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A DFT-Based Algorithm for n-Order Singular State Space Systems

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Abstract. The discrete Fourier transform (DFT) is used for determining the coefficients of a transfer function for n-order singular linear systems, $\mathbf{E}x^{(n)} = \sum_{i=1}^{n} \mathbf{A}_i x^{(n-1)} + \mathbf{B}u$, where $\mathbf{E}$ may be singular. The algorithm is straight forward and easily can be implemented. Three step-by-step examples illustrating the application of the algorithm are presented.

Key words: n-order singular systems, generalized systems, state space, transfer function, Fourier transform.

1. Introduction

The study of sigular or generalized or semistate linear systems has experienced a great deal of interest in recent years. First-order generalized systems are applied in engineering as well as in biology, artificial neural networks and economics (Campbell, 1982; Dai, 1989; Luenberger and Arbel, 1977). Moreover second-order generalized systems, applied to power systems, were introduced by Campbell and Rose (1982). Also these systems were used to model flexible beams (Marszalek and Unbehauen, 1992).

The representation of a transfer function with a state space model is of considerable utility in system and control theory. This is due to the fact that knowing the state space representation of a system, various problems, such as feedback control, observers, LQG control, etc., can be studied using state space techniques (Dai, 1989; Paraskevopoulos *et al.*, 1984).

In this paper a new algorithm is presented for the computation of the transfer functions of generalized n-order systems, using the discrete Fourier transform (DFT). The DFT has been used for computing the transfer functions for regular, first and second order generalized systems (Lee, 1976; Antoniou *et al.*, 1989) and (Antoniou, 2001).

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2. Background

A n-order generalized linear system is described by the following state space equations:

$$\mathbf{E}x^{(n)} = \mathbf{A}_1 x^{(n-1)} + \cdots + \mathbf{A}_n x^{(0)} + \mathbf{B}u = \sum_{i=1}^{n} \mathbf{A}_i x^{(n-i)} + \mathbf{B}u, \tag{1}$$

$$y = \mathbf{C}x, \tag{2}$$

where $x^{(i)}$ denotes the ith derivative of x with respect to time, $x \in \mathcal{R}^\lambda$, $u \in \mathcal{R}^p$, $y \in \mathcal{R}^m$. Matrices $\mathbf{E}$, $\mathbf{A}_i$, $i = 1, \ldots, n$ are real with dimensions $\lambda \times \lambda$, and matrix $\mathbf{E}$ may be singular with rank μ. Applying the Laplace transform to (1, 2), with zero initial conditions, the corresponding transfer function is found to be

$$T(s) = \mathbf{C}(s^n\mathbf{E} - s^{n-1}\mathbf{A}_1 - \cdots - s\mathbf{A}_{n-1} - \mathbf{A}_n)^{-1}\mathbf{B}, \tag{3}$$

where, regularity is assumed, $\det[s^n\mathbf{E} - s^{n-1}\mathbf{A}_1 - \cdots - s\mathbf{A}_{n-1} - \mathbf{A}_n] \neq 0$.

In the following section an interpolative approach is developed for determining the transfer function $T(s)$, given the matrices $\mathbf{E}, \mathbf{A}_i, i = 1, \ldots, n,$ $\mathbf{B}$ and $\mathbf{C}$, using the DFT. For the sake of completeness a brief description of the DFT follows.

Given the finite sequences $X(k)$ and $\widetilde{X}(r)$, $k = 0, \ldots, N$, the following relationships are necessary in order for the sequences to constitute a DFT pair (Mitra, 2000):

$$\widetilde{X}(r) = \sum_{k=0}^{N} X(k) W^{-kr}, \tag{4}$$

$$X(k) = \frac{1}{(N+1)} \sum_{r=0}^{N} \widetilde{X}(r) W^{kr}, \tag{5}$$

where $r = 0, \ldots, N,\quad k = 0, \ldots, N$. $X = \quad \widetilde{X}$, are discrete argument matrix valued functions with dimensions $p \times m$, and

$$W = \mathrm{e}^{(2\pi j)/(N+1)}. \tag{6}$$

In the following section an interpolative approach is developed for determining the transfer function $T(s)$, given the matrices $\mathbf{E}, \mathbf{A}_i, i = 1, \ldots, n,\quad \mathbf{B}$ and $\mathbf{C}$, using the DFT.

3. Main Result – Algorithm

Let the transfer function, $T(s)$, of the n-order system be defined as

$$T(z, w) = \frac{\mathbf{N}(s)}{d(s)}, \tag{7}$$

where

$$\mathbf{N}(s) = \mathbf{C}\ \mathrm{adj}\ [s^n\mathbf{E} - s^{n-1}\mathbf{A}_1 - \cdots - \mathbf{A}_n]\mathbf{B}, \tag{8}$$

$$d(s) = \det\ [s^n\mathbf{E} - s^{n-1}\mathbf{A}_1 - \cdots - \mathbf{A}_n]. \tag{9}$$

Taking into consideration that the degree of the characteristic polynomial $\deg[d(s)] = \deg(\det\ [s^n\mathbf{E} - s^{n-1}\mathbf{A}_1 - \cdots - s\mathbf{A}_{n-1} - \mathbf{A}_n])\ \leqslant n \times \lambda - 1 = r$, Eqs. 8 and 9 can be written in polynomial form as follows:

$$\mathbf{N}(s) = \sum_{k=0}^{r} \mathbf{P}_k s^k, \tag{10}$$

$$d(s) = \sum_{k=0}^{r} q_k s^k, \tag{11}$$

where $\mathbf{P}_k$ are matrices with dimensions $(p \times m)$, while q_k are scalars.

The numerator polynomial matrix $\mathbf{N}(s)$ and the denominator polynomial $d(s)$ can be numerically computed at $(r+1)$ points, equally spaced on the unit circle,

$$v(i) = W^{-i}, \quad \forall i = 0, \ldots, r, \tag{12}$$

where

$$W = \mathrm{e}^{(2\pi j)/(r+1)}. \tag{13}$$

The values of the transfer function (7) at the $(r+1)$ points form its corresponding DFT coefficients.

3.1. *Denominator Polynomial*

To evaluate the denominator coefficients q_k, define

$$a_i = \det\,[\mathbf{E}v^n(i) - \mathbf{A}_1 v^{n-1}(i) - \cdots - \mathbf{A}_n]. \tag{14}$$

Using Eqs. 9 and 14, a_i can also be defined as

$$a_i = d[v(i)] \tag{15}$$

provided that at least one of $a_i \neq 0$.

Eqs. 11, 12 and 15 yield

$$a_i = \sum_{k=0}^{r} q_k W^{-ik}. \tag{16}$$

In the above Eq. 16, $[a_i]$ and $[q_k]$ form a DFT pair. Therefore the coefficients q_k can be computed using the inverse DFT, as follows:

$$q_k = \frac{1}{(r+1)} \sum_{i=0}^{r} a_i W^{ik}, \tag{17}$$

where $k = 0, \dots, r$.

3.2. *Numerator Polynomial*

To evaluate the numerator matrix polynomial $\mathbf{P}_k$, define

$$\mathbf{F}_i = \mathbf{C}\,\mathrm{adj}\,[\mathbf{E}v^n(i) - v^{n-1}(i)\mathbf{A}_1 - \cdots - \mathbf{A}_n]\mathbf{B}, \tag{18}$$

provided that at least one of $\mathbf{F}_i \neq 0$.

Using Eqs. 8 and 18, $\mathbf{F}_i$ can also be defined as

$$\mathbf{F}_i = \mathbf{N}[v(i)]. \tag{19}$$

Eqs. 10, 12 and 19 yield

$$\mathbf{F}_i = \sum_{k=0}^{r} \mathbf{P}_k W^{-ik}. \tag{20}$$

In the above Eq. 20, $[\mathbf{F}_i]$ and $[\mathbf{P}_k]$ form a DFT pair. Therefore the coefficients $\mathbf{P}_k$ can be computed using the inverse DFT, as follows:

$$\mathbf{P}_k = \frac{1}{(r+1)} \sum_{i=0}^{r} \mathbf{F}_i W^{ik}, \tag{21}$$

where $k = 0, \dots, r$.

Finally, the transfer function sought is

$$T(s) = \frac{\mathbf{N}(s)}{d(s)} = \frac{\sum_{k=0}^{r} \mathbf{P}_k s^k}{\sum_{k=0}^{r} q_k s^k}, \tag{22}$$

where $\mathbf{P}_k$ and q_k are given in (17) and (21).

Three salient examples, simple yet illustrative of the theoretical concepts presented in this work, follow below.

4. Examples

4.1. *Third Order System: Two-Inputs Two-Outputs*

Consider the following third-order generalized system:

$$\begin{aligned} &\mathbf{E}x^{(3)} = \mathbf{A}_1 x^{(2)} + \mathbf{A}_2 x^{(1)} + \mathbf{A}_3 + \mathbf{B}u, \\ &y = \mathbf{C}x, \end{aligned} \tag{23}$$

where

$$\mathbf{E} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

We would like to determine the transfer function for this system using the technique of Section 3. Note that the system satisfies the regularity condition. In this example $dim(\mathbf{E}) = \lambda = 2$, therefore $r = 3\lambda - 1 = 5$. The direct application of the proposed algorithm, for $r + 1 = 6$ points, yields

$$\begin{aligned} &W^0 = +1, \quad W^{-1} = +0.5 - 0.866j, \quad W^{-2} = -0.5 - 0.8660j, \\ &W^{-3} = -1, \quad W^{-4} = -0.5 + 0.8660j, \quad W^{-5} = +0.5 + 0.8660j, \end{aligned}$$

and

$$\begin{aligned} &a_0 = -14, \quad a_1 = -4.5 + 11.2581j, \quad a_2 = +2.5 - 2.5981j, \\ &a_3 = 0, \quad a_4 = +2.5 - 2.5981j, \quad a_5 = -4.5 - 11.2581j, \end{aligned}$$

$$\mathbf{F}_0 = \begin{bmatrix} 6 & -1 \\ 2 & 2 \end{bmatrix}, \quad \mathbf{F}_1 = \begin{bmatrix} 2.5 - 2.5981j & -1 + 1.7321j \\ 3 & 1.5 + 0.8660j \end{bmatrix},$$
$$\mathbf{F}_2 = \begin{bmatrix} 1.5 + 0.8660j & 2 \\ 2 - 1.7321j & 0.5 - 0.8660j \end{bmatrix}, \quad \mathbf{F}_3 = \begin{bmatrix} 4 & -1 \\ 0 & 0 \end{bmatrix},$$
$$\mathbf{F}_4 = \begin{bmatrix} 1.5 - 0.8660j & 2 \\ 2 + 1.7321j & 0.5 + 0.8660j \end{bmatrix},$$
$$\mathbf{F}_5 = \begin{bmatrix} 2.5 - 2.5981j & -1 - 1.7321j \\ 3 & 1.5 + 0.8660j \end{bmatrix},$$

$$q_0 = -3, \quad q_1 = -6, \quad q_2 = -6, \quad q_3 = 0, \quad q_4 = +2, \quad q_5 = -1,$$

$$\mathbf{P}_0 = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{P}_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{P}_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Once the denominator and the adjoint matrix have been computed, Eq. 22 can be utilized to obtain the transfer function $T(s)$. Therefore, we obtain

$$T(s) = \frac{\mathbf{P}_3 s^3 + \mathbf{P}_2 s^2 + \mathbf{P}_1 s + \mathbf{P}_0}{q_5 s^5 + q_4 s^4 + q_3 s^3 + q_2 s^2 + q_1 s + q_0},$$

or

$$T(s) = \frac{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} s^3 - \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} s + \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}}{-s^5 + 2s^4 + 0s^3 - 6s^2 - 6s - 3}.$$

The correctness of the above result can be verified using (3),

$$T(s) = \mathbf{C}(s^3\mathbf{E} - s^2\mathbf{A}_1 - s\mathbf{A}_2 - \mathbf{A}_3)^{-1}\mathbf{B}.$$

4.2. *Second Order System: Two-Inputs Two-Outputs*

Consider the following second-order generalized system:

$$\begin{aligned} \mathbf{E}x^{(2)} &= \mathbf{A}_1 x + \mathbf{A}_2 + \mathbf{B}u, \\ y &= \mathbf{C}x, \end{aligned} \tag{24}$$

where

$$\mathbf{E} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}_0 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We would like to determine the transfer function for this system using the technique of Section 3. Note that the system satisfies the regularity condition. In this example $dim(\mathbf{E}) = \lambda = 2$, therefore $r = 2\lambda - 1 = 3$. The direct application of the proposed algorithm, for $r + 1 = 4$ points, yields

$$W^0 = +1, \quad W^{-1} = -j, \quad W^{-2} = -1, \quad W^{-3} = +j,$$

and

$$a_0 = +1, \quad a_1 = -4 - j, \quad a_2 = -1, \quad a_3 = -4 + j,$$
$$\mathbf{F}_0 = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{F}_1 = \begin{bmatrix} j & 3+j \\ 1 & j \end{bmatrix}, \quad \mathbf{F}_2 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{F}_3 = \begin{bmatrix} -j & 3-j \\ 1 & -j \end{bmatrix},$$
$$q_0 = -2, \quad q_1 = 1, \quad q_2 = 2, \quad q_3 = 0,$$
$$\mathbf{P}_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{P}_1 = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{P}_3 = 0.$$

Once the denominator and the adjoint matrix have been computed, Eq. 22 can be utilized to obtain the transfer function $T(s)$. Therefore, we obtain

$$T(s) = \frac{\mathbf{P}_2 s^2 + \mathbf{P}_1 s + \mathbf{P}_0}{q_2 s^2 + q_1 s + q_0}, \tag{25}$$

or

$$T(s) = \frac{\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} s^2 + \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} s + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}}{2s^2 + s - 2},$$

or

$$T(s) = \frac{\begin{bmatrix} -s & 2s^2 - s - 1 \\ s^2 & -s \end{bmatrix}}{2s^2 + s - 2}.$$

The correctness of the above result can be verified using (3),

$$T(s) = \mathbf{C}(s^2\mathbf{E} - s\mathbf{A}_1 - \mathbf{A}_2)^{-1}\mathbf{B}.$$

4.3. *Second Order System: Single-Input Single-Output*

Consider the following second-order generalized system:

$$\begin{aligned} \mathbf{E}x^{(2)} &= \mathbf{A}_1 x + \mathbf{A}_2 + \mathbf{B}u, \\ y &= \mathbf{C}x, \end{aligned} \tag{26}$$

where

$$\mathbf{E} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}_0 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

We would like to determine the transfer function for this system using the technique of Section 3. Note that the system satisfies the regularity condition. In this example $dim(\mathbf{E}) = \lambda = 2$, therefore $r = 2\lambda - 1 = 3$. The direct application of the proposed algorithm, for $r + 1 = 4$ points, yields

$$W^0 = +1, \quad W^{-1} = -j, \quad W^{-2} = -1, \quad W^{-3} = +j,$$

and

$$a_0 = +1, \quad a_1 = -4 - j, \quad a_2 = -1, \quad a_3 = -4 + j,$$

$$F_0 = 1, \quad F_1 = 3, \quad F_2 = 1, \quad F_3 = 3,$$
$$q_0 = -2, \quad q_1 = 1, \quad q_2 = 2, \quad q_3 = 0,$$
$$P_0 = 2, \quad P_1 = 0, \quad P_2 = -1, \quad P_3 = 0.$$

Once the denominator and the adjoint matrix have been computed, Eq. 22 can be utilized to obtain the transfer function $T(s)$. Therefore, we obtain

$$T(s) = \frac{P_2 s^2 + P_0}{q_2 s^2 + q_1 s + q_0}, \tag{27}$$

or

$$T(s) = \frac{-s^2 + 2}{2s^2 + s - 2}. \tag{28}$$

The correctness of the above result can be verified using (3),

$$T(s) = \mathbf{C}(s^2\mathbf{E} - s\mathbf{A}_1 - \mathbf{A}_2)^{-1}\mathbf{B}.$$

5. Conclusions

A new algorithm was presented for the computation of the transfer function for n-order singular state space systems. The technique is based on the DFT algorithm and has been implemented with the software package MATLAB. To improve the computational speed of the algorithm, the DFT can be implemented using fast Fourier techniques.

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References

Antoniou, G.E., G.O.A. Glentis, S.J. Varoufakis and D.A. Karras (1989). Transfer function determination of singular systems using the DFT. *IEEE Transactions on Circuit and Systems*, **36**(8), 1140–1142.

Antoniou, G.E. (2001). Second-order generalized systems: DFT algorithm for transfer function determination of singular systems using the DFT. In *5th International Conference on Circuits, Systems and Communications*. Athens, Greece. pp. 135–139.

Campbell, S.L. (1982). *Singular Systems of Differential Equations*. Pittman, London.

Campbell, S.L., and N.J. Rose (1982). A second order singular linear system arising in electric power analysis. *Int. J. Systems Science*, **13**(1), 101–108.

Dai, L. (1989). *Singular Control Systems*. Lecture notes in control and information sciences. Springer-Verlag, Berlin.

Lee, T. (1976). A simple method to determine the characteristic function $f(s) = |s\mathbf{I} - \mathbf{A}|$. *IEEE Transactions on Circuits and Systems*, **23**(2), 242.

Luenberger, D.G., and A. Arbel (1977). Singular dynamic Leondieff systems. *Econometrica*, **45**(9), 991.

Marszalek, W., and H. Unbehauen (1992). Second order generalized linear systems arising in analysis of flexible beams. In *Proceedings of the 31st Conference on Decision and Control*. Arizona, USA. pp. 3514–3518.

Mitra, S.K. (2000). *Digital Signal Processing*. McGraw-Hill, New York.

Paraskevopoulos, P.N., M.A. Christodoulou and A.K. Boglou (1984). An algorithm for the computation of the transfer function matrix for singular systems. *Automatica*, **20**(2), 259–260.

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DFT besiremiantis algoritmas *n*-tos eilės išsigimusioms būsenų erdvės sistemoms

George E. ANTONIOU

Diskretinė Furjė transformacija (DFT) naudojama n-tos eilės išsigimusių tiesinių sistemų $\mathbf{E}x^{(n)} = \sum_{i=1}^{n} \mathbf{A}_i x^{(n-1)} + \mathbf{B}u$, kur $\mathbf{E}$ gali būti išsigimusioji, perdavimo funkcijos koeficientų nustatymui. Algoritmas yra lengvai realizuojamas. Pateikiami trys pavyzdžiai, iliustruojantys algoritmo darbingumą.