INFORMATICA, 2017, Vol. 28, No. 3, 547–564

DOI: http://dx.doi.org/10.15388/Informatica.2017.144

Some Cosine Similarity Measures for Picture Fuzzy Sets and Their Applications to Strategic Decision Making

Guiwu WEI
School of Business, Sichuan Normal University, Chengdu, 610101, China
e-mail: weiguiwu@163.com

Received: October 2016; accepted: March 2017

Abstract. In this paper, we presented another form of eight similarity measures between PFSs based on the cosine function between PFSs by considering the degree of positive membership, degree of neutral membership, degree of negative membership and degree of refusal membership in PFSs. Then, we applied these weighted cosine function similarity measures between PFSs to strategic decision making. Finally, an illustrative example for selecting the optimal production strategy is given to demonstrate the efficiency of the similarity measures for strategic decision making problem.

Key words: strategic decision making; picture fuzzy set, cosine function; cosine similarity measure, optimal production strategy.

1. Introduction

The similarity measures are important and useful tools for determining the degree of similarity between two objects. Measures of similarity between fuzzy sets have gained attention from researchers for their wide applications in various fields, such as pattern recognition, machine learning, decision making and image processing, many measures of similarity between fuzzy sets have been proposed and researched in recent years (see, Bustince *et al.*, 2006, 2007, 2008; Lee *et al.*, 2009). Fuzzy set theory, introduced by Zadeh (1965), has been widely used to model uncertainty present in real-world applications. Atanassov (1986) extended fuzzy sets to Atanassov's intuitionistic fuzzy sets (IFSs), many different similarity measures between IFSs have been investigated in Li *et al.* (2007). Li and Cheng (2002) proposed a suitable similarity measure between IFSs and applied it to pattern recognition problems. Liang and Shi (2003) defined some similarity measures to differentiate different IFSs and discussed the relationships between them. Furthermore, Mitchell (2003) modified Li and Cheng's measures. Based on the extension of the Hamming distance on fuzzy sets, Szmidt and Kacprzyk (2000) developed a similarity measure between IFSs based on the Hamming distance. Hung and Yang (2004) calculated the distance between IFSs based on the Hausdorff distance and generated some similarity measures between IFSs. Liu (2005) developed some new similarity measures between IFSs and between elements. Hung and Yang (2007) proposed a similarity measure between IFSs based on the Lp metric. Xu and Xia (2010) defined the geometric distance and similarity measures of IFSs for group decision making problems. Ye (2011) proposed the cosine similarity measure between IFSs. Hung (2012) developed the likelihood-based measurement of IFSs for the medical diagnosis and bacteria classification problems. Shi and Ye (2013) further improved the cosine similarity measure of IFSs. Tian *et al.* (2013) proposed the cotangent similarity measure between IFSs for medical diagnosis. Rajarajeswari and Uma (2013) further introduced the cotangent similarity measure which considers membership, nonmembership and hesitation degrees in IFSs. Furthermore, Szmidt (2014) discussed distances between IFSs and introduced a family of similarity measures which considered the membership, nonmembership and hesitation degrees described in IFSs. Ye (2016) proposed two new cosine similarity measures and weighted cosine similarity measures based on cosine function and the information carried by the membership degrees, nonmembership degree and hesitancy degree in intuitionistic fuzzy sets (IFSs). Son and Phong (2016) gave the intuitionistic vector similarity measures for medical diagnosis.

Recently, Cuong (2014) proposed picture fuzzy set (PFS) and investigated some basic operations and properties of PFS. The picture fuzzy set is characterized by three functions expressing the degree of membership, the degree of neutral membership and the degree of nonmembership. The only constraint is that the sum of the three degrees must not exceed 1. Basically, PFS based models can be applied to situations requiring human opinions involving more answers of types: yes, abstain, no, refusal, which can't be accurately expressed in the traditional FS and IFS. Until now, some progress has been made in the research of the PFS theory. Singh (2014) investigated the correlation coefficients for picture fuzzy set and applied the correlation coefficient to clustering analysis with picture fuzzy information. Son etc. introduced several novel fuzzy clustering algorithms on the basis of picture fuzzy sets and applications to time series forecasting and weather forecasting (see, Son, 2015; Thong and Son, 2015). Thong (2015) developed a novel hybrid model between picture fuzzy clustering and intuitionistic fuzzy recommender systems for medical diagnosis and application to health care support systems. Wei (2016b) proposed picture fuzzy cross-entropy model for multiple attribute decision making problems.

Although Atanassov's intuitionistic fuzzy set theory and similarity measures have been successfully applied in different areas (see Tang *et al.*, 2017; Wei, 2015, 2008, 2009, 2010a, 2010b, 2011a; Wei *et al.*, 2011, 2013a, 2013b; Wei and Zhao, 2012b; Zhao and Wei, 2013; Zhao *et al.*, 2014), but there are situations in real life which can't be represented by Atanassov's intuitionistic fuzzy sets. Voting can be a good example of such situation as the human voters may be divided into four groups of those who: vote for, abstain, refuse to vote. Basically, picture fuzzy sets (see Cuong, 2014) based models may be adequate in situations when we face human opinions involving more answers of the type: yes, abstain, no, refusal. Therefore, in order to deal with these types of situations, in this paper we introduce the concept of similarity measures for picture fuzzy sets based on the cosine functions, which is a new extension of the similarity measure of IFSs based on the cosine functions. In order to do so, the remainder of this paper is set out as follows. In the next section, we introduce some basic concepts related to intuitionistic fuzzy set and some similarity measure between IFSs and picture fuzzy sets. In Section 3, we shall propose some similarity measure and some weighted similarity measure between PFSs based on the concept of the cosine function. In Section 4, the similarity measures for PFSs are applied to strategic decision making problem for selecting the optimal production strategy. Section 5 concludes the paper with some remarks.

2. Preliminaries

In the following, we introduce some basic concepts related to intuitionistic fuzzy sets and some similarity measure between IFSs.

Definition 1. (See Atanassov, 1986, 1989.) An IFS is given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}, \tag{1}$$

where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$, where, $0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1$, $\forall x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x to the set A.

Definition 2. (See Atanassov, 1989.) For each IFS A in X, if

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \quad \forall x \in X. \tag{2}$$

Then $\pi_A(x)$ is called the degree of indeterminacy of x to A.

Suppose that there are two IFSs:

$$A = \{\langle x_j, \mu_A(x_j), \nu_A(x_j)\rangle \mid x_j \in X\}$$

and

$$B = \{\langle x_j, \mu_B(x_j), \nu_B(x_j)\rangle \mid x_j \in X\}$$

in the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$.

Ye (2011) proposed the cosine similarity measure between IFSs and as following:

$$IFC^1(A, B) = \frac{1}{n}\sum_{j=1}^{n} \frac{\mu_A(x_j)\mu_B(x_j) + \nu_A(x_j)\nu_B(x_j)}{\sqrt{\mu_A^2(x_j) + \nu_A^2(x_j)}\sqrt{\mu_B^2(x_j) + \nu_B^2(x_j)}}. \tag{3}$$

Shi and Ye (2013) further presented the cosine similarity measure by considering membership degree, nonmembership degree and hesitancy degree in IFSs as the vector space of the three terms:

$$IFC^2(A, B) = \frac{1}{n}\sum_{j=1}^{n} \frac{\mu_A(x_j)\mu_B(x_j) + \nu_A(x_j)\nu_B(x_j) + \pi_A(x_j)\pi_B(x_j)}{\sqrt{\mu_A^2(x_j) + \nu_A^2(x_j) + \pi_A^2(x_j)}\sqrt{\mu_B^2(x_j) + \nu_B^2(x_j) + \pi_B^2(x_j)}}. \tag{4}$$

Based on cosine function, Ye (2016) proposed two cosine similarity measures between IFSs A and B.

$$IFCS^1(A,B) = \frac{1}{n}\sum_{j=1}^{n}\cos\left\{\frac{\pi}{2}\left[\left|\mu_A(x_j)-\mu_B(x_j)\right| \vee \left|\nu_A(x_j)-\nu_B(x_j)\right| \vee \left|\pi_A(x_j)-\pi_B(x_j)\right|\right]\right\}, \quad (5)$$

$$IFCS^1(A,B) = \frac{1}{n}\sum_{j=1}^{n}\cos\left\{\frac{\pi}{4}\left[\left|\mu_A(x_j)-\mu_B(x_j)\right| + \left|\nu_A(x_j)-\nu_B(x_j)\right| + \left|\pi_A(x_j)-\pi_B(x_j)\right|\right]\right\}. \quad (6)$$

On the other hand, Tian *et al.* (2013) proposed a cotangent similarity measure between IFSs and as following:

$$IFCT^1(A,B) = \frac{1}{n}\sum_{j=1}^{n}\cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(\left|\mu_A(x_j)-\mu_B(x_j)\right| \vee \left|\nu_A(x_j)-\nu_B(x_j)\right|\right)\right], \quad (7)$$

where the symbol "$\vee$" is the maximum operation. When the three terms like membership degree, nonmembership degree and hesitancy degree are considered in IFSs, Rajarajeswari and Uma (2013) defined the cotangent similarity measure of IFSs:

$$IFCT^2(A,B) = \frac{1}{n}\sum_{j=1}^{n}\cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(\left|\mu_A(x_j)-\mu_B(x_j)\right| \vee \left|\nu_A(x_j)-\nu_B(x_j)\right| \vee \left|\pi_A(x_j)-\pi_B(x_j)\right|\right)\right]. \quad (8)$$

In the following, we introduced the weighted cosine and cotangent similarity measures between IFSs and, respectively (see Ye, 2011; Shi and Ye, 2013; Rajarajeswari and Uma, 2013; Ye, 2016):

$$IFC^1(A,B) = \sum_{j=1}^{n}\omega_j\frac{\mu_A(x_j)\mu_B(x_j)+\nu_A(x_j)\nu_B(x_j)}{\sqrt{\mu_A^2(x_j)+\nu_A^2(x_j)}\sqrt{\mu_B^2(x_j)+\nu_B^2(x_j)}}, \quad (9)$$

$$IFC^2(A,B) = \sum_{j=1}^{n}\omega_j\frac{\mu_A(x_j)\mu_B(x_j)+\nu_A(x_j)\nu_B(x_j)+\pi_A(x_j)\pi_B(x_j)}{\sqrt{\mu_A^2(x_j)+\nu_A^2(x_j)+\pi_A^2(x_j)}\sqrt{\mu_B^2(x_j)+\nu_B^2(x_j)+\pi_B^2(x_j)}}, \quad (10)$$

$$WIFCS^1(A,B) = \sum_{j=1}^{n}\omega_j\cos\left\{\frac{\pi}{2}\left[\left|\mu_A(x_j)-\mu_B(x_j)\right| \vee \left|\nu_A(x_j)-\nu_B(x_j)\right| \vee \left|\pi_A(x_j)-\pi_B(x_j)\right|\right]\right\}, \quad (11)$$

$$WIFCS^2(A,B) = \sum_{j=1}^{n} \omega_j \cos\left\{\frac{\pi}{4}\left[\left|\mu_A(x_j)-\mu_B(x_j)\right| + \left|\nu_A(x_j)-\nu_B(x_j)\right| + \left|\pi_A(x_j)-\pi_B(x_j)\right|\right]\right\}, \tag{12}$$

$$WIFCT^1(A,B) = \sum_{j=1}^{n} \omega_j \cot\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\left|\mu_A(x_j)-\mu_B(x_j)\right| \vee \left|\nu_A(x_j)-\nu_B(x_j)\right|\right)\right], \tag{13}$$

$$WIFCT^2(A,B) = \sum_{j=1}^{n} \omega_j \cot\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\left|\mu_A(x_j)-\mu_B(x_j)\right| \vee \left|\nu_A(x_j)-\nu_B(x_j)\right| \vee \left|\pi_A(x_j)-\pi_B(x_j)\right|\right)\right], \tag{14}$$

where ω_j $(j = 1, 2, \dots, n)$ is the weight of an element x_j, $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} = 1$ and the symbol "$\vee$" is the maximum operation.

3. Some Similarity Measure Based on Cosine Function for Picture Fuzzy Sets

Although Atanassov's intuitionistic fuzzy set theory (see Atanassov, 1986, 1989) has been successfully applied in different areas, there are situations in real life which can't be represented by Atanassov's intuitionistic fuzzy sets. Picture fuzzy sets are extension of Atanassov's intuitionistic fuzzy sets. Picture fuzzy set (see Cuong, 2014) based models may be adequate in situations when we face human opinions involving more answers of types: yes, abstain, no, refusal. It can be considered as a powerful tool to represent the uncertain information in the process of patterns recognition and cluster analysis.

Definition 3. (See Cuong, 2014.) A picture fuzzy set (PFS) A on the universes an object of the form

$$A = \left\{\left\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\right\rangle \,\middle|\, x \in X\right\}, \tag{15}$$

where $\mu_A(x) \in [0, 1]$ is called the "degree of positive membership of A", $\eta_A(x)$ is called the "degree of neutral membership of A" and $\mu_A(x)$ is called the "degree of negative membership of A", and $\mu_A(x), \eta_A(x), \nu_A(x)$ satisfy the following condition:

$$0 \leqslant \mu_A(x) + \eta_A(x) + \nu_A(x) \leqslant 1, \quad \forall x \in X.$$

Then for $x \in X$,

$$\rho_A(x) = 1 - \left(\mu_A(x) + \eta_A(x) + \nu_A(x)\right)$$

could be called the degree of refusal membership of x in A.

3.1. *Cosine Similarity Measure for Picture Fuzzy Sets*

Let A be a PFS in an universe of discourse $X = \{x\}$, the PFS is characterized by the degree of positive membership $\mu_A(x)$, the degree of neutral membership $\eta_A(x)$ and the degree of negative membership $\nu_A(x)$ which can be considered as a vector representation with the three elements. Therefore, a cosine similarity measure and a weighted cosine similarity measure for PFSs are proposed in an analogous manner to the cosine similarity measure based on Bhattacharya's distance (see Salton and Mcgill, 1983; Bhattacharya, 1946) and cosine similarity measure for intuitionistic fuzzy set (see Ye, 2011).

Suppose that there are two PFSs:

$$A = \{\langle x_j, \mu_A(x_j), \eta_A(x_j), \nu_A(x_j)\rangle \mid x_j \in X\}$$

and

$$B = \{\langle x_j, \mu_B(x_j), \eta_B(x_j), \nu_B(x_j)\rangle \mid x_j \in X\}$$

in the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$.

A cosine similarity measure between PIFSs and is proposed as follows:

$$PFC^1(A,B) = \frac{1}{n}\sum_{j=1}^{n} \frac{\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j)}{\sqrt{\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j)}\sqrt{\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j)}}. \tag{16}$$

If we take $n = 1$, then the cosine similarity measure between PFSs A and B becomes the correlation coefficient between PFSs A and B, i.e. $C_{PFS}(A,B) = K_{PFS}(A,B)$. Therefore, the cosine similarity measure between PFSs A and B also satisfies the following properties:

(1) $0 \leqslant PFC^1(A,B) \leqslant 1$;
(2) $PFC^1(A,B) = PFC^1(B,A)$;
(3) $PFC^1(A,B) = 1$, if $A = B$, $i = 1, 2, \dots, n$.

Proof.

(1) It is obvious that the proposition is true according to the cosine value.
(2) It is obvious that the proposition is true.
(3) When $A = B$, there are $\mu_A(x_j) = \mu_B(x_j)$, $\eta_A(x_j) = \eta_B(x_j)$ and $\nu_A(x_j) = \nu_B(x_j)$ for $j = 1, 2, \dots, n$. So $C_{PFS}^1(A,B) = 1$. Therefore, we have finished the proofs. □

In the following, we shall investigate the distance measure of the angle as $d(A,B) = \arccos(C_{PFS}^1(A,B))$. It satisfies the following properties:

(1) $d(A,B) \geqslant 0$, if $0 \leqslant C_{PFS}(A,B) \leqslant 1$;
(2) $d(A,B) = \arccos(1) = 0$, if $C_{PFS}(A,A) = 1$;

(3) $d(A,B)=d(B,A)$, if $C_{PFS}(A,B)=C_{PFS}(B,A)$;
(4) $d(A,C)\leqslant d(A,B)+d(B,C)$, if $A\subseteq B\subseteq C$ for any PFS C.

Proof. Obviously, $d(A,B)$ satisfies the properties (1)–(3). In the following, $d(A,B)$ will be proved to satisfy the property (4).

For any $C=\{\langle x_j,\mu_C(x_j),\eta_C(x_j),\nu_C(x_j)\rangle \mid x_j\in X\}$, $A\subseteq B\subseteq C$, Since Eq. (16) is the sum of terms, let us investigate the distance measures of the angle between the vectors:

$$\begin{aligned} d_j\big(A(x_j),B(x_j)\big) &= \arccos\big(PFC_i^1\big(A(x_i),B(x_i)\big)\big),\\ d_j\big(B(x_j),C(x_j)\big) &= \arccos\big(PFC_i^1\big(B(x_i),C(x_i)\big)\big),\\ d_j\big(A(x_j),C(x_j)\big) &= \arccos\big(PFC_i^1\big(A(x_i),C(x_i)\big)\big),\end{aligned}$$

for $j=1,2,\dots,n$, where

$$\begin{aligned} PFC_j^1(A,B) &= \frac{\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j)}{\sqrt{\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j)}\sqrt{\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j)}},\\ PFC_j^1(B,C) &= \frac{\mu_B(x_j)\mu_C(x_j)+\eta_B(x_j)\eta_C(x_j)+\nu_B(x_j)\nu_C(x_j)}{\sqrt{\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j)}\sqrt{\mu_C^2(x_j)+\eta_C^2(x_j)+\nu_C^2(x_j)}},\\ PFC_j^1(A,C) &= \frac{\mu_A(x_j)\mu_C(x_j)+\eta_A(x_j)\eta_C(x_j)+\nu_A(x_j)\nu_C(x_j)}{\sqrt{\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j)}\sqrt{\mu_C^2(x_j)+\eta_C^2(x_j)+\nu_C^2(x_j)}}.\end{aligned}$$

For three vectors

$$\begin{aligned} A(x_j)&=\big\langle\mu_A(x_j),\eta_A(x_j),\nu_A(x_j)\big\rangle,\\ B(x_j)&=\big\langle\mu_B(x_j),\eta_B(x_j),\nu_B(x_j)\big\rangle,\\ C(x_j)&=\big\langle\mu_C(x_j),\eta_C(x_j),\nu_C(x_j)\big\rangle\end{aligned}$$

in one plane, if $A(x_j)\subseteq B(x_j)\subseteq C(x_j)$, $j=1,2,\dots,n$. Then, it is obvious that

$$d_j\big(A(x_j),C(x_j)\big)\leqslant d_j\big(A(x_j),B(x_j)\big)+d_j\big(B(x_j),C(x_j)\big),$$

according to the triangle inequality. Combining the inequality with Eq. (16), we can obtain

$$d(A,C)\leqslant d(A,B)+d(B,C).$$

Thus $d(A,B)$ satisfies the property (4). So we finished the proof. □

If we consider the weights of x_j, a weighted cosine similarity measure between PFSs A and B is proposed as follows:

$$WPFC^1(A,B)=\sum_{j=1}^{n}\omega_j\frac{\mu_A(x_j)\mu_B(x_j)+\eta_A(x_j)\eta_B(x_j)+\nu_A(x_j)\nu_B(x_j)}{\sqrt{\mu_A^2(x_j)+\eta_A^2(x_j)+\nu_A^2(x_j)}\sqrt{\mu_B^2(x_j)+\eta_B^2(x_j)+\nu_B^2(x_j)}},\quad (17)$$

where $\omega = (\omega_1, \omega_1, \ldots, \omega_n)^T$ is the weight vector of x_j $(j = 1, 2, \ldots, n)$, with $\omega_j \in [0, 1]$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^{n} \omega_j = 1$. In particular, if $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then the weighted cosine similarity measure reduces to cosine similarity measure. That's to say, if we take $\omega_i = 1/n$, $i = 1, 2, \ldots, n$, then there is $WPFC^1(A, B) = PFC^1(A, B)$.

Obviously, the weighted cosine similarity measure of two PFSs A and B also satisfies the following properties:

(1) $0 \leqslant WPFC^1(A, B) \leqslant 1$;
(2) $WPFC^1(A, B) = WPFC^1(B, A)$;
(3) $WPFC^1(A, B) = 1$, if $A = B$, $i = 1, 2, \ldots, n$.

Similar to the previous proof method, we can prove the above three properties.

When the four terms like degree of positive membership, degree of neutral membership, degree of negative membership and degree of refusal membership are considered in PFSs, we further propose the cosine similarity measure and weighted cosine similarity measure between PFSs as follows:

$$PFC^2(A, B) = \frac{1}{n}\sum_{j=1}^{n} \frac{\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j) + \rho_A(x_j)\rho_B(x_j)}{\sqrt{\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j) + \rho_A^2(x_j)}\sqrt{\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j) + \rho_B^2(x_j)}}, \tag{18}$$

$$WPFC^2(A, B) = \sum_{j=1}^{n} \omega_j \frac{\mu_A(x_j)\mu_B(x_j) + \eta_A(x_j)\eta_B(x_j) + \nu_A(x_j)\nu_B(x_j) + \rho_A(x_j)\rho_B(x_j)}{\sqrt{\mu_A^2(x_j) + \eta_A^2(x_j) + \nu_A^2(x_j) + \rho_A^2(x_j)}\sqrt{\mu_B^2(x_j) + \eta_B^2(x_j) + \nu_B^2(x_j) + \rho_B^2(x_j)}}, \tag{19}$$

where $\omega = (\omega_1, \omega_1, \ldots, \omega_n)^T$ is the weight vector of x_i $(i = 1, 2, \ldots, n)$, with $\omega_j \in [0, 1]$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^{n} \omega_j = 1$.

3.2. *Similarity Measures of Picture Fuzzy Sets Based on Cosine Function*

Based on the cosine function, in this section, we shall propose two cosine similarity measures between PFSs and analyse their properties.

Definition 4. Let

$$A = \{\langle x_j, (\mu_A(x_j), \eta_A(x_j), \nu_A(x_j))\rangle \mid x_j \in X\}$$

and

$$B = \{\langle x_j, (\mu_B(x_j), \eta_B(x_j), \nu_B(x_j))\rangle \mid x_j \in X\}$$

be any two PFSs in $X = \{x_1, x_2, \ldots, x_n\}$. Then, we shall define four cosine similarity measures between PFSs, respectively, as follows:

$$PFCS^1(A, B) = \frac{1}{n}\sum_{j=1}^{n}\cos\left\{\frac{\pi}{2}\left[\left|\mu_A(x_j) - \mu_B(x_j)\right| \vee \left|\eta_A(x_j) - \eta_B(x_j)\right| \vee \left|\nu_A(x_j) - \nu_B(x_j)\right|\right]\right\}, \tag{20}$$

$$PFCS^2(A, B) = \frac{1}{n}\sum_{j=1}^{n}\cos\left\{\frac{\pi}{4}\left[\left|\mu_A(x_j) - \mu_B(x_j)\right| + \left|\eta_A(x_j) - \eta_B(x_j)\right| + \left|\nu_A(x_j) - \nu_B(x_j)\right|\right]\right\}, \tag{21}$$

where the symbol "$\vee$" is the maximum operation.

When the four terms like degree of positive membership, degree of neutral membership, degree of negative membership and degree of refusal membership are considered in PFSs, we further propose two cosine similarity measures between PFSs as follows:

$$PFCS^3(A, B) = \frac{1}{n}\sum_{j=1}^{n}\cos\left\{\frac{\pi}{2}\begin{pmatrix} |\mu_A(x_j) - \mu_B(x_j)| \vee |\eta_A(x_j) - \eta_B(x_j)| \vee \\ |\nu_A(x_j) - \nu_B(x_j)| \vee |\rho_A(x_j) - \rho_B(x_j)| \end{pmatrix}\right\}, \tag{22}$$

$$PFCS^4(A, B) = \frac{1}{n}\sum_{j=1}^{n}\cos\left\{\frac{\pi}{4}\begin{pmatrix} |\mu_A(x_j) - \mu_B(x_j)| + |\eta_A(x_j) - \eta_B(x_j)| + \\ |\nu_A(x_j) - \nu_B(x_j)| + |\rho_A(x_j) - \rho_B(x_j)| \end{pmatrix}\right\}. \tag{23}$$

Proposition 1. *For two PFSs A and B in $X = \{x_1, x_2, \ldots, x_n\}$, the cosine similarity measures*

$$PFCS^k(A, B), \quad k = 1, 2, 3, 4,$$

should satisfy the following properties (1)–(4):

(1) $0 \leqslant PFCS^k(A, B) \leqslant 1$;
(2) $PFCS^k(A, B) = 1$ *if and only if* $A = B$;
(3) $PFCS^k(A, B) = PFCS^k(B, A)$;
(4) *If C is a PFS in X and $A \subseteq B \subseteq C$, then*

$$PFCS^k(A, C) \leqslant PFCS^k(A, B) \quad \textit{and} \quad PFCS^k(A, C) \leqslant PFCS^k(B, C).$$

Proof. (1) Since the value of the cosine function is within [0, 1], the similarity measure based on the cosine function is also within [0, 1]. Thus, there is $0 \leqslant PFCS^k(A, B) \leqslant 1$.

(2) For two PFSs A and B in $X = \{x_1, x_2, \ldots, x_n\}$, if $A = B$, then $\mu_A(x_j) = \mu_B(x_j)$, $\eta_A(x_j) = \eta_B(x_j)$, $\nu_A(x_j) = \nu_B(x_j)$, $\rho_A(x_j) = \rho_B(x_j)$ for $j = 1, 2, \ldots, n$. Thus,

$|\mu_A(x_j) - \mu_B(x_j)| = 0$, $|\eta_A(x_j) - \eta_B(x_j)| = 0$, $|\nu_A(x_j) - \nu_B(x_j)| = 0$, $|\rho_A(x_j) - \rho_B(x_j)| = 0$. So, $PFCS^k(A, B) = 1$, $k = 1, 2, 3, 4$.

If $PFCS^k(A, B) = 1$, $k = 1, 2, 3, 4$, this implies

$$\begin{aligned}
&\left|\mu_A(x_j) - \mu_B(x_j)\right| = 0, \quad \left|\eta_A(x_j) - \eta_B(x_j)\right| = 0, \\
&\left|\nu_A(x_j) - \nu_B(x_j)\right| = 0, \quad \left|\rho_A(x_j) - \rho_B(x_j)\right| = 0,
\end{aligned}$$

for $j = 1, 2, 3, 4$. Since $\cos(0) = 1$. Then, there are

$$\mu_A(x_j) = \mu_B(x_j), \quad \eta_A(x_j) = \eta_B(x_j), \quad \nu_A(x_j) = \nu_B(x_j), \quad \rho_A(x_j) = \rho_B(x_j),$$

for $j = 1, 2, 3, 4$. Hence $A = B$.

(3) Proof is straightforward.

(4) If $A \subseteq B \subseteq C$, then there are

$$\begin{aligned}
&\mu_A(x_j) \leqslant \mu_B(x_j) \leqslant \mu_C(x_j), \quad \eta_A(x_j) \leqslant \eta_B(x_j) \leqslant \eta_C(x_j), \\
&\nu_A(x_j) \geqslant \nu_B(x_j) \geqslant \nu_C(x_j),
\end{aligned}$$

for $j = 1, 2, \ldots, n$. Then, we have

$$\begin{aligned}
&\left|\mu_A(x_j) - \mu_B(x_j)\right| \leqslant \left|\mu_A(x_j) - \mu_C(x_j)\right|, \\
&\left|\mu_B(x_j) - \mu_C(x_j)\right| \leqslant \left|\mu_A(x_j) - \mu_C(x_j)\right|, \\
&\left|\eta_A(x_j) - \eta_B(x_j)\right| \leqslant \left|\eta_A(x_j) - \eta_C(x_j)\right|, \\
&\left|\eta_B(x_j) - \eta_C(x_j)\right| \leqslant \left|\eta_A(x_j) - \eta_C(x_j)\right|, \\
&\left|\nu_A(x_j) - \nu_B(x_j)\right| \leqslant \left|\nu_A(x_j) - \nu_C(x_j)\right|, \\
&\left|\nu_B(x_j) - \nu_B(x_j)\right| \leqslant \left|\nu_A(x_j) - \nu_C(x_j)\right|, \\
&\left|\rho_A(x_j) - \rho_B(x_j)\right| \leqslant \left|\rho_A(x_j) - \rho_C(x_j)\right|, \\
&\left|\rho_B(x_j) - \rho_C(x_j)\right| \leqslant \left|\rho_A(x_j) - \rho_C(x_j)\right|.
\end{aligned}$$

Hence, $PFCS^k(A, C) \leqslant PFCS^k(A, B)$ and $PFCS^k(A, C) \leqslant PFCS^k(B, C)$ for $k = 1, 2, 3, 4$ as the cosine function is a decreasing function with the interval $[0, \pi/2]$. Thus, the proofs of these properties are completed. □

In many situations, the weight of the elements $x_j \in X$ should be taken into account. For example, in multiple attribute decision making, the considered attributes usually have different importance, and thus need to be assigned different weights. As a result, four weighted cosine similarity measure between PFSs A and B is proposed as follows:

$$\begin{aligned}
&WPFCS^1(A, B) \\
&\quad = \sum_{j=1}^{n} \omega_j \cos\left\{\frac{\pi}{2}\left[\left|\mu_A(x_j) - \mu_B(x_j)\right| \vee \left|\eta_A(x_j) - \eta_B(x_j)\right| \vee \left|\nu_A(x_j) - \nu_B(x_j)\right|\right]\right\}, \quad (24)
\end{aligned}$$

$$WPFCS^2(A,B) = \sum_{j=1}^{n} \omega_j \cos\left\{\frac{\pi}{4}\left[\left|\mu_A(x_j)-\mu_B(x_j)\right|+\left|\eta_A(x_j)-\eta_B(x_j)\right|+\left|\nu_A(x_j)-\nu_B(x_j)\right|\right]\right\}, \quad (25)$$

$$WPFCS^3(A,B) = \sum_{j=1}^{n} \omega_j \cos\left\{\frac{\pi}{2}\begin{pmatrix} |\mu_A(x_j)-\mu_B(x_j)| \vee |\eta_A(x_j)-\eta_B(x_j)| \vee \\ |\nu_A(x_j)-\nu_B(x_j)| \vee |\rho_A(x_j)-\rho_B(x_j)| \end{pmatrix}\right\}, \quad (26)$$

$$WPFCS^4(A,B) = \sum_{j=1}^{n} \omega_j \cos\left\{\frac{\pi}{4}\begin{pmatrix} |\mu_A(x_j)-\mu_B(x_j)| + |\eta_A(x_j)-\eta_B(x_j)| + \\ |\nu_A(x_j)-\nu_B(x_j)| + |\rho_A(x_j)-\rho_B(x_j)| \end{pmatrix}\right\}, \quad (27)$$

where $\omega = (\omega_1, \omega_1, \dots, \omega_n)^T$ is the weight vector of x_i, $i = 1, 2, \dots, n$, with $\omega_j \in [0, 1]$, $j = 1, 2, \dots, n$, $\sum_{j=1}^{n} \omega_j = 1$ and the symbol "$\vee$" is the maximum operation. In particular, if $\omega = (1/n, 1/n, \dots, 1/n)^T$, then the weighted cosine similarity measure reduces to cosine similarity measure. That's to say, if we take $\omega_j = 1/n$, $j = 1, 2, \dots, n$, then there is $WPFCS^k(A, B) = PFCS^k(B, A)$, $k = 1, 2, 3, 4$. Obviously, the weighted cosine similarity measures also satisfy the axiomatic requirements of similarity measures in Proposition 2.

Proposition 2. *For two PFSs A and B in $X = \{x_1, x_2, \dots, x_n\}$, the weighted cosine similarity measures $WPFCS^k(A, B)$, $k = 1, 2, 3, 4$, satisfy the following properties* (1)–(4):

(1) $0 \leqslant WPFCS^k(A, B) \leqslant 1$;
(2) $WPFCS^k(A, B) = 1$ *if and only if* $A = B$;
(3) $WPFCS^k(A, B) = WPFCS^k(B, A)$;
(4) *If C is a PFS in X and $A \subseteq B \subseteq C$, then*

$$WPFCS^k(A, C) \leqslant WPFCS^k(A, B) \ \ \textit{and} \ \ WPFCS^k(A, C) \leqslant WPFCS^k(B, C).$$

By using similar proof in proposition 1, we can give the proofs of these properties (1)–(4).

3.3. *Similarity Measures of Picture Fuzzy Sets Based on Cotangent Function*

In this section, we shall propose a cotangent similarity measures between PFSs as follows:

$$PFCT^1(A,B) = \frac{1}{n}\sum_{j=1}^{n} \cot\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\left|\mu_A(x_j)-\mu_B(x_j)\right| \vee \left|\eta_A(x_j)-\eta_B(x_j)\right| \vee \left|\nu_A(x_j)-\nu_B(x_j)\right|\right)\right], \quad (28)$$

where the symbol "$\vee$" is the maximum operation. When the four terms like degree of positive membership, degree of neutral membership, degree of negative membership and degree of refusal membership are considered in PFSs, we further propose a cotangent similarity measures between PFSs as follows:

$$PFCT^2(A,B) = \frac{1}{n}\sum_{j=1}^{n}\cot\left[\frac{\pi}{4}+\frac{\pi}{4}\begin{pmatrix}\left|\mu_A(x_j)-\mu_B(x_j)\right|\vee\left|\eta_A(x_j)-\eta_B(x_j)\right|\vee\\ \left|\nu_A(x_j)-\nu_B(x_j)\right|\vee\left|\rho_A(x_j)-\rho_B(x_j)\right|\end{pmatrix}\right]. \tag{29}$$

In many situations, the weight of the elements $x_i \in X$ should be taken into account. For example, in multiple attribute decision making, the considered attributes usually have different importance, and thus need to be assigned different weights. As a result, four weighted cotangent similarity measure between PFSs A and B is proposed as follows:

$$WPFCT^1(A,B) = \sum_{j=1}^{n}\omega_j\cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(\left|\mu_A(x_j)-\mu_B(x_j)\right|\vee\left|\eta_A(x_j)-\eta_B(x_j)\right|\vee\left|\nu_A(x_j)-\nu_B(x_j)\right|\right)\right], \tag{30}$$

$$WPFCT^2(A,B) = \sum_{j=1}^{n}\omega_j\cot\left[\frac{\pi}{4}+\frac{\pi}{4}\begin{pmatrix}\left|\mu_A(x_j)-\mu_B(x_j)\right|\vee\left|\eta_A(x_j)-\eta_B(x_j)\right|\vee\\ \left|\nu_A(x_j)-\nu_B(x_j)\right|\vee\left|\rho_A(x_j)-\rho_B(x_j)\right|\end{pmatrix}\right], \tag{31}$$

where $\omega = (\omega_1, \omega_1, \ldots, \omega_n)^T$ is the weight vector of x_i, $i = 1, 2, \ldots, n$,with $\omega_j \in [0, 1]$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^{n}\omega_j = 1$ and the symbol "$\vee$" is the maximum operation. In particular, if $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then the weighted cotangent similarity measure reduces to cotangent similarity measure.

4. Numerical Example

In this section, the cosine similarity measures for PFSs are applied to strategic decision making problems (adapted from Wei and Merigó, 2012). In the following, we shall analyse a strategic decision-making problem about the selection of the optimal production strategy. Assume a company wants to create a new product and they are analysing the optimal target in order to obtain the highest benefits. After analysing the market they consider four possible strategies to follow: ①A_1: create a new product oriented to the rich customers; ②A_2: create a new product oriented to the mid-level and low-level customers; ③A_3: create a new product adapted to all the customers; ④A_4: do not create any product. After careful review of the information, the decision makers have summarized the information of the strategies in six general characteristics: ①S_1: benefits in the short term; ②S_2: benefits in the mid term; ③S_3: benefits in the long term; ④S_4: risk of the production strategy; ⑤S_5: potential market and market risk; ⑥S_6: industrialization infrastructure, human resources and financial conditions. The decision makers are required to evaluate the four possible

Table 1
The data on production strategies.

	A_1	A_2	A_3	A_4	A
S_1	(0.53,0.33,0.09)	(1.00,0.00,0.00)	(0.91,0.03,0.02)	(0.85,0.09,0.05)	(0.90,0.05,0.02)
S_2	(0.89,0.08,0.03)	(0.13,0.64,0.21)	(0.07,0.09,0.05)	(0.74,0.16,0.10)	(0.68,0.08,0.21)
S_3	(0.42,0.35,0.18)	(0.03,0.82,0.13)	(0.04,0.85,0.10)	(0.02,0.89,0.05)	(0.05,0.87,0.06)
S_4	(0.08,0.89,0.02)	(0.73,0.15,0.08)	(0.68,0.26,0.06)	(0.08,0.84,0.06)	(0.13,0.75,0.09)
S_5	(0.33,0.51,0.12)	(0.52,0.31,0.16)	(0.15,0.76,0.07)	(0.16,0.71,0.05)	(0.15,0.73,0.08)
S_6	(0.17,0.53,0.13)	(0.51,0.24,0.21)	(0.31,0.39,0.25)	(1.00,0.00,0.00)	(0.91,0.03,0.05)

Table 2
The similarity measures between A_i $(i = 1, 2, 3, 4)$ and A.

Similarity measures	(A_1, A)	(A_2, A)	(A_3, A)	(A_4, A)
$WPFC^1(A_i, A)$	0.813	0.656	0.787	**0.994**
$WPFC^2(A_i, A)$	0.810	0.656	0.638	**0.993**
$WPFCS^1(A_i, A)$	0.813	0.765	0.762	**0.992**
$WPFCS^2(A_i, A)$	0.840	0.765	0.831	**0.991**
$WPFCS^3(A_i, A)$	0.813	0.765	0.709	**0.992**
$WPFCS^4(A_i, A)$	0.813	0.757	0.707	**0.989**
$WPFCT^2(A_i, A)$	0.486	0.442	0.469	**0.666**
$WPFCT^2(A_i, A)$	0.486	0.442	0.440	**0.665**

production strategies A_i $(i = 1, 2, 3, 4)$ under six general characteristics and the decision information is represented by PFSs and is presented in Table 1. Each of which is featured by the content of six characteristics in the feature space $S = \{S_1, S_2, S_3, S_4, S_5, S_6\}$. The weight vector of S_i $(i = 1, 2, \dots, 6)$ is: $\omega = (0.12, 0.25, 0.09, 0.16, 0.20, 0.18)^T$.

Now, we consider another kind of unknown production strategy A, with data as listed in Table 1. Based on the weight vector and the data in Table 1, we can use the above similarity measures to identify to which type the unknown production strategy A should belong.

From the above numerical results in Table 2, we know that the degree of similarity between A_4 and A is the largest one as derived by eight similarity measures. That is, all the eight similarity measures assign the unknown production strategy A to the class of production strategy A_4 according to the principle of the maximum degree of similarity between PFSs. Yet, there exist two slightly different ranking results: for the similarity measures $WPFC^1(A_i, A)$, $WPFCS^2(A_i, A)$ and $WPFCT^1(A_i, A)$, $i = 1, 2, 3, 4$, all these three similarity measures derive the same ranking of the production strategies, in which the degree of similarity between A_1 and A ranks the second, the degree of similarity between A_3 and A ranks the third, the degree of similarity between A_2 and A is the smallest one. While for the other five similarity measures, the degree of similarity between A_1 and A ranks the second, the degree of similarity between A_2 and A ranks the third, the degree of similarity between A_3 and A is the smallest one.

5. Conclusion

In this paper, we presented another form of eight similarity measures between PFSs based on the cosine function between PFSs by considering the degree of positive membership, degree of neutral membership, degree of negative membership and degree of refusal membership in PFSs. Then, we applied these weighted cosine function similarity measures between PFSs to strategic decision making problem. Finally, an illustrative example for selection of the optimal production strategy is given to demonstrate the efficiency of the similarity measures for strategic decision making problem. In the future, the application of the proposed cosine similarity measures of PFSs needs to be explored in complex group decision making, risk analysis and many other fields under uncertain environments, such as dual hesitant fuzzy linguistic sets, dual hesitant fuzzy uncertain linguistic sets, interval-valued dual hesitant fuzzy linguistic sets, and so on (see Wei *et al*., 2016a; Lu and Wei, 2016; Wei *et al*., 2016b; Zhou *et al*., 2013; Lin *et al*., 2014; Wei and Zhao, 2012a; Wei, 2016a, 2012, 2011c, 2011b; Park *et al*., 2009; Ye, 2010; Wu and Chiclana, 2014; Chen, 2014; Liu *et al*., 2015; Wei *et al*., 2013c; Meng *et al*., 2016; Wei *et al*., 2017; Wei and Wang, 2017; Lu *et al*., 2017a, 2017b; Wei, 2017a, 2017b).

Acknowledgements. The work was supported by the National Natural Science Foundation of China under Grant Nos. 61174149 and 71571128 and the Humanities and Social Sciences Foundation of Ministry of Education of the People's Republic of China (No. 15YJCZH138) and the construction plan of scientific research innovation team for colleges and universities in Sichuan Province (15TD0004).

References

Atanassov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20, 87–96.

Atanassov, K. (1989). More on intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 33, 37–46.

Bhattacharya, A. (1946). On a measure of divergence of two multinomial populations. *Sankhya*, 7, 401–406.

Bustince, H., Barrenechea, E., Pagola, M. (2006). Restricted equivalence functions. *Fuzzy Sets and Systems*, 157, 2333–2346.

Bustince, H., Barrenechea, E., Pagola, M. (2007). Image thresholding using restricted equivalence functions and maximizing the measures of similarity. *Fuzzy Sets and Systems*, 158, 496–516.

Bustince, H., Barrenechea, E., Pagola, M. (2008). Relationship between restricted dissimilarity functions, restricted equivalence functions and normal E–N functions: image thresholding invariant. *Pattern Recognition Letters*, 29, 525–536.

Chen, T.Y. (2014). Interval-valued intuitionistic fuzzy QUALIFLEX method with a likelihood-based comparison approach for multiple criteria decision analysis. *Information Sciences*, 261, 149–169.

Cuong, B.C. (2014). Picture fuzzy sets. *Journal of Computer Science and Cybernetics*, 30(4), 409–420.

Hung, K.C. (2012). Applications of medical information: Using an enhanced likelihood measured approach based on intuitionistic fuzzy sets. *IIE Transactions on Healthcare Systems Engineering*, 2, 224–231.

Hung, W.L., Yang, M.S. (2004). Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance. *Pattern Recognition Letters*, 25, 1603–1611.

Hung, W.L., Yang, M.S. (2007). Similarity measures of intuitionistic fuzzy sets based on Lp metric. *International Journal of Approximate Reasoning*, 46, 120–136.

Lee, S.H., Pedrycz, W., Sohn, G. (2009). Design of similarity and dissimilarity measures for fuzzy sets on the basis of distance measure. *International Journal of Fuzzy Systems*, 11, 67–72.

Li, D.F., Cheng, C. (2002). New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions. *Pattern Recognition Letters*, 23, 221–225.

Li, Y.H., Olson, D.L., Zheng, Q. (2007). Similarity measures between intuitionistic fuzzy (vague) sets: a comparative analysis. *Pattern Recognition Letters*, 28, 278–285.

Liang, Z., Shi, P. (2003). Similarity measures on intuitionistic fuzzy sets. *Pattern Recognition Letters*, 24, 2687–2693.

Lin, R., Zhao, X.F., Wei, G.W. (2014). Models for selecting an ERP system with hesitant fuzzy linguistic information. *Journal of Intelligent and Fuzzy Systems*, 26, 2155–2165.

Liu, H.W. (2005). New similarity measures between intuitionistic fuzzy sets and between elements. *Mathematical and Computer Modelling*, 42, 61–70.

Liu, B.S., Shen, Y.H., Zhang, W. (2015). An interval-valued intuitionistic fuzzy principal component analysis model-based method for complex multi-attribute large-group decision-making. *European Journal of Operational Research*, 245, 209–225.

Lu, M., Wei, G.W. (2016). Models for multiple attribute decision making with dual hesitant fuzzy uncertain linguistic information. *International Journal of Knowledge-Based and Intelligent Engineering Systems*, 20, 217–227.

Lu, M., Wei, G.W., Alsaadi, F.E., Hayat, T., Alsaedi, A. (2017a). Hesitant pythagorean fuzzy hamacher aggregation operators and their application to multiple attribute decision making. *Journal of Intelligent and Fuzzy Systems*, 33(2), 1105–1117.

Lu, M., Wei, G.W., Alsaadi, F.E., Hayat, T., Alsaedi, A. (2017b). Bipolar 2-tuple linguistic aggregation operators in multiple attribute decision making. *Journal of Intelligent and Fuzzy Systems*, 33(2), 1197–1207.

Meng, F.Y., Zhou, D., Chen, X.H. (2016). An approach to hesitant fuzzy group decision making with multi-granularity linguistic information. *Informatica*, 27, 767–798.

Mitchell, H.B. (2003). On the Dengfeng–Chuntian similarity measure and its application to pattern recognition. *Pattern Recognition Letters*, 24, 3101–3104.

Park, J.H., Park, Y., Young, C.K. (2009). Correlation coefficient of interval-valued intuitionistic fuzzy sets and its application to multiple attribute group decision making problems. *Mathematical and Computer Modeling*, 50, 1279–1293.

Rajarajeswari, P., Uma, N. (2013). Intuitionistic fuzzy multi similarity measure based on cotangent function. *International Journal of Engineering Research and Technology*, 2, 1323–1329.

Salton, G., Mcgill, M.J. (1983). *Introduction to Modern Information Retrieval*. McGrawpHill.

Shi, L.L., Ye, J. (2013). Study on fault diagnosis of turbine using an improved cosine similarity measure for vague sets. *Journal of Applied Sciences*, 13, 1781–1786.

Singh, P. (2014) Correlation coefficients for picture fuzzy sets. *Journal of Intelligent and Fuzzy Systems*, 27, 2857–2868.

Son, L. (2015). DPFCM: a novel distributed picture fuzzy clustering method on picture fuzzy sets. *Expert System with Applications*, 2, 51–66.

Son, L.H., Phong, P.H. (2016). On the performance evaluation of intuitionistic vector similarity measures for medical diagnosis. *Journal of Intelligent and Fuzzy Systems*, 31, 1597–1608.

Szmidt, E. (2014). *Distances and Similarities in Intuitionistic Fuzzy Sets*, Vol. 307. Springer International Publishing Switzerland.

Szmidt, E., Kacprzyk, J. (2000). Distances between intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 114, 505–518.

Tang, Y. Wen, L.L., Wei, G.W. (2017). Approaches to multiple attribute group decision making based on the generalized Dice similarity measures with intuitionistic fuzzy information. *International Journal of Knowledge-Based and Intelligent Engineering Systems*, 21, 85–95.

Thong, N.T. (2015). HIFCF: an effective hybrid model between picture fuzzy clustering and intuitionistic fuzzy recommender systems for medical diagnosis expert systems with applications. *Expert System with Applications*, 42, 3682–3701.

Thong, P.H., Son, L.H. (2015). A new approach to multi-variable fuzzy forecasting using picture fuzzy clustering and picture fuzzy rule interpolation method. *Advances in Intelligent Systems and Computing*, 326, 679–690.

Tian, M.Y. (2013). A new fuzzy similarity based on cotangent function for medical diagnosis. *Advanced Modeling and Optimization*, 15, 151–156.

Wei, G.W. (2008). Maximizing deviation method for multiple attribute decision making in intuitionistic fuzzy setting. *Knowledge-Based Systems*, 21, 833–836.

Wei, G.W. (2009). Some geometric aggregation functions and their application to dynamic multiple attribute decision making in intuitionistic fuzzy setting. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 17, 179–196.

Wei, G.W. (2010a). GRA method for multiple attribute decision making with incomplete weight information in intuitionistic fuzzy setting. *Knowledge-Based Systems*, 23, 243–247.

Wei, G.W. (2010b). Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making. *Applied Soft Computing*, 10, 423–431.

Wei, G.W. (2011a). Gray relational analysis method for intuitionistic fuzzy multiple attribute decision making. *Expert Systems with Applications*, 38, 11671–11677.

Wei, G.W. (2011b). Grey relational analysis method for 2-tuple linguistic multiple attribute group decision making with incomplete weight information. *Expert Systems with Applications*, 38, 4824–4828.

Wei, G.W. (2011c). Grey relational analysis model for dynamic hybrid multiple attribute decision making. *Knowledge-Based Systems*, 24, 672–679.

Wei, G.W. (2012). Hesitant Fuzzy prioritized operators and their application to multiple attribute group decision making. *Knowledge-Based Systems*, 31, 176–182.

Wei, G.W. (2015). Approaches to interval intuitionistic trapezoidal fuzzy multiple attribute decision making with incomplete weight information. *International Journal of Fuzzy Systems*, 17, 484–489.

Wei, G.W. (2016a). Interval valued hesitant fuzzy uncertain linguistic aggregation operators in multiple attribute decision making. *International Journal of Machine Learning and Cybernetics*, 7, 1093–1114.

Wei, G.W. (2016b). Picture fuzzy cross-entropy for multiple attribute decision making problems. *Journal of Business Economics and Management*, 17, 491–502.

Wei, G.W. (2017a). Interval-valued dual hesitant fuzzy uncertain linguistic aggregation operators in multiple attribute decision making. *Journal of Intelligent and Fuzzy Systems*, 33(3), 1881–1893.

Wei, G.W. (2017b). Picture 2-tuple linguistic Bonferroni mean operators and their application to multiple attribute decision making. *International Journal of Fuzzy System*, 19(4), 997–1010.

Wei, G.W., Merigó, J.M. (2012). Methods for strategic decision making problems with immediate probabilities in intuitionistic fuzzy setting. *Scientia Iranica E*, 19, 1936–1946.

Wei, G.W., Zhao, X.F. (2012a). Some dependent aggregation operators with 2-tuple linguistic information and their application to multiple attribute group decision making. *Expert Systems with Applications*, 39, 5881–5886.

Wei, G.W., Zhao, X.F. (2012b). Some induced correlated aggregating operators with intuitionistic fuzzy information and their application to multiple attribute group decision making. *Expert Systems with Applications*, 39, 2026–2034.

Wei, G.W., Wang, J.M. (2017). A comparative study of robust efficiency analysis and data envelopment analysis with imprecise data. *Expert Systems with Applications*, 81, 28–38.

Wei, G.W., Wang, H.J., Lin, R. (2011). Application of correlation coefficient to interval-valued intuitionistic fuzzy multiple attribute decision making with incomplete weight information. *Knowledge and Information Systems*, 26, 337–349.

Wei, G.W., Wang, J.M., Chen, J. (2013a). Potential optimality and robust optimality in multiattribute decision analysis with incomplete information: A comparative study. *Decision Support Systems*, 55, 679–684.

Wei, G.W., Zhao, X.F., Lin, R. (2013b). Some hesitant interval-valued fuzzy aggregation operators and their applications to multiple attribute decision making. *Knowledge-Based Systems*, 46, 43–53.

Wei, G.W., Zhao, X.F., Lin, R. (2013c). Uncertain linguistic Bonferroni mean operators and their application to multiple attribute decision making. *Applied Mathematical Modelling*, 37, 5277–5285.

Wei, G.W., Alsaadi, F.E., Hayat, T., Alsaedi, A. (2016a). Hesitant fuzzy linguistic arithmetic aggregation operators in multiple attribute decision making. *Iranian Journal of Fuzzy Systems*, 13, 1–16.

Wei, G.W., Xu, X.R., Deng, D.X. (2016b). Interval-valued dual hesitant fuzzy linguistic geometric aggregation operators in multiple attribute decision making. *International Journal of Knowledge-based and Intelligent Engineering Systems*, 20, 189–196.

Wei, G.W., Alsaadi, F.E., Hayat, T. (2017). A linear assignment method for multiple criteria decision analysis with hesitant fuzzy sets based on fuzzy measure. *International Journal of Fuzzy Systems*, 19, 607–614.

Wu, J., Chiclana, F. (2014). A risk attitudinal ranking method for interval-valued intuitionistic fuzzy numbers based on novel attitudinal expected score and accuracy functions. *Applied Soft Computing*, 22, 272–286.

Xu, Z.S., Xia, M.M. (2010). Some new similarity measures for intuitionistic fuzzy values and their application in group decision making. *Journal of System Science and Engineering*, 19, 430–452.

Ye, J. (2010). Fuzzy decision-making method based on the weighted correlation coefficient under intuitionistic fuzzy environment. *European Journal of Operational Research*, 205, 202–204.

Ye, J. (2011). Cosine similarity measures for intuitionistic fuzzy sets and their applications. *Mathematical and Computer Modelling*, 53, 91–97.

Ye, J. (2016). Similarity measures of intuitionistic fuzzy sets based on cosine function for the decision making of mechanical design schemes. *Journal of Intelligent and Fuzzy Systems*, 30, 151–158.

Zadeh, L.A. (1965). Fuzzy sets. *Information and Control*, 8, 338–356.

Zhao, X.F., Wei, G.W. (2013). Some Intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute decision making. *Knowledge-Based Systems*, 37, 472–479.

Zhao, X.F., Lin, R., Wei, G.W. (2014). Hesitant triangular fuzzy information aggregation based on Einstein operations and their application to multiple attribute decision making. *Expert Systems with Applications*, 41, 1086–1094.

Zhou, L.Y., Lin, R., Zhao, X.F. (2013). Uncertain linguistic prioritized aggregation operators and their application to multiple attribute group decision making. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 21, 603–627.

G.W. Wei has an MSc and a PhD degree in applied mathematics from SouthWest Petroleum University, Business Administration from school of Economics and Management at SouthWest Jiaotong University, China, respectively. From May 2010 to April 2012, he was a postdoctoral researcher with the School of Economics and Management, Tsinghua University, Beijing, China. He is a professor in the School of Business at Sichuan Normal University. He has published more than 90 papers in journals, books and conference proceedings including journals such as *Omega*, *Decision Support Systems*, *Expert Systems with Applications*, *Applied Soft Computing*, *Knowledge and Information Systems*, *Computers & Industrial Engineering*, *Knowledge-Based Systems*, *International Journal of Uncertainty*, *Fuzziness and Knowledge-Based Systems*, *International Journal of Computational Intelligence Systems and Information: An International Interdisciplinary Journal*. He has published 1 book. He has participated in several scientific committees and serves as a reviewer in a wide range of journals including *Computers & Industrial Engineering*, *International Journal of Information Technology and Decision Making*, *Knowledge-Based Systems*, *Information Sciences*, *International Journal of Computational Intelligence Systems and European Journal of Operational Research*. He is currently interested in aggregation operators, decision making and computing with words.