table 1.

Note. The given fuzzy sets are $A(a_1,\, a_2,\, a_3)$ and $B(b_1,\, b_2,\, b_3)$. It will be considered that the fuzzy set A is "*wider*" than the fuzzy set B if it is $a_i \geqslant b_i$ for every $i \in \{1, 2, 3\}$ (if for every $i \in \{1, 2, 3\}$ refers to the equality, then the fuzzy sets A and B are equal). The fuzzy set B will be considered "*narrower*" than the fuzzy set A.

Table 2 reviews the fuzzy conclusions derived from the use of T_M, T_P, T_L and H-norm. It can be noticed that the fuzzy conclusions derived from the use of T_M-norm are wider than the fuzzy conclusions derived for T_P-norm, whereas the fuzzy conclusions derived from the use of T_P-norm are wider than the fuzzy conclusions derived from the use of T_L-norm. The fuzzy conclusions derived from H-norm are narrower than the ones derived from the use of T_M-norm and wider than fuzzy conclusions derived from the use of T_L-norms. When compared to the fuzzy conclusions derived from the use of T_P-norm, the fuzzy conclusions within H-norm can be wider (2, 4, 5, 6, 7, 9) or narrower (1, 3, 8).

4. Comparison of Finite Fuzzy Sets by Use of Coherence Measure

To define the similarities between the two finite fuzzy sets, a coherence measure was used, based on the maximum distance between the elements of the two fuzzy sets. The summary of the definitions and the characteristics of coherence measures is given below (Royo and Verdegay, 2000).

DEFINITION 3. Let $X = \{x_1, \ldots, x_m\}$ be a finite set and $P^f(X)$, the set of fuzzy sets on X, we say that $\text{cohe}\colon\ P^f(X) \times P^f(X) \to [0, 1]$ is a *coherence measure* on $P^f(X)$ iff holds:

Table 1
Fuzzy rule base

Ordinal number	Fuzzy rule base	Fuzzy sets
1	Initial fact: A0 Rule 1: A1→ B1 Fuzzy conclusion: B.	A0(0.6,1,0.3) A1(1,0.7,0.5) B1(0.3,0.8,1)
2	Initial fact: A0 Rule 1: notA1→ B1 Fuzzy conclusion: B.	A0(0.1,0.2,0.3) A1(0.4,0.3,0.5) B1(0.7,0.2,0.1)
3	Initial facts: A0 B0 Rule 1: A1 AND A2 → B1 Fuzzy conclusion: B	A0(0.1,0.2,0.3) A1(0.4,0.3,0.5) B0(1,0.5,0) A2(0.2,0.4,0.8) B1(0.7,0.2,0.1)
4	Initial facts: A0 B0 Rule 1: notA1 AND A2 → B1 Fuzzy conclusion: B	A0(0.1,0.2,0.3) A1(0.4,0.3,0.5) B0(1,0.5,0) A2(0.2,0.4,0.8) B1(0.7,0.2,0.1)
5	Initial facts: A0 Rule 1: A1→ B1 Rule 2: A2→ B2 Fuzzy conclusion: B	A0(0.6,1,0.3) A1(1,0.7,0.5) B1(0.3,0.8,1) A2(0.1,0.2,0.3) B2(0.4,0.5,0.6)
6	Initial facts: A0 Rule 1: A1→ B1 Rule 2: A2→ B2 Rule 3: A3→ B3 Fuzzy conclusion: B	A0(0.6,1,0.3) A1(1,0.7,0.5) B1(0.3,0.8,1) A2(0.1,0.2,0.3) B2(0.4,0.5,0.6) A3(0.7,0.3,0.5) B3(0.4,0.6,0.1)
7	Initial facts:A0 B0 Rule 1: notA1 → B1 Rule 2: A1 AND A2 → B2 Rule 3: A3 → B3 Fuzzy conclusion: B	A0(0.5,0.7,1) A1(0.6,0.4,0.5) B1(1,0.2,0.4) B0(0.4,0.7,0.2) A2(0.9,0.8,0.3) B2(1,0.5,0.8) A3(0.5,0.2,0.9) B3(0.4,1,0.8)
8	Initial facts: A0 B0 Rule 1: A1 OR notA2 → B1 Rule 2: notA1 AND A3 → B2 Fuzzy conclusion: B	A0(0.5,0.7,1) A1(0.6,0.4,0.5) B0(0.3,1,0.9) A2(0.9,0.8,0.3) B1(1,0.2,0.4) B2(1,0.5,0.8) A3(0.5,0.2,0.9)
9	Initial facts: A0 Rule 1: A1 → B1 Rule 2: A2 → B2 Rule 3: A3 → B3 Rule 4: A4 → B4 Fuzzy conclusion: B	A0(0.6,1,0.3) A1(1,0.7,0.5) B1(0.3,0.8,1) A2(0.4,0.6,0.7) B2(0.8,0.5,0.9) A3(0.1,0.8,1) B3(0.8,0.2,0.3) A4(0.9,0.2,0.9) B4(0.5,0.6,0.7)

Table 2

Fuzzy conclusions derived from the fuzzy rule base

Example	T_M norm	T_P norm	T_L norm	H norm
1	B (0.3, 0.7, 0.7)	B(0.21, 0.56, 0.7)	B(0, 0.5, 0.7)	B(0, 0.5, 0.7)
2	B (0.3, 0.2, 0.1)	B(0.105, 0.03, 0.015)	B(0, 0, 0)	B(0.2, 0.2, 0.067)
3	B (0.3, 0.2, 0.1)	B(0.021, 0.006, 0.003)	B(0, 0, 0)	B(0, 0, 0)
4	B (0.3, 0.2, 0.1)	B(0.021, 0.006, 0.003)	B(0, 0, 0)	B(0.2, 0.2, 0.067)
5	B (0.3, 0.7, 0.7)	B(0.21, 0.56, 0.7)	B(0, 0.5, 0.7)	B(0.267, 0.5, 0.7)
6	B (0.4, 0.7, 0.7)	B(0.21, 0.56, 0.7)	B(0, 0.5, 0.7)	B(0.267, 0.566, 0.7)
7	B (0.6, 0.9, 0.8)	B(0.5, 0.9, 0.72)	B(0.5, 0.9, 0.7)	B(0.567, 0.9, 0.767)
8	B (0.7, 0.5, 0.6)	B(0.815, 0.415, 0.515)	B(1, 0.2, 0.4)	B(0.633, 0.5, 0.501)
9	B (0.8, 0.7, 0.7)	B(0.64, 0.56, 0.7)	B(0.6, 0.5, 0.7)	B(0.8, 0.5, 0.7)

C1. $\text{cohe}\,(A, B) = \text{cohe}\,(B, A)$,
C2. $\text{cohe}\,(A, B^c) = 1 - \text{cohe}\,(A, B)$,
C3. $\text{cohe}\,(\varnothing, X) = 0$.

Lemma 3. *Let* $\text{cohe}: P^f(X) \times P^f(X) \to [0, 1]$ *be a coherence measure, then:*
a) $\text{cohe}\,(A^c, B^c) = \text{cohe}\,(A, B)$;
b) $\text{cohe}\,(\varnothing, \varnothing) = \text{cohe}\,(X, X) = 1$;
c) *If* $A^*(x) = 0.5 \forall x$, *then* $\forall A \in P^f(X) \text{cohe}\,(A, A^*) = 0.5$.
Coherence measures are not monotonous.

Lemma 4. *Let* $\text{cohe}: P^f(X) \times P^f(X) \to [0, 1]$ *be a coherence measure. Then, it is not true that:*
(1) $\forall A, B, C, D \in P^f(X)$

$$\left.\begin{matrix} A \subseteq B \\ C \subseteq D \end{matrix}\right\} \to \text{cohe}\,(A, C) \leqslant \text{cohe}\,(B, D)$$

equally, it is not true either that:
(2) $\forall A, B, C, D \in P^f(X)$

$$\left.\begin{matrix} A \subseteq B \\ C \subseteq D \end{matrix}\right\} \to \text{cohe}\,(A, C) \geqslant \text{cohe}\,(B, D).$$

The following lemma presents the method of coherence measure construction starting from metrics. Elements of $P^f(X)$ are mappings from X to [0, 1], so that it can be established directly isomorphism $P^f(X) \approx [0, 1]^m$.

Lemma 5. *Let X be a finite set of m elements, $P^f(X)$ the set of fuzzy subsets on X, let d: $P^f(X)xP^f(X) \to [0,1]$ be a bounded metric defined by:*

$$d(A,B) = \left(\sum_{i=1}^{m} h(a_i, b_i)\right)^{1/r}, \quad r \geqslant 1.$$

Then starting from d, a coherence measure can be constructed as:

$$\beta(A,B) = \frac{1 + d(A,B^c) - d(A,B)}{2}$$

if and only if:

a) $h(0,1) = (1/m)$,

b) $h(a, 1-b) = h(1-a, b) \;\; \forall a, b \in [0,1]$.

The previous lemma also shows the existence of coherence measures. By applying this lemma to r-metrics $(r \geqslant 1)$ on $P^f(X) \cong [0,1]^m$, various coherence measures can be obtained:

$$d(A,B) = \left(\frac{1}{m}\sum_{i=1}^{m} |a_i - b_i|^r\right)^{1/r}.$$

5. Comparison of Fuzzy Conclusions

For comparison of fuzzy conclusions derived from the fuzzy rule base given in the mentioned examples (Table 1), the coherence measure $\beta_{r\infty}$ derived from the metrics is used:

$$d(A,B) = \max_{1 \leqslant i \leqslant m} |a_i - b_i| \tag{5}$$

and the formula

$$\beta(A,B) = \frac{1 + d(A,B^c) - d(A,B)}{2}, \tag{6}$$

where

$$d(A,B^C) = \max_{1 \leqslant i \leqslant m} |a_i - (1 - b_i)| = \max_{1 \leqslant i \leqslant m} |a_i + b_i - 1|. \tag{7}$$

The coherence measure $\beta_{r\infty}$ is defined for every two corresponding fuzzy conclusions from Table 2, which is derived from different norms, and whose measures are shown in Table 3.

Table 3 presents means of derived coherence measures of fuzzy conclusions within every two norms, and average measures of similarity for every two norms $\bar{\beta}^{ij}_{r\infty}$. It is

Table 3

Coherence measures of fuzzy conclusions

	$T_M; T_P$ $\beta_{r\infty}^{MP}$	$T_M; T_L$ $\beta_{r\infty}^{ML}$	$T_M; H$ $\beta_{r\infty}^{MH}$	$T_P; T_L$ $\beta_{r\infty}^{PL}$	$T_P; H$ $\beta_{r\infty}^{PH}$	$T_L; H$ $\beta_{r\infty}^{LH}$
1	0.675	0.7	0.7	0.79	0.79	1
2	0.845	0.8	0.8665	0.94	0.874	0.8665
3	0.809	0.8	0.8	0.988	0.988	1
4	0.836	0.8	0.8865	0.952	0.868	0.8665
5	0.675	0.7	0.6165	0.79	0.7315	0.733
6	0.605	0.6	0.633	0.79	0.733	0.733
7	0.85	0.85	0.8835	0.89	0.8665	0.8665
8	0.7	0.7	0.617	0.8	0.633	0.633
9	0.64	0.6	0.7	0.67	0.64	0.6
$\bar{\beta}_{r\infty}^{ij}$	0.737	0.727	0.742	0.845	0.7915	0.8109

evident that inequality applies:

$$\bar{\beta}_{r\infty}^{PL} > \bar{\beta}_{r\infty}^{LH} > \bar{\beta}_{r\infty}^{PH} > \bar{\beta}_{r\infty}^{MH} > \bar{\beta}_{r\infty}^{MP} > \bar{\beta}_{r\infty}^{ML}. \tag{8}$$

Thus, the most similar are the fuzzy conclusions derived within T_P and T_L-norms, while within H-norm are found fuzzy conclusions which are most similar to fuzzy conclusions derived within T_L-norm, and then within T_P and T_M-norm. According to (8), there is a bigger similarity of fuzzy conclusions for H-norm to the fuzzy conclusions for all the three t-norms than the similarity of the fuzzy conclusions within T_M-norms to the fuzzy conclusions within T_P and T_L-norm.

Defining the average measure of similarity of the norm to other norms

At this point, the average measure of similarity of one norm to the other norms is defined:

1. Average measure of similarity of T_M-norm to T_P, T_L and H-norm:

$$\beta_{T_M} = \frac{\bar{\beta}_{r\infty}^{MP} + \bar{\beta}_{r\infty}^{ML} + \bar{\beta}_{r\infty}^{MH}}{3} = \frac{0.737 + 0.727 + 0.742}{3} = 0.735.$$

2. Average measure of similarity of T_P-norm to T_M, T_L and H-norm:

$$\beta_{T_P} = \frac{\bar{\beta}_{r\infty}^{MP} + \bar{\beta}_{r\infty}^{PL} + \bar{\beta}_{r\infty}^{PH}}{3} = \frac{0.737 + 0.845 + 0.7915}{3} = 0.791.$$

3. Average measure of similarity of T_L-norm to T_M, T_P and H-norm

$$\beta_{T_L} = \frac{\bar{\beta}_{r\infty}^{ML} + \bar{\beta}_{r\infty}^{PL} + \bar{\beta}_{r\infty}^{LH}}{3} = \frac{0.727 + 0.845 + 0.8109}{3} = 0.794.$$

4. Average measure of similarity of H-norm to T_M, T_P, T_L-norm

$$\beta_H = \frac{\bar{\beta}_{r\infty}^{MPh} + \bar{\beta}_{r\infty}^{PPh} + \bar{\beta}_{r\infty}^{LH}}{3} = \frac{0.742 + 0.7915 + 0.8109}{3} = 0.7808.$$

Consequently, the following order of average measures of similarity applies:

$$\beta_{T_M} < \beta_H < \beta_{T_P} < \beta_{T_L}. \tag{9}$$

The smallest average measure of similarity is β_{T_M} of the T_M-norm to other norms. The average measure of similarity of H-norm to other norms, β_H , is bigger than β_{T_M}. The biggest is the average measure of similarity β_{T_L} of T_L-norm to other norms.

6. Conclusion

The similarity of fuzzy conclusions derived by the use of H-norm with the fuzzy conclusions derived by use of t-norms is determined by the coherence measure of similarity. In inequality (8), the order between means of coherence measures is defined within each two norms. According to this inequality, the average measure of similarity of H-norm and T_L-norm is greater than the average measure of similarity of H-norm and T_P-norm, and the average measure of similarity of H-norm and T_P-norm is greater than the average measure of similarity of H-norm and T_M-norm. In other words, the fuzzy conclusions derived by use of H-norm are the most similar to the fuzzy conclusions derived from T_L-norm, then T_P-norm and the least similarity is with the fuzzy conclusions derived by the use of T_M-norm (Fig. 2).

The average measures of similarity of H-norm with every individual t-norm (T_P, T_L, T_M) are greater than the average measures of similarity of T_M-norm with T_P or T_L-norm (8). Thus, fuzzy conclusions obtained by H-norm are more similar to the fuzzy conclusions derived by the use of T_P-norm than the fuzzy conclusions obtained by the use of T_M-norm.

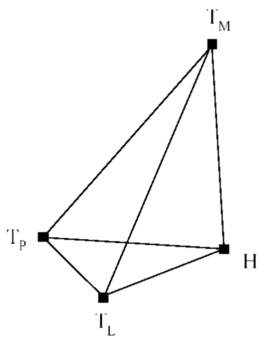

Fig. 2. Average measure of similarity for every two norms (Nikolić, 2002).

By forming the order between average measures of similarity of one norm with the other norms (9) it was established that the average measure of similarity of H-norm with T_P, T_L and T_M-norm is greater than the average measure of similarity of T_M-norm with T_P, T_L and H-norm.

It should be underlined here that the research was performed on a small number of examples, so that it has to be confirmed on a greater number of examples.

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H-loginių normų palyginimas su kai kuriomis *t*-normomis

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Straipsnyje pateiktas neryškiųjų (angl. fuzzy) išvadų, gautų naudojant t-normas ir H-normą, palyginimas. H-normą 1975 m. pasiūlė vienas iš straipsnio autorių P. Hotomski. H-norma yra nemonotoninė, o t-normos, naudojamos neryškiajame samprotavime, yra monotoninės. Neryškiosios išvados buvo palygintos naudojant koherentiškumo matą. Tyrimas atliktas naudojant palyginus mažai testinių pavyzdžių (devynis). Jis parodė, kad artimiausios neryškiosioms išvadoms, gautoms pagal H-normą, buvo neryškiosios išvados, gautos pagal TL-normą.