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CALCULATION OF THE WEIGHT FUNCTION OF A RANDOM SPACE-TIME AUTOREGRESSIVE FIELD IN SPACE R^3

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Abstract. The structure of the weight function of a random space-time autoregressive field, existing in three-dimensional space and time, is considered. The two weight coefficients calculation algorithms are proposed here.

Key words: random field, autoregressive field, weight function of field.

1. Introduction. The weight function (WF) is an important characteristic of an autoregressive (AR) field. The other characteristics can be derived from it (Rytov, 1978; Krivickas, 1984). The autocovariances, for example, can be expressed by the weight coefficients and then calculated. The stability criterion can be formulated by those coefficients and then stability searching is possible. Therefore the developing of the weight coefficients calculation algorithm is an important problem. A similar problem has been considered in the recent papers (Kapustinskas, 1986, 1989, 1991) for a space-time AR field in one-dimensional and two-dimensional space R^1, R^2.

In this paper the two weight coefficients calculation algorithms for a space-time AR field, existing in three-dimensional space R^3 and time, are proposed.

2. The problem. The space-time AR field in a space R^3 is described by the following difference equation

$$\xi_t^{xyz} = \sum_{k=1}^{n_t} \sum_{i_x=-n'_x}^{n''_x} \sum_{i_y=-n'_y}^{n_y''} \sum_{i_z=-n'_z}^{n_z''} a_k^{i_x,i_y,i_z} \xi_{t-k}^{x+i_x,y+i_y,z+i_z} + g_t^{xyz}, \qquad (1)$$

where t are discrete time moments ($t \in (-\infty,\infty)$), x, y, z are the discrete values of the space coordinates ($x, y, z \in (-\infty,\infty)$), ξ_t^{xyz} is the value of the field at a point (x, y, z) and a moment t, $a_k^{i_x,i_y,i_z}$ are the parameters of the field, n_t is the order of the field with regard to a coordinate t, $\left\{n'_x,\ n''_x\right\}$, $\left\{n'_y,\ n''_y\right\}$, $\left\{n'_z,\ n''_z\right\}$ is the order of the field with regard to space coordinates x, y, z, $\left\{g_t^{xyz}\right\}$ is the sequence of the independent normal random values with zero average and finite dispersion σ_g^2 (white noise field).

Let the weight coefficients are h_t^{xyz}. It is necessary to develop a weight coefficients calculation algorithm at the time moments $t = 1, 2, \ldots, T$ for the field (1), the order and the parameters of it being known.

3. Structure of WF. The WF is a reaction of the model (1) to a unit pulse input signal at the point $x, y, z = 0$ and the moment $t = 1$, i.e.,

$$\delta_t^{xyz} = \begin{cases} \delta_1^{000} = 1 & (x, y, z = 0, \quad t = 1), \\ 0 & (x, y, z \neq 0, \quad t \neq 1). \end{cases} \tag{2}$$

It is supposed that there exist following zero initial conditions

$$\xi_t^{xyz} = 0 \quad (x, y, z = 0, \pm 1, \ldots, \quad t = 0, -1, \ldots). \tag{3}$$

The recurrent equation of weight coefficients can be derived from (1) by replacing ξ_t^{xyz} by h_t^{xyz} and g_t^{xyz} by δ_t^{xyz}:

$$h_t^{xyz} = \sum_{k=1}^{n_t} \sum_{i_x=-n'_x}^{n''_x} \sum_{i_y=-n'_y}^{n''_y} \sum_{i_z=-n'_z}^{n''_z} a_k^{i_x,i_y,i_z}\, h_{t-k}^{x+i_x,y+i_y,z+i_z} + \delta_t^{xyz}, \tag{4}$$

$$h_t^{xyz} = 0 \quad (x, y, z = 0, \pm 1, \ldots, \quad t = 0, -1, \ldots, -(n_t - 1)). \tag{5}$$

The WF of the field (1) exists in a four-dimensional space (x, y, z, t). The weight coefficients h_t^{xyz} are distributed at the points from the intervals $-\infty \leqslant x, y, z \leqslant \infty$, $1 \leqslant t \leqslant T$. It is obvious that the number of them is infinite. The same is with the case when the time interval is limited ($1 \leqslant t \leqslant T$). It is clear that infinite quantity of the weight coefficients can be calculated by no algorithm.

However, many of the coefficients really are equal to zero and the number of the nonzero weight coefficients is finite the time interval of the WF being limited. Then a weight coefficients algorithm is possible. First of all a structure of the WF, must be considered, i.e., it must be determined at which points (x, y, z) the nonzero weight coefficients of the field (1) are distributed. The similar problem for an AR field in R^1 and R^2 has been considered in the papers (Kapustinskas, 1986; 1989; 1990).

Any weight coefficient h_t^{xyz}, as follows from the equation (4), depends on a certain sequence $\left\{h_{t-k}^{x+i_x,y+i_y,z+i_z}\right\}$ of the weight coefficients at the past time moments $(k = \overline{1,n_t})$. The weight coefficient h_t^{xyz} is a nonzero one or more nonzero coefficients being in that sequence only. The moment $t = 1$ is a particular one. Any $h_{t-k}^{x+i_x,y+i_y,z+i_z}$ at this moment is equal to zero as follows from the initial conditions (5). Therefore the equation (4) turns into such

$$h_t^{xyz} = \delta_t^{xyz} \qquad (t = 1). \tag{6}$$

However $\delta_t^{xyz} \neq 0$ only at the point $x, y, z = 0$. Hence it follows that the single nonzero coefficient is at this moment at the point $x, y, z = 0$, i.e.,

$$h_1^{xyz} = \begin{cases} 1 \neq 0 & (x, y, z = 0), \\ 0 & \text{(in other cases)}. \end{cases} \tag{7}$$

Only the single nonzero coefficient h_1^{000} can be in the sequence $\left\{h_{t-k}^{x+i_x,y+i_y,z+i_y}\right\}$ at the moment $t = 2$. Therefore nonzero h_t^{xyz} are distributed at the points (x, y, z) for which

$$h_{t-k}^{x+i_x,y+i_y,z+i_z} = h_1^{000}. \tag{8}$$

There are many nonzero weight coefficients at this moment. They are distributed at the points from the intervals $x' \leqslant x \leqslant x''$, $y' \leqslant y \leqslant y''$, $z' \leqslant z \leqslant z''$, where $(.)'$, $(.)''$ are minimal and maximal values of the interval. These values are determined from the following conditions

$$x', y', z' : \quad h_1^{\max(x+i_x),\max(y+i_y),\max(z+i_z)} = h_1^{000}, \tag{9}$$

$$x'', y'', z'' : \quad h_1^{\min(x+i_x),\min(y+i_y),\min(z+i_z)} = h_1^{000}, \tag{10}$$

i.e.,

$$x': \quad \max(x+i_x)=0, \qquad x'': \quad \min(x+i_x)=0, \tag{11}$$
$$y': \quad \max(y+i_y)=0, \qquad y'': \quad \min(y+i_y)=0, \tag{12}$$
$$z': \quad \max(z+i_z)=0, \qquad z'': \quad \min(z+i_z)=0. \tag{13}$$

The index $i_{(.)}$ changes within the interval $-n'_{(.)} \leqslant i_{(.)} \leqslant n''_{(.)}$ and therefore

$$\min(x+i_x)=x-n'_x, \qquad \max(x+i_x)=x+n''_x, \tag{14}$$
$$\min(y+i_y)=y-n'_y, \qquad \max(y+i_y)=y+n''_y, \tag{15}$$
$$\min(z+i_z)=z-n'_z, \qquad \max(z+i_z)=z+n''_z, \tag{16}$$

so that

$$x': \quad x-n'_x=0, \qquad x'': \quad x+n''_x=0, \tag{17}$$
$$y': \quad y-n'_y=0, \qquad y'': \quad y+n''_y=0, \tag{18}$$
$$z': \quad z-n'_z=0, \qquad z'': \quad z+n''_z=0. \tag{19}$$

Hence it follows that nonzero h_2^{xyz} are distributed at the points from such intervals

$$-n''_x \leqslant x \leqslant n'_x, \quad -n''_y \leqslant y \leqslant n'_y, \quad -n''_z \leqslant z \leqslant n'_z. \tag{20}$$

Therefore

$$h_2^{xyz} = \begin{cases} \neq 0 & (-n''_x \leqslant x \leqslant n'_x,\ -n''_y \leqslant y \leqslant n'_y,\ -n''_z \leqslant z \leqslant n'_z), \\ 0 & \text{(in other cases)}. \end{cases} \tag{21}$$

At the moment $t = 3$ only the weight coefficients h_1^{xyz} and h_2^{xyz} can be in the sequence $\left\{h_{t-k}^{x+i_x,y+i_y,z+i_z}\right\}$. The limit values of the point intervals of the nonzero h_3^{xyz} are determined from such conditions

$$x',y',z': \quad h_2^{\max(x+i_x),\max(y+i_y),\max(z+i_z)} = h_2^{-n''_x,-n''_y,-n''_z}, \tag{22}$$
$$x'',y'',z'': \quad h_2^{\min(x+i_x),\min(y+i_y),\min(z+i_z)} = h_2^{n'_x,n'_y,n'_z}, \tag{23}$$

i.e.,

$$x': \quad \max(x+i_x) = -n''_x, \qquad x'': \quad \min(x+i_x) = n'_x, \tag{24}$$
$$y': \quad \max(y+i_y) = -n''_y, \qquad y'': \quad \min(y+i_y) = n'_y, \tag{25}$$
$$z': \quad \max(z+i_z) = -n''_z, \qquad z'': \quad \min(z+i_z) = n'_z. \tag{26}$$

It follows from (14) – (16) and (24) – (26), that the nonzero h_3^{xyz} are distributed at the points from the intervals

$$-2n''_x \leqslant x \leqslant 2n'_x, \quad -2n''_y \leqslant y \leqslant 2n'_y, \quad -2n''_z \leqslant z \leqslant 2n'_z. \tag{27}$$

Therefore

$$h_3^{xyz} = \begin{cases} \neq 0 & (-2n''_x \leqslant x \leqslant 2n'_x, \; -2n''_y \leqslant y \leqslant 2n'_y, \\ & \;-2n''_z \leqslant z \leqslant 2n'_z), \\ 0 & \text{(in other cases)}. \end{cases} \tag{28}$$

Finally it can be shown in a similar way that at any moment t

$$h_t^{xyz} = \begin{cases} \neq 0 & (x' \leqslant x \leqslant x'', \; y' \leqslant y \leqslant y'', \; z' \leqslant z \leqslant z''), \\ 0, & \text{(in other cases)}, \end{cases} \tag{29}$$

where

$$x' = -n''_x(t-1), \qquad x'' = n'_x(t-1), \tag{30}$$
$$y' = -n''_y(t-1), \qquad y'' = n'_y(t-1), \tag{31}$$
$$z' = -n''_z(t-1), \qquad z'' = n'_z(t-1). \tag{32}$$

The equation (29) defines the structure of the WF of the AR field (1). Hence it follows that the nonzero weight coefficients at any moment $t \geqslant 2$ are distributed at the interval points of a rectangle parallelepiped D_t. It decreases to the point $x, y, z = 0$ at the moment $t = 1$. The coordinates of vertexes of D_t in the space (x, y, z, t) are

$$\begin{aligned} &(x', y', z', t), \; (x', y', z'', t), \; (x', y'', z', t), \\ &(x', y'', z'', t), \; (x'', y', z', t), \; (x'', y'', z', t), \\ &(x'', y', z'', t), \; (x'', y'', z'', t). \end{aligned} \tag{33}$$

A polyhedral angle in the space (x, y, z, t) can be formed by joing the vertexes of $D_t (t = 1, 2, \ldots)$ with straight lines. The nonzero weight coefficients are distributed at internal points of it.

4. The WF calculation algorithm using the main memory only. Calculation of the weight coefficients h_t^{xyz} is based on the equation (4). For this purpose a four-dimensional array D is formed in the main memory. This array can be represented as a parallelepiped in the space (x, y, z), divided into first, second, $\ldots$, $(n_t + 1)$ layers. The weight coefficients h_t^{xyz} at the current moment t and at the points from the intervals $-n_x''(T-1)-n_x' \leqslant x \leqslant n_x'(T-1)+n_x''$, $-n_y''(T-1)-n_y' \leqslant y \leqslant n_y'(T-1)+n_y''$, $-n_z''(T-1)-n_z' \leqslant z \leqslant n_z'(T-1)+n_z''$ are distributed in first layer, coefficients $h_{t-k}^{xyz} (k = \overline{1, n_t})$ – in following layers. The capacity of the array D is

$$V_D = \Big[\Big(n_x' + n_x'' \Big) T + 1 \Big] \Big[\Big(n_y' + n_y'' \Big) T + 1 \Big] \times \Big[\Big(n_z' + n_z'' \Big) T + 1 \Big] \Big(n_t + 1 \Big) . \tag{34}$$

Calculations of the nonzero h_t^{xyz} are realized by the algorithm described below.

Step 0. The array D is cleaned.

Step 1. A unit value is assigned to the variable t. The value h_1^{000} is stored to the point $x, y, z = 0$ of the first layer of D and is led out to the printer.

Step 2. The value t is incremented by one.

Step 3. The n_t-layer of D is transferred to (n_t+1)-layer, (n_t-1)-layer – to n_t-layer, etc. The first layer is cleaned.

Step 4. The values h_t^{xyz} at a current time moment and at points from the interval $-n_x''(t-1) \leqslant x \leqslant n_x'(t-1)$, $-n_y''(t-1) \leqslant y \leqslant n_y'(t-1)$, $-n_z''(t-1) \leqslant z \leqslant n_z'(t-1)$ are calculated by (4) and transferred to the corresponding points of the first layer of array D.

Step 5. The weight coefficients from the above interval of the first layer are led out to the printer.

Step 6. If $t < T$, then we return to Step 2. In other cases calculations are ended.

The advantage of this algorithm is that the all weight coefficients h_{t-k}^{xyz} necessary for calculation are stored in the main memory.

Therefore this algorithm is a fast calculating one. However it requires a significant amount of the main memory. As follows from the expression (34) the capacity of the array D depends on the order of the field (1) in the fourth degree and the length T of the time interval in the third degree. Therefore the developed algorithm can be used for calculation of the WF at a comparatively short time interval and of the comparatively small order models of the field (1). For example, if a computer's main memory 1 Mb and $n'_{(.)}, n''_{(.)}, n_t \doteq 2$, the weight coefficients can be calculated at the time interval aproximatly $1 \leqslant T \leqslant 20$.

5. The algorithm using external memory. The equation (4) can be rewritten in the following way

$$h^{xyz}_{t,i_x,i_y,k} = \sum_{i_z=-n'_z}^{n''_z} a_k^{i_x,i_y,i_z} h_{t-k}^{x+i_x,y+i_y,z+i_z} + \delta_t^{xyz}$$

$$(i_x = \overline{-n'_x, n''_x},\ i_y = \overline{-n'_y, n''_y},\ k = \overline{1, n_t}), \tag{35}$$

$$h^{xyz}_{t,i_x,i_y} = \sum_{k=1}^{n_t} h^{xyz}_{t,i_x,i_y,k} \quad (i_x = \overline{-n'_x, n''_x},\ i_y = \overline{-n'_y, n''_y}), \tag{36}$$

$$h^{xyz}_{t,i_x} = \sum_{i_y=-n'_y}^{n''_y} h^{xyz}_{t,i_x,i_y} \quad (i_x = \overline{-n'_x, n''_x}), \tag{37}$$

$$h^{xyz}_{t} = \sum_{i_x=-n'_x}^{n''_x} h^{xyz}_{t,i_x}. \tag{38}$$

The array W, similar to D, is formed in the external memory. The capacities of W and D are equal. The distribution of the weight coefficients in the array W is the same as in D. The array W is divided into smaller arrays W_{l_x,l_y,l_t} $(l_x = \overline{l'_x, l''_x},\ l_y = \overline{l'_y, l''_y},\ l_t = \overline{1, n_t+1})$, where $l'_x = -n''_x(T-1) - n'_x$, $l''_x = n_x(T-1) + n''_x$, $l'_y = -n''_y(T-1) - n'_y$, $l''_y = n_y(T-1) + n''_y$. The capacity of any of these arrays is $(n'_z + n''_z)T + 1$, and the full number of them – $\left[\left(n'_x + n''_x\right)T+1\right]\left[\left(n'_y+n''_y\right)T+1\right]\left(n_t+1\right)$. The two-dimensional array D_2, divided into first and second layer, is formed in the main memory.

The every layer is identical to any of the array W_{l_x,l_y,l_t}. Therefore the capacity of the array D_2 is

$$V_{D_2} = 2\Big[\Big(n'_z + n''_z\Big)T + 1\Big]. \tag{39}$$

Calculations are based on the equations (35) – (38). The first layer is reserved for the storing calculated values $h^{xyz}_{t,i_x,i_y,k}$, h^{xyz}_{t,i_x,i_y}, h^{xyz}_{t,i_x}, h^{xyz}_t, the second – for necessary values transferred from the external memory array W_{l_x,l_y,l_t}. Calculations are realized by such algorithm.

Step 0. The array W and D_2 are cleaned.

Step 1. A unit value is assigned to the variable t. The value $h^{000}_{\cdot}$ is stored up to the point $z = 0$ of the first layer of D_2 and is led out to the printer.

Step 2. The first layer of D_2 is transferred to array W_{001}.

Step 3. The value t is incremented by one. A value $-n''_x(T-1)-1$ is assigned to the variable l_x.

Step 4. The value l_x is incremented by one. A value $-n''_y(T-1)-1$ is assigned to the variable l_y.

Step 5. The value l_y is incremented by one. A value $-n'_x - 1$ is assigned to the variable i_x. The first-layer is cleaned.

Step 6. The value i_x is incremented by one. A value $-n'_y - 1$ is assigned to the variable i_y.

Step 7. The value i_y is incremented by one. A unit value is assigned to the variable l_t.

Step 8. The value l_t is incremented by one.

Step 9. The array $W_{l_x+i_x,l_y+i_y,l_t}$ is transferred to the second layer of D_2.

Step 10. The values $h^{xyz}_{t,i_x,i_y,k}$ at the points from interval $-n''_z(t-1) \leqslant z \leqslant n'_z(t-1)$ are calculated by the equation (35). They are summed with the values from the first layer of D_2 and the results are transferred to the same layer.

Step 11. If $l_t < n_t + 1$, then we return to Step 8, in other cases – to Step 12.

Step 12. If $i_y < n''_y$, then we return to Step 7, in other cases – to Step 13.

Step 13. If $i_x < n''_x$, then we return to Step 6, in other cases – to Step 14.

Step 14. The values h_t^{xyz} at a current $t, x = l_x, y = l_y$ and points $-n''_z(t-1) \leqslant z \leqslant n'_z(t-1)$ from the first layer of D_2 are transferred to the array $W_{l_x,l_y,1}$ of the external memory and are led out to the printer also.

Step 15. If $l_y < n'_y(t-1)$, then we return to Step 5, in other cases – to Step 16.

Step 16. If $l_x < n'_x(t-1)$, then we return to Step 4, in other cases – to Step 17.

Step 17. The every array W_{l_x,l_y,l_t} is transferred to array

$$W_{l_x,l_y,l_t+1}(l_t = \overline{n_t, n_t - 1}, \quad l_x = \overline{-n''_x(t-1), n'_x(t-1)}, \quad l_y = \overline{-n''_y(t-1), n'_y(t-1)}.$$

Step 18. If $t < T$, then we return to Step 2. In other cases calculations are ended.

The developed algorithm calculates slower then the first one. However it uses significantly smaller part of the main memory for that. The ratio of the capacities ratio of the arrays D and D_2 is such

$$\lambda = V_D/V_{D_2} = \left[\left(n'_x + n''_x\right)T + 1\right]\left[\left(n'_y + n''_y\right)T + 1\right]\left(n_t + 1\right)/2, \quad (40)$$

i.e., the developed algorithm requires λ times less amount of the main memory then the first one. It follows from the expression (39) that the capacity of D_2 depends on the order and on the time interval length in first degree. Therefore this algorithm can be used for the calculation of the WF at a longer time interval and for the higher order models of the field (1) then the first one.

6. Conclusions. The weight coefficients of the space-time AR field (1) in space R^3 are determined by the recurrent equation (4). The structure of the weight function is described by the expression (29). Many coefficients of the weight function are equal to zero and the total number of the nonzero weight coefficients is finite. The nonzero weight coefficients are distributed at internal points of a certain polyhedral angle in a four-dimensional space.

The two weight coefficients calculation algorithms, based on equation (4), are proposed. The first algorithm is a fast calculating one, but it requires a significant amount of the main memory of a computer. The second calculates slower, but requires a significantly less amount of the main memory. The required amount of the main memory for the first algorithm depends on the time interval length and on the order of the field in the first degree. That for second depends on those in third and fourth degree. The second algorithm enables to calculate the weight function at a longer time interval and for the higer order models of a field then the first one.

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