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THE OVERLAPPINGLY DECOMPOSED NETWORKS

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Abstract. The special class of networks are presented. Based on unreachable parts of subgraphs the overlappingly decomposed networks are defined. The special decomposition scheme of those networks is applicated for shortest path problem, dynamic programming and synthetic neural nets architecture.

Key words: networks connectivity, unreachable nodes, subnetwork, shortest path, matroid, synthetic neural network.

1. Introduction. There are networks that may be covered by overlappingly subnetworks in such a way that one subnetwork differs from another in elements which can not reach elements from other neighbour subnetwork and this may be established on basis of topological properties of the network. For the overlappingly decomposed networks or OD-networks we have suggested the special scheme for finding shortest path between the fixed nodes. It is interesting that operative storage requirements for this procedure depend only on topological properties of the networks.

If the problem of discrete programming may be interpreted as the shortest path problem in large scale OD-network then presented method enables to solve the problem with smaller storage requirements than using other methods (Richter, 1982; Hu, 1968; Tufekci, 1983). The another area of OD-networks applications is in large scale full connected back-propogating synthetic neural nets architecture (Walker *et al.*, 1990). If we can present neural net as conjunction of some OD-nets then we have shown that it is possible to realize large scale full connected between layers neural net in the some number smaller independent neural nets locally connected between layers which have only one common last output layer (Walker *et al.*, 1990).

2. The unreachable elements in subnetworks. Let $G = (V, E)$ be a directed network. Let us define sets:

$D(v,G) := \{w \in V: w$ is reached from v or v is reached from w in $G\}$;

$D(A,V) := \bigcup_{v\in A} D(v,G)$;

$G(A)$ is a subnetwork which is generated by set $A \subset V$, i.e., $G(A) := (A, E(A))$, where $E(A) := \{(v,w) \in E: v, w \in A\}$.

We shall define:

$P(s,t)$ – the set of nodes in directed path from s to t in G;

$P^*(s,t)$ – the set of nodes in the path from s to t in $G(A)$;

$P_A^*(s,t)$ – the set of nodes in shortest path from s to t in $G(A)$;

$c(P(s,t))$ – the length (cost) of path $P(s,t)$;

$\bar{A} := V \backslash A$.

Let us assume that in network G there are no more paths the length of which is equal to the length $P^*(s,t)$, where s, t – are fixed nodes.

Lemma 1. $(\exists v \in A\backslash P_A^*(s,t): D(v,G) \cap \bar{A} = \varnothing) \to (v \notin P^*(s,t))$.

Proof. Proof follows from the fact that all the paths which contain v belong to $G(A)$.

The nodes which satisfy Lemma 1 are shown in Fig. 1. Those nodes will be named *isolated nodes* in subnetwork $G(A)$.

Lemma 2. $(\exists v \in D(\bar{A},G) \cap (A\backslash P_A^*(s,t)) \backslash D(\bar{A}, G(V\backslash P_A^*(s,t))) \to (v \notin P^*(s,t))$.

Proof. Let $v \in P(s,t)$ and $P(s,t) \cap \bar{A} \neq \varnothing$. We shall show that it is possible to construct the path which does not contain v and is shorter than $P(s,t)$.

Let $\exists w \in P(s,t) \cap \bar{A}$. It follows from condition of the lemma that

$$P(s,t) \cap P_A^*(s,t) \backslash \{s,t\} = \{s_1, s_2, \ldots, s_k\} \neq \varnothing.$$

When $k = 1$, then $P(s,t)$ is comparison of two parts $P(s, s_1)$ and $P(s_1, t)$, one of which necessarily belongs to $G(A)$ and v belongs

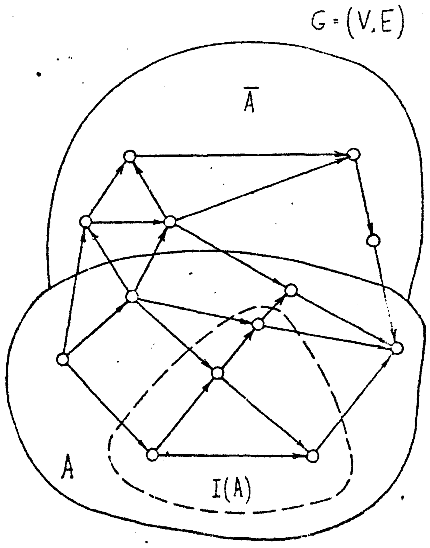

Fig. 1. $I(A)$ – the isolated nodes in subnetwork $G(A)$.

to this part. Let $v \in P(s,s_1) \subset A$. As $P_A^*(s,t)$ is the shortest in subnetwork $G(A)$, so $P_A^*(s,s_1)$ is shorter than $P(s,s_1)$. Therefore after substitution of part $P(s,s_1)$ by part $P_A^*(s,s_1)$ we shall obtain a shorter path which doesn't contain node v.

Let $k \geqslant 2$. In this case it is possible to select $u,w \in P(s,t) \cap P_A^*(s,t) \setminus \{s,t\}$ so that $P(s,t)$ crosses $P_A^*(s,t)$ at these nodes in contrary directions, and so that $v \in P(u,w)$ and between u and w there are no more nodes from $P_A^*(s,t)$ in the part $P(u,w)$.

If it is not possible to select such nodes u and w, then v is found in the path $P(s,t)$ either before $P(s,t)$ crosses $P_A^*(s,t)$ for a first time in node s_1 and $P(s,s_1) \subset A$, or after $P(s,t)$ crosses $P_A^*(s,t)$ for the last time in node s_k and $P(s_k,t) \in A$. In this case we can use the same way of thinking as when $k = 1$.

Let us say, that the above mentioned u and w exist. Then $P(u,w) \subset A$ and in the path $P(s,t)$ the part $P(u,w)$ is substituted by the part $P_A^*(u,w)$. The obtained new path will be shorter and will not contain node v.

In such a way for each path $P(s,t){:}\, v \in P(s,t)$ and $P(s,t) \cap \bar{A} \neq \varnothing$

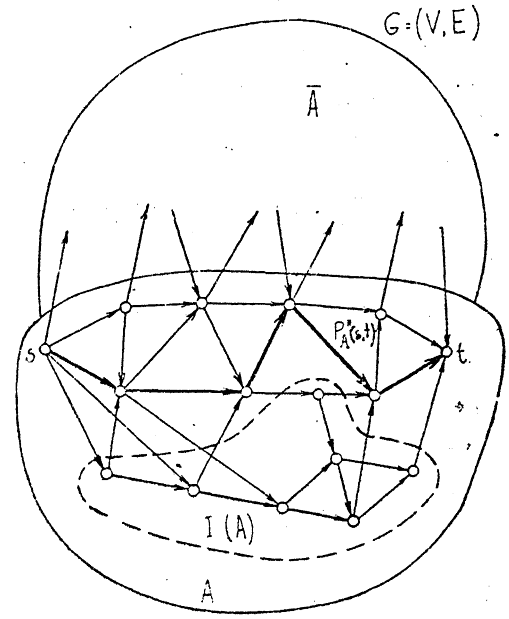

Fig. 2. $I(A) = \{v: v$ satisfy the condition of Lemma 2$\}$.

we can construct a shorter path, which doesn't contain node v. As according to the assumption we consider $P^*(s,t)$ to be the only one path of such length, then $v \notin P^*(s,t)$. Lemma is proved.

The example of nodes which satisfied condition from Lemma 2 is shown in Fig. 2.

3. The overlappingly decomposed families. Lemmas 1 and 2 show the existence of network elements which can not belong to the shortest path and this may be established on basis of topological properties of these subnetworks. So, there exist networks, that may be covered by subnetworks in such a way, that one differs from another in unperspective elements.

DEFINITION 1. The family of sets $\{A_0, A_1, \ldots, A_n : A_j \subset V\}$ will be named a overlappingly decomposed family of network $G = (V, E)$, if the following conditions are satisfied (nodes s, t – are fixed):

(a) $\bigcup_{j=0}^{n} A_j = V$;

(b) $\exists P(s,t) \subset A_j, \quad j = 0, 1, \ldots, n$;

(c) $A_j \backslash A_{j+1} \subset M(A_j) \neq \varnothing, \quad j = 0, 1, \ldots, n-1$,

where:

$$M(A_j) := A_j \backslash D(B_j, G) \cup (D(B_j, G) \cap (A_j \backslash P^*_{A_j}(s,t)) \backslash D\big(B_j, G(V \backslash P^*_{A_j}(s,t))\big),$$

$$B_j := V \backslash (\bigcup_{i=0}^{j} A_i).$$

Network G in this case will be named *overlappingly decomposed* network.

Theorem 1. *Let* $\{A_0, A_1, \ldots, A_n\}$ *– be named overlappingly decomposed. Then*

$$c\big(P^*_{A_0}(s,t)\big) \geqslant c\big(P^*_{A_1}(s,t)\big) \geqslant \ldots \geqslant c\big(P^*_{A_n}(s,t)\big) = c\big(P^*(s,t)\big);$$

and

$$P^*(s,t) \in \{P^*_{A_0}(s,t),\ P^*_{A_1}(s,t), \ldots, P^*_{A_n}(s,t)\}.$$

Proof. We shall notice, that $M(A_j)$ $(j = 0, 1, \ldots, n-1)$ – are the nodes, satisfying the conditions of Lemmas 1, 2 in the formulation of which A is substituted by Aj, and G – by $G(A_j \cup B_j)$.

The correctness of the inequalities follows from

$$P^*_{A_j}(s,t) \subset A_{j+1}, \quad j = 0, 1, \ldots, n-1.$$

Thus $c\big(P^*_{A_j}(s,t)\big) \geqslant c\big(P^*_{A_{j+1}}(s,t)\big)$.

Let $\exists \widehat{P}(s,t) \notin \{P^*_{A_0}, \ldots, P^*_{A_n}(s,t)\}$ and $c\big(\widehat{P}(s,t)\big) < c\big(P^*(s,t)\big)$. Then $\exists \xi, \nu: 0 \leqslant \xi < \nu \leqslant n$ are such as

$$\widehat{P}(s,t) \cap A_\xi \backslash \{s,t\} \neq \varnothing \qquad \text{and} \qquad \widehat{P}(s,t) \cap A_\nu \backslash \{s,t\} \neq \varnothing.$$

Two cases are possible:

(i) $A_\xi \cap A_\nu \backslash \{s,t\} = \varnothing$;

(ii) $A_\xi \cap A_\nu \neq \varnothing$.

In the first case it follows from the definition, that $\exists \eta: \xi < \eta < \nu: A_\eta \cap A_\nu \backslash \{s,t\} \neq \varnothing$ and $\widehat{P}(s,t) \cap A_\eta \backslash \{s,t\} \neq \varnothing$,, so it is enough to consider only the second case.

If $\widehat{P}(s,t) \cap M(A_\xi) \neq \varnothing$, then doesn't exist the node $v \in \widehat{P}(s,t) \cap M(A_\xi)$, which could be isolated in subnetwork $G(A_\xi)$, because $\widehat{P}(s,t) \cap A_\nu \backslash \{s,t\} \neq \varnothing$. So the set $\widehat{P}(s,t) \cap M(A_\xi)$ contains nodes, satisfying

Lemma 2. It follows, that $c(P^*_{A_\xi}(s,t)) < c(\widehat{P}(s,t))$, which contradicts to the assumption, that the shortest path is $\widehat{P}(s,t)$.

If $\widehat{P}(s,t) \cap A_x i \backslash \{s,t\} \neq \varnothing$, but $\widehat{P}(s,t) \cap M(A_\xi) = \varnothing$, then appears $\eta: \nu > \eta > \xi$ and $\widehat{P}(s,t) \cap M(A_\eta) \neq \varnothing$. In this case Lemma 2 is applied to subnetwork $G(A_\eta)$ in general network $G(A_\eta \cup B_\eta)$. The theorem is proved.

For a connected network G we can suggest the following formal procedure for constructing the overlappingly decomposed family and finding the shortest path between the fixed nodes s and t.

Procedure ODF.

Step 1. Let us take $A_0 \subset V: |A_0| < |V|$ and $M(A_0) \neq \varnothing$;

$X := A_0$ – the considered set of nodes at a given moment;

$S := V \backslash A_0$ – the set of unconsidered nodes;

$i := 0$ – the counter of paths, which do not cross each other nodes in network $G(X)$, the maximum value q;

$l := 0$ – the number of possible ways of selecting nodes from the set S, these nodes will be used for constructing the following subnetwork in the overlappingly decomposed family, the maximum value of the counter is p;

$J(X) := \varnothing$; $K(X) := \varnothing$ – additional sets.

Step 2. $i := i + 1$;

if $i > q$ or there are no i-th paths, which cross nodes of other paths, then we proceed to Step 4;

with the help of known algorithm we find $P^{*i}_X(s,t)$ – the i-th shortest path between the nodes which do not cross nodes of other paths;

if $S = 0$ we proceed to Step 6.

Step 3. We form a set of unperspective nodes $M(X)$; if $|M(X)| \leqslant \varepsilon$, we proceed to Step 2, otherwise $S := S \backslash J(X)$.

Step 4. $l := l + 1$;

if $l > p$ then we proceed to Step 6; otherwise we form a set $K(X) \subset S$ in the l-th way.

Step 5. If $i > q$ then $X := X \backslash J(X) \cup K(X)$, otherwise $X := X \backslash M(X) \cup K(X)$ and $l := 0$;

$i := 0$;

$J(X) := K(X)$;

go to Step 2.

Step 6. If $S = 0$ then $P_X^{*i}(s,t)$ – is one of the shortest paths (if there are several of them) in network G, otherwise we state that the constructing of overlappingly decomposed family was unsuccessful.

It is obvious that this general procedure requires $O(|X|)$ operative storage. The time necessary for the procedure ODF, if to assume that A_0 is the given set and do not take into account time, which is used for the relation with the outside storage, may be evaluated $O(p|V|^2/r)$, where $r := \min\{|M(X)| > 0\}$.

It is interesting that the operative storage requirements in this procedure depend not on the size of network, but only on its topological properties, which condition the size of subset X.

4. One class of overlappingly decomposed networks.

DEFINITION 2. By (m,k,r) – network we mean a k-level ($k \geqslant 3$) directed network $G = (V,E)$, in every level of which there are m successively from 1 to m enumerated nodes (let us say, that levels are also enumerated from 1 to k) and the condition is satisfied:

$$(u,w) \in E \Leftrightarrow |a(u) - a(w)| \leqslant r, \quad r = 1,2,3,\ldots,m/2,$$

where $a(v)$ means the number of node v in the level.

The examples of (m,k,r)-networks are demonstrated in Fig. 3.

We shall define (m,k,r)-subnetwork in (m,k,r)-network, then the subnetwork, containing nodes, numbers of which in levels are from 1 to μ.

Let us indicate $\mu_0 := \min\{\mu :$ all nodes in (m,k,r)-subnetwork v which $a(v) = 1$ – are isolated $\}$.

Let us indicate $\varkappa(v)$ number of layer (level) to which depend node v in (m,k,r)-network or (μ,k,r)-subnetwork and $\rho(\mu)$ – the number of isolated nodes from (μ,k,r)-subnetwork.

Theorem 2. *Let $G = (V,E)$ is (m,k,r)-network. Then the following conditions are satisfied:*

(1) *If v and u there are nodes from (m,k,r)-network, then*

$$v \in D(v,G) \Leftrightarrow |a(u) - a(v)| \leqslant r|\varkappa(u) - \varkappa(v)|.$$

(2) $\mu_0 = 1 + r(k-1)$.

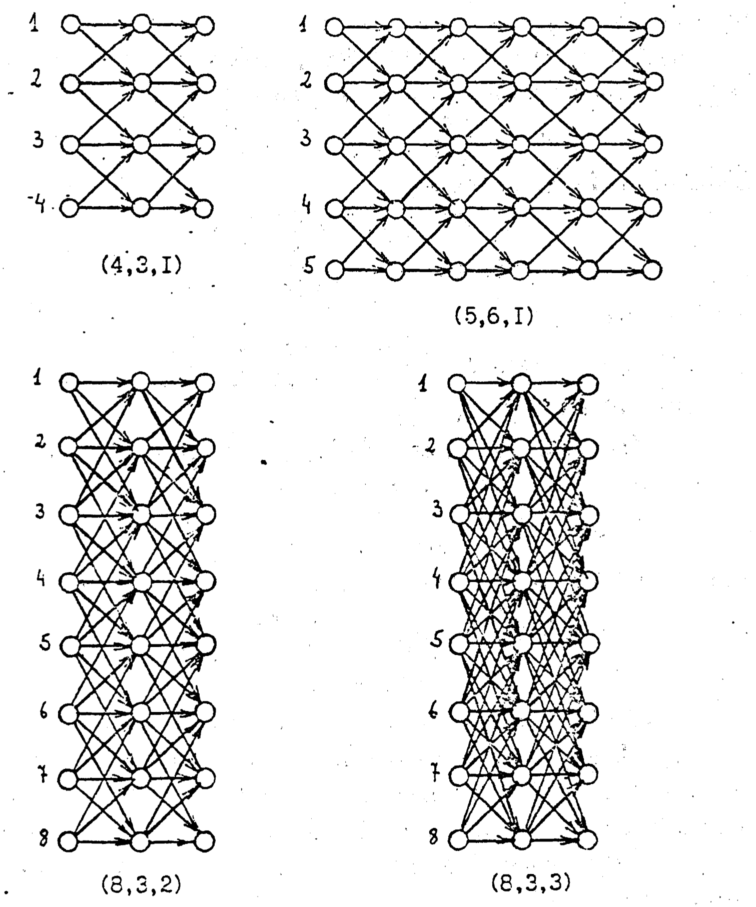

Fig. 3. (m, k, r)-networks.

(3) *Let v is isolated node from (μ, k, r)-subnetwork (the set of nodes let us indicate X),* and *let* $\mu \geqslant \mu_0$ *and*

$$\varkappa(v) \leqslant \begin{cases} k/2, & k \bmod 2 = 0, \\ (k-1)/2 + 1, & k \bmod 2 \neq 0. \end{cases}$$

Then all nodes from the set

$$\{w \in V : w \text{ is reachable from } v\}$$
$$\cap \left\{ u \in X : \varkappa(u) \leqslant \begin{cases} k/2, & k \bmod 2 = 0, \\ (k-1)/2, & k \bmod 2 \neq 0. \end{cases} \right\}$$

are isolated.

(4) *Let X be the set of nodes (μ, k, r)-subnetwork and $\mu \geqslant \mu_0$. Let v is isolated node from this subnetwork and*

$$\varkappa(v) \geqslant \begin{cases} k/2, & k \bmod 2 = 0, \\ (k-1)/2+1, & k \bmod 2 \neq 0. \end{cases}$$

Then all nodes from the set

$$\{w \in V : v \text{ is reachable from } w\}$$
$$\cap \left\{ u \in X : \varkappa(u) \geqslant \begin{cases} k/2, & k \bmod 2 = 0, \\ (k-1)/2+1, & k \bmod 2 \neq 0. \end{cases} \right\}$$

are isolated.

(5) $$\rho(\mu_0) = \begin{cases} k + k(k-2)r/4, & k \bmod 2 = 0, \\ k + (k-1)^2 r/4, & k \bmod 2 \neq 0. \end{cases}$$

(6) $\rho(\mu) = \rho(\mu_0) + k(\mu - \mu_0), \quad (\mu \geqslant \mu_0).$

(7) *The node v is isolated in (μ, k, r)-subnetwork then and only then, when next conditions are satisfied;*

if $$\varkappa(v) \leqslant \begin{cases} k/2, & k \bmod 2 = 0, \\ (k-1)/2+1, & k \bmod 2 \neq 0, \end{cases}$$

then $a(v) \leqslant 1 + (\varkappa(v) - 1)r + (\mu - \mu_0)$;

if $$\varkappa(v) \geqslant \begin{cases} k/2, & k \bmod 2 = 0, \\ (k-1)/2+1, & k \bmod 2 \neq 0, \end{cases}$$

then $a(v) \leqslant 1 + (k - \varkappa(v))r + (\mu - \mu_0)$.

Proof. (1) Follows from (m, k, r)-networks definition.

(2) From μ_0 definition follows that first node in first level (or layer) in (m_0, k, r)-subnetwork is isolated. Then follows that all nodes in level k which are reachable from such node must depend (μ_0, k, r)-subnetwork. The maximal number such node in level k is $1 + (k-1)r$.

(3) Let u is any node from indicated set. We shall proof that it is isolated. We shall show that $D^+(u, G) \cup D^-(u, G) \subset X$, where $D^+(u, G) := \{w \in V : w \text{ is reachable from } u\}$, $D^-(u, G) := \{w \in V : u \text{ is reachable from } w\}$.$D^+(u, G) \subset D^+(v, G)$, because u is reachable from v, but $D^+(v, G) \subset X$, because v is isolated. Those $D^+(u, G) \subset X$. We must only show that $D^-(u, G) \subset X$. Let us say that exist $z \notin X$ and $z \in D^-(u, G)$. From property (1) and Definition 2 follows that $\mu \geqslant a(v) + (k - \varkappa(v))r$. Therefore $\mu \geqslant \mu_0$ then $\mu \geqslant a(v) + (k - \ )r$. The node u is reachable from v. Then from (1) follows that $a(u) \leqslant$

$a(v)+(\varkappa(u)-\varkappa(v))r$. But u is also reachable from z. Then using last inequality we receive:

$$a(z) \leqslant a(u)+(\varkappa(u)-\varkappa(z))r \leqslant a(v)+(\varkappa(u)-\varkappa(z))r$$
$$\leqslant a(v)+\begin{cases}(k-2)r, & k \bmod 2=0,\\ (k-1)r, & k \bmod 2 \neq 0.\end{cases}$$

From assumption that $z \notin X$ follows $a(z)>\mu \geqslant a(v)+(k-1)r$. Therefore last two inequalities with $a(z)$ we receive contradicts each other. Than follows $D^-(u,G) \subset X$ and u is isolated.

(4) Proof follows from (3) in the same network with interchanged orientation.

(5) We must detect the number of isolated nodes in (μ_0,k,r)-subnetwork. The set of isolated nodes cosist from conjunction the sets defined in properties (3) and (4). We will detect the number of nodes reachable from $v{:}\,\varkappa(v)=1,\ a(v)=1$ and the number of nodes from which is reachable the node $w{:}\,\varkappa(w)=k,\ a(w)=1$. From (1) follows that the number of nodes which are reachable from v and depend to level j, where

$$j \geqslant \begin{cases}k/2, & k \bmod 2=0,\\ (k-1)/2+1, & k \bmod 2\neq 0\end{cases}$$

is equal $1+(j-1)r$. The same we can detect about the number of nodes in level l, where

$$l \geqslant \begin{cases}k/2, & k \bmod 2=0,\\ (k-1)/2+1, & k \bmod 2\neq 0\end{cases}$$

from which the node w is reachable. Then general number of isolated nodes is:

$$\rho(\mu_0)=\begin{cases}2\sum\limits_{j=1}^{k/2}(1+(j-1)r), & k \bmod 2=0,\\ 2\sum\limits_{j=1}^{(k-1)/2}(1+(j-1)r)+(1+(k-1)r/2), & k \bmod 2\neq 0.\end{cases}$$

Using formula about sum of arithmetical progression we receive final equation.

(6) Proof follows from fact that all nodes v, which $a(v)<\mu-\mu_0$ are isolated.

(7) Proof follows from properties (1), (2), (3).

Let $G=(V,E)$ be such network that $G(V\backslash\{s,t\})$ is (m,k,r)-network, where s and t are fixed nodes. Such network G is overlappingly decomposed network, because it may be covered by subnetworks satisfying the Definition 1. One of the rules of subnetwork selection is as follows:

(i0) $G(A_0)$ is selected so that $G(A_0\backslash\{s,t\})$ would make (μ,k,r)-subnetwork, where $\mu > \mu_0$.

(i1) $G(A_j)$, where $j = 1,\ldots,n$ and is constructed according to the formula:

$A_j := A_{j-1}\backslash\{v \in A_{j-1}: a(v) \leqslant j\delta\} \cup \{v: a(v) \leqslant m + j\delta\}, \delta := \mu - \mu_0 + 1.$

In order to establish the unperspectiveness of a node it is enough to verify, whether a certain inequality from Theorem 2 is satisfied. For such network G we can suggest the following procedure, which is a certain variant of ODF procedure.

Procedure ODFMKR.

Step 1. We select A_0 according to the Rule (i0);

$$X := A_0; \quad j := 1; \quad S := V\backslash A_0;$$

Step 2. We find $P_X^*(s,t)$ in network $G(X \cup \{s,t\})$; if $S = \varnothing$ we proceed to Step 6.

Step 3. We form $M(X)$ – unperspective nodes according to the Rule (i1).

Step 4. We form $J(X)$ – joined nodes according to the Rule (i1).

Step 5. $X := X\backslash M(X) \cup J(X)$; $S := S\backslash J(X)$; $j := j+1$; go to Step 2.

Step 6. The output $P_X^*(s,t)$ – the shortest path in network G.

This procedure calls for a following number of operations $O\Big(lk\big((\mu-2r)(2r+1)+r(3r+1)\big)\Big)$, where $l := [(m-\mu)/\delta]+1$ and the necessary storage requirements may be equal $O(k\mu)$. When using a simple nondecompositional algorithm: number of operations – $O\big(k\mu(2r+1)\big)$, storage requirements – $O(k\mu)$. The comparison of computational times of procedure ODFMKR and nondecompositional algorithm is presented in Table 1.

More detailed ODF procedure applications for solving discrete optimization problems was presented in publications (Garliauskas, Lašinskas, 1990; Lašinskas, 1990).

Table 1. The comparison of computational times of procedure ODFMKR and nondecompositional algorithm

m	k	r	μ	Nondecompositional algorithm time (sec)	ODFMKR time (sec)
30	3	1	3	0.234	0.422
"	"	"	5	0.266	0.313
"	"	"	10	0.219	0.281
50	3	1	3	0.375	0.719
"	"	"	5	0.266	0.437
"	"	"	10	0.313	0.375
"	"	"	15	0.328	0.328
50	3	2	5	0.437	1.450
"	"	"	10	0.453	0.719
"	"	"	15	0.375	0.562
60	3	1	3	0.344	0.875
"	"	"	5	0.375	0.609
"	"	"	10	0.406	0.437
60	3	2	5	0.578	2.190
"	"	"	10	0.531	0.687
60	3	3	7	0.641	3.250
"	"	"	12	0.562	1.05
60	5	1	5	0.5	2.16
"	"	"	10	0.453	0.719
60	5	2	9	0.609	4.72
"	"	"	20	0.766	1.02
80	3	1	3	0.406	1.08
"	"	"	10	0.437	0.641
"	"	"	20	0.391	0.531
80	3	5	15	1.06	2.91
"	"	"	30	1.34	1.84
80	3	6	15	1.37	5.94
"	"	"	30	1.34	1.56
"	"	"	40	1.23	1.58

5. The overlappingly decomposed networks and matroids. It is naturally to think that some theoretical results of matroid theory (Aigner, 1969) may be usefully applied in systems modeling connected with properties which makes possible to decompose the systems and investigate them with the help of some parallel procedure. We have proved theorem which shows connection between matroids and overlappingly decomposed families discussed in publications (Garliauskas, Lašinskas, 1990; Lašinskas, 1990) (early we have used concept "successively decomposed" but it is not exact because the possibilities of parallel search of graph are hidden; therefore there is more acceptable to use concept "overlappingly decomposed").

Let $G = (V, E)$ be a directed network. We shall define: $I(A) = \{v \in A : v$ is unreachable from the nodes $V \backslash A$ in $G\}$, $D(a) = \{x \in V : x$ is reachable from a in $G\}$.

A directed graph G is weakly connected if there is at least one directed path between every pair of nodes in G, i.e., the undirected graph obtained by ignoring the edge directions in G is connected.

DEFINITION 3. The family F of subsets S is said to be a family of independent sets of matroid on S if the following conditions are satisfied:

(1) $\varnothing \in F$;

(2) $A \in F \& B \subseteq A \rightarrow B \in F$;

(3) $A,\ B \in F \& |A| = |B| + 1 \rightarrow \exists x \in A \backslash B : B \cup \{x\} \in F$.

Theorem 3. *The family of sets* $\tau = \{B_0, B_1, \ldots, B_m : B_j \subset V, I(B_j) = \varnothing, \forall C \subset B_j \rightarrow G(C)$ *is weakly connected,* $j = 0, 1, \ldots, m\}$ *in the network* $G = (V, E)$ *is the family of independent sets of a matroid on* V.

Proof. The conditions (1) and (2) are automatically satisfied. Let $A,\ B \in \tau$ and $|A| = |B| + 1$. We must proof that $\exists x \in A \backslash B : B \cup \{x\} \in \tau$. There are following cases (a) $B \subset A$; (b) $B \cap A = \varnothing$; (c) $B \cap A \neq \varnothing \& B \backslash A \neq \varnothing$.

Case (a) The proof follows from the second condition (2).

Case (b) Since $G(A)$ is weekly connected $\exists a : D(a) \neq \varnothing$ in subgraph $G(A)$. Let us form a subset $B \cup \{a\}$. Then $I(B \cup \{a\}) = \varnothing$ since $I(B) = \varnothing$ and a is connected at least with one vertice from $A \backslash (B \cup \{a\})$.

Case (c) Let $\forall x \in A \backslash B \rightarrow I(B \cup \{x\}) \neq \varnothing$. We will show that it is impossible.

Let $y \in I(B \cup \{x\})$. Then $y \in A \backslash B$. So $x \equiv y$ and $\exists B^* \subset B$ which separate x and $V \backslash (B \cup \{x\})$. Since $\forall x \in A \backslash B \to x \in I(B \cup \{x\})$ and $\forall C \subset AG(C)$ is weekly connected then it follows $\exists y \in A \cap B : y \in D(x)$. But it is easy to see that for every subset $S \subset V : x \in I(S) \& \exists y \in D(x) \to y \in I(S)$. So it follows that $\exists y \in I(B \cup \{x\}) \cap B$ but that contradicts with a fact that $I(B) = \emptyset$, i.e., $B \in \tau$. The theorem is proved.

6. The overlappingly decomposed networks application by synthetic neural network architecture (an example). There is problem to decrease the number of intersections between connections in neural net. We shall show how this problem could be solved using overlappingly decomposed networks and adding additional nodes and layers. This idea may be applicate by back-propogation neural net projection.

Let we have the net as in Fig. 4:

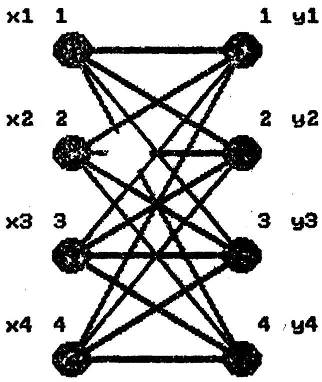

Fig. 4. The full connected two-layered neural network.

This net has 72 intersections. Lets make the next nets transformation:

(1) the net "duplication": the "duplicated" net in second layer has the same neurons but neurons pairs (1, 2) and (3, 4) exchange places:

(2) the additional layer leading in to net: this layer consist from four neurons;

(3) the connections between last layer and additional layer are organized so that in each neuron from additional layer come edges from "duplicated" neurons with same number (Fig. 5.)

The weights between first and second layer correspond weights matrix. The outputs in second layer are next:

$$y_1^{(1)} = \text{sign}\Big(\sum_{j=1}^{2} t_{j1}x_j\Big), \qquad y_2^{(1)} = \text{sign}\Big(\sum_{j=1}^{3} t_{j2}x_j\Big),$$

$$y_3^{(1)} = \text{sign}\Big(\sum_{j=2}^{4} t_{j3}x_j\Big), \qquad y_4^{(1)} = \text{sign}\Big(\sum_{j=3}^{4} t_{j4}x_j\Big).$$

$$y_1^{(2)} = \text{sign}\Big(\sum_{j=3}^{4} t_{j1}x_j\Big), \qquad y_2^{(2)} = \text{sign}\Big(\sum_{j=4}^{4} t_{j2}x_j\Big),$$

$$y_3^{(2)} = \text{sign}\Big(\sum_{j=1}^{1} t_{j3}x_j\Big), \qquad y_4^{(2)} = \text{sign}\Big(\sum_{j=1}^{2} t_{j4}x_j\Big).$$

Index (1) and (2) correspond different "duplicated" neurons. Now we shall show what connection must be between second layer and additional layer that final outputs in transformed net (Fig. 5) will be same as in the initial net (Fig. 4).

Let a_i $(i = 1, \dots, 4)$ be weights between neurons from second layer in first "duplicated" net and last additional layer and let b_i $(i = 1, \dots, 4)$ be weights between neurons from second layer in second "duplicated" net and last layer. Then the outputs in transformed net and initial net will be equal if next condition will satisfied:

$$\text{sign}(a_i y_i^{(1)} + b_i y_i^{(2)}) = y_i \qquad (i = 1, \dots, 4), \tag{$*$}$$

where $y_i = \text{sign}\Big(\sum\limits_{j=1}^{4} t_{ji}x_j\Big)$ – outputs in the initial net.

Let indicate $u_i = \arg(y_i^{(1)})$, $s_i = \arg(y_i^{(2)})$ then $(*)$ is equivalent

$$\text{sign}\big(a_i \text{sign}(u_i) + b_i \text{sign}(s_i)\big) = \text{sign}(u_i + s_i), \qquad i = 1, \dots, 4.$$

Now it is easy to see that this equation is satisfied when $a_i = |u_i|$ and $b_i = |s_i|$.

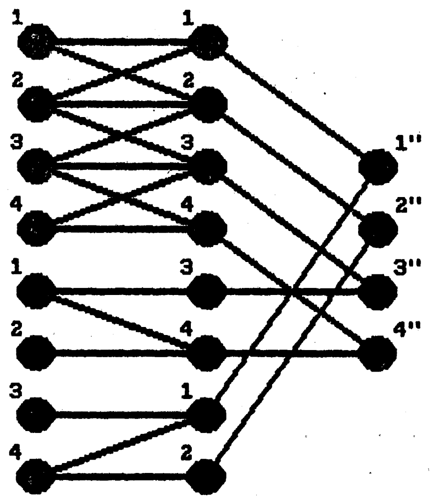

Fig. 5. The neural network received after the transformation.

So we receive that transformed network work as initial network but intersections number in transformed net is only 13.

This example may be easy to generalize when we have two layered network consist from m neurons in each layer. The generalization for three layered net is more complicated.

7. Conclusions. The procedure ODF enable to find the shortest path in large-scale overlappingly decomposed network when the known methods of decomposition are not easily directly applied because of large-scale cut sets.

It is known that the method of dynamic programming takes large storage requirements for solving problems of discrete programming. If the problem may be interpreted as the shortest path problem in overlappingly decomposed network then in subnetworks from overlappingly decomposed family it is possible to apply the method of dynamic programming. This enables to solve the problem with small storage requirements using some parallel independent process.

The overlappingly decomposed networks are close with some properties of matroids. Some theoretical results of matroid theory (Aigner, 1969) may be usefully applied in systems modeling connected with properties which makes possible to decompose the systems and investigate them with the help of some parallel procedure.

The properties of overlappingly decomposed networks may be applied for solving synthetic neural networks architecture problems.

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