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# An Approach to Decision Making with Interval-Valued Intuitionistic Hesitant Fuzzy Information Based on the 2-Additive Shapley Function

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**Abstract.** Interval-valued intuitionistic hesitant fuzzy sets (IVIHFSs) are useful to denote the decision makers' interval preferred, interval non-preferred and hesitant opinions simultaneously. Considering the application of IVIHFSs, this paper introduces a new decision-making method with interval-valued intuitionistic hesitant fuzzy information that extends the application scopes. To do this, the interval-valued intuitionistic hesitant fuzzy hybrid Shapley weighted averaging (IVIHFHSWA) operator and the interval-valued intuitionistic hesitant fuzzy hybrid Shapley weighted geometric (IVIHFHSWG) operator are defined to aggregate the collective attribute values of alternatives. To reflect the interactions and reduce the complexity of calculating the weights, the 2-additive measures are used to define these two hybrid Shapley weighted operators. To derive the exact weight information of attributes and ordered positions, the associated programming models for determining the optimal 2-additive measures are constructed that are based on the defined Hamming distance measure. To show the feasibility and efficiency of the new method, a practical decision-making problem is offered, which is also used to compare with the previous methods.

**Key words:** decision making, interval-valued intuitionistic hesitant fuzzy set, Choquet integral, hamming distance, Shapley function.

## 1. Introduction

With the socioeconomic development, the complexity of decision-making problems is constantly increasing. To denote the fuzzy and uncertain information in decision making, researchers applied fuzzy sets introduced by Zadeh (1965) to cope with this situation. Later, Atanassov (1983) noted that fuzzy sets can only express the decision makers' preference information and introduced the concept of intuitionistic fuzzy sets (IFSs),

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which are expressed by two real values in [0, 1] to denote the membership and non-membership degrees, respectively. However, due to various kinds of reasons, it is not an easy thing to give the exact values of the membership and non-membership degrees. Thus, Atanassov and Gargov (1989) further gave the concept of interval-valued intuitionistic fuzzy sets (IVIFSs) that apply two intervals in [0, 1] to give the uncertain membership and non-membership information. Following the original works of Atanassov (1983) and Atanassov and Gargov (1989), the theory and application of decision making with (interval-valued) intuitionistic fuzzy sets are developed and becomes a hot researching topic (Beliakov and Janes, 2013; Chen and Huang, 2017; Gou and Xu, 2017; Garg, 2017; Liu *et al.*, 2017a; Meng and Chen, 2017; Meng *et al.*, 2017d; Ureña *et al.*, 2015; Wang and Chen, 2017).

Recently, Torra (2010) found that there might be several values for a judgment rather than only one. To address this issue, Torra (2010) introduced the definition of hesitant fuzzy sets (HFSs), which are denoted by several values in [0, 1]. To endow the decision makers with more rights to denote their hesitant information, Chen *et al.* (2013) proposed interval-valued hesitant fuzzy sets (IVHFSs) that use several intervals in [0, 1] to express the decision makers' uncertain hesitancy. To calculate the collective attribute values, the authors introduced two aggregation operators: the generalized interval-valued hesitant fuzzy hybrid averaging (IVHFWA) operator and the generalized interval-valued hesitant fuzzy hybrid geometric (GIVHFWG) operator. Furthermore, He *et al.* (2016) developed an approach to group decision making with interval-valued hesitant fuzzy information using the interval-valued hesitant fuzzy weighted power Bonferroni mean (IVHFWPBM) operator. Jin *et al.* (2016) studied the cross-entropy and similarity measures of interval-valued hesitant fuzzy elements (IVHFEs) and illustrated their application for evaluating emergency risk management (ERM). Note that the cross-entropy and similarity measures in Jin *et al.* (2016) require the considered IVHFEs to have the same length; otherwise, it needs to extend the IVHFEs with the shorter length. Meng *et al.* (2016) considered the correlation coefficients of IVHFEs, which permit IVHFEs to have the different numbers of intervals. To address the situations where the weights of attributes are interactive, Meng and Chen (2014) defined the induced generalized interval-valued hesitant fuzzy hybrid Shapley averaging (IG-IVHFHSWA) operator and the induced generalized interval-valued hesitant fuzzy hybrid Shapley geometric mean (IG-IVHFHSGM) operator, which are general cases of the Chen *et al.*'s aggregation operators.

From the concepts of interval-valued intuitionistic fuzzy sets and interval-valued hesitant fuzzy sets, one can find that the former denotes the interval preferred and non-preferred degrees of the decision makers, while the latter indicates the decision makers' hesitancy. However, they cannot express the decision makers' interval preferred, interval non-preferred and hesitant opinions simultaneously. Thus, Zhang (2013) presented interval-valued intuitionistic hesitant fuzzy elements (IVIHFEs) that use several interval-valued intuitionistic fuzzy values (IVIFVs) (Xu and Chen, 2007) to denote the decision makers' opinions. Then, the author defined a series of aggregation operators to rank objects. Later, Joshi and Kumar (2016) developed a method to multi-criteria decision making with interactive characteristics, which is based on the interval-valued intuitionistic hesitant fuzzy Choquet integral (IVIHFCI) operator with respect to the $\lambda$-fuzzy measures and the hamming distance on IVIHFEs. The IVIHFCI operator can be seen as an extension of the ordered weighted averaging (OWA) operator in the setting of interactions. However, the Yager's OWA operator is associated with the weights of the ordered positions, while the IVIHFCI operator considers the weights of the criteria. This means that the objects' comprehensive criteria values calculated by using the IVIHFCI operator are based on the different criteria weight vectors, namely, the different evaluation standards are used for the different objects. Furthermore, the $\lambda$-fuzzy measures can only reflect the complementary or redundant interactions between elements in a set, but they cannot deal with these two aspects simultaneously. Nevertheless, when the importance of the elements is interdependent, we cannot guarantee that there are only complementary or redundant interactions between them. Moreover, the hamming distance on IVIHFEs is unreasonable either. Just as Meng *et al*. (2016) noted, it is not suitable to extend the shorter IVIHFE to the length of the longer IVIHFE by adding some IVIFV several times since it drives a different IVIHFE. Note that the methods in Zhang (2013), Joshi and Kumar (2016) are both based on the operational laws in (Xu and Da, 2002), which have some undesirable properties.

This paper continues to study decision making with interval-valued intuitionistic hesitant fuzzy information and introduces a new method that can address the situations where the weight information is incompletely known and has interactive characteristics. To do this, the interval-valued intuitionistic hesitant fuzzy hybrid Shapley weighted averaging (IVIHFHSWA) operator and the interval-valued intuitionistic hesitant fuzzy hybrid Shapley weighted geometric mean (IVIHFHSWGM) operator are defined. Note that when the applied fuzzy measures are 2-additive measures, we derive the IVIHFHSWA and IVIHFHSWGM operators with respect to 2-additive measures. Using the Shapley function, models for determining the optimal fuzzy measures and 2-additive measures on the attribute set and the ordered position set are constructed. The rest of this paper is organized as follows: Section 2 first reviews several basic concepts including fuzzy measures, Choquet integral, $\lambda$-fuzzy measures and IVIHFEs. Then, the IVIHFCI operator (Joshi and Kumar, 2016) is listed. Meanwhile, it analyses the limitations of the IVIHFCI operator. Section 3 defines two new interval-valued intuitionistic hesitant fuzzy aggregation operators by using the Shapley function and briefly studies several special cases. Section 4 first recalls the hamming distance in Joshi and Kumar (2016) and points out its limitations. Then, a new hamming distance measure is defined that avoids the limitations in the Joshi and Kumar's hamming distance. When the weight information is not exactly known, models for the optimal fuzzy measures and 2-additive measures on the criteria set and the ordered position set are constructed, respectively. Section 5 gives a new group decision making with interval-valued intuitionistic hesitant fuzzy information. Furthermore, a practical decision-making problem about the development of large projects is offered to show the concrete application of the new method. Meanwhile, the comparison analysis is made. Conclusions and future remarks are offered in the last section.

## 2. Several Concepts

This section contains three parts. The first part reviews the concepts of fuzzy measures and the Choquet integral. The second section recalls interval-valued intuitionistic hesitant fuzzy elements (IVIHFEs), several operations and a ranking order relationship. The last section lists three previous aggregation operators.

### 2.1. *Fuzzy Measures and the Choquet Integral*

Just as researchers (Beliakov and Janes, 2011; Grabisch, 1995, 1996, 1997; Fujimoto *et al*., 2006; Tan, 2011; Tan *et al*., 2011; Xu, 2010) noted, the independence of the weights of criteria in a decision-making problem is usually violated. To deal with this case, fuzzy measures introduced by Sugeno (1974) are good choices, which not only give the importance of each element but also consider the weights of all their combinations.

Definition 1 (Sugeno, 1974). A fuzzy measure on finite set $X = \{x_1, x_2, \ldots, x_n\}$ is a set function $\mu : P(X) \to [0, 1]$ satisfying

(i) $\mu(\emptyset) = 0, \mu(X) = 1$,
(ii) For all $A, B \in P(X)$ with $A \subseteq B$, we have $\mu(A) \leqslant \mu(B)$, where $P(X)$ is the power set of $X$.

When $X$ denotes the criteria set in a decision-making problem, $\mu(A)$ can be viewed as the importance of the criteria in $A$. Just as some researchers noted, there are three cases for the interactions between the weights of criteria. Let $A$ and $B$ be any two subsets in $X$ such that $A \cap B = \emptyset$. When $\mu(A) + \mu(B) = \mu(A \cup B)$, then there is no interaction between the weights of the criteria in subsets $A$ and $B$. When $\mu(A) + \mu(B) < \mu(A \cup B)$, then the complementary interaction exists between their weights. Furthermore, when $\mu(A) + \mu(B) > \mu(A \cup B)$, we know that their important interaction is redundant.

Although fuzzy measures are powerful tools to reflect the interactions between the weights of criteria, they define on the power set. This means that we need $2^{n-2}$ coefficients to determine a fuzzy measure on a set with $n$ elements. To address this issue, some special types of fuzzy measures are introduced such as λ-fuzzy measures (Sugeno, 1974) and $k$-additive measures (Grabisch, 1997).

Definition 2 (Sugeno, 1974). Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set. A fuzzy measure $g_\lambda$ on $X$ is called a λ- fuzzy measure if it satisfies

$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B), \tag{1}$$

where $\lambda > -1$, and $A, B \subseteq X$ with $A \cap B = \emptyset$.

Remark 1. One can check that when $\lambda = 0$, then $g_\lambda$ is an additive measure. This means that the weights of the elements in subsets $A$ and $B$ are independent. When $\lambda > 0$, there is a complementary interaction between their weights, and the interaction between their weights is redundant for $-1 < \lambda < 0$.

From formula (1), one can check that the λ-fuzzy measure $g_\lambda$ can be equivalently expressed as:

$$g_\lambda(A) = \begin{cases} \frac{1}{\lambda}(\prod_{i\in A}[1+\lambda g_\lambda(i)] - 1) & \text{if } \lambda \neq 0, \\ \sum_{i\in A} g_\lambda(i) & \text{if } \lambda = 0. \end{cases} \tag{2}$$

From $\mu(X) = 1$, we know that λ can be determined as:

$$\prod_{i\in N}\left[1+\lambda g_\lambda(i)\right] = 1+\lambda. \tag{3}$$

Thus, when the importance of each element in $X$ is known, we can use formula (3) to determine the value of λ. Then, we can further apply formula (2) to get the λ-fuzzy measure $g_\lambda$.

Considering decision-making problems with interactive characteristics, it needs some other tools to obtain the comprehensive criteria values of the objects. The Choquet integral on discrete sets introduced by Grabisch (1996) is one of the most applied tools, which is explicitly defined as follows:

Definition 3 (Grabisch, 1996). Let $f$ be a positive real-valued function on $X = \{x_1, x_2, \ldots, x_n\}$, and $\mu$ be a fuzzy measure on $X$. The discrete Choquet integral of $f$ with respect to $\mu$ is defined as:

$$C_\mu\big(f(x_{(1)}), f(x_{(2)}), \ldots, f(x_{(n)})\big) = \sum_{i=1}^{n} f(x_{(i)})\big(\mu(A_{(i)}) - \mu(A_{(i+1)})\big), \tag{4}$$

where $(\cdot)$ indicates a permutation on $N$ such that $f(x_{(1)}) \leqslant f(x_{(2)}) \leqslant \cdots \leqslant f(x_{(n)})$, and $A_{(i)} = \{x_{(i)}, \ldots, x_{(n)}\}$ with $A_{(n+1)} = \emptyset$.

After the pioneer work of Grabisch (1996), many decision-making methods based on the Choquet integral are developed (Beliakov, 2005; Xu, 2010; Tan, 2011; Tan *et al.*, 2011; Joshi and Kumar, 2016).

## 2.2. *The Concept of IVIHFEs*

With the constant increasing complexity of decision making problems, many extended types of fuzzy sets are proposed. To express the uncertain membership and non-membership information as well as the hesitancy of the decision makers, Zhang (2013) presented the concept of IVIHFEs that are composed by several interval-valued intuitionistic fuzzy values (IVIFVs) (Xu and Chen, 2007).

Definition 4 (Zhang, 2013). Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set. An interval-valued intuitionistic fuzzy set (IVIHFS) $\tilde{E}$ on $X$ is defined in terms of a function that when applied to $X$ it returns a subset of the set of all IVIFVs, denoted by $\tilde{E} = \{\langle x_i, h_{\tilde{E}}(x)\rangle, x_i \in X\}$, where $h_{\tilde{E}}(x)$ is a set of several IVIFVs denoting the possible interval membership and non-membership degrees of the element $x_i \in X$ to the set $\tilde{E}$. For simplicity, $\tilde{h} = h_{\tilde{E}}(x)$ is called the interval-valued intuitionistic hesitant fuzzy element (IVIHFE), and $\tilde{H}$ is the set of all IVIHFEs. Any $\tilde{\alpha} \in \tilde{h}$ is an IVIFV, denoted by $\tilde{\alpha} = ([\mu_l, \mu_u], [v_l, v_u])$.

Remark 2. From Definition 4, one can check that IVIHFEs can be viewed as an extension of several types of fuzzy sets including hesitant fuzzy values (HFVs) (Torra, 2010), interval-valued intuitionistic fuzzy values (IVIFVs) (Atanassov and Gargov, 1989) and interval-valued hesitant fuzzy values (IVHFVs) (Chen *et al.*, 2013).

Following the operations on IVIFVs (Xu and Chen, 2007), Zhang (2013) defined the following operational laws on IVIHFEs. Let $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$ be any three IVIHFEs in $\tilde{H}$. Then,

(i) $\tilde{h}_1 \oplus \tilde{h}_2 = \{\tilde{\alpha}_i \oplus \tilde{\alpha}_j | \tilde{\alpha}_i \in \tilde{h}_1, \tilde{\alpha}_j \in \tilde{h}_2\} = \{([\mu_l^i + \mu_l^j - \mu_l^i\mu_l^j, \mu_u^i + \mu_u^j - \mu_u^i\mu_u^j], [v_l^i v_l^j, v_u^i v_u^j]) | \tilde{\alpha}_i \in \tilde{h}_1, \tilde{\alpha}_j \in \tilde{h}_2\}$;
(ii) $\tilde{h}_1 \otimes \tilde{h}_2 = \{\tilde{\alpha}_i \otimes \tilde{\alpha}_j | \tilde{\alpha}_i \in \tilde{h}_1, \tilde{\alpha}_j \in \tilde{h}_2\} = \{([\mu_l^i\mu_l^j, \mu_u^i\mu_u^j], [v_l^i + v_l^j - v_l^i v_l^j, v_u^i + v_u^j - v_u^i v_u^j]) | \tilde{\alpha}_i \in \tilde{h}_1, \tilde{\alpha}_j \in \tilde{h}_2\}$;
(iii) $\lambda\tilde{h} = \{\lambda\tilde{\alpha} | \tilde{\alpha} \in \tilde{h}\} = \{([1-(1-\mu_l)^\lambda, 1-(1-\mu_u)^\lambda], [v_l^\lambda, v_u^\lambda]) | \tilde{\alpha} \in \tilde{h}\}, \lambda \in [0,1]$;
(iv) $\tilde{h}^\lambda = \{\tilde{\alpha}^\lambda | \tilde{\alpha} \in \tilde{h}\} = \{([\mu_l^\lambda, \mu_u^\lambda], [1-(1-v_l)^\lambda, 1-(1-v_u)^\lambda]) | \tilde{\alpha} \in \tilde{h}\}, \lambda \in [0,1]$.

Considering the order relationship between IVIHFEs, similar to Xu and Chen (2007), Zhang (2013) introduced the concepts of the score function and the accuracy function on IVIHFEs. Let $\tilde{h}$ be an IVIHFE in $\tilde{H}$, then its score function is given as:

$$s(\tilde{h}) = \frac{1}{\#\tilde{h}} \sum_{\tilde{\alpha} \in \tilde{h}} \frac{\mu_l + \mu_u - v_l - v_u}{2} \tag{5}$$

and the accuracy function is defined as follows:

$$a(\tilde{h}) = \frac{1}{\#\tilde{h}} \sum_{\tilde{\alpha} \in \tilde{h}} \frac{\mu_l + \mu_u + v_l + v_u}{2} \tag{6}$$

where $\#\tilde{h}$ denotes the number of IVIFVs in $\tilde{h}$.

Let $\tilde{h}_1$ and $\tilde{h}_2$ be any two IVIHFEs in $\tilde{H}$. Then, their order relationship is listed as follows:
If $s(\tilde{h}_1) < s(\tilde{h}_2)$, then $s\tilde{h}_1 < \tilde{h}_2$.
If $s(\tilde{h}_1) = s(\tilde{h}_2)$, then $\begin{cases} a(\tilde{h}_1) < a(\tilde{h}_2) \Rightarrow \tilde{h}_1 < \tilde{h}_2, \\ a(\tilde{h}_1) = a(\tilde{h}_2) \Rightarrow \tilde{h}_1 = \tilde{h}_2. \end{cases}$

### 2.3. *Several Interval-Valued Intuitionistic Hesitant Fuzzy Aggregation Operators*

To compute the comprehensive values of alternatives, Zhang (2013) introduced the following two interval-valued intuitionistic hesitant fuzzy hybrid aggregation operators:

DEFINITION 5 (Zhang, 2013). Let $\tilde{h}_i,\ i = 1, 2, \ldots, n,$ be a collection of IVIHFEs, which has an associated weight vector $w = (w_1, w_2, \ldots, w_n)^T$ on the ordered set $N = \{1, 2, \ldots, n\}$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Then,

(i) the interval-valued intuitionistic hesitant fuzzy hybrid averaging (IVIHFHA) operator is defined as follows:

$$\mathrm{IVIHFHA}_{w,\omega}(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \bigoplus_{i=1}^{n} w_i \tilde{h}'_{\sigma(i)} \tag{7}$$

where $\tilde{h}'_{\sigma(i)}$ is the $i$th largest value of is the weighted arguments $n\omega_j \tilde{h}_j,\ j = 1, 2, \ldots, n,\ \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector on $\{\tilde{h}_i\}_{i\in\{1,2,\ldots,n\}}$ with $\omega_i > 0$ and $\sum_{i=1}^{n} \omega_i = 1$, and $n$ is the balancing coefficient;

(ii) the interval-valued intuitionistic hesitant fuzzy hybrid geometric (IVIHFHG) operator is defined as follows:

$$\mathrm{IVIHFHG}_{w,\omega}(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \bigotimes_{i=1}^{n} (\tilde{h}_{\sigma(i)})^{w_i} \tag{8}$$

where $\tilde{h}_{\sigma(i)}$ is the $i$th largest value of is the weighted arguments $\tilde{h}_j^{n\omega_j},\ j = 1, 2, \ldots, n$, and the other notations as shown in formula (7).

Recently, Joshi and Kumar (2016) noted that the Zhang's aggregation operators are based on the assumption that the importance of criteria is independent. To extend the application of IVIHFEs in decision making, Joshi and Kumar (2016) used the Choquet integral with respect to the λ-fuzzy measures to define the following interval-valued intuitionistic hesitant fuzzy Choquet integral (IVIHFCI) operator:

DEFINITION 6 (Joshi and Kumar, 2016). Let $\mu$ be a fuzzy measure on $X = \{x_1, x_2, \ldots, x_n\}$, and $\tilde{h}_i$, $i = 1, 2, \ldots, n$, be a collection of IVIHFEs on $X$. The IVIHFCI operator is a mapping IVIHFCI: $\tilde{H}^n \to \tilde{H}$, defined as:

$$\begin{aligned}
&\mathrm{IVIHFCI}(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n)\\
&\quad = \textstyle\bigotimes_{i=1}^{n} (\tilde{h}_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\\
&\qquad \Big\{\Big[\textstyle\prod_{i=1}^{n} \big(\mu_l^{(i)}\big)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{n} \big(\mu_u^{(i)}\big)^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big],\\
&\qquad \Big[1 - \textstyle\prod_{i=1}^{n} \big(1 - v_l^{(i)}\big)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1 - \prod_{i=1}^{n} \big(1 - v_u^{(i)}\big)^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Big]\\
&\qquad \Big|\tilde{\alpha}_{(i)} \in \tilde{h}_{(i)},\ i = 1, 2, \ldots, N\Big\},
\end{aligned} \tag{9}$$

where $(\cdot)$ is a permutation on the subscripts of the elements in $X$ such that $\tilde{h}_{(1)} \leqslant \tilde{h}_{(2)} \leqslant \cdots \leqslant \tilde{h}_{(n)}$, $A_{(i)} = \{\tilde{h}_{(i)}, \ldots, \tilde{h}_{(n)}\}$ and $A_{(n+1)} = \emptyset$.

REMARK 3. Although the IVIHFCI operator can cope with the situations where the importance of criteria in a decision-making problem is interactive, it is not a good tool to calculate the comprehensive values of the objects. As well known, the fundamental principle of the Yager's OWA operator is to rearrange the considered elements, and their weights are only related to the ordered positions. The Choquet integral, as an extension of the OWA operator, also has this property, namely, the fuzzy measure should be defined on the ordered position set rather than on the criteria set. Otherwise, when the Choquet integral operator is applied to calculate the objects' comprehensive values, it may endow the same criterion with the different weights just because the objects' criteria values are different. This seems to be undesirable.

EXAMPLE 1 (Joshi and Kumar, 2016). Let us consider the decision-making problem about selecting the project manager, where the IVIHFSs $\tilde{E}_{A_1}$, $\tilde{E}_{A_2}$ and $\tilde{E}_{A_3}$ for the candidates $A_1$, $A_2$ and $A_3$ with respect to the criteria $\{C_1$: knowledge; $C_2$: reliability; $C_3$: demanding $\}$ are defined as follows:

$$\begin{aligned}\tilde{E}_{A_1} = {} & \big\{\big\langle C_1, \big\{\big([0.7, 0.9], [0.1, 0.1]\big), \big([0.6, 0.8], [0.1, 0.2]\big), \big([0.3, 0.4].[0.6, 0.6]\big)\big\}\big\rangle, \\ & \langle C_2\big\{\big([0.5, 0.6], [0.2, 0.3]\big), \big([0.1, 0.1], [0.8, 0.9]\big)\big\}\rangle, \\ & \langle C_3, \big\{\big([0.8, 0.9], [0.1, 0.1]\big)\big\}\rangle\big\}, \\ \tilde{E}_{A_2} = {} & \big\{\big\langle C_1, \big\{\big([0.3, 0.5], [0.4, 0.5]\big)\big\}\big\rangle, \\ & \big\langle C_2, \big\{\big([0.8, 0.9], [0.1, 0.1]\big), \big([0.5, 0.7], [0.1, 0.2]\big)\big\}\big\rangle, \\ & \big\langle C_3, \big\{\big([0.2, 0.3], [0.5, 0.6]\big), \big([0.1, 0.3], [0.6, 0.6]\big)\big\}\big\rangle\big\}\end{aligned}$$

and

$$\begin{aligned}\tilde{E}_{A_3} = {} & \big\{\big\langle C_1, \big\{\big([0.2, 0.4], [0.3, 0.5]\big), \big([0.5, 0.7], [0.1, 0.2]\big)\big\}\big\rangle, \\ & \big\langle C_2, \big\{\big([0.1, 0.1], [0.7, 0.9]\big)\big\}\big\rangle, \big\langle C_3, \big\{\big([0.1, 0.3], [0.6, 0.7]\big), \\ & \big([0.2, 0.2], [0.7, 0.8]\big), \big([0.3, 0.4], [0.6, 0.6]\big)\big\}\rangle\big\}.\end{aligned}$$

According to the ranking method on IVIHFEs, we derive $\tilde{h}_2(A_1) < \tilde{h}_1(A_1) < \tilde{h}_3(A_1)$, $\tilde{h}_3(A_3) < \tilde{h}_1(A_3) < \tilde{h}_2(A_3)$ and $\tilde{h}_2(A_4) < \tilde{h}_3(A_4) < \tilde{h}_1(A_4)$ with respect to $\tilde{E}_{A_1}$, $\tilde{E}_{A_2}$ and $\tilde{E}_{A_3}$, respectively. When $\mu(C_1) = \mu(C_3) = 0.4$ and $\mu(C_2) = 0.3$, using formula (3) we derive $\lambda = -0.258$. Thus, the fuzzy measures are

$$\mu(C_1, C_2) = \mu(C_2, C_3) = 0.669, \qquad \mu(C_1, C_3) = 0.769, \qquad \mu(C_1, C_2, C_3) = 1.$$

From the IVIHFCI operator, we know that the weights of the criteria $C_1$, $C_2$, and $C_3$ are 0.369, 0.231 and 0.4 for the candidate $A_1$, their weights are respective of 0.369, 0.3 and 0.331 for the candidate $A_2$, while they are 0.4, 0.231 and 0.369 for the candidate $A_3$ (see Table 1). None of them are the same for these candidates. This means that the evaluations of these three candidates are based on the different standards. It is unreasonable to give their ranking orders by using the comprehensive IVIHFEs obtained from the IVIHFCI operator.

Table 1
The weights of the criteria with respect to the different candidates.

| The candidates | The weights of the criteria | | |
|---|---|---|---|
| | $C_1$ | $C_2$ | $C_3$ |
| $A_1$ | 0.369 | 0.231 | 0.400 |
| $A_2$ | 0.369 | 0.300 | 0.331 |
| $A_3$ | 0.400 | 0.231 | 0.369 |

Remark 4. Although the $\lambda$-fuzzy measures can simplify the complexity of determining fuzzy measures, especially, when the set has a large number of elements, there are some limitations. From formula (3), one can verify that $-1 < \lambda < 0$, when the sum of each element's weight is bigger than one; If the sum of each element's weight is smaller than one, we derive $\lambda > 0$, and we have $\lambda = 0$ when the sum of each element's weight is equal to one. This shows that their interactions are completely determined by their respective weights.

Furthermore, the $\lambda$-fuzzy measures can only reflect the complementary interactions, redundant interactions or independency between the elements' weights. The question is when the exact fuzzy measure on an element set is unknown, how can we ensure that there are only complementary interactions, redundant interactions or independency between their weights? Furthermore, the interactive characteristics should be determined by the characteristics of criteria themselves rather than their respective weights.

These properties indicate that the $\lambda$-fuzzy measures are unsuitable to give the interactions between the weights of criteria. For example, Grabisch (1995) introduced the well-known example to show the application of the Choquet integral, which is about the evaluation of the students in relation to three subjects: {mathematics, physics, literature}, where the importance is respectively defined by 0.375, 0.375 and 0.3. On the other hand, more importance is attributed to science-related subjects than to literature, but some advantages are given to the students that are good both in literature and in any of the science-related subjects. In this situation, the $\lambda$-fuzzy measures seem to be helpless since $\lambda = 0$ by formula (3), which cannot reflect interactions between the weights of these three subjects.

In this example, one can find that the interactions between the weights of the criteria have no direct relationship with respect to the sum of each criterion's weight. Furthermore, there may be complementary and redundant interactions simultaneously.

## 3. New Interval-Valued Intuitionistic Hesitant Fuzzy Aggregation Operators

Considering the previous aggregation operators that cannot well address the situation where the weights of elements in a set are interactive, the section defines several new ones. Before defining the new aggregation operators, let us first introduce the following two new operations on IVIHFEs:

Definition 7. Let $\tilde{h}_1$ and $\tilde{h}_2$ be any two IVIHFEs in $\tilde{H}$. Then,

(i) $\lambda_1\tilde{h}_1 \oplus \lambda_2\tilde{h}_2 = \{\lambda_1\tilde{\alpha}_i \oplus \lambda_2\tilde{\alpha}_j|\tilde{\alpha}_i \in \tilde{h}_1, \tilde{\alpha}_j \in \tilde{h}_2\} = \{([\lambda_1\mu_l^i + \lambda_2\mu_l^j, \lambda_1\mu_u^i + \lambda_2\mu_u^j], [\lambda_1 v_l^i + \lambda_2 v_l^j, \lambda_1 v_u^i + \lambda_2 v_u^j])|\tilde{\alpha}_i \in \tilde{h}_1, \tilde{\alpha}_j \in \tilde{h}_2\}$,

(ii) $\tilde{h}_1^{\lambda_1} \otimes \tilde{h}_2^{\lambda_2} = \{\tilde{\alpha}_i^{\lambda_1} \otimes \tilde{\alpha}_j^{\lambda_2} | \tilde{\alpha}_i \in \tilde{h}_1, \tilde{\alpha}_j \in \tilde{h}_2\} = \{([(\mu_l^i)^{\lambda_1}(\mu_l^j)^{\lambda_2}, (\mu_u^i)^{\lambda_1}(\mu_u^j)^{\lambda_2}], [(v_l^i)^{\lambda_1}(v_l^j)^{\lambda_2}, (v_u^i)^{\lambda_1}(v_u^j)^{\lambda_2}]) | \tilde{\alpha}_i \in \tilde{h}_1, \tilde{\alpha}_j \in \tilde{h}_2\}$, where $\lambda_1, \lambda_2 \in [0,1]$ with $\lambda_1 + \lambda_2 \leqslant 1$.

Remark 5. The above new operations can avoid some limitations that are listed in Section 2.2. Let $\tilde{h}_1$ and $\tilde{h}_2$ be any two IVIHFEs, if we have $\tilde{h}_1 < \tilde{h}_2$ according to formulae (5) and (6). Then, we might derive $\begin{cases} \tilde{h}_1 \oplus \tilde{h} > \tilde{h}_2 \oplus \tilde{h} \\ \tilde{h}_1 \otimes \tilde{h} > \tilde{h}_2 \otimes \tilde{h} \end{cases}$ for some IVIHFE $\tilde{h}$ and $\begin{cases} \lambda\tilde{h}_1 > \lambda\tilde{h}_2 \\ \tilde{h}_1^{\lambda} > \tilde{h}_2^{\lambda} \end{cases}$ for $\lambda \in [0,1]$. For example, let $\tilde{h}_1 = \{[0.3, 0.4], [0.4, 0.6], [0.4, 0.6], [0.3, 0.4]\}$, $\tilde{h}_2 = \{([0.2, 0.3], [0.2, 0.3], ([0.4, 0.5], [0.3, 0.5])\}$ and $\tilde{h} = \{([0.6, 0.7], [0.1, 0.2])\}$. According to formula (5), we derive $\tilde{h}_1 < \tilde{h}_2$ for $s(\tilde{h}_1) = 0$ and $s(\tilde{h}_1) = 0.05$. However, we derive $\tilde{h}_1 \oplus \tilde{h} > \tilde{h}_2 \oplus \tilde{h}$ according to the first operational law listed in Section 2.2, where $s(\tilde{h}_1 \oplus \tilde{h}) = 0.7275 > s(\tilde{h}_2 \oplus \tilde{h}) = 0.7125$ with $\tilde{h}_1 \oplus \tilde{h} = \{([0.72, 0.82], [0.04, 0.12]), ([0.76, 0.88], [0.03, 0.08])\}$ and $\tilde{h}_2 \oplus \tilde{h} = \{([0.68, 0.79], [0.02, 0.06])([0.76, 0.85], [0.03, 0.01])\}$. Furthermore, let $\lambda = 0.3$, we get $\tilde{h}_1^{\lambda} > \tilde{h}_2^{\lambda}$ using the fourth operational law listed in Section 2.2, where $s(\tilde{h}_1^{\lambda}) = 0.6120 > s(\tilde{h}_2^{\lambda}) = 0.6076$ with $\tilde{h}_1^{\lambda} = \{([0.69, 0.76], [0.14, 0.24]), ([0.76, 0.86], [0.10, 0.14])\}$ and $\tilde{h}_2^{\lambda} = \{([0.62, 0.69], [0.06, 0.10]), ([0.76, 0.81], [0.10, 0.19])\}$. For the other two operations, one can similarly derive the above conclusions.

Remark 6. Without loss of generality, all of the following operations use the operational laws offered in Definition 7.

From the analysis in Section 2.3, we know that it is unreasonable to calculate the compared objects' comprehensive IVIHFEs by using the IVIHFCI operator. To reflect the interactions between the weights of the criteria as well as to ensure the weights of criteria are the same for all objects, we apply the Shapley function in game theory (Shapley, 1953) to define two hybrid aggregation operators on IVIHFEs.

The Shapley function (Shapley, 1953) is one of the most important payoff indices in cooperative game theory, which satisfies many desirable properties. When the Shapley function is restricted in the setting of fuzzy measures, we derive:

$$Sh_{x_i}(\mu, X) = \sum_{S \subseteq X \setminus x_i} \frac{(n-s-1)!s!}{n!}\big(\mu(S \cup x_i) - \mu(S)\big), \quad \forall x_i \in X, \tag{10}$$

where $\mu$ is a fuzzy measure on $X = \{x_1, x_2, \ldots, x_n\}$, $n$ and $s$ denote the numbers of the elements in $X$ and $S$, respectively.

Remark 7. From formula (10), we know that the Shapley function gives the global interactions between each element $x_i$ and all coalitions in $X_i$. When $X$ denotes the criteria set in a decision-making problem, $\mu(S \cup x_i) - \mu(S)$ can be viewed as the contribution of the $x_i$'s importance to the importance of the coalition $S$.

PROPERTY 1. Let $\mu$ be a fuzzy measure on $X = \{x_1, x_2, \dots, x_n\}$, and $Sh$ be the Shapley function as shown in formula (10).

(i) Let $x_i$ and $x_j$ be two elements in $X$. If we have $\mu(S \cup x_i) = \mu(S \cup x_j)$ for all $S \subseteq X \backslash \{x_i, x_j\}$, then $Sh_{x_i}(\mu, X) = Sh_{x_j}(\mu, X)$;
(ii) Let $x_i$ be an element in $X$. If we have $\mu(S \cup x_i) = \mu(S)$ for all $S \subseteq X \backslash x_i$, then $Sh_{x_i}(\mu, X) = 0$;
(iii) Let $x_i$ be an element in $X$. If we have $\mu(S \cup x_i) = \mu(S) + \mu(x_i)$ for all $S \subseteq X \backslash x_i$, then $Sh_{x_i}(\mu, X) = \mu(x_i)$;
(iv) The Shapley value vector $\{Sh_{x_i}(\mu, X)\}_{x_i \in X}$ is a normalized weight vector on $X$, namely, $Sh_{x_i}(\mu, X) \geqslant 0$ for all $x_i \in X$ and $\sum_{x_i \in X} Sh_{x_i}(\mu, X) = 1$.

*Proof.* From formula (10), one can easily derive the conclusions listed in Property 1. □

REMARK 8. The first conclusion shows that when two elements have the same contribution to all coalitions in $X \backslash \{x_i, x_j\}$, then they have the same importance. Especially, when $\mu$ is an additive measure, we derive $\mu(x_i) = \mu(x_j)$. The second conclusion shows that when the element $x_i$ has no contribution to the importance of all coalitions in $X \backslash x_i$, then its importance is zero. The third conclusion indicates that when the contribution of the element $x_i$ to the importance of all coalitions in $X \backslash x_i$ is $\mu(x_i)$, then its importance equals to itself. The last conclusion shows that the vector composed by the elements' Shapley values is a weight vector. All these conclusions can be seen as a natural extension of additive measures.

Now, let us define the following interval-valued intuitionistic hesitant fuzzy Shapley weighted averaging (IVIHFSWA) operator and interval-valued intuitionistic hesitant fuzzy Shapley weighted geometric mean (IVIHFSWGM) operator.

DEFINITION 8. Let $\mu$ be a fuzzy measure on $X = \{x_1, x_2, \dots, x_n\}$, and $\tilde{h}_i$, $i = 1, 2, \dots, n$, be a collection of IVIHFEs for $X$.

(i) The IVIHFSWA operator is defined as follows:

$$\begin{aligned} &\text{IVIHFSWA}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) \\ &\quad = \bigoplus_{i=1}^n (Sh_{x_i}(\mu, X)\tilde{h}_i \\ &\quad = \Big\{\Big[\sum_{i=1}^n Sh_{x_i}(\mu, X)\mu_l^i, \sum_{i=1}^n Sh_{x_i}(\mu, X)\mu_u^i\Big], \\ &\qquad \Big[\sum_{i=1}^n Sh_{x_i}(\mu, X)v_l^i, \sum_{i=1}^n Sh_{x_i}(\mu, X)v_u^i\Big] \Big| \tilde{\alpha}_i \in \tilde{h}_i, \imath = 1, 2, \dots, n\Big\}; \end{aligned} \tag{11}$$

(ii) The IVIHFSWGM operator is defined as follows:

$$\begin{aligned} &\text{IVIHFSWGM}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) \\ &\quad = \bigotimes_{i=1}^n (\tilde{h}_i)^{Sh_{x_i}(\mu, X)} \\ &\quad = \Big\{\Big[\prod_{i=1}^n (\mu_l^i)^{Sh_{x_i}(\mu, X)}, \prod_{i=1}^n (\mu_u^i)^{Sh_{x_i}(\mu, X)}\Big], \\ &\qquad \Big[\prod_{i=1}^n (v_l^i)^{Sh_{x_i}(\mu, X)}, \prod_{i=1}^n (v_u^i)^{Sh_{x_i}(\mu, X)}\Big] \Big| \tilde{\alpha}_i \in \tilde{h}_i,\ i = 1, 2, \dots, n\Big\}, \end{aligned} \tag{12}$$

where $Sh_{x_i}(\mu, X)$ is the Shapley value of the element $x_i$ with respect to the fuzzy measure $\mu$.

Remark 9. One easily shows that all properties for the IVIHFCI operator still hold for the IVIHFSWA and IVIHFSWGM operators.

In EXAMPLE 1, when we apply the criteria's Shapley values as their weights, we derive

$$Sh_{C_1}(\mu, C) = Sh_{C_3}(\mu, C) = 0.367, \qquad Sh_{C_2}(\mu, C) = 0.267,$$

which addresses the issue in the IVIHFCI operator.

Just as some researchers (Lin and Jiang, 2014; Merigo and Casanovas, 2009; Xu and Da, 2002; Xu, 2004) noted, the IVIHFSWA and IVIHFSWGM operators only consider the importance of the elements, but the importance of the ordered positions is not enclosed. To deal with this issue, we define the interval-valued intuitionistic hesitant fuzzy hybrid Shapley weighted averaging (IVIHFHSWA) operator and the interval-valued intuitionistic hesitant fuzzy hybrid Shapley weighted geometric mean (IVIHFHSWGM) operator as follows:

Definition 9. Let $\mu$ be a fuzzy measure on $X = \{x_1, x_2, \dots, x_n\}$, let $v$ be a fuzzy measure on the ordered position set $N = \{1, 2, \dots, n\}$, and let $\tilde{h}_i$ $(i = 1, 2, \dots, n)$ be a collection of IVIHFEs on $X$.

(i) The IVIHFHSWA operator is defined as follows:

$$\begin{aligned} &\mathrm{IVIHFHSWA}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) \\ &= \frac{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)\tilde{h}_{(j)}}{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)} \\ &= \Bigg\{\Bigg[\frac{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)\mu_l^{(j)}}{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)}, \frac{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)\mu_u^{(j)}}{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)}\Bigg], \\ &\Bigg[\frac{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)v_l^{(j)}}{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)}, \frac{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)v_u^{(j)}}{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)}\Bigg] \\ &\Bigg| \tilde{\alpha}_i \in \tilde{h}_i, \imath = 1, 2, \dots, n\Bigg\}; \end{aligned} \tag{13}$$

(ii) The IVIHFHSWGM operator is defined as follows:

$$\begin{aligned} &\mathrm{IVIHFHSWA}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) \\ &= \bigotimes_{i=1}^{n} (\tilde{h}_{(j)})^{\frac{Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)}{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)}} \\ &= \Bigg\{\Bigg[\prod_{j=1}^{n} \big(\mu_l^{(j)}\big)^{\frac{Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)}{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)}}, \prod_{j=1}^{n} \big(\mu_u^{(j)}\big)^{\frac{Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)}{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)}}\Bigg], \\ &\prod_{j=1}^{n} \big(v_l^{(j)}\big)^{\frac{Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)}{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)}}, \prod_{j=1}^{n} \big(v_u^{(j)}\big)^{\frac{Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)}{\sum_{j=1}^{n} Sh_j(v,N) Sh_{x_{(j)}}(\mu,X)}} \\ &\Bigg| \tilde{\alpha}_i \in \tilde{h}_i,\ i = 1, 2, \dots, n\Bigg\}, \end{aligned} \tag{14}$$

where $(\cdot)$ is a permutation on $Sh_{x_i}(\mu, X)\tilde{h}_i$, $i = 1, 2, \dots, n$, such that $Sh_{x_{(j)}}(\mu, X)\tilde{h}_{(j)}$ is the $j$th smallest value of $Sh_{x_i}(\mu, X)\tilde{h}_i$, $i = 1, 2, \dots, n$ for the IVIHFHSWA operator, and $(\cdot)$ is a permutation on $\tilde{h}_i^{Sh_{x_i}(\mu,X)}$, $i = 1, 2, \dots, n$, such that $\tilde{h}_{(j)}^{Sh_{x_{(j)}}(\mu,X)}$ is the $j$ th smallest value of $\tilde{h}_i^{Sh_{x_i}(\mu,X)}$, $i = 1, 2, \dots, n$, for the IVIHFHSWGM operator, $Sh_j(v, N)$ is the Shapley value of the $j$th ordered position with respect to the fuzzy measure $v$, and $Sh_{x_i}(\mu, X)$ is the Shapley value of the element $x_i$ for the fuzzy measure $G$.

Remark 10. The IVIHFHSWA and IVIHFHSWGM operators not only consider the importance of the elements and the ordered positions, but also reflect their respective interactions. Furthermore, these two operators can be seen as extensions of many interval-valued intuitionistic hesitant fuzzy aggregation operators.

(i) When there are no interactions between the weights of the ordered positions in $N$ as well as between the weights of the elements in X, then the IVIHFHSWA operator degenerates to the interval-valued intuitionistic hesitant fuzzy hybrid weighted averaging (IVIHFHWA) operator

$$\begin{aligned}
&\mathrm{IVIHFHWA}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \frac{\sum_{j=1}^n w_j \omega_{x_{(j)}} \tilde{h}_{(j)}}{\sum_{j=1}^n w_j \omega_{x_{(j)}}} \\
&= \Biggl\{ \Biggl[ \frac{\sum_{j=1}^n w_j \omega_{x_{(j)}} \mu_l^{(j)}}{\sum_{j=1}^n w_j \omega_{x_{(j)}}}, \frac{\sum_{j=1}^n w_j \omega_{x_{(j)}} \mu_u^{(j)}}{\sum_{j=1}^n w_j \omega_{x_{(j)}}} \Biggr], \Biggl[ \frac{\sum_{j=1}^n w_j \omega_{x_{(j)}} v_l^{(j)}}{\sum_{j=1}^n w_j \omega_{x_{(j)}}}, \frac{\sum_{j=1}^n w_j \omega_{x_{(j)}} v_u^{(j)}}{\sum_{j=1}^n w_j \omega_{x_{(j)}}} \Biggr] \\
&\tilde{\alpha}_i \in \tilde{h}_i,\ i = 1, 2, \dots, n \Biggr\},
\end{aligned}$$

and the IVIHFHSWGM operator degenerates to the interval-valued intuitionistic hesitant fuzzy hybrid weighted geometric mean (IVIHFHWGM) operator

$$\begin{aligned}
&\mathrm{IVIHFHWGM}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \bigotimes_{i=1}^n (\tilde{h}_{(j)})^{\frac{w_j \omega_{x_{(j)}}}{\sum_{j=1}^n w_j \omega_{x_{(j)}}}} \\
&= \Biggl\{ \Biggl[ \prod_{j=1}^n \bigl(\mu_l^{(j)}\bigr)^{\frac{w_j \omega_{x_{(j)}}}{\sum_{j=1}^n w_j \omega_{x_{(j)}}}}, \bigl(\mu_u^{(j)}\bigr)^{\frac{w_j \omega_{x_{(j)}}}{\sum_{j=1}^n w_j \omega_{x_{(j)}}}} \Biggr], \\
&\quad \Biggl[ \bigl(v_l^{(j)}\bigr)^{\frac{w_j \omega_{x_{(j)}}}{\sum_{j=1}^n w_j \omega_{x_{(j)}}}}, \bigl(v_u^{(j)}\bigr)^{\frac{w_j \omega_{x_{(j)}}}{\sum_{j=1}^n w_j \omega_{x_{(j)}}}} \Biggr] \Bigg| \tilde{\alpha}_i \in \tilde{h}_i, i = 1, 2, \dots, n \Biggr\},
\end{aligned}$$

where $\omega = (\omega_{x_1}, \omega_{x_2}, \dots, \omega_{x_n})$ is an additive weight vector on $X$, $w = (w_1, w_2, \dots, w_n)$ is an additive weight vector on $N$, $(\cdot)$ is a permutation on $\omega_{x_i}\tilde{h}_i$, $i = 1, 2, \dots, n$, such that $\omega_{x_{(j)}}\tilde{h}_{(j)}$ is the $j$th smallest value of $\omega_{x_i}\tilde{h}_i$, $i = 1, 2, \dots, n$, for the IVIHFHWA operator, and $(\cdot)$ is a permutation on $\tilde{h}_i^{\omega_{x_i}}$, $i = 1, 2, \dots, n$, such that $\tilde{h}_{(j)}^{\omega_{x_{(j)}}}$ is the $j$th smallest value of $\tilde{h}_i^{\omega_{x_i}}$, $i = 1, 2, \dots, n$, for the IVIHFHWGM operator.

(ii) When there are no interactions between the weights of the ordered positions in $N$ and $w = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, then the IVIHFHSWA operator degenerates to the IVIHFSWA operator, and the IVIHFHSWGM operator degenerates to the IVIHFSWGM operator.

(iii) When there are no interactions between the weights of the elements in $X$ and $\omega = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, then the IVIHFHSWA operator degenerates to the interval-valued intuitionistic hesitant fuzzy ordered Shapley weighted averaging (IVIHFOSWA) operator

$$\begin{aligned}&\mathrm{IVIHFOSWA}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \textstyle\sum_{j=1}^n Sh_j(v, N)\tilde{h}_{(j)}\\ &\quad = \bigl\{\bigl[\textstyle\sum_{j=1}^n Sh_j(v, N)\mu_l^{(j)}, \sum_{j=1}^n Sh_j(v, N)\mu_u^{(j)}\bigr],\\ &\qquad \bigl[\textstyle\sum_{j=1}^n Sh_j(v, N)v_l^{(j)}, \sum_{j=1}^n Sh_j(v, N)v_u^{(j)}\bigr]\big|\tilde{\alpha}_i \in \tilde{h}_i,\ i = 1, 2, \dots, n\bigr\},\end{aligned}$$

and the IVIHFHSWGM operator degenerates to the interval-valued intuitionistic hesitant fuzzy ordered Shapley weighted geometric mean (IVIHFOSWGM) operator

$$\begin{aligned}&\mathrm{IVIHFOSWGM}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \textstyle\bigotimes_{i=1}^n \tilde{h}_{(j)}^{Sh_j(v,N)}\\ &\quad = \bigl\{\bigl[\textstyle\prod_{j=1}^n (\mu_l^{(j)})^{Sh_j(v,N)}, \prod_{j=1}^n (\mu_u^{(j)})^{Sh_j(v,N)}\bigr],\\ &\qquad \bigl[\textstyle\prod_{j=1}^n (v_l^{(j)})^{Sh_j(v,N)}, \prod_{j=1}^n (v_u^{(j)})^{Sh_j(v,N)}\bigr]\big|\tilde{\alpha}_i \in \tilde{h}_i,\ i = 1, 2, \dots, n\bigr\},\end{aligned}$$

where $(\cdot)$ is a permutation on the IVIHFEs $\tilde{h}_i$, $i = 1, 2, \dots, n$, such that $\tilde{h}_{(j)}$ is the $j$th smallest value of $\tilde{h}_i$, $i = 1, 2, \dots, n$.

Furthermore, when there are no interactions between the weights of the ordered positions in $N$, then the IVIHFOSWA operator reduces to the interval-valued intuitionistic hesitant fuzzy ordered weighted averaging (IVIHFOWA) operator

$$\begin{aligned}&\mathrm{IVIHFOWA}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \textstyle\sum_{j=1}^n w_j\tilde{h}_{(j)}\\ &\quad = \bigl\{\bigl[\textstyle\sum_{j=1}^n w_j\mu_l^{(j)}, \sum_{j=1}^n w_j\mu_u^{(j)}\bigr],\\ &\qquad \bigl[\textstyle\sum_{j=1}^n w_j v_l^{(j)}, \sum_{j=1}^n w_j v_u^{(j)}\bigr]\big|\tilde{\alpha}_i \in \tilde{h}_i,\ i = 1, 2, \dots, n\bigr\},\end{aligned}$$

and the IVIHFOSWGM operator reduces to the interval-valued intuitionistic hesitant fuzzy ordered weighted geometric mean (IVIHFOWGM) operator

$$\begin{aligned}&\mathrm{IVIHFOWGM}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \textstyle\bigotimes_{i=1}^n \tilde{h}_{(j)}^{w_j}\\ &\quad = \bigl\{\bigl[\textstyle\prod_{j=1}^n (\mu_l^{(j)})^{w_j}, \prod_{j=1}^n (\mu_u^{(j)})^{w_j}\bigr], \bigl[\prod_{j=1}^n (v_l^{(j)})^{w_j}, \prod_{j=1}^n (v_u^{(j)})^{w_j}\bigr]\\ &\qquad \big|\tilde{\alpha}_i \in \tilde{h}_i,\ i = 1, 2, \dots, n\bigr\},\end{aligned}$$

where $w = (w_1, w_2, \dots, w_n)$ is an additive weight vector on $N$, and $(\cdot)$ is a permutation on the IVIHFEs $\tilde{h}_i$, $i = 1, 2, \dots, n$, such that $\tilde{h}_{(j)}$ is the $j$th smallest value of $\tilde{h}_i$, $i = 1, 2, \dots, n$.

To overcome the limitations of the λ-fuzzy measures, 2-additive measures introduced by Grabisch (1997) are good choices to reduce the complexity of determining fuzzy measures.

Definition 10 (Grabisch, 1997). A fuzzy measure $\mu$ on $N = \{1, 2, \dots, n\}$ is said to be a 2-additive measure, if, for any $S \subseteq N$ with $s \geqslant 2$, we have

$$\mu(S) = \sum_{\{i,j\} \subseteq S} \mu(i, j) - (s-2) \sum_{i \in S} \mu(i) \tag{15}$$

where $s$ is the cardinality of $S$.

From the concept of 2-additive measures, it only needs $n(n-1)/2$ coefficients to determine a fuzzy measure on a set with $n$ elements. Furthermore, we have the following conclusion.

**Theorem 1** (Grabisch, 1997). *Let $\mu$ be a fuzzy measure on $N = \{1, 2, \dots, n\}$, then $\mu$ is a 2-additive measure if and only if there exist coefficients $\mu(i)$ and $\mu(i, j)$ for all $i, j \in N$ that satisfy the following conditions:*

(i) $\mu(i) \geqslant 0$, $i \in N$,
(ii) $\sum_{\{i,j\} \subseteq N} \mu(i, j) - (n-2) \sum\limits_{i \in N} \mu(i) = 1$,
(iii) $\sum_{i \subseteq S \setminus k} (\mu(i, k) - \mu(i)) \geqslant (s-2)\mu(k)$ $\forall S \in N$ s.t. $k \in S$ *with* $s \geqslant 2$,

*where $s$ and $n$ denote the cardinalities of $S$ and $N$, respectively.*

When $\mu$ is a 2-additive measure, Meng and Tang (2013) gave the following conclusion:

**Theorem 2** (Meng and Tang, 2013). *Let $\mu$ be a 2-additive measure defined on $N = \{1, 2, \dots, n\}$, then the Shapley function $Sh$ can be expressed as follows:*

$$Sh_i(\mu, N) = \frac{3-n}{2}\mu(i) + \frac{1}{2} \sum_{j \in N \setminus i} (\mu(i, j) - \mu(j)), \quad \forall i \in N. \tag{16}$$

Remark 11. From formula (16), we know that $n(n+1)/2$ coefficients are needed to determine a 2-additive measure on a set with $n$ elements. Although to determine a 2-additive measure demands more coefficients than to derive a $\lambda$-fuzzy measure, 2-additive measures can reflect the complementary interactions, redundant interactions and independency between the weights of the elements simultaneously. Furthermore, their interactive characteristics have no relationship to the sum of each criterion's weight in the setting of 2-additive measures. Thus, it is more reasonable to use 2-additive measures than to apply $\lambda$-fuzzy measures.

When the fuzzy measure $\mu$ and the fuzzy measure $v$ are both a 2-additive measure, then we derive the associated IVIHFHSWA and IVIHFHSWGM operators with respect to 2-additive measures. It is worth noting that both the IVIHFSWG operator and the IVIHFHSWG operator overcome the issues in the IVIHFCI operator.

## 4. Hamming Distance Based Models for the Optimal Fuzzy Measures

This section focuses on how to determine the optimal fuzzy measures when the weight information is partly known. Similarly, the 2-additive measures can also be derived by using the built models.

### 4.1. *A New Hamming Distance Measure on IVIHFEs*

Joshi and Kumar (2016) defined the hamming distance on IVIHFEs to rank objects, but this hamming distance requires the calculated IVIHFEs to have the same length; otherwise, we need to add some IVIFVs in the shorter IVIHFE several times until its length equals to the longer one.

Example 2. Let $\tilde{h}_1$ and $\tilde{h}_2$ be two IVIHFEs, where

$$\begin{aligned}\tilde{h}_1 &= \big\{\tilde{\alpha}_1 = \big([0.3, 0.4], [0.1, 0.2]\big), \tilde{\alpha}_2 = \big([0.5, 0.6], [0.2, 0.3]\big), \\ &\quad \tilde{\alpha}_3 = \big([0.6, 0.7], [0.1, 0.2]\big)\big\}\end{aligned}$$

and

$$\tilde{h}_2 = \big\{\tilde{\beta}_1 = \big([0.1, 0.2], [0.4, 0.5]\big), \tilde{\beta}_2 = \big([0.3, 0.4], [0.1, 0.2]\big)\big\}.$$

According to Definition 2.5 in Joshi and Kumar (2016), we need to add one IVIFV into $\tilde{h}_2$. When we add the first IVIFV in $\tilde{h}_2$, we derive $\tilde{h}'_2 = \{\tilde{\beta}_1, \tilde{\beta}_1, \tilde{\beta}_2\}$. Then, we can apply formula (8) in (Joshi and Kumar, 2016) to calculate the hamming distance between $\tilde{h}_1$ and $\tilde{h}_2$, which is in fact the hamming distance between $\tilde{h}_1$ and $\tilde{h}'_2$. However, one can check that $\tilde{h}'_2$ and $\tilde{h}_2$ are not equivalent. Using formula (5), we obtain their scores $s(\tilde{h}_2) = -0.05$ and $s(\tilde{h}'_2) = -0.13$, by which we get $\tilde{h}_2 > \tilde{h}'_2$. On the other hand, when we add the second IVIFV in $\tilde{h}_2$, we have $\tilde{h}''_2 = \{\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_2\}$. Using formula (5), we obtain its score $s(\tilde{h}''_2) = -0.03$, by which we get $\tilde{h}_2 < \tilde{h}''_2$. Furthermore, according to DEFINITION 2.5, the hamming distance between $\tilde{h}_1$ and $\tilde{h}_2$ is

$$d(\tilde{h}_1, \tilde{h}_2) = \frac{1}{3 \times 4}\big(|\tilde{\alpha}_1 - \tilde{\beta}_1| + |\tilde{\alpha}_2 - \tilde{\beta}_1| + |\tilde{\alpha}_3 - \tilde{\beta}_2|\big) = 0.23$$

for $\tilde{h}'_2$, and it is

$$d(\tilde{h}_1, \tilde{h}_2) = \frac{1}{3 \times 4}\big(|\tilde{\alpha}_1 - \tilde{\beta}_1| + |\tilde{\alpha}_2 - \tilde{\beta}_2| + |\tilde{\alpha}_3 - \tilde{\beta}_2|\big) = 0.18$$

for $\tilde{h}''_2$, where $|\tilde{\alpha}_i - \tilde{\beta}_j| = |\mu_l^i - \mu_l^j| + |\mu_u^i - \mu_u^j| + |v_l^i - v_l^j| + |v_u^i - v_u^j|$ with $i = 1, 2, 3$; $j = 1, 2$. Thus, it is unreasonable to add IVIFVs into IVIHFEs subjectively.

For $\tilde{\alpha}_1 = ([0.3, 0.4], [0.1, 0.2])$, we know $|\tilde{\alpha}_1 - \tilde{\beta}_1| = 1$ according to formula (8) in Joshi and Kumar (2016). However, we have $|\tilde{\alpha}_1 - \tilde{\beta}_2| = 0$ for $\tilde{\alpha}_1 = \tilde{\beta}_2$. In this case, we should choose the latter for $\tilde{\alpha}_1 \in \tilde{h}_2$. This also shows that it is undesirable to use formula (8) in (Joshi and Kumar, 2016) to calculate the hamming distance between IVIHFEs.

Next, we introduce a new hamming distance measure on IVIHFEs, which needn't consider the length of IVIHFEs and the arrangement of their possible IVIFVs.

Definition 11. Let $\tilde{h}_1$ and $\tilde{h}_2$ be two IVIHFEs. Without loss of generality, suppose that $\tilde{h}_1 = \{\tilde{\alpha}_i\}_{i=1,2,\dots,m}$ and $\tilde{h}_2 = \{\tilde{\beta}_j\}_{j=1,2,\dots,n}$, where $\tilde{\alpha}_i = ([\mu_l^i, \mu_u^i], [v_l^i, v_u^i])$ and $\tilde{\beta}_j = ([\eta_l^j, \eta_u^j], [\kappa_l^j, \kappa_u^j])$ for all $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$. Then, the distance measure from $\tilde{\alpha}_i$ to $\tilde{h}_2$ is defined as follows:

$$\overrightarrow{D(\tilde{\alpha}_i, \tilde{h}_2)} = \min_{\tilde{\beta}_j \in \tilde{h}_2} \frac{|\mu_l^i - \eta_l^j| + |\mu_u^i - \eta_u^j| + |v_l^i - \kappa_l^j| + |v_u^i - \kappa_u^j|}{4}. \tag{17}$$

Definition 12. Let $\tilde{h}_1$ and $\tilde{h}_2$ be two IVIHFEs, where $\tilde{h}_1 = \{\tilde{\alpha}_i\}_{i=1,2,\dots,m}$ and $\tilde{h}_2 = \{\tilde{\beta}_j\}_{j=1,2,\dots,n}$. Then, the hamming distance measure between $\tilde{h}_1$ and $\tilde{h}_2$ is defined as:

$$D(\tilde{h}_1, \tilde{h}_2) = \frac{\overrightarrow{D(\tilde{h}_1, \tilde{h}_2)} + \overrightarrow{D(\tilde{h}_2, \tilde{h}_1)}}{2}, \tag{18}$$

where $\overrightarrow{D(\tilde{h}_1, \tilde{h}_2)} = \frac{1}{4m}\sum_{i=1}^{m} \overrightarrow{D(\tilde{\alpha}_i, \tilde{h}_2)}$ and $\overrightarrow{D(\tilde{h}_2, \tilde{h}_1)} = \frac{1}{4n}\sum_{j=1}^{n} \overrightarrow{D(\tilde{\beta}_j, \tilde{h}_1)}$.

Property 2. Let $\tilde{h}_1$ and $\tilde{h}_2$ be any two IVIHFEs, where $\tilde{h}_1 = \{\tilde{\alpha}_i\}_{i=1,2,\dots,m}$ and $\tilde{h}_2 = \{\tilde{\beta}_j\}_{j=1,2,\dots,n}$. Then,

- (i) $D(\tilde{h}_1, \tilde{h}_2) = 0$ if and only if $\tilde{h}_1 = \tilde{h}_2$, namely, there exists $\tilde{\beta}_j$ such that $\tilde{\alpha}_i = \tilde{\beta}_j$ for all $\tilde{\alpha}_i \in \tilde{h}_1$, and there is $\tilde{\alpha}_i$ such that $\tilde{\beta}_j = \tilde{\alpha}_i$ for all $\tilde{\beta}_j \in \tilde{h}_2$;
- (ii) $D(\tilde{h}_1, \tilde{h}_2) = 1$ if and only if $\tilde{h}_1 = \{([1,1],[0,0])\}$ and $\tilde{h}_2 = \{([0,0],[1,1])\}$ or $\tilde{h}_1 = \{([0,0],[1,1])\}$ and $\tilde{h}_1 = \{([1,1],[0,0])\}$. Otherwise, we have $0 < D(\tilde{h}_1, \tilde{h}_2) < 1$ with $\tilde{h}_1 \neq \tilde{h}_2$;
- (iii) $D(\tilde{h}_1, \tilde{h}_2) = D(\tilde{h}_1, \tilde{h}_2)$.

*Proof.* From formulae (17) and (18), it is not difficult to derive the results in Property 2. □

In Example 2, when the new hamming distance measure on IVIHFEs is applied, we have

$$\overrightarrow{D(\tilde{h}_1, \tilde{h}_2)} = \frac{1}{12}\big(|\tilde{\alpha}_1 - \tilde{\beta}_2| + |\tilde{\alpha}_2 - \tilde{\beta}_2| + |\tilde{\alpha}_3 - \tilde{\beta}_2|\big) = \frac{1}{12}(0 + 0.6 + 0.6) = 0.1$$

and

$$\overrightarrow{D(\tilde{h}_1, \tilde{h}_2)} = \frac{1}{8}\big(|\tilde{\beta}_1 - \tilde{\alpha}_1| + |\tilde{\beta}_2 - \tilde{\alpha}_1|\big) = \frac{1}{8}(1 + 0) = 0.125,$$

by which the hamming distance between $\tilde{h}_1$ and $\tilde{h}_2$ is $D(\tilde{h}_1, \tilde{h}_2) = 0.1125$. It is different to the values obtained from Joshi and Kumar's hamming distance.

From the example given in Joshi and Kumar (2016), when we use the IVIHFSWA and IVIHFSWGM operators as well as the new hamming distance measure, the ranking order is $A_1 > A_3 > A_2 > A_4$, which is different from that obtained by using the Joshi and Kumar's method.

### 4.2. *Models for the Optimal Fuzzy Measures and 2-Additive Measures*

Considering a multi-criteria decision making problem, let $A = \{A_1, A_2, \dots, A_m\}$ be the set of alternatives, which are evaluated by a decision maker team according to the criteria set $C = \{C_1, C_2, \dots, C_n\}$ by using IVIFVs. When the decision makers have different opinions for the alternatives' criteria values, there can be several IVIFVs to denote the criteria values, which are denoted using IVIHFEs. Suppose that the evaluation of the alternative $A_i$ with respect to the attribute $C_j$ is an IVIHFE $\tilde{h}_{ij}$ $(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$. By $\tilde{H} = (\tilde{h}_{ij})_{m\times n}$, we denote the IVIHFE matrix given by the decision maker team.

When all criteria $C_j$, $j = 1, 2, \dots, n$, are benefit (i.e. the larger the greater preference), then the criteria values needn't normalization; otherwise, we normalize the IVIHFE matrix $\tilde{H} = (\tilde{h}_{ij})_{m\times n}$ into $\tilde{H}' = (\tilde{h}'_{ij})_{m\times n}$, where $\tilde{h}'_{ij} = \begin{cases} \tilde{h}_{ij} & \text{for benefit criteria } C_j \\ (\tilde{h}_{ij})^c & \text{for cost criteria } C_j \end{cases}$ with $(\tilde{h}_{ij})^c = \{(\tilde{\alpha}_{ij}^k)^c = ([v_{lij}^k, v_{uij}^k], [\mu_{lij}^k, \mu_{uij}^k]) | \tilde{\alpha}_{ij}^k = ([\mu_{lij}^k, \mu_{uij}^k], [v_{lij}^k, v_{uij}^k]) \in \tilde{h}_{ij}, k = 1, 2, \dots, \tilde{h}_{ij}\}$ for all $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$.

When the fuzzy measure $\mu$ on the criteria set $C$ and the fuzzy measure $v$ on the ordered position set $N$ are exactly known, we can adopt the aggregation operator given in Section 3 to calculate the collective values. However, when we do not own the fully weight information, it needs to first determine the weight vectors on them. Next, we construct Hamming distance measure based models to determine the optimal fuzzy measures and 2-additive measures on the criteria set $C$ and the ordered position set $N$, which can be seen as the extensions of the additive measures.

Because we cannot guarantee the importance of the criteria is independent, we apply the Shapley function to give the weights of the criteria. With respect to the normalized IVIHFE matrix $\tilde{H}' = (\tilde{h}'_{ij})_{m\times n}$, let

$$\tilde{h}'^{+}_{j} = \max_{1\leqslant i\leqslant m} \left\{ \max_{1\leqslant k\leqslant \#\tilde{h}_{ij}} \tilde{\alpha}'^{k}_{ij} \middle| \tilde{\alpha}'^{k}_{ij} \in \tilde{h}'_{ij} \right\}$$

and

$$\tilde{h}'^{-}_{j} = \min_{1\leqslant i\leqslant m} \left\{ \min_{1\leqslant k\leqslant \#\tilde{h}_{ij}} \tilde{\alpha}'^{k}_{ij} \middle| \tilde{\alpha}'^{k}_{ij} \in \tilde{h}'_{ij} \right\}$$

for each $j = 1, 2, \dots, n$.

Because the weight information makes the criteria values the bigger the better, when the weight information on the criteria set $C$ is not exactly known, we construct the following model to determine the optimal fuzzy measure $\mu$.

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{D(\tilde{h}'_{ij},\tilde{h}'^{+}_{j})}{D(\tilde{h}'_{ij},\tilde{h}'^{+}_{j})+D(\tilde{h}'_{ij},\tilde{h}'^{-}_{j})} Sh_{C_j}(\mu, C)$$
$$\text{s.t.}\begin{cases} \mu(S) \leqslant \mu(T), & \forall S \subseteq T \subseteq C,\\ \mu(C_i) \in W_{C_i}, & i=1,2,\ldots,n,\\ \mu(C_i) \geqslant 0, & i=1,2,\ldots,n, \end{cases} \tag{19}$$

where $W_{\mathrm{C}_i}$ is the known weight information of the criterion $C_j$, $j=1,2,\ldots,n$.

When the optimal fuzzy measure $\mu$ is a 2-additive measure, model (19) can be equivalently transformed into the following model:

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{D(\tilde{h}'_{ij},\tilde{h}'^{+}_{j})}{D(\tilde{h}'_{ij},\tilde{h}'^{+}_{j})+D(\tilde{h}'_{ij},\tilde{h}'^{-}_{j})} Sh_{C_j}(\mu, C)$$
$$\text{s.t.}\begin{cases} \sum_{C_j \in S\setminus C_i} \big(\mu(C_i, C_j) - \mu(C_j)\big) \geqslant (s-2)\mu(C_i), \forall S \subseteq C, \quad \forall C_i \in S,\ s \geqslant 2,\\ \sum\limits_{\{C_i, C_j\}\subseteq C} \mu(C_i, C_j) - (n-2) \sum\limits_{C_i \in C} \mu(C_i) = 1,\\ \mu(C_i) \in W_{C_i}, \quad i=1,2,\ldots,n,\\ \mu(C_i) \geqslant 0, \quad i=1,2,\ldots,n, \end{cases} \tag{20}$$

where the notations as listed in model (19).

From formula (16), we further have the following model for the optimal 2-additive measure $\mu$:

$$\min \tfrac{3-n}{2} \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{D(\tilde{h}'_{ij},\tilde{h}'^{+}_{j})}{D(\tilde{h}'_{ij},\tilde{h}'^{+}_{j})+D(\tilde{h}'_{ij},\tilde{h}'^{-}_{j})} \mu(C_j)$$
$$+ \frac{1}{2} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1,k\neq j}^{n} \frac{D(\tilde{h}'_{ij},\tilde{h}'^{+}_{j})}{D(\tilde{h}'_{ij},\tilde{h}'^{+}_{j})+D(\tilde{h}'_{ij},\tilde{h}'^{-}_{j})} \big(\mu(C_j, C_k) - \mu(C_k)\big)$$
$$\text{s.t.}\begin{cases} \sum_{\mathrm{C}_j \in \mathrm{S}\setminus \mathrm{C}_i} \big(\mu(\mathrm{C}_i, \mathrm{C}_j) - \mu(\mathrm{C}_j)\big) \geqslant (s-2)\mu(\mathrm{C}_i), \forall S \subseteq \mathrm{C}, \quad \forall \mathrm{C}_i \in S,\ s \geqslant 2,\\ \sum_{\{\mathrm{C}_i, \mathrm{C}_j\}\subseteq \mathrm{C}} \mu(\mathrm{C}_i, \mathrm{C}_j) - (n-2) \sum\limits_{\mathrm{C}_i \in \mathrm{C}} \mu(\mathrm{C}_i) = 1,\\ \mu(\mathrm{C}_i) \in W_{\mathrm{C}_i}, \quad i=1,2,\ldots,n,\\ \mu(\mathrm{C}_i) \geqslant 0, \quad i=1,2,\ldots,n. \end{cases} \tag{21}$$

To reduce the influence of the unduly high or low evaluation values induced by the decision makers' limited knowledge or decision expertise, when the weight information on the ordered position set $N$ is incompletely known, we construct the following model to determine the optimal fuzzy measure $v$ on the ordered position set $N$:

$$\begin{aligned}&\min \sum_{i=1}^{m}\left(\sum_{j=1}^{\mathrm{mid}(n)}(1-R_{i(j)})Sh_j(\mu,N)+\sum_{j=\mathrm{mid}(n)+1}^{n}R_{i(j)}Sh_j(\mu,N)\right)\\&\text{s.t.}\begin{cases}\mu(S)\leqslant\mu(T), & \forall S\subseteq T\subseteq N,\\ \mu(i)\in W_i, & i=1,2,\ldots,n,\\ \mu(i)\geqslant 0, & i=1,2,\ldots,n,\end{cases}\end{aligned}\tag{22}$$

where $R_{i(j)}=\frac{D(\tilde{h}_{i(j)},\tilde{h}^{+}_{(j)})}{D(\tilde{h}_{i(j)},\tilde{h}^{+}_{(j)})+D(\tilde{h}_{i(j)},\tilde{h}^{-}_{(j)})}$ is the $j$th smallest value for $R_{ij}=\frac{D(\tilde{h}_{ij},\tilde{h}^{+}_{j})}{D(\tilde{h}_{ij},\tilde{h}^{+}_{j})+D(\tilde{h}_{ij},\tilde{h}^{-}_{j})}$, $j=1,2,\ldots,n$, for each $i=1,2,\ldots,m$, $\mathrm{mid}(n)=\begin{cases}\frac{n}{2} & n \text{ is an even number,}\\ \frac{n+1}{2} & n \text{ is an odd number,}\end{cases}$ and $W_i$ is the known weight information of the $j$th ordered position, $j=1,2,\ldots,n$.

When the optimal fuzzy measure $v$ is a 2-additive measure, model (22) can be equivalently transformed into the following model:

$$\begin{aligned}&\min \frac{3-n}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{\mathrm{mid}(n)}\left(1-R_{i(j)}\right)v(j)+\sum_{j=\mathrm{mid}(n)+1}^{n}R_{i(j)}v(j)\right)\\&+\frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{\mathrm{mid}(n)}\sum_{k=1,k\neq j}^{n}(1-R_{i(j)})(v(j,k)-\mu(k))\right.\\&\left.+\sum_{j=\mathrm{mid}(n)+1}^{n}\sum_{k=1,k\neq j}^{n}R_{i(j)}(v(j,k)-\mu(k))\right)\\&s.t.\begin{cases}\sum_{j\in N\setminus i}(v(i,j)-v(j))\geqslant(s-2)v(i),\forall S\subseteq N,\forall i\in S, s\geqslant 2,\\ \sum_{\{i,j\}\in N}v(i,j)-(n-2)\sum_{i\in N}v(i)=1,\\ v(i)\in W_i,\quad i=1,2,\ldots,n,\\ v(i)\geqslant 0,\quad i=1,2,\ldots,n,\end{cases}\end{aligned}\tag{23}$$

where the notations as listed in model (22).

When we further require the 2-additive measure $v$ on the ordered position set $N$ to be symmetric for the middle position mid($n$), then we establish the following model for the optimal symmetric 2-additive measure $v$:

$$\begin{aligned}&\min \frac{3-n}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{\mathrm{mid}(n)}(1-R_{i(j)})v(j)+\sum_{j=\mathrm{mid}(n)+1}^{n}R_{i(j)}v(j)\right)\\&\quad+\frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{\mathrm{mid}(n)}\sum_{k=1,k\neq j}^{n}(1-R_{i(j)})(v(j,k)-\mu(k))\right.\\&\quad\left.+\sum_{j=\mathrm{mid}(n)+1}^{n}\sum_{k=1,k\neq j}^{n}R_{i(j)}(v(j,k)-\mu(k))\right)\end{aligned}$$

$$
\text{s.t.} \begin{cases} \sum_{j\in N\backslash i}(v(i,j)-v(j))\geqslant (s-2)v(i), & \forall S\subseteq N, \forall i\in S, s\geqslant 2,\\ \sum_{\{i,j\}\subseteq N} v(i,j)-(n-2)\sum\limits_{i\in N} v(i)=1, & \\ v(i)=v(n-i+1), & \forall i\in N,\\ v(i,j)=v(n-i+1,n-j+1), & \forall i,j\in N,\\ v(i)\in W_i, & i=1,2,\ldots,n,\\ v(i)\geqslant 0, & i=1,2,\ldots,n, \end{cases} \tag{24}
$$

where the notations as shown in model (22).

Note that when there is no interaction between the weights of the ordered positions and that of criteria, models (19) and (22) reduce to the optimal additive measures on them, respectively.

## 5. A New Approach to Decision Making with IVIHFEs

Because of the complexity of decision-making problems, it is difficult or even impossible to require a decision maker to consider all aspects of a decision-making problem. Thus, group decision making attracts considerable attention from researchers (Beliakov *et al.*, 2014; Liu *et al.*, 2017b; Meng *et al.*, 2017a, 2017b, 2017c; Pérez *et al.*, 2010, 2014; Perez *et al.*, 2016; Wu *et al.*, 2017).

Based on the new defined aggregation operators with respect to 2-additive measures and the built programming models, this section gives a new group decision-making method with interval-valued intuitionistic hesitant fuzzy information.

With respect to the decision-making problem listed in Section 4.2, the following procedure is needed to rank objects:

**Step 1:** Let $\tilde{H}'=(\tilde{h}'_{ij})_{m\times n}$ be the normalized IVIHFE matrix of $\tilde{H}=(\tilde{h}_{ij})_{m\times n}$, let $\mu$ be the fuzzy measure on the criteria set $C$, and let $v$ be the fuzzy measure on the ordered position set $N=\{1,2,\ldots,n\}$;

**Step 2:** When the fuzzy measures $\mu$ and $v$ are not exactly known, we adopt models (21) and (24) to determine the associated optimal 2-additive measures, otherwise, go to the next step;

**Step 3:** We use the IVIHFHSWA or IVIHFHSWGM operator to calculate the alternatives' comprehensive IVIHFEs $\tilde{h}_i$, $i=1,2,\ldots,m$;

**Step 4:** With respect to the comprehensive IVIHFEs $\tilde{h}_i$, $i=1,2,\ldots,m$, we apply the score function in Zhang (2013) to rank objects $A_i$, $i=1,2,\ldots,m$, where $s(\tilde{h}_i)=\frac{1}{2*\#\tilde{h}_i}\sum_{\tilde{\alpha}=([\mu_l,\mu_u],[v_l,v_u])\in\tilde{h}_i}(\mu_l+\mu_u-v_l-v_u)$ with $\#\tilde{h}_i$ being the number of IVIFVs in $\tilde{h}_i$, $i=1,2,\ldots,m$.

To show the concrete application of the above algorithm, we offer the following decision-making problem about the development of large projects.

Table 2
IVIHFE matrix $\tilde{H}$.

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $A_1$ | $\{([0.2, 0.3], [0.4, 0.5]), ([0.4, 0.45], [0.3, 0.4])\}$ | $\{([0.3, 0.5], [0.3, 0.4]), ([0.6, 0.8], [0.1, 0.2])\}$ | $\{([0.4, 0.6], [0.3, 0.4])\}$ | $\{([0.3, 0.5], [0.4, 0.5]), ([0.6, 0.7], [0.2, 0.3])\}$ |
| $A_2$ | $\{([0.2, 0.4], [0.5, 0.6]), ([0.6, 0.8], [0.1, 0.2])\}$ | $\{([0.2, 0.3], [0.5, 0.6]), ([0.4, 0.5], [0.3, 0.4])\}$ | $\{([0.3, 0.4], [0.4, 0.5]), ([0.6, 0.7], [0.2, 0.3])\}$ | $\{([0.5, 0.7], [0.1, 0.3]), ([0.8, 0.9], [0.1, 0.1])\}$ |
| $A_3$ | $\{([0.3, 0.5], [0.3, 0.4]), ([0.6, 0.7], [0.2, 0.3])\}$ | $\{([0.6, 0.8], [0.1, 0.2])\}$ | $\{([0.2, 0.4], [0.4, 0.5]), ([0.5, 0.6], [0.2, 0.3])\}$ | $\{([0.2, 0.5], [0.3, 0.4]), ([0.6, 0.7], [0.2, 0.3])\}$ |
| $A_4$ | $\{([0.2, 0.4], [0.5, 0.6]), ([0.5, 0.7], [0.1, 0.3])\}$ | $\{([0.3, 0.4], [0.4, 0.5])\}$ | $\{([0.2, 0.3], [0.4, 0.6]), ([0.4, 0.5], [0.3, 0.5]), ([0.7, 0.8], [0.1, 0.2])\}$ | $\{([0.6, 0.8], [0.1, 0.2])\}$ |

Example 3. The enterprise's board of directors is to plan the development of large projects strategy initiatives for the following five years. There are four possible projects $A_i$, $i = 1, 2, 3, 4$, to be evaluated. It is necessary to compare these projects to select the most important one as well as order them from the point of view of their importance, taking into account four attributes suggested by the Balanced Scorecard methodology (it should be noted that all of them are of the maximization type): $C_1$: financial perspective, $C_2$: the customer satisfaction, $C_3$: internal business process perspective, and $C_4$: learning and growth perspective. To avoid influencing each other, the decision makers are required to provide their preferences in anonymity and the IVIHFE matrix $\tilde{H} = (\tilde{h}_{ij})_{4\times 4}$ is presented in Table 2, where $\tilde{h}_{ij}$, $i, j = 1, 2, 3, 4$, is in the form of IVIHFEs.

Assume that the attribute weights are defined by $W_{C_1} = [0.1, 0.3]$, $W_{C_2} = [0.2, 0.4]$, $W_{C_3} = [0.05, 0.25]$ and $W_{C_3} = [0.25, 0.45]$, respectively. Furthermore, the weights on the ordered positions are given as follows: $W_1 = [0.1, 0.3]$, $W_2 = [0.2, 0.4]$, $W_3 = [0.2, 0.4]$ and $W_4 = [0.1, 0.3]$.

To rank these four possible projects, the following procedure is needed:

*Step 1:* Using model (21), the optimal 2-additive measure $\mu$ on the criteria set $C$ can be derived, by which the Shapley values are $Sh_{C_1}(\mu, C) = 0.025$, $Sh_{C_2}(\mu, C) = 0.400$, $Sh_{C_3}(\mu, C) = 0.475$, $Sh_{C_4}(\mu, C) = 0.100$.
*Step 2:* Using model (24), we can obtain the optimal symmetric 2-additive measure $v$ on the ordered set $N$ and the associated Shapley values, where $Sh_1(v, N) = Sh_2(v, N) = Sh_3(v, N) = Sh_4(v, N) = 0.25$.
*Step 3:* Adopting the IVIHFHSWA operator, we can derive the comprehensive IVIHFEs of the projects. Taking the collective IVIHFE of the project $a_1$, for example, we have:

$$\begin{aligned}\tilde{h}_1 = \big\{&\big([0.3450, 0.5425], [0.3125, 0.4125]\big), \big([0.3500, 0.5463], [0.3100, 0.4100]\big),\\ &\big([0.4650, 0.6625], [0.2325, 0.3325]\big), \big([0.4700, 0.6663], [0.2300, 0.3300]\big),\\ &\big([0.3750, 0.5625], [0.2925, 0.3925]\big), \big([0.3800, 0.5663], [0.2900, 0.3900]\big),\\ &\big([0.4950, 0.6825], [0.2125, 0.3125]\big), \big([0.5000, 0.6863], [0.2100, 0.3100]\big)\big\}.\end{aligned}$$

*Step 4:* With respect to the comprehensive IVIHFEs $\tilde{h}_i$, $i = 1, 2, 3, 4$, the scores are derived as follows:

$$s(\tilde{h}_1) = 0.2072, \qquad s(\tilde{h}_2) = 0.1075, \qquad s(\tilde{h}_3) = 0.2813, \qquad s(\tilde{h}_4) = 0.0802.$$

Thus, the ranking order of projects is $A_3 \succ A_1 \succ A_2 \succ A_4$.

In this example, when the IVIHFHSWGM operator is adopted. The scores of the projects are

$$s(\tilde{h}_1) = 0.2110, \qquad s(\tilde{h}_2) = 0.0997, \qquad s(\tilde{h}_3) = 0.2789, \qquad s(\tilde{h}_4) = 0.0848,$$

and the ranking order of projects is $A_3 \succ A_1 \succ A_4 \succ A_2$, which is different from the above ranking order. However, the same optimal choice is derived.

In this example, when Joshi and Kumar's method (Joshi and Kumar, 2016) is used, similar to Joshi and Kumar (2016), let $W_{C_1} = 0.4$, $W_{C_2} = 0.3$, $W_{C_3} = 0.3$, and $W_{C_4} = 0.4$. From formula (3), we get $\lambda = -0.6359$. According to formula (2), we can derive the value of any coalition. Using Joshi and Kumar's algorithm, we derive the following closeness coefficients of the projects:

$$Cc_1 = 0.8653, \qquad Cc_2 = 0.8843, \qquad Cc_3 = 0.8810, \qquad Cc_4 = 0.8779,$$

by which the ranking order is $A_2 \succ A_3 \succ A_4 \succ A_1$, where the ranking order as well as the best choice are different from that obtained above.

These two methods both consider the interactive characteristics between the weights of elements in a set. When we assume that there is no interaction, using models (19) and (22) in the setting of additive measures, the weight vectors on the criteria set and on the ordered set are obtained as follows:

$$W_C = (0.1, 0.4, 0.25, 0.25) \quad \text{and} \quad W_N = (0.1, 0.4, 0.4, 0.1).$$

Then, we can use the defined aggregation operators in (Zhang, 2013) to calculate the comprehensives and to rank projects. To show the ranking values and the ranking orders obtained from different methods, please see Table 3.

**Note:** the IVIHFWA operator: the interval-valued intuitionistic hesitant fuzzy weighted averaging operator, the IVIHFWG operator: the interval-valued intuitionistic hesitant fuzzy weighted geometric operator, the IVIHFOWA operator: the interval-valued intuitionistic hesitant fuzzy ordered weighted averaging operator, IVIHFOWG operator: the interval-valued intuitionistic hesitant fuzzy ordered weighted geometric operator, the IVIHFHA operator: the interval-valued intuitionistic hesitant fuzzy hybrid averaging operator, the IVIHFHG operator: the interval-valued intuitionistic hesitant fuzzy hybrid geometric operator.

To further show the application of the new method and to compare with the previous ones, we provide the following decision making problem.

Table 3
Ranking values and ranking orders obtained from different methods.

| Methods | Ranking values $A_1, A_2, A_3, A_4$ | Ranking orders | | | |
|---|---|---|---|---|---|
| Our method using the IVIHFHSWA operator | 0.2072 | 0.1075 | 0.2813 | 0.0802 | $A_3 \succ A_1 \succ A_2 \succ A_4$ |
| Our method using the IVIHFHSWGM operator | 0.2110 | 0.0997 | 0.2789 | 0.0848 | $A_3 \succ A_1 \succ A_4 \succ A_2$ |
| Joshi and Kumar's method (Joshi and Kumar, 2016) | 0.8653 | 0.8843 | 0.8810 | 0.8779 | $A_2 \succ A_3 \succ A_4 \succ A_1$ |
| Zhang's method using the IVIHFWA operator (Zhang, 2013) | 0.2243 | 0.2646 | 0.3562 | 0.2129 | $A_3 \succ A_2 \succ A_1 \succ A_4$ |
| Zhang's method using the IVIHFWG operator (Zhang, 2013) | 0.1754 | 0.1142 | 0.2783 | 0.0932 | $A_3 \succ A_1 \succ A_2 \succ A_4$ |
| Zhang's method using the IVIHFOWA operator (Zhang, 2013) | 0.1732 | 0.2513 | 0.2583 | 0.1858 | $A_3 \succ A_2 \succ A_4 \succ A_1$ |
| Zhang's method using the IVIHFOWG operator (Zhang, 2013) | 0.1389 | 0.1265 | 0.2110 | 0.0894 | $A_3 \succ A_1 \succ A_2 \succ A_4$ |
| Zhang's method using the IVIHFHA operator (Zhang, 2013) | 0.1960 | 0.2410 | 0.2431 | 0.2051 | $A_3 \succ A_2 \succ A_4 \succ A_1$ |
| Zhang's method using the IVIHFHG operator (Zhang, 2013) | -0.1969 | -0.2394 | -0.2217 | -0.2470 | $A_1 \succ A_3 \succ A_2 \succ A_4$ |

Table 4
IVIHFE matrix $\tilde{H}$.

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $A_1$ | $\{([0.5, 0.6], [0.2, 0.3]), ([0.6, 0.7], [0.2, 0.3])\}$ | $\{([0.3, 0.5], [0.3, 0.4]), ([0.6, 0.7], [0.2, 0.3])\}$ | $\{([0.4, 0.5], [0.2, 0.4])\}$ | $\{([0.3, 0.4], [0.3, 0.5])\}$ |
| $A_2$ | $\{([0.6, 0.8], [0.1, 0.2])\}$ | $\{([0.2, 0.3], [0.4, 0.5]), ([0.4, 0.5], [0.3, 0.4])\}$ | $\{([0.5, 0.7], [0.1, 0.3])\}$ | $\{([0.4, 0.5], [0.2, 0.4]), ([0.6, 0.7], [0.1, 0.2])\}$ |
| $A_3$ | $\{([0.5, 0.7], [0.2, 0.3])\}$ | $\{([0.3, 0.5], [0.2, 0.3])\}$ | $\{([0.2, 0.4], [0.3, 0.5]), ([0.4, 0.5], [0.3, 0.5])\}, ([0.6, 0.7], [0.2, 0.3])\}$ | $\{([0.4, 0.6], [0.3, 0.4])\}$ |
| $A_4$ | $\{([0.3, 0.4], [0.4, 0.5]), ([0.5, 0.6], [0.3, 0.4])\}$ | $\{([0.4, 0.6], [0.1, 0.3])\}$ | $\{([0.5, 0.6], [0.2, 0.3]), ([0.7, 0.8], [0.1, 0.2])\}$ | $\{([0.3, 0.4], [0.3, 0.5]), ([0.5, 0.6], [0.2, 0.4])\}$ |

Example 4. Let us consider an investment company that wants to invest a sum of money in the best option (Liu and Jin, 2012). There is a panel with four possible alternatives in which to invest the money: $A_1$ is a car company; $A_2$ is a computer company; $A_3$ is a TV company; $A_4$ is a food company. The investment company must make a decision according to the following four attributes: the risk index $C_1$; the growth index $C_2$; the social-political impact index $C_3$; the environmental impact index $C_4$. The evaluating IVIHFE matrix $\tilde{H} = (\tilde{h}_{ij})_{4\times 4}$ is offered as shown in Table 4.

Let $W_C = ([0.3, 0.4], [0.15, 0.25], [0.2, 0.25], [0.25, 0.3])$ be the known weight information on the attribute set, and let $W_N = ([0.2, 0.3], [0.25, 0.4], [0.25, 0.4], [0.2, 0.3])$ be the known weight information on the ordered set.

Because the criterion $c_2$ is benefit, and the criteria $c_1$, $c_3$, and $c_4$ are cost, we need to normalize the IVIHFE matrix $\tilde{H}$ into the following IVIHFE $\tilde{H}'$, please see Table 5.

Similar to Example 3, with respect to different methods the ranking results are obtained as shown in Table 6.

Examples 3 and 4 both show that different ranking orders might be derived with respect to the different methods that are based on the different aggregation operators and the different operational laws. Thus, when the decision makers make decisions for some prob-

Table 5
IVIHFE matrix $\tilde{H}'$.

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $A_1$ | $\{([0.2, 0.3], [0.5, 0.6]), ([0.2, 0.3], [0.6, 0.7])\}$ | $\{([0.3, 0.5], [0.3, 0.4]), ([0.6, 0.7], [0.2, 0.3])\}$ | $\{([0.2, 0.4], [0.4, 0.5])\}$ | $\{([0.3, 0.5], [0.3, 0.4])\}$ |
| $A_2$ | $\{([0.1, 0.2], [0.6, 0.7])\}$ | $\{([0.2, 0.3], [0.4, 0.5]), ([0.4, 0.5], [0.3, 0.4])\}$ | $\{([0.1, 0.3], [0.5, 0.7])\}$ | $\{([0.2, 0.4], [0.4, 0.5]), ([0.1, 0.2], [0.6, 0.7])\}$ |
| $A_3$ | $\{([0.2, 0.3], [0.5, 0.7])\}$ | $\{([0.3, 0.5], [0.2, 0.3])\}$ | $\{([0.3, 0.5], [0.2, 0.4]), ([0.3, 0.5], [0.4, 0.5]), ([0.2, 0.3], [0.6, 0.7])\}$ | $\{([0.3, 0.4], [0.3, 0.6])\}$ |
| $A_4$ | $\{([0.4, 0.5], [0.3, 0.4]), ([0.3, 0.4], [0.5, 0.6])\}$ | $\{([0.4, 0.6], [0.1, 0.3])\}$ | $\{([0.2, 0.3], [0.5, 0.6]), ([0.1, 0.2], [0.7, 0.8])\}$ | $\{([0.3, 0.5], [0.3, 0.4]), ([0.2, 0.4], [0.5, 0.6])\}$ |

Table 6
Ranking values and ranking orders obtained from different methods.

| Methods | Ranking values $A_1, A_2, A_3, A_4$ | Ranking orders | | | |
|---|---|---|---|---|---|
| Our method using the IVIHFHSWA operator | $-0.0030$ | $-0.1146$ | $-0.0318$ | $-0.0059$ | $A_1 \succ A_4 \succ A_3 \succ A_2$ |
| Our method using the IVIHFHSWGM operator | 0.2110 | 0.0997 | 0.2789 | 0.0848 | $A_1 \succ A_4 \succ A_3 \succ A_2$ |
| Joshi and Kumar's method (Joshi and Kumar, 2016) | 0.8653 | 0.8843 | 0.8810 | 0.8779 | $A_1 \succ A_4 \succ A_3 \succ A_2$ |
| Zhang's method using the IVIHFWA operator (Zhang, 2013) | 0.2243 | 0.2646 | 0.3562 | 0.2129 | $A_4 \succ A_1 \succ A_3 \succ A_2$ |
| Zhang's method using the IVIHFWG operator (Zhang, 2013) | 0.1754 | 0.1142 | 0.2783 | 0.0932 | $A_1 \succ A_4 \succ A_3 \succ A_2$ |
| Zhang's method using the IVIHFOWA operator (Zhang, 2013) | 0.1732 | 0.2513 | 0.2583 | 0.1858 | $A_1 \succ A_4 \succ A_3 \succ A_2$ |
| Zhang's method using the IVIHFOWG operator (Zhang, 2013) | 0.1389 | 0.1265 | 0.2110 | 0.0894 | $A_1 \succ A_3 \succ A_4 \succ A_2$ |
| Zhang's method using the IVIHFHA operator (Zhang, 2013) | 0.1960 | 0.2410 | 0.2431 | 0.2051 | $A_4 \succ A_1 \succ A_3 \succ A_2$ |
| Zhang's method using the IVIHFHG operator (Zhang, 2013) | -0.1969 | -0.2394 | -0.2217 | -0.2470 | $A_1 \succ A_4 \succ A_3 \succ A_2$ |

lem, they need to first choose the used method. Because the new operational laws avoid the limitations defined in Section 2.2 and the new aggregation operators globally consider the interactions between the weights of the criteria and the weights of the ordered positions, we suggest the decision makers to apply the new method that can also address the situation where the weight information is not exactly known.

To show the differences between our method and two previous ones intuitively, according to their principles, Table 7 is offered.

## 6. Conclusion

As well known, for a given decision making problem, there are main three procedures that need to be solved: determining the weight information, calculating the alternatives' comprehensive values, and ranking the alternatives. Considering the previous researches about decision making with interval-valued intuitionistic hesitant fuzzy sets, we wrote this paper and proposed two new aggregation operators that overall consider the interactions between elements in a set. To cope with the situation where the weight information is not exactly known, models for the optimal fuzzy measures and 2-additive measures are

Table 7
The comparison of our method and two previous ones.

| | Our method | Method in Joshi and Kumar (2016) | Method in Zhang (2013) |
|---|---|---|---|
| Can the used operational laws preserve the monotonicity? | Yes | No | No |
| Are the weights of criteria for the different alternatives the same? | Yes | No | Yes |
| Are the interactive characteristics between the weights considered? | Yes | Yes | No |
| Can the complementary, redundant and independent characteristics between the weights of elements be reflected simultaneously? | Yes | No | No |
| Is the situation where the weighting information is incompletely known considered? | Yes | No | No |

constructed. Then, we introduced a new decision-making method and offered a practical decision-making problem to show the concrete application of the new theoretical results. Meanwhile, comparison analysis is performed.

This paper focuses on the theoretical research, and we will continue to study the application of the new approach in some other fields including the application in new product screening, propulsion system selection problem, selecting suitable hotels, economic production problem, evaluating machine tool, social media, green supplier selection, and evaluation of the professor in a university. Furthermore, we shall research decision making with other types of hesitant fuzzy sets and dynamic decision making models in the setting of hesitant fuzzy environment.

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