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# The Generalized Dice Similarity Measures for Picture Fuzzy Sets and Their Applications

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**Abstract.** The picture fuzzy set is characterized by three functions expressing the degree of membership, the degree of neutral membership and the degree of non-membership. It was proposed as a generalization of an intuitionistic fuzzy set in order to deal with indeterminate and inconsistent information. In this work, we shall present some novel Dice similarity measures of picture fuzzy sets and the generalized Dice similarity measures of picture fuzzy sets and indicate that the Dice similarity measures and asymmetric measures (projection measures) are the special cases of the generalized Dice similarity measures in some parameter values. Then, we propose the generalized Dice similarity measures-based patterns recognition models with picture fuzzy information. Then, we apply the generalized Dice similarity measures between picture fuzzy sets to building material recognition. Finally, an illustrative example is given to demonstrate the efficiency of the similarity measures for building material recognition.

**Key words:** generalized Dice similarity measures, Dice similarity measures, picture fuzzy set, asymmetric measures, projection measures, patterns recognition, building material recognition.

## 1. Introduction

Multiple attribute decision making is a main branch of decision theory, where (PFS) introduced by Cuong (2013) has been successfully applied in recent years. The picture fuzzy set is a generalization of an intuitionistic fuzzy set (IFS) (see Atanassov, 1986, 1989; Xu and Cai, 2008; Wei, 2011; Xu and Chen, 2008; Ye, 2011; Chen, 2016; Wei, 2015; Zhang and Xu, 2015; Li and Ren, 2015; Wei, 2009; Wei, 2010). The picture fuzzy set (Cuong, 2013) is characterized by three functions expressing the degree of membership, the degree of neutral membership and the degree of non-membership. The only constraint is that the sum of the three degrees must not exceed 1. Basically, PFS based models can be applied to situations requiring human opinions involving more answers of types: yes, abstain, no, refusal, which cannot be accurately expressed in the traditional FS and IFS. Recently, many researchers have applied PFSs to the decision-making problems. Various methods have been developed to solve the multiple attribute decision-making problems with picture fuzzy information. For example, Singh (2014) investigated the correlation

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coefficients for picture fuzzy set and applied the correlation coefficient to clustering analysis with picture fuzzy information. Son (2015) proposed a novel distributed picture fuzzy clustering method with picture fuzzy information. Thong and Son (2015) developed a new approach to multi-variables fuzzy forecasting by using picture fuzzy clustering and picture fuzzy rules interpolation method. Thong (2015) developed a novel hybrid model between picture fuzzy clustering and intuitionistic fuzzy recommender systems for medical diagnosis and application to health care support systems. Wei (2016b) proposed picture fuzzy cross-entropy model for multiple attribute decision making problems. Wei (2017c) proposed some aggregating operators for picture fuzzy set. Wei *et al.* (2016a) defined the projection models for multiple attribute decision making with picture fuzzy information. Wei (2017e) developed some cosine similarity measures for picture fuzzy sets. Wei (2018) proposed some picture 2-tuple linguistic aggregation operators for multiple attribute decision making. Wei (2017b) developed some picture 2-tuple linguistic Bonferroni mean operators for multiple attribute decision making. Wu and Wei (2017) gave some picture uncertain linguistic aggregation operators for multiple attribute decision making.

The similarity measure is one of the important and useful tools for degree of similarity between objects (see Szmidt and Kacprzyk, 2000; Liu, 2005; Hung and Yang, 2007; Xu and Xia, 2010; Ye, 2011; Tian, 2013; Rajarajeswari and Uma, 2013; Szmidt, 2014; Ye, 2016b). Functions expressing the degree of similarity of items or sets are used in physical anthropology, automatic classification, ecology, psychology, citation analysis, information retrieval, patterns recognition and numerical taxonomy (Ye, 2012b). In fact, the degree of similarity or dissimilarity between the objects under study plays an important role. In vector space, especially the Jaccard, Dice, and cosine similarity measures (Jaccard, 1901; Dice, 1945a; Salton and McGill, 1987) are often used in information retrieval, citation analysis, and automatic classification. Therefore, Ye (2012b) proposed the Jaccard, Dice, and cosine similarity measures between trapezoidal intuitionistic fuzzy numbers (TIFNs) and applied them to group decision-making problems. Ye (2012a) proposed the multi-criteria decision making models by using the Dice similarity measure between expected intervals of trapezoidal fuzzy numbers. Ye (2014) developed the Dice measures for simplified neutrosophic sets. Ye (2016a) proposed the generalized Dice measures for multiple attribute decision making under simplified neutrosophic environments.

However, these similarity measures do not deal with the similarity measures for picture fuzzy information. Therefore, it is necessary to extend the Dice measure to picture fuzzy set to handle patterns recognition, citation analysis, information retrieval and multiple attribute decision making problems to satisfy the requirements of decision makers' preference and flexible decision making. In order to do so, the main purposes of this paper are: (1) to propose two forms of the Dice measures of PFSs, (2) to present the generalized Dice measures of PFSs, and (3) to develop the generalized Dice measures-based patterns recognition methods with picture fuzzy information. In the patterns recognition process, the main advantage of the proposed methods is more general and more flexible than existing patterns recognition methods with picture fuzzy information to satisfy the practical requirements.

In order to do so, the remainder of this paper is set out as follows. In the next section, we introduce some basic concepts related to intuitionistic fuzzy set and picture fuzzy sets.

In Section 3, we shall propose some Dice similarity measure and some weighted Dice similarity measure between PFSs. In Section 4, the Dice similarity measures for PFSs are applied to building material recognition and minerals field recognition. Section 5 concludes the paper with some remarks.

## 2. Preliminaries

In the following, we introduce some basic concepts related to intuitionistic fuzzy sets and picture fuzzy sets.

Definition 1 (Atanassov, 1986, 1989). An IFS is given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}, \tag{1}$$

where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$, where, $0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1$, $\forall x \in X$. The number $\mu_A(x)$ and $\nu_A(x)$ represents, respectively, the membership degree and non-membership degree of the element $x$ to the set $A$.

Definition 2 (Atanassov, 1986, 1989). For each IFS $A$ in $X$, if

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \quad \forall x \in X. \tag{2}$$

Then $\pi_A(x)$ is called the degree of indeterminacy of $x$ to $A$.

Although Atanassov's intuitionistic fuzzy set theory (Atanassov, 1986, 1989) has been successfully applied in different areas, there are situations in real life which cannot be represented by Atanassov's intuitionistic fuzzy sets. Picture fuzzy sets (Cuong, 2013) are an extension of Atanassov's intuitionistic fuzzy sets (Atanassov, 1986, 1989). Picture fuzzy set (Cuong, 2013) based models may be adequate in situations when we face human opinions involving more answers of types: yes, abstain, no, refusal. It can be considered as a powerful tool representing the uncertain information in the process of patterns recognition and cluster analysis.

Definition 3 (Cuong, 2013). A picture fuzzy set (PFS) $A$ on the universe is $X$ an object of the form

$$A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle \mid x \in X\}, \tag{3}$$

where $\mu_A(x) \in [0, 1]$ is called the "degree of positive membership of $A$", $\eta_A(x) \in [0, 1]$ is called the "degree of neutral membership of $A$" and $\eta_A(x) \in [0, 1]$ is called the "degree of negative membership of $A$", and $\mu_A(x)$, $\eta_A(x)$, $\nu_A(x)$, satisfy the following condition: $0 \leqslant \mu_A(x) + \eta_A(x) + \nu_A(x) \leqslant 1, \forall x \in X$. Then for $x \in X$, $\rho_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ could be called the degree of refusal membership of $x$ in $A$.

## 3. Some Dice Similarity Measures for Picture Fuzzy Sets

The Dice similarity measure cannot be computed in this undefined situation when one vector is zero, which overcomes the disadvantage of the cosine similarity measure (Dice, 1945b). Therefore, the concept of the Dice similarity measure is introduced in this section (Dice, 1945b).

Let $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ be two vectors of length where all the coordinates are positive real numbers. Then the Dice similarity measure [41] is defined as follows:

$$D(X, Y) = \frac{2X \cdot Y}{\|X\|_2^2 + \|Y\|_2^2} = \frac{2\sum_{j=1}^{n} x_j y_j}{\sum_{j=1}^{n} x_j^2 + \sum_{j=1}^{n} y_j^2} \tag{4}$$

where $X \cdot Y = \sum_{j=1}^{n} x_j y_j$ is called the inner product of the vector $X$ and $Y$ and $\|X\|_2 = \sqrt{\sum_{j=1}^{n} x_j^2}$ and $\|Y\|_2 = \sqrt{\sum_{j=1}^{n} y_j^2}$ are the Euclidean norms of $X$ and $Y$ (also called the $L_2$ norms).

The Dice similarity measure takes value in the interval [0, 1]. However, it is undefined if $x_j = y_j = 0$ $(j = 1, 2, \dots, n)$. In this case, let the Dice measure value be zero when $x_j = y_j = 0$ $(j = 1, 2, \dots, n)$.

### 3.1. *Dice Similarity Measures for Picture Fuzzy Sets*

Let $A$ be an PFS in an universe of discourse $X = \{x\}$, the PFS is characterized by the degree of positive membership $\mu_A(x)$, the degree of neutral membership $\eta_A(x)$ and the degree of negative membership $\nu_A(x)$ which can be considered as a vector representation with the three elements.

In this section, we shall propose some Dice similarity measures and some weighted Dice similarity measures between PFSs based on the concept of the Dice similarity measure (Ye, 2014, 2016a).

Let $A = (\mu_A(x_j), \eta_A(x_j), \nu_A(x_j))$ and $B = (\mu_B(x_j), \eta_B(x_j), \nu_B(x_j))$, $j = 1, 2, \dots, n$, be two groups of picture fuzzy numbers, a Dice similarity measure between PFSs $A$ and $B$ is proposed as follows:

$$D_{PFS}^1(A, B) = \frac{1}{n}\sum_{j=1}^{n} \frac{2(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j))}{(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + (\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))}. \tag{5}$$

The Dice similarity measure between PFSs $A$ and $B$ also satisfies the following properties:

(1) $0 \leqslant D_{PFS}^1(A, B) \leqslant 1$;
(2) $D_{PFS}^1(A, B) = D_{PFS}^1(B, A)$;
(3) $D_{PFS}^1(A, B) = 1$ if $A = B$ i.e. $\mu_A(x_j) = \mu_B(x_j)$, $\eta_A(x_j) = \eta_B(x_j)$, $\nu_A(x_j) = \nu_B(x_j)$, $j = 1, 2, \dots, n$.

*Proof.* (1) Let us consider the $j$th item of the summation in Eq. (5):

$$D^1_{PFS}(A_j, B_j) = \frac{1}{n}\sum_{j=1}^{n} \frac{2(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j))}{(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + (\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))}.$$

It is obvious that $D^1_{PFS}(A, B) \geqslant 0$, and

$$\big(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)\big) + \big(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j)\big) \geqslant 2\big(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j)\big)$$

according to the inequality $a^2 + b^2 \geqslant 2ab$. Thus, $0 \leqslant D^1_{PFS}(A_j, B_j) \leqslant 1$. From Eq. (5), the summation of $n$ terms is $0 \leqslant D^1_{PFS}(A, B) \leqslant 1$.

(2) It is obvious that the proposition is true.

(3) When $A = B$, there are $\mu_A(x_j) = \mu_B(x_j)$, $\eta_A(x_j) = \eta_B(x_j)$, and $\nu_A(x_j) = \nu_B(x_j)$, for $j = 1, 2, \ldots, n$. So, there is

$$\begin{aligned} D^1_{PFS}(A, B) &= \frac{1}{n}\sum_{j=1}^{n} \frac{2(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j))}{(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + (\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))} \\ &= \frac{1}{n}\sum_{j=1}^{n} \frac{2(\mu_A(x_j)\mu_A(x_j) + \eta_A(x_j)\eta_A(x_j) + \nu_A(x_j)\nu_A(x_j))}{(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + (\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j))} \\ &= \frac{1}{n}\sum_{j=1}^{n} \frac{2(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j))}{2(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j))} \\ &= 1. \end{aligned}$$

Therefore, we have finished the proofs. □

If we consider the weights of $x_j$, a weighted Dice similarity measure between PFSs $A$ and $B$ is proposed as follows:

$$WD^1_{PFS}(A, B) = \sum_{j=1}^{n} \omega_j \frac{2(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j))}{(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + (\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))}, \tag{6}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $x_j$ ($j = 1, 2, \ldots, n$), with $w_j \in [0, 1]$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^{n} w_j = 1$.

In particular, if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the weighted Dice similarity measure reduces to Dice similarity measure. That's to say, if we take $\omega_j = \frac{1}{n}$, $j = 1, 2, \ldots, n$, then there is $WD^1_{PFS}(A, B) = D^1_{PFS}(A, B)$. Obviously, the weighted Dice similarity measure of two PFSs $A$ and $B$ also satisfies the following properties:

(1) $0 \leqslant WD^1_{PFS}(A, B) \leqslant 1$;
(2) $WD^1_{PFS}(A, B) = WD^1_{PFS}(B, A)$;
(3) $WD^1_{PFS}(A, B) = 1$ if $A = B$ i.e. $\mu_A(x_j) = \mu_B(x_j)$, $\eta_A(x_j) = \eta_B(x_j)$, $\nu_A(x_j) = \nu_B(x_j)$, $j = 1, 2, \ldots, n$.

Similar to the previous proof method, we can prove the above three properties.

When the four terms like degree of positive membership, degree of neutral membership, degree of negative membership and degree of refusal membership are considered in PFSs, we further propose the Dice similarity measure and weighted Dice similarity measure between PFSs as follows:

$$D^2_{\mathrm{PFS}}(A, B) = \frac{1}{n}\sum_{j=1}^{n} \frac{2(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j) + \rho_A(x_j)\rho_B(x_j))}{\begin{pmatrix} (\mu^2_A(x_j) + \eta^2_A(x_j) + \nu^2_A(x_j) + \rho^2_A(x_j)) \\ +(\mu^2_B(x_j) + \eta^2_B(x_j) + \nu^2_B(x_j) + \rho^2_B(x_j)) \end{pmatrix}}, \tag{7}$$

$$WD^2_{PFS}(A, B) = \sum_{j=1}^{n} \omega_j \frac{2(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j) + \rho_A(x_j)\rho_B(x_j))}{\begin{pmatrix} \sum_{j=1}^{n}(\mu^2_A(x_j) + \eta^2_A(x_j) + \nu^2_A(x_j) + \rho^2_A(x_j)) \\ +\sum_{j=1}^{n}(\mu^2_B(x_j) + \eta^2_B(x_j) + \nu^2_B(x_j) + \rho^2_B(x_j))) \end{pmatrix}}, \tag{8}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $x_j$ $(j = 1, 2, \ldots, n)$, with $\omega_j \in [0, 1]$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^{n} \omega_j = 1$.

### 3.2. *Another Form of the Dice Similarity Measure for Picture Fuzzy Sets*

In this section, we shall develop another form of Dice similarity measure for picture fuzzy sets, which is defined as follows.

Definition 4. Let $A = (\mu_A(x_j), \eta_A(x_j), \nu_A(x_j))$ and $B = (\mu_B(x_j), \eta_B(x_j), \nu_B(x_j))$, $j = 1, 2, \ldots, n$, be two groups of picture fuzzy numbers, then a Dice similarity measure between PFSs $A$ and $B$ is proposed as follows:

$$D^3_{PFS}(A, B) = \frac{\sum_{j=1}^{n} 2(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j))}{\sum_{j=1}^{n}(\mu^2_A(x_j) + \eta^2_A(x_j) + \nu^2_A(x_j)) + \sum_{j=1}^{n}(\mu^2_B(x_j) + \eta^2_B(x_j) + \nu^2_B(x_j))}. \tag{9}$$

The Dice similarity measure between PFSs $A$ and $B$ also satisfies the following properties:

(1) $0 \leqslant D^3_{PFS}(A, B) \leqslant 1$;
(2) $D^3_{PFS}(A, B) = D^3_{PFS}(B, A)$;
(3) $D^3_{PFS}(A, B) = 1$ if $A = B$, i.e. $\mu_A(x_j) = \mu_B(x_j)$, $\eta_A(x_j) = \eta_B(x_j)$, $\nu_A(x_j) = \nu_B(x_j)$, $j = 1, 2, \ldots, n$.

Similar to the previous proof method, we can prove the above three properties.

If we consider the weights of $x_j$, a weighted Dice similarity measure between PFSs $A$ and $B$ is proposed as follows:

$$WD^3_{PFS}(A, B) = \frac{2\sum_{j=1}^n \omega_j^2(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j))}{\sum_{j=1}^n \omega_j^2(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + \sum_{j=1}^n \omega_j^2(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))}, \tag{10}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $x_j$ ($j = 1, 2, \ldots, n$), with $w_j \in [0, 1]$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^n w_j = 1$. In particular, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, then the weighted Dice similarity measure reduces to Dice similarity measure. That's to say, if we take $\omega_j = \frac{1}{n}$, $j = 1, 2, \ldots, n$, then there is $WD^3_{PFS}(A, B) = D^3_{PFS}(A, B)$.

Obviously, the weighted Dice similarity measure of two PFSs $A$ and $B$ also satisfies the following properties:

(1) $0 \leqslant WD^3_{PFS}(A, B) \leqslant 1$;
(2) $WD^3_{PFS}(A, B) = WD^3_{PFS}(B, A)$;
(3) $WD^3_{PFS}(A, B) = 1$ if $A = B$ i.e. $\mu_A(x_j) = \mu_B(x_j)$, $\eta_A(x_j) = \eta_B(x_j)$, $\nu_A(x_j) = \nu_B(x_j)$, $j = 1, 2, \ldots, n$.

When the four terms like degree of positive membership, degree of neutral membership, degree of negative membership and degree of refusal membership are considered in PFSs, we further propose the another form of Dice similarity measure and weighted Dice similarity measure between PFSs as follows:

$$D^4_{PFS}(A, B) = \frac{2\sum_{j=1}^n (\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j) + \rho_A(x_j)\rho_B(x_j))}{\begin{pmatrix} \sum_{j=1}^n (\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j) + \rho_A^2(x_j)) \\ + \sum_{j=1}^n (\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j) + \rho_B^2(x_j)) \end{pmatrix}}, \tag{11}$$

$$WD^4_{PFS}(A, B) = \frac{2\sum_{j=1}^n \omega_j^2(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j) + \rho_A(x_j)\rho_B(x_j))}{\begin{pmatrix} \sum_{j=1}^n \omega_j^2(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j) + \rho_A^2(x_j)) \\ + \sum_{j=1}^n \omega_j^2(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j) + \rho_B^2(x_j)) \end{pmatrix}}, \tag{12}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $x_j$ $(j = 1, 2, \ldots, n)$, with $\omega_j \in [0, 1]$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^{n} \omega_j = 1$.

### 3.3. *The Generalized Dice Similarity Measure for Picture Fuzzy Sets*

In this section, we develop the generalized Dice similarity measure for picture fuzzy sets. As the generalization of the Dice similarity measure for picture fuzzy sets, the generalized Dice similarity measure for picture fuzzy sets are defined below.

Definition 5. Let $A = (\mu_A(x_j), \eta_A(x_j), \nu_A(x_j))$ and $B = (\mu_B(x_j), \eta_B(x_j), \nu_B(x_j))$, $j = 1, 2, \ldots, n$, be two groups of picture fuzzy numbers, then the generalized Dice similarity measure between PFSs $A$ and $B$ is defined as follows:

$$GD_{PFS}^{1}(A, B) = \frac{1}{n}\sum_{j=1}^{n} \frac{(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j))}{\lambda(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + (1-\lambda)(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))}, \tag{13}$$

$$GD_{PFS}^{2}(A, B) = \frac{\sum_{j=1}^{n}(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j))}{\lambda\sum_{j=1}^{n}(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + (1-\lambda)\sum_{j=1}^{n}(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))}, \tag{14}$$

where $\lambda$ is a positive parameter for $0 \leqslant \lambda \leqslant 1$.

Then, the generalized Dice similarity measure includes some special cases by altering the parameter value $\lambda$.

If $\lambda = 0.5$, the two generalized Dice similarity measures (13) and (14) reduced to Dice similarity measures (9) and (10):

$$\begin{aligned} GD_{PFS}^{1}(A, B) &= \frac{1}{n}\sum_{j=1}^{n} \frac{(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j))}{\lambda(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + (1-\lambda)(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))} \\ &= \frac{1}{n}\sum_{j=1}^{n} \frac{(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j))}{0.5(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + (1-0.5)(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))} \\ &= \frac{1}{n}\sum_{j=1}^{n} \frac{2(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j))}{(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + (\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))}, \end{aligned} \tag{15}$$

$$
\begin{aligned}
&GD^2_{PFS}(A,B)\\
&= \frac{\sum_{j=1}^n(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\lambda\sum_{j=1}^n(\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j))+(1-\lambda)\sum_{j=1}^n(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))}\\
&= \frac{\sum_{j=1}^n(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{0.5\sum_{j=1}^n(\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j))+(1-0.5)\sum_{j=1}^n(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))}\\
&= \frac{2\sum_{j=1}^n(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\sum_{j=1}^n(\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j))+\sum_{j=1}^n(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))}.
\end{aligned}
\tag{16}
$$

If $\lambda = 0, 1$, the two generalized Dice similarity measures reduced to the following asymmetric similarity measures respectively:

$$
\begin{aligned}
&GD^1_{PFS}(A,B)\\
&= \frac{1}{n}\sum_{j=1}^n \frac{(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\lambda(\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j))+(1-\lambda)(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))}\\
&= \frac{1}{n}\sum_{j=1}^n \frac{(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j)} \quad \text{for } \lambda = 0,
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&GD^1_{PFS}(A,B)\\
&= \frac{1}{n}\sum_{j=1}^n \frac{(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\lambda(\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j))+(1-\lambda)(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))}\\
&= \frac{1}{n}\sum_{j=1}^n \frac{(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j)} \quad \text{for } \lambda = 1,
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&GD^2_{PFS}(A,B)\\
&= \frac{\sum_{j=1}^n(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\lambda\sum_{j=1}^n(\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j))+(1-\lambda)\sum_{j=1}^n(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))}\\
&= \frac{\sum_{j=1}^n(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\sum_{j=1}^n(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))} \quad \text{for } \lambda = 0,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&GD^2_{PFS}(A,B)\\
&= \frac{\sum_{j=1}^n(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\lambda\sum_{j=1}^n(\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j))+(1-\lambda)\sum_{j=1}^n(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))}
\end{aligned}
$$

$$= \frac{\sum_{j=1}^{n}(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j))}{\sum_{j=1}^{n}(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j))} \quad \text{for } \lambda = 1. \tag{20}$$

From above analysis, it can be seen that the above four asymmetric similarity measures are the extension of the relative projection measure of the picture fuzzy numbers.

In many situations, the weight of the elements $x_j \in X$ should be taken into account. For example, in multiple attribute decision making, the considered attributes usually have different importance, and thus need to be assigned different weights. Thus, we further propose the following two weighted generalized Dice similarity measures for PFSs, respectively, as follows:

$$WD_{PFS}^1(A,B) = \sum_{j=1}^{n}\omega_j \frac{\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j)}{\lambda(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + (1-\lambda)(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))}, \tag{21}$$

$$WD_{PFS}^2(A,B) = \frac{\sum_{j=1}^{n}\omega_j^2(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j))}{\begin{pmatrix}\lambda\sum_{j=1}^{n}\omega_j^2(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) \\ (1-\lambda)\sum_{j=1}^{n}\omega_j^2(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))\end{pmatrix}}, \tag{22}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $x_j$ $(j = 1, 2, \ldots, n)$, with $\omega_j \in [0, 1]$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^{n}\omega_j w_j = 1$. In particular, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, then the weighted Dice similarity measure reduces to Dice similarity measure. That's to say, if we take $\omega_j = \frac{1}{n}$, $j = 1, 2, \ldots, n$, then there is $WGD_{PFS}^1(A,B) = D_{PFS}^1(A,B)$, $WGD_{PFS}^2(A,B) = GD_{PFS}^2(A,B)$.

Then, the weighted generalized Dice similarity measure includes some special cases by altering the parameter value $\lambda$. If $\lambda = 0.5$, the two weighted generalized Dice similarity measures (21) and (22) reduced to weighted Dice similarity measures (6) and (10):

$$\begin{aligned} WD_{PFS}^1(A,B) &= \sum_{j=1}^{n}\omega_j \frac{\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j)}{\lambda(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + (1-\lambda)(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))} \\ &= \sum_{j=1}^{n}\omega_j \frac{\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j)}{0.5(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + (1-0.5)(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))} \\ &= \sum_{j=1}^{n}\omega_j \frac{2(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j))}{(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)) + (\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j))}, \end{aligned} \tag{23}$$

$$WD^2_{PFS}(A,B) = \frac{\sum_{j=1}^{n}\omega_j^2(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\lambda\sum_{j=1}^{n}\omega_j^2(\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j))+(1-\lambda)\sum_{j=1}^{n}\omega_j^2(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))}$$
$$= \frac{\sum_{j=1}^{n}\omega_j^2(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{0.5\sum_{j=1}^{n}\omega_j^2(\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j))+(1-0.5)\sum_{j=1}^{n}\omega_j^2(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))}$$
$$= \frac{2\sum_{j=1}^{n}\omega_j^2(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\sum_{j=1}^{n}\omega_j^2(\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j))+\sum_{j=1}^{n}\omega_j^2(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))}. \tag{24}$$

If $\lambda = 0, 1$, the two weighted generalized Dice similarity measures reduced to the following asymmetric weighted similarity measures respectively:

$$WGD^1_{PFS}(A,B) = \sum_{j=1}^{n}\omega_j\frac{\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j)}{\lambda(\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j))+(1-\lambda)(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))}$$
$$= \sum_{j=1}^{n}\omega_j\frac{\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j)}{\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j)} \quad \text{for } \lambda = 0, \tag{25}$$

$$WGD^1_{PFS}(A,B) = \sum_{j=1}^{n}\omega_j\frac{\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j)}{\lambda(\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j))+(1-\lambda)(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))}$$
$$= \sum_{j=1}^{n}\omega_j\frac{\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j)}{\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j)} \quad \text{for } \lambda = 1, \tag{26}$$

$$WGD^2_{PFS}(A,B) = \frac{\sum_{j=1}^{n}\omega_j^2(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\lambda\sum_{j=1}^{n}\omega_j^2(\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j))+(1-\lambda)\sum_{j=1}^{n}\omega_j^2(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))}$$
$$= \frac{\sum_{j=1}^{n}\omega_j^2(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j)} \quad \text{for } \lambda = 0, \tag{27}$$

$$WGD^2_{PFS}(A,B) = \frac{\sum_{j=1}^{n}\omega_j^2(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\lambda\sum_{j=1}^{n}\omega_j^2(\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j))+(1-\lambda)\sum_{j=1}^{n}\omega_j^2(\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j))}$$
$$= \frac{\sum_{j=1}^{n}\omega_j^2(\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j))}{\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j)} \quad \text{for } \lambda = 1. \tag{28}$$

From above analysis, it can be seen that the above four asymmetric weighted similarity measures are the extension of the relative weighted projection measure of the picture fuzzy numbers. When the four terms like degree of positive membership, degree of neutral membership, degree of negative membership and degree of refusal membership are considered in PFSs, we further propose the generalized Dice similarity measure and weighted generalized Dice similarity measure between PFSs as follows:

$$GD_{PFS}^{3}(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j) + \rho_A(x_j)\rho_B(x_j))}{\left(\begin{array}{c}\lambda(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j) + \rho_A^2(x_j)) \\ +(1-\lambda)(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j) + \rho_B^2(x_j))\end{array}\right)}, \tag{29}$$

$$GD_{PFS}^{4}(A,B) = \frac{\sum_{j=1}^{n}(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j) + \rho_A(x_j)\rho_B(x_j))}{\left(\begin{array}{c}\lambda\sum_{j=1}^{n}(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j) + \rho_A^2(x_j)) \\ +(1-\lambda)\sum_{j=1}^{n}(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j) + \rho_B^2(x_j))\end{array}\right)}, \tag{30}$$

$$WGD_{PFS}^{3}(A,B) = \sum_{j=1}^{n}\omega_j \frac{(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j) + \rho_A(x_j)\rho_B(x_j))}{\left(\begin{array}{c}\lambda(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j) + \rho_A^2(x_j)) \\ +(1-\lambda)(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j) + \rho_B^2(x_j))\end{array}\right)}, \tag{31}$$

$$WGD_{PFS}^{4}(A,B) = \frac{\sum_{j=1}^{n}\omega_j^2(\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j) + \rho_A(x_j)\rho_B(x_j))}{\left(\begin{array}{c}\lambda\sum_{j=1}^{n}\omega_j^2(\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j) + \rho_A^2(x_j)) \\ +(1-\lambda)\sum_{j=1}^{n}\omega_j^2(\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j) + \rho_B^2(x_j))\end{array}\right)}, \tag{32}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weight vector of $x_j$ $(j = 1, 2, \dots, n)$, with $\omega_j \in [0, 1]$, $j = 1, 2, \dots, n$, $\sum_{j=1}^{n} w_j = 1$, and $\lambda$ is a positive parameter for $0 \leqslant \lambda \leqslant 1$.

## 4. Applications

In this section, the Dice similarity measures for PFSs are applied to building material recognition (adapted from Xu and Cai, 2008). Let us consider four building materials: sealant, floor varnish, wall paint and polyvinyl chloride flooring, which are represented

Table 1
The data on building materials.

| | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A$ |
|---|---|---|---|---|---|
| $x_1$ | (0.17,0.53,0.13) | (0.51,0.24,0.21) | (0.31,0.39,0.25) | (1.00,0.00,0.00) | (0.91,0.03,0.05) |
| $x_2$ | (0.89,0.08,0.03) | (0.13,0.64,0.21) | (0.07,0.09,0.05) | (0.74,0.16,0.10) | (0.68,0.08,0.21) |
| $x_3$ | (0.53,0.33,0.09) | (1.00,0.00,0.00) | (0.91,0.03,0.02) | (0.85,0.09,0.05) | (0.90,0.05,0.02) |
| $x_4$ | (0.89,0.08,0.03) | (0.13,0.64,0.21) | (0.07,0.09,0.05) | (0.74,0.16,0.10) | (0.68,0.08,0.21) |
| $x_5$ | (0.42,0.35,0.18) | (0.03,0.82,0.13) | (0.04,0.85,0.10) | (0.02,0.89,0.05) | (0.05,0.87,0.06) |
| $x_6$ | (0.08,0.89,0.02) | (0.73,0.15,0.08) | (0.68,0.26,0.06) | (0.08,0.84,0.06) | (0.13,0.75,0.09) |
| $x_7$ | (0.33,0.51,0.12) | (0.52,0.31,0.16) | (0.15,0.76,0.07) | (0.16,0.71,0.05) | (0.15,0.73,0.08) |

Table 2
The generalized Dice similarity measures of Eq. (21) and ranking orders.

| $\lambda$ | $WGD(A_1, B)$ | $WGD(A_1, B)$ | $WGD(A_1, B)$ | $WGD(A_1, B)$ | Ranking orders |
|---|---|---|---|---|---|
| 0 | 0.658 | 0.675 | 0.708 | 1.059 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 0.2 | 0.661 | 0.700 | 0.726 | 1.027 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 0.5 | 0.688 | 0.749 | 0.767 | 0.988 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 0.7 | 0.726 | 0.791 | 0.816 | 0.967 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 1.0 | 0.834 | 0.883 | 1.521 | 0.939 | $A_3 \succ A_4 \succ A_2 \succ A_1$ |

Table 3
The generalized Dice similarity measures of Eq. (22) and ranking orders.

| $\lambda$ | $WGD(A_1, B)$ | $WGD(A_1, B)$ | $WGD(A_1, B)$ | $WGD(A_1, B)$ | Ranking orders |
|---|---|---|---|---|---|
| 0 | 0.604 | 0.713 | 0.754 | 1.055 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 0.2 | 0.633 | 0.743 | 0.794 | 1.027 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 0.5 | 0.683 | 0.792 | 0.860 | 0.988 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 0.7 | 0.722 | 0.829 | 0.911 | 0.964 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 1.0 | 0.788 | 0.891 | 1.001 | 0.930 | $A_3 \succ A_4 \succ A_2 \succ A_1$ |

by the PFSs $A_i$ $(i = 1, 2, 3, 4)$ in the feature space $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$. The weight vector of $x_i$ $(i = 1, 2, \dots, 7)$ is:

$$w = (0.12, 0.15, 0.09, 0.16, 0.20, 0.10, 0.18)^T.$$

Now, we consider another kind of unknown building material $A$, with data as listed in Table 1. Based on the weight vector $w$ and the data in Table 1, we can use the above similarity measures to identify to which type the unknown material $A$ belongs.

According to Eqs. (21), (22), (31), (32) and different values of the parameter $\lambda$, the weighted generalized Dice measure values between $A_i$ $(i = 1, 2, 3, 4)$ can be obtained, which are shown in Tables 2, 3, 4 and 5 respectively.

From the Tables 2, 3, 4 and 5, different ranking orders are shown by taking different values of $\lambda$ and different Dice similarity measures. Then the building material $A$ should belong to the class of building material $A_3$ or $A_4$ according to the principle of the maximum degree of Dice similarity measures between PFSs.

Table 4
The generalized Dice similarity measures of Eq. (31) and ranking orders.

| $\lambda$ | $WGD(A_1, B)$ | $WGD(A_1, B)$ | $WGD(A_1, B)$ | $WGD(A_1, B)$ | Ranking orders |
|---|---|---|---|---|---|
| 0 | 0.658 | 0.676 | 0.715 | 1.059 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 0.2 | 0.661 | 0.700 | 0.727 | 1.027 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 0.5 | 0.688 | 0.745 | 0.752 | 0.988 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 0.7 | 0.724 | 0.786 | 0.776 | 0.966 | $A_4 \succ A_2 \succ A_3 \succ A_1$ |
| 1.0 | 0.827 | 0.870 | 0.832 | 0.938 | $A_4 \succ A_2 \succ A_3 \succ A_1$ |

Table 5
The generalized Dice similarity measures of Eq. (32) and ranking orders.

| $\lambda$ | $WGD(A_1, B)$ | $WGD(A_1, B)$ | $WGD(A_1, B)$ | $WGD(A_1, B)$ | Ranking orders |
|---|---|---|---|---|---|
| 0 | 0.604 | 0.714 | 0.760 | 1.055 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 0.2 | 0.633 | 0.743 | 0.773 | 1.027 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 0.5 | 0.682 | 0.790 | 0.794 | 0.988 | $A_4 \succ A_3 \succ A_2 \succ A_1$ |
| 0.7 | 0.719 | 0.825 | 0.809 | 0.963 | $A_4 \succ A_2 \succ A_3 \succ A_1$ |
| 1.0 | 0.783 | 0.884 | 0.831 | 0.929 | $A_4 \succ A_2 \succ A_3 \succ A_1$ |

Furthermore, for the special cases of the four generalized Dice measures we obtain the following results:

► When $\lambda = 0$, the four weighted generalized Dice measures are reduced to the weighted projection measures of $A_i$ $(i = 1, 2, 3, 4)$ on $A$. Thus, the building material $A$ should belong to the class of building material $A_4$ according to the principle of the maximum degree of Dice similarity measures between PFSs.

► When $\lambda = 0$, the four weighted generalized Dice measures are reduced to the weighted Dice similarity measures of $A_i$ $(i = 1, 2, 3, 4)$ on $A$. Thus, the building material $A$ should belong to the class of building material $A_4$ according to the principle of the maximum degree of Dice similarity measures between PFSs.

► When $\lambda = 0$, the four weighted generalized Dice measures are reduced to the weighted projection measures of $A_i$ $(i = 1, 2, 3, 4)$ on $A$. Thus, the building material $A$ should belong to the class of building material $A_3$ or $A_4$ according to the principle of the maximum degree of Dice similarity measures between PFSs.

Therefore, according to different Dice similarity measures and different values of the parameter $\lambda$, ranking orders may be also different. Thus the proposed patterns recognition methods can be assigned some value of $\lambda$ and some measure to satisfy the real requirements.

Obviously, the patterns recognition methods based on the Dice measures and the projection measures are the special cases of the proposed patterns recognition models based on generalized Dice measures. Therefore, in the patterns recognition process, the patterns recognition models developed in this paper are more general and more flexible than existing patterns recognition models under picture fuzzy environment.

## 5. Conclusion

The picture fuzzy set is characterized by three functions expressing the degree of membership, the degree of neutral membership and the degree of non-membership. It was proposed as a generalization of an intuitionistic fuzzy set in order to deal with indeterminate and inconsistent information. In this work, we shall present some novel Dice similarity measures of picture fuzzy sets and the generalized Dice similarity measures of picture fuzzy sets and indicate that the Dice similarity measures and asymmetric measures (projection measures) are the special cases of the generalized Dice similarity measures in some parameter values. Then, we propose the generalized Dice similarity measures-based patterns recognition models with picture fuzzy information. Then, we apply the generalized Dice similarity measures between picture fuzzy sets to building material recognition. Finally, an illustrative example is given to demonstrate the efficiency of the similarity measures for building material recognition. In the future, the application of the proposed Dice similarity measure of PFSs needs to be explored in decision making, risk analysis and many other fields under uncertain environment (see Wei *et al.*, 2017a; Wei and Lu, 2018; Wei *et al.*, 2017c; Zeng *et al.*, 2016; Lu *et al.*, 2017b; Zhang and Xu, 2014; Lu *et al.*, 2017a; Zeng *et al.*, 2017; Wei *et al.*, 2017b; Hu *et al.*, 2013; Peng and Yang, 2015; Wei, 2016a; Xu and Ma, 2016; Wei *et al.*, 2016b; Garg, 2016; Wei, 2017e; Wei and Lu, 2017; Wei, 2017d; Wei, 2017a; Wei and Wang, 2017).

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