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Distance Measure and Correlation Coefficient for Linguistic Hesitant Fuzzy Sets and Their Application

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Abstract. Linguistic hesitant fuzzy sets (LHFSs) permit the decision maker to apply several linguistic terms with each having several membership degrees to denote his/her preference of one thing. This type of fuzzy sets can well address the qualitative and quantitative cognitions of the decision maker as well as reflect his/her hesitancy, uncertainty and inconsistency. This paper introduces a distance measure between any two LHFSs and then defines a correlation coefficient of LHFSs. Considering the application of LHFSs, the weighted distance measure and the weighted correlation coefficient of LHFSs are defined. To address the interactions between elements in a set, the Shapley weighted distance measure and the Shapley weighted correlation coefficient are presented. It is worth noting that when the elements are independent, they degenerate to the associated weighted distance measure and the weighted correlation coefficient, respectively. After that, their application to pattern recognition is studied. Furthermore, an approach to multi-attribute decision making under linguistic hesitant fuzzy environment is developed. Meanwhile, numerical examples are offered to show the concrete application of the developed procedure.

Key words: decision making; linguistic hesitant fuzzy set; correlation coefficient; TOPSIS method; the Shapley function.

1. Introduction

According to the attribute values of alternatives, decision-making theory can be classified into two types. One type is the stochastic decision making, where the attribute values are stochastic variables; the other is the fuzzy decision making, where the attribute values are fuzzy variables. It is worth noting that fuzzy decision-making theory has some advantages to cope with uncertain information. Since Zadeh (1965) first introduced fuzzy set theory, decision making based on fuzzy sets has been successfully applied in many fields, such as recommender system (Tejeda-Lorente *et al.*, 2014; Martínez-Cruz *et al.*, 2015; Yager,

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2003, 2004), education (Bryson and Mobolurin, 1995), medical care (James and Dolan, 2010), engineering (Chen and Weng, 2006; Lennon *et al.*, 2013; Meng *et al.*, 2016c), economics (Ölçer *et al.*, 2006; Vaidogas and Sakenaite, 2011; Meng *et al.*, 2017a, 2016d), reservoir flood control (Fu, 2008), facility location selection (Kahraman *et al.*, 2003), new product development (NDP) project screening (Meng and Chen, 2017a), and supplier selection (Meng *et al.*, 2017b). With the increasing complexity of the decision-making problems, researchers found that it is insufficient to address decision-making problems by using fuzzy sets, which only permit the decision maker to apply one fuzzy number to denote the uncertainty. Furthermore, fuzzy sets can only express the decision maker's positive judgment. Thus, several types of generalized fuzzy sets are proposed, such as intuitionistic fuzzy sets (Atanassov, 1986; Atanassov and Gargov, 1989), type-2 fuzzy sets (Zadeh, 1973) and hesitant fuzzy sets (Chen *et al.*, 2013a; Torra, 2010).

However, all these types of fuzzy sets can only denote the decision maker's quantitative cognitions. As Zadeh (1975) noted, there are many situations, where the decision-making problems are too complex or too ill-defined to use quantitative expressions. To address this issue, Zadeh (1975) introduced the concept of linguistic variables, which permit the decision maker to use linguistic variables rather than quantitative fuzzy variables to express the judgment. Since then, many studies about decision making based on linguistic variables are developed (Cai *et al.*, 2014a, 2014b, 2015; Dong *et al.*, 2009, 2016; Gou and Xu, 2016; Herrera and Martínez, 2000; Herrera *et al.*, 2000; Ju *et al.*, 2016; Li *et al.*, 2017; Massanet *et al.*, 2014; Morente-Molinera *et al.*, 2015; Martínez and Herrera, 2012; Meng *et al.*, 2016a; Meng and Chen, 2016b; Pedrycz, 2013; Wei, 2011; Wu and Xu, 2016; Xu, 2004a, 2007; Ye, 2016a). Just as quantitative fuzzy variables, it is still not an easy thing to require a decision maker to apply one linguistic variable to express his/her qualitative judgment. Thus, Xu (2004b) introduced the concept of uncertain linguistic variables, which permit the decision maker to use an interval linguistic variable rather than one exact linguistic variable to denote information. However, uncertain linguistic variables are inadequate to denote the decision maker's hesitancy and irresolution. Hesitant fuzzy linguistic term sets (HFLTSs) introduced by Rodríguez *et al.* (2012) can well address this issue, which are composed by several linguistic terms.

All of the above mentioned fuzzy sets can denote either the decision maker's quantitative or qualitative information. However, none of them can denote these two aspects simultaneously. Following the works of Atanassov (1986) and Zadeh (1975), Wang and Li (2009) presented intuitionistic linguistic sets (ILSs), which are composed by one linguistic variable and an intuitionistic fuzzy variable. Using this type of fuzzy sets, the decision maker can apply one linguistic variable to denote his/her qualitative judgment as well as use an intuitionistic fuzzy variable to show the membership and non-membership degrees about the qualitative judgment. Meng *et al.* (2016e) developed a group decision-making method with intuitionistic linguistic preference relations (ILPRs), where the elements in ILPRs are intuitionistic linguistic fuzzy variables. Later, Liu and Jin (2012) and Liu (2013) introduced intuitionistic uncertain linguistic sets (IULSs) and interval-valued intuitionistic uncertain linguistic sets (IVIULSs), respectively. Such generalizations further endow the decision makers with more rights to express their judgments. As researchers (Rodríguez *et al.*, 2012; Torra, 2010; Ye, 2015;

Meng and An, 2017) noted that the difficulty of expressing the judgments does not arise: there is a margin of error or some possibility distribution on the possibility values but there are several possible values. Recently, Meng *et al.* (2014) presented a new type of fuzzy sets called linguistic hesitant fuzzy sets (LHFSs). This kind of fuzzy sets permits the decision maker to apply several linguistic variables with each having several membership degrees to denote the judgment of one thing. Meanwhile, this type of fuzzy sets can express the qualitative and quantitative cognitions of the decision makers and reflect their hesitancy and inconsistency.

Considering the application of LHFSs, Meng *et al.* (2014) defined several operational laws of LHFSs and then gave a ranking method. After that, the authors developed a method to linguistic hesitant fuzzy multi-attribute decision making with interactive characteristics and incomplete weight information. However, one can see that this method is based on the defined aggregation operators, this makes the process of decision making seem to be complex. Especially, the calculation of the comprehensive attribute values will be very complex with the increase of the number of linguistic hesitant fuzzy sets. Later, Zhou *et al.* (2015) applied a special example to show that the ranking order offered in Meng *et al.* (2014) is unreasonable, and introduced a new ranking method. However, Zhou *et al.*'s ranking method is illogical. Furthermore, the Hamming distance on LHFSs offered by Zhou *et al.* (2015) is wrong. Recently, Zhu *et al.* (2016) developed a cloud model method to linguistic hesitant fuzzy multi-attribute decision making and extended the power operators to linguistic hesitant fuzzy environment. However, this method seems also to be complex.

To address the above listed issues in previous researches, the paper continues to research the application of LHFSs. To do this, we first introduce a distance measure between LHFSs that can be seen as an extension of Hamming distance on real numbers. One can check that the new distance measure addresses the issues in Zhou *et al.* (2015). To discriminate the importance of features or attributes, several additive weighted distance measures are defined that are used to calculate the comprehensive ranking values of objects. Meanwhile, a correlation coefficient on LHFSs is provided, and several weighted correlation coefficients are offered. Considering the interactive characteristics and the complexity of fuzzy numbers, Shapley-based distance measures and correlation coefficients with 2-additive measures are provided, which can be seen as extensions of weighted distance measures and correlation coefficients, respectively. Then, an approach to pattern recognition and to multi-attribute decision making with LHFSs is performed, respectively. Meanwhile, associated practical application is offered.

This paper is organized as follows: Section 2 reviews several basic concepts related to LHFSs. Section 3 introduces a distance measure and a correlation coefficient of LHFSs. Section 4 defines two types of hybrid weighted distance measures and correlation coefficients of LHFSs. One is based on additive measures, and the other uses the Shapley function with respect to 2-additive measures. Section 5 develops an approach to pattern recognition and to multi-attribute decision making by using the defined distance measures and correlation coefficients, and then comparison analysis with the existing methods is made. The last section is the conclusion.

2. Some Basic Concepts

Considering the hesitancy and inconsistency of the decision makers, Torra (2010) defined hesitant fuzzy sets that permit the decision makers to apply several possible values in [0, 1] to denote the membership degree of one thing.

Definition 1. (See Torra, 2010.) Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set. A hesitant fuzzy set (HFS) in X is expressed in terms of a function such that when applied to X it returns a subset of [0, 1], denoted by $E = (\langle x_i, h_E(x_i)\rangle | x_i \in X)$, where $h_E(x_i)$ is a set of some values in [0, 1] denoting the possible membership degrees of the element $x_i \in X$ to the set E.

Sometimes, it is not easy for the decision makers to estimate their information using quantitative values. In this case, linguistic variables are more suitable to only provide the decision makers with the qualitative values. The linguistic reasoning is a technique that represents qualitative aspect using linguistic variables. Let $S = \{s_i \mid i = 1, 2, \ldots, t\}$ be a linguistic term set with odd cardinality. Any label s_i represents a possible value for a linguistic variable, and it should satisfy the following characteristics: (i) The set is ordered: $s_i > s_j$, if $i > j$; (ii) Max operator: $\max(s_i, s_j) = s_i$, if $s_i \geqslant s_j$; (iii) Min operator: $\min(s_i, s_j) = s_i$, if $s_i \leqslant s_j$. For example, a linguistic term set S may be expressed by $S = \{s_1$: very poor, s_2: poor, s_3: slightly poor, s_4: fair, s_5: slightly good, s_6: good, s_7: very good$\}$.

Similar to hesitant fuzzy sets, Rodríguez *et al.* (2012) introduced the following concept of hesitant fuzzy linguistic term sets (HFLTSs) that permit a qualitative reference to have several linguistic terms.

Definition 2. (See Rodríguez *et al.*, 2012.) An HFLTS, H_S, is an ordered finite subset of consecutive linguistic terms of S, where $S = \{s_1, \ldots, s_t\}$ is a linguistic term set.

For example, let S be a linguistic term set as shown above, and let Q be a qualitative reference. An HFLTS could be $H_S(Q) = \{s_2, s_3, s_4\}$.

As pointed out in introduction, HFLTSs only denote the hesitancy and inconsistency of the decision makers' qualitative references, and it is based on the assumption that the decision makers have the same cognition degrees of the given linguistic terms in an HFLTS. However, this might be not true. For instance, to evaluate the quietness of the refrigerator, the decision maker might hesitate to give the value 15% or 20% for slightly good, the value 30%, 40% or 50% for good, and the value 15% for very good. To address this situation, HFLTSs and HFSs seem to be insufficient. Linguistic hesitant fuzzy sets (LHFSs) introduced in Meng *et al.* (2014) can well address this problem.

Definition 3. (See Meng *et al.*, 2014.) Let $S = \{s_1, \ldots, s_t\}$ be a linguistic term set. A LHFS in S is a set that when applied to the linguistic terms of S it returns a subset of S with several values in [0, 1], denoted by $LH = \{(s_{\theta(i)}, lh(s_{\theta(i)}) \mid s_{\theta(i)} \in S)\}$, where $lh(s_{\theta(i)}) = \{r_1, r_2, \ldots, r_{m_i}\}$ is a set with m_i values in [0, 1] denoting the possible membership degrees of the element $s_{\theta(i)} \in S$ to the set LH.

In the example of evaluating the quietness of the refrigerator, the decision maker's judgment can be expressed by a LHFS $LH = \{(s_5, 0.15, 0.2), (s_6, 0.3, 0.4, 0.5), (s_7, 0.15)\}$. To compare LHFSs, Meng *et al.* (2014) introduced the following method:

Definition 4. (See Meng *et al.*, 2014.) Let LH_1 be a LHFS for the predefined linguistic term set $S = \{s_1, \dots, s_t\}$. Suppose that $lh_i = (s_{\theta(i)}, lh(s_{\theta(i)})) \in LH_1$ with $lh(s_{\theta(i)}) = \{r_{i_1}, r_{i_2}, \dots, r_{i_m}\}$, then the expectation value of lh_i is defined by $E(lh_i) = \frac{\theta(i)\sum_{k=1}^{m} r_{i_k}}{m}$, and its variance is given as $V(lh_i) = \frac{\sum_{k=1}^{m}(\theta(i)r_{i_k} - E(lh_i))^2}{m}$.

Let LH_1 be a LHFS for the predefined linguistic term set $S = \{s_1, \dots, s_t\}$. Suppose that $lh_i = (s_{\theta(i)}, lh(s_{\theta(i)})) \in LH_1$ with $lh(s_{\theta(i)}) = \{r_{i_1}, r_{i_2}, \dots, r_{i_m}\}$ and $lh_j = (s_{\theta(j)}, lh(s_{\theta(j)})) \in LH_1$ with $lh(s_{\theta(j)}) = \{r_{j_1}, r_{j_2}, \dots, r_{j_n}\}$. Then, their order relationship is defined as follows:

If $E(lh_i) \leqslant E(lh_j)$, then $lh_i \leqslant lh_j$;

If $E(lh_i) = E(lh_i)$, then $\begin{cases} V(lh_i) > V(lh_j), & lh_i < lh_j, \\ V(lh_i) < V(lh_j), & lh_i > lh_j, \\ V(lh_i) = V(lh_j), & lh_i = lh_j. \end{cases}$

3. Distance Measure and Correlation Coefficient of LHFSs

Distance measure and correlation coefficient are two useful tools to decision making, by which we can obtain the best choice or rank the objects. This section introduces a distance measure and a correlation coefficient of LHFSs.

3.1. *A Distance Measure of LHFSs*

Distance measure is an effective tool to measure the deviations of different arguments, which is applied in many fields, such as ranking fuzzy numbers (Tran and Duckstein, 2002), determining the weights (Yue, 2011), decision making (Cabrerizo *et al.*, 2015; Gong *et al.*, 2016; Peng *et al.*, 2013; Xu, 2010b), economics (Merigó and Casanovas, 2011), pattern recognition (Hung and Yang, 2004; Zeng *et al.*, 2016), and cluster analysis (Yang and Lin, 2009). Similar to the OWA operator (Yager, 1988), Xu and Chen (2008) defined the ordered weighted distance measure (OWDM), which can decrease the influence of extreme values. Later, Zeng and Su (2011) introduced an OWDM on intuitionistic fuzzy sets. However, the OWDM only considers the importance of the ordered positions, but it does not give the importance of the elements. It is worth noting that a distance measure corresponds to a similarity measure (Hung and Yang, 2004; Xu and Xia, 2011; Yang and Lin, 2009). This subsection gives a distance measure of LHFSs.

Definition 5. Let $S = \{s_1, \dots, s_t\}$ be a linguistic term set, and let LH_1 and LH_2 be any two LHFSs for the predefined linguistic term set S. Without loss of generality, suppose that $lh_i = (s_{\theta(i)}, lh(s_{\theta(i)})) \in LH_1$ with $lh(s_{\theta(i)}) = \{r_{i_1}, r_{i_2}, \dots, r_{i_m}\}$ and $lh_j =$

$(s_{\theta(j)}, lh(s_{\theta(j)})) \in LH_2$ with $lh(s_{\theta(j)}) = \{r_{j_1}, r_{j_2}, \ldots, r_{j_n}\}$. The distance measure from lh_i to lh_j is defined as follows:

$$\overrightarrow{d(lh_i, lh_j)} = \frac{1}{mt}\sum_{k=1}^{m}\Big(\min_{r_{jp} \in lh(s_{\theta(j)})}\big|\theta(i)r_{i_k} - \theta(j)r_{j_p}\big|\Big), \tag{1}$$

and the distance measure from lhj to lhi is defined as follows:

$$\overrightarrow{d(lh_j, lh_i)} = \frac{1}{nt}\sum_{k=1}^{n}\Big(\min_{r_{i_k} \in lh(s_{\theta(i)})}\big|\theta(j)r_{j_k} - \theta(i)r_{i_p}\big|\Big). \tag{2}$$

Furthermore, the distance measure between lh_i and lh_j is defined as follows:

$$d(lh_i, lh_j) = \frac{\overrightarrow{d(lh_i, lh_j)} + \overrightarrow{d(lh_j, lh_i)}}{2}. \tag{3}$$

Property 1. Let LH_1 and LH_2 be any two LHFSs for the predefined linguistic term set $S = \{s_1, \ldots, s_t\}$. Suppose that $lh_i = (s_{\theta(i)}, lh(s_{\theta(i)})) \in LH_1$ and $lh_j = (s_{\theta(j)}, lh(s_{\theta(j)})) \in LH_2$ are given as shown in Definition 5. Then, we have:

(i) $d(lh_i, lh_j) = 0$ if and only if there is $r_{j_p} \in lh(s_{\theta(j)})$ such that $\theta(j)r_{j_p} = \theta(i)r_{i_k}$ for all $k = 1, 2, \ldots, m$, and there is $r_{i_k} \in lh(s_{\theta(i)})$ such that $\theta(i)r_{i_k} = \theta(j)r_{j_p}$ for all $p = 1, 2, \ldots, n$;

(ii) $0 \leqslant d(lh_i, lh_j) \leqslant 1$;

(iii) $d(lh_i, lh_j) = d(lh_j, lh_i)$;

(iv) Let LH_3 be another LHFS with $lhg = (s_{\theta(g)}, lh(s_{\theta(g)})) \in LH_3$. If we have

$$\begin{cases} \min_{r_{i_z} \in lh(s_{\theta(g)})}\big|\theta(i)r_{i_k} - \theta(g)r_{i_z}\big| \leqslant \min_{r_{jp} \in lh(s_{\theta(j)})}\big|\theta(i)r_{i_k} - \theta(j)r_{j_p}\big|, \\ \min_{r_{i_z} \in lh(s_{\theta(g)})}\big|\theta(g)r_{i_z} - \theta(i)r_{i_k}\big| \leqslant \min_{r_{i_k} \in lh(s_{\theta(i)})}\big|\theta(j)r_{j_p} - \theta(i)r_{i_k}\big| \end{cases}$$

for all $r_{j_p} \in lh(s_{\theta(j)})$, $r_{i_k} \in lh(s_{\theta(i)})$ and $r_{i_z} \in lh(s_{\theta(g)})$, then $d(lh_i, lh_g) \leqslant d(lh_i, lh_j)$.

Proof. For (i): when we have $d(lh_i, lh_j) = 0$, by the equation (3) we have $\overrightarrow{d(lh_i, lh_j)} = \overrightarrow{d(lh_j, lh_i)} = 0$. According to the equations (1) and (2), we get $\min_{r_{j_p} \in lh(s_{\theta(j)})}|\theta(i)r_{i_k} - \theta(j)r_{j_p}| = \min_{r_{i_p} \in lh(s_{\theta(i)})}|\theta(j)r_{j_k} - \theta(i)r_{i_p}| = 0$. Thus, there is $r_{j_p} \in lh(s_{\theta(j)})$ such that $\theta(j)r_{j_p} = \theta(i)r_{i_k}$ for all $k = 1, 2, \ldots, m$, and there is $r_{i_k} \in lh(s_{\theta(i)})$ such that $\theta(i)r_{i_k} = \theta(j)r_{j_p}$ for all $p = 1, 2, \ldots, n$. On the other hand, one can easily derive that $d(lh_i, lh_j) = 0$ from the equations (1)–(3) according to the listed conditions in (i).

For (ii): from the equations (1) and (3), we have $0 \leqslant \overrightarrow{d(lh_i, lh_j)}, \overrightarrow{d(lh_j, lh_i)} \leqslant 1$ by $0 \leqslant \min_{r_{i_k} \in lh(s_{\theta(i)})}|\theta(j)r_{j_k} - \theta(i)r_{i_p}|, \min_{r_{j_p} \in lh(s_{\theta(j)})}|\theta(i)r_{i_k} - \theta(j)r_{j_p}| \leqslant t$ for each $k = 1, 2, \ldots, m$ and each $p = 1, 2, \ldots, n$. According to the equation (3), we get $0 \leqslant d(lh_i, lh_j) \leqslant 1$.

For (iii): from the equation (3), we obtain $d(lh_i, lh_j) = \frac{\overrightarrow{d(lh_i, lh_j)} + \overrightarrow{d(lh_j, lh_i)}}{2} = \frac{\overrightarrow{d(lh_j, lh_i)} + \overrightarrow{d(lh_i, lh_j)}}{2} = d(lh_j, lh_i)$.

For (iv): from $\begin{cases} \min\limits_{r_{i_z} \in lh(s_{\theta(g)})} \left|\theta(i)r_{i_k} - \theta(g)r_{i_z}\right| \leqslant \min\limits_{r_{j_p} \in lh(s_{\theta(j)})} \left|\theta(i)r_{i_k} - \theta(j)r_{j_p}\right| \\ \min\limits_{r_{i_z} \in lh(s_{\theta(g)})} \left|\theta(g)r_{i_z} - \theta(i)r_{i_k}\right| \leqslant \min\limits_{r_{i_k} \in lh(s_{\theta(i)})} \left|\theta(j)r_{j_p} - \theta(i)r_{i_k}\right| \end{cases}$ for all $r_{j_p} \in lh(s_{\theta(j)})$, $r_{i_k} \in lh(s_{\theta(i)})$ and $r_{i_z} \in lh(s_{\theta(g)})$, we have

$$\begin{cases} \overrightarrow{d(lh_i, lh_g)} \leqslant \overrightarrow{d(lh_i, lh_j)}, \\ \overrightarrow{d(lh_g, lh_i)} \leqslant \overrightarrow{d(lh_j, lh_i)}. \end{cases}$$

From the equation (3), we derive $d(lh_i, lh_g) \leqslant d(lh_i, lh_j)$. □

Definition 6. Let $S = \{s_1, \ldots, s_t\}$ be a linguistic term set, and let LH_1 and LH_2 be any two LHFSs for the predefined linguistic term set S. Without loss of generality, suppose that $lh_i = (s_{\theta(i)}, lh(s_{\theta(i)})) \in LH_1$ and $lh_j = (s_{\theta(j)}, lh(s_{\theta(j)})) \in LH_2$. Then, the distance measure from lh_i to LH_2 is defined as follows:

$$\overrightarrow{d(lh_i, LH_2)} = \min_{lh_j \in LH_2} \overrightarrow{d(lh_i, lh_j)}, \tag{4}$$

and the distance measure from lh_j to LH_1 is defined as follows:

$$\overrightarrow{d(lh_j, LH_1)} = \min_{lh_i \in LH_1} \overrightarrow{d(lh_j, lh_i)}. \tag{5}$$

Remark 1. Let $S = \{s_1, \ldots, s_t\}$ be a linguistic term set, and let LH_1 and LH_2 be any two LHFSs for the predefined linguistic term set S. Let $lh_i = (s_{\theta(i)}, lh(s_{\theta(i)})) \in LH_1$, if we have $\overrightarrow{d(lh_i, LH_2)} = d(lh_i, lh_j)$ with $lh_j \in LH_2$, then we denote lh_j as $lh^i_j = (s_{\theta^i(j)}, lh(s_{\theta^i(j)}))$.

Definition 7. Let $S = \{s_1, \ldots, s_t\}$ be a linguistic term set, and let LH_1 and LH_2 be any two LHFSs for the predefined linguistic term set S. Then, the distance measure between LH_1 and LH_2 is defined as follows:

$$d(LH_1, LH_2) = \frac{\overrightarrow{d(LH_1, LH_2)} + \overrightarrow{d(LH_2, LH_1)}}{2}, \tag{6}$$

where $\overrightarrow{d(LH_1, LH_2)} = \frac{1}{|LH_1|} \sum_{i=1}^{|LH_1|} \overrightarrow{d(lh_i, LH_2)}$ and $\overrightarrow{d(LH_2, LH_1)} = \frac{1}{|LH_2|} \sum_{j=1}^{|LH_2|} \overrightarrow{d(lh_j, LH_1)}$ with $lh_i \in LH_1$ and $lh_j \in LH_2$, $|LH_1|$ and $|LH_2|$ denote the cardinalities of the linguistic variables in LH_1 and LH_2, respectively.

Corollary 1. *Let LH_1 and LH_2 be any two LHFSs for the predefined linguistic term set $S = \{s_1, \ldots, s_t\}$. Then, the distance measure $d(LH_1, LH_2)$ between LH_1 and LH_2 satisfies*:

(i) $d(LH_1, LH_2) = 0$ *if and only if there is* $lh_j \in LH_2$ *such that* $\overrightarrow{d(lh_i, lh_j)} = 0$ *for any* $lh_i \in LH_1$*, and there is* $lh_i \in LH_1$ *such that* $\overrightarrow{d(lh_j, lh_i)} = 0$ *for any* $lh_j \in LH_2$;

(ii) $0 \leqslant d(LH_1, LH_2) \leqslant 1$;

(iii) $d(LH_1, LH_2) = d(LH_2, LH_1)$;

(iv) *Let* LH_3 *be another LHFS with* $lh_g = (s_{\theta(g)}, lh(s_{\theta(g)})) \in LH_3$*. If we have* $\begin{cases} \min\limits_{lh_g \in LH_3} \overrightarrow{d(lh_i, lh_g)} \leqslant \min\limits_{lh_j \in LH_2} \overrightarrow{d(lh_i, lh_j)} \\ \min\limits_{lh_i \in LH} \overrightarrow{d(lh_g, lh_i)} \leqslant \min\limits_{lh_i \in LH_1} \overrightarrow{d(lh_j, lh_i)} \end{cases}$ *for all* $lh_i \in LH_1$, $lh_j \in LH_2$ *and* $lh_g \in LH_3$ *, then* $d(LH_1, LH_3) \leqslant d(LH_1, LH_2)$.

Proof. Similar to Property 1, one can easily derive the conclusions. □

For example, let $LH_1 = (s_2, 0.2, 0.3), (s_3, 0.3, 0.5, 0.7), (s_4, 0.1)$ and $LH_2 = (s_4, 0.5, 0.6), (s_5, 0.1, 0.2)$ be two LHFSs for the predefined linguistic term set $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$. Then, the distance measure between $lh_1^1 = (s_2, 0.2, 0.3)$ and $lh_1^2 = (s_4, 0.5, 0.6)$ is

$$\begin{aligned} d(lh_1^1, lh_1^2) &= \frac{\overrightarrow{d(lh_1^1, lh_1^2)} + \overrightarrow{d(lh_1^2, lh_1^1)}}{2} \\ &= \frac{1}{2} \times \left(\frac{1}{7 \times 2} \big(|2 \times 0.2 - 4 \times 0.5| + |2 \times 0.3 - 4 \times 0.5| \big) \right. \\ &\quad \left. + \frac{1}{7 \times 2} \big(|4 \times 0.5 - 2 \times 0.3| + |2 \times 0.3 - 4 \times 0.6| \big) \right) \\ &= 0.22, \end{aligned}$$

and the distance between $lh_1^1 = (s_2, 0.2, 0.3)$ and LH_2 is

$$d\big(lh_1^1, LH_2\big) = \min\big\{d\big(lh_1^1, lh_1^2\big), d\big(lh_1^1, lh_2^2\big)\big\} = \min\{0.22, 0.025\} = 0.025.$$

Furthermore, the distance between LH_1 and LH_2 is $d(LH_1, LH_2) = 0.034$.

Remark 2. Let LH_1 and LH_2 be any two LHFSs for the predefined linguistic term set $S = \{s_1, \dots, s_t\}$. Let

$$S(LH_1, LH_2) = 1 - d(LH_1, LH_2).$$

Then, $S(LH_1, LH_2)$ is a similarity measure between LH_1 and LH_2, which satisfies: (i) $S(LH_1, LH_2) = 1$ if and only if $d(LH_1, LH_2) = 0$; (ii) $0 \leqslant S(LH_1, LH_2) \leqslant 1$; (iii) $S(LH_1, LH_2) = S(LH_2, LH_1)$; (iv) Let LH_3 be another LHFS. If $d(LH_1, LH_2) \leqslant d(LH_1, LH_3)$, then $S(LH_1, LH_2) \geqslant S(LH_1, LH_3)$.

3.2. *A Correlation Coefficient of LHFSs*

Correlation coefficient is a powerful tool to measure the linear relation between stochastic variables. Recently, researchers applied the correlation coefficient to measure the similarity between fuzzy variables and discussed their application in digital image processing (Van der Weken *et al.*, 2004), clustering analysis (Chen *et al.*, 2013b; Xu *et al.*, 2008), pattern recognition (Liang and Shi, 2003), artificial intelligence (Zhang *et al.*, 2013; Zhang *et al.*, 2014), and multi-attribute decision making (Park *et al.*, 2009; Wei *et al.*, 2011; Ye, 2016b; Tong and Yu, 2016). Recently, Meng and Chen (2015) noted the issues of the correlation coefficient of hesitant fuzzy sets in Chen *et al.* (2013b) and defined several new ones, which need not consider the lengths of HFEs and the arrangement of the possible values. Furthermore, Meng *et al.* (2016b) defined several correlation coefficients of interval-valued hesitant fuzzy sets in a similar way to Meng and Chen (2015). Following the work of Meng and Chen (2015), this section introduces a correlation coefficient of LHFSs.

Definition 8. Let $S = \{s_1, \ldots, s_t\}$ be a linguistic term set, and let LH_1 and LH_2 be any two LHFSs for the predefined linguistic term set S. The correlation coefficient between LH_1 and LH_2 is defined as follows:

$$CC(LH_1, LH_2) = \frac{\overrightarrow{C(LH_1, LH_2)} + \overrightarrow{C(LH_2, LH_1)}}{\max\left\{D(LH_1), D(LH_2^{LH_1})\right\} + \max\left\{D(LH_2), D(LH_1^{LH_2})\right\}}, \tag{7}$$

where $\overrightarrow{C(LH_1, LH_2)}$ is the correlation of LH_1 with respect to LH_2 defined as follows:

$$\begin{aligned}&\overrightarrow{C(LH_1, LH_2)}\\ &\quad = \frac{1}{|LH_1|}\sum_{lh_i=(s_{\theta(i)}, lh(s_{\theta(i)}))\in LH_1}\frac{1}{|lh(s_{\theta(i)})|}\sum_{\substack{r_{i_k}\in lh(s_{\theta(i)}): r_{j_p}^{i_k}\in lh(s_{\theta^i(j)}),\\ lh_j^i=(s_{\theta^i(j)}, lh(s_{\theta^i(j)}))\in LH_2}}\theta(i)r_{i_k}\theta^i(j)r_{j_p}^{i_k},\end{aligned} \tag{8}$$

with $|\theta(i)r_{i_k} - \theta^i(j)r_{j_p}^{i_k}| = \min_{r_{j_p}^i\in lh(s_{\theta^i(j)})}|\theta(i)r_{i_k} - \theta^i(j)r_{j_p}^i|$ and $|lh(s_{\theta(i)})|$ being the cardinality of $lh(s_{\theta(i)})$. $\overrightarrow{C(LH_2, LH_1)}$ is the correlation of LH_2 with respect to LH_1 defined as follows:

$$\begin{aligned}&\overrightarrow{C(LH_2, LH_1)}\\ &\quad = \frac{1}{|LH_2|}\sum_{lh_j=(s_{\theta(j)}, lh(s_{\theta(j)}))\in LH_2}\frac{1}{|lh(s_{\theta(j)})|}\sum_{\substack{r_{j_p}\in lh(s_{\theta(j)}): r_{i_k}^{j_p}\in h(s_{\theta^j(i)}),\\ lh_i^j=(s_{\theta^j(i)}, lh(s_{\theta^j(i)}))\in LH_1}}\theta(j)r_{j_p}\theta^j(i)r_{i_k}^{j_p},\end{aligned} \tag{9}$$

with $|\theta(j)r_{j_p} - \theta^j(i)r_{i_k}^{j_p}| = \min_{r_{i_k}^j\in lh(s_{\theta^j(i)})}|\theta(j)r_{j_p} - \theta^j(i)r_{i_k}^j|$ and $|lh(s_{\theta(j)})|$ being the cardinality of $lh(s_{\theta(j)})$.

Furthermore,

$$D(LH_1) = \frac{1}{|LH_1|} \sum_{lh_i=(s_{\theta(i)}, lh(s_{\theta(i)}))\in LH_1} \frac{1}{|lh(s_{\theta(i)})|} \sum_{r_{i_k}\in lh(s_{\theta(i)})} \big(\theta(i) r_{i_k}\big)^2, \tag{10}$$

$$D\big(LH_2^{LH_1}\big) = \frac{1}{|LH_1|} \sum_{\substack{lh^i_j=(s_{\theta^i(j)}, lh(s_{\theta^i(j)}))\in LH_2:\\ lh_i=(s_{\theta(i)}, lh(s_{\theta(i)}))\in LH_1}} \frac{1}{|lh(s_{\theta(i)})|} \sum_{r^{i_k}_{j_p}\in lh(s_{\theta^i(j)}): r_{i_k}\in lh(s_{\theta(i)})} \big(\theta^i(j) r^{i_k}_{j_p}\big)^2, \tag{11}$$

$$D(LH_2) = \frac{1}{|LH_2|} \sum_{lh_j=(s_{\theta(j)}, lh(s_{\theta(j)}))\in LH_2} \frac{1}{|lh(s_{\theta(j)})|} \sum_{r_{j_p}\in lh(s_{\theta(j)})} \big(\theta(j) r_{j_p}\big)^2, \tag{12}$$

$$D\big(LH_1^{LH_2}\big) = \frac{1}{|LH_2|} \sum_{\substack{lh^j_i=(s_{\theta^j(i)}, lh(s_{\theta^j(i)}))\in LH_1:\\ lh_j=(s_{\theta(j)}, lh(s_{\theta(j)}))\in LH_2}} \frac{1}{|lh(s_{\theta(j)})|} \sum_{r^{j_p}_{i_k}\in lh(s_{\theta^j(i)}): r_{j_p}\in lh(s_{\theta(j)})} \big(\theta^j(i) r^{j_p}_{i_k}\big)^2. \tag{13}$$

Property 2. Let $S = \{s_1, \dots, s_t\}$ be a linguistic term set, and let LH_1 and LH_2 be any two LHFSs for the predefined linguistic term set S. The correlation coefficient between LH_1 and LH_2 satisfies:

(i) $CC(LH_1, LH_1) = 1$;
(ii) $CC(LH_1, LH_2) = CC(LH_2, LH_1)$;
(iii) $0 \leqslant CC(LH_1, LH_2) \leqslant 1$.

Proof. For (i): From the equation (11), we have the following:

$$\begin{aligned} D\big(LH_1^{LH_1}\big) &= \frac{1}{|LH_1|} \sum_{\substack{lh^i_j=(s_{\theta^i(j)}, lh(s_{\theta^i(j)}))\in LH_1:\\ lh_i=(s_{\theta(i)}, lh(s_{\theta(i)}))\in LH_1}} \frac{1}{|lh(s_{\theta(i)})|} \sum_{r^{i_k}_{j_p}\in lh(s_{\theta^i(j)}): r_{i_k}\in lh(s_{\theta(i)})} \big(\theta^i(j) r^{i_k}_{j_p}\big)^2 \\ &= \frac{1}{|LH_1|} \sum_{lh_i=(s_{\theta(i)}, lh(s_{\theta(i)}))\in LH_1} \frac{1}{|lh(s_{\theta(i)})|} \sum_{r_{i_k}\in lh(s_{\theta(i)})} \big(\theta(i) r_{i_k}\big)^2 \\ &= D(LH_1). \end{aligned}$$

On the other hand, from $\overrightarrow{d(lh_i, LH_1)} = d(lh_i, lh_i) = 0$, we derive the following:

$$lh^i_j = \big(s_{\theta^i(j)}, lh(s_{\theta^i(j)})\big) = \big(s_{\theta(i)}, lh(s_{\theta(i)})\big) = lh_i,$$

namely, $|\theta(i)r_{i_k} - \theta^i(j)r_{j_p}^{i_k}| = \min_{r_{j_p}^i \in lh(s_{\theta^i(j)})} |\theta(i)r_{i_k} - \theta^i(j)r_{j_p}^i| = \min_{r_{i_l} \in lh(s_{\theta(i)})} |\theta(i)r_{i_k} - \theta(i)r_{i_l}| = |\theta(i)r_{i_k} - \theta(i)r_{i_k}| = 0$. Thus, $\theta^i(j)r_{j_p}^{i_k} = \theta(i)r_{i_k}$.

According to the equation (8), we obtain the following:

$$\begin{aligned}
&\overrightarrow{C(LH_1, LH_1)} \\
&\quad = \frac{1}{|LH_1|} \sum_{lh_i=(s_{\theta(i)}, lh(s_{\theta(i)}))\in LH_1} \frac{1}{|lh(s_{\theta(i)})|} \sum_{r_{i_k}\in lh(s_{\theta(i)})} \big(\theta(i)r_{i_k}\big)^2 \\
&\quad = D(LH_1).
\end{aligned}$$

Thus, $CC(LH_1, LH_1) = \frac{D(LH_1)}{D(LH_1)} = 1$.

For (ii): From the equation (8), one can easily check that this conclusion holds.

For (iii): From $CC(LH_1, LH_2) \geqslant 0$, we only need to show $CC(LH_1, LH_2) \leqslant 1$. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&\big(\overrightarrow{C(LH_1, LH_2)}\big)^2 \\
&\quad = \Bigg(\frac{1}{|LH_1|} \sum_{lh_i=(s_{\theta(i)}, lh(s_{\theta(i)}))\in LH_1} \frac{1}{|lh(s_{\theta(i)})|} \sum_{\substack{r_{i_k}\in lh(s_{\theta(i)}):r_{j_p}^{i_k}\in lh(s_{\theta^i(j)}),\\ (s_{\theta^i(j)}, lh(s_{\theta^i(j)}))\in LH_2}} \theta(i)r_{i_k}\theta^i(j)r_{j_p}^{i_k}\Bigg)^2 \\
&\quad = \Bigg(\sum_{lh_i=(s_{\theta(i)}, lh(s_{\theta(i)}))\in LH_1} \sum_{\substack{r_{i_k}\in lh(s_{\theta(i)}):r_{j_p}^{i_k}\in lh(s_{\theta^i(j)}),\\ (s_{\theta^i(j)}, lh(s_{\theta^i(j)}))\in LH_2}} \frac{\theta(i)r_{i_k}\theta^i(j)r_{j_p}^{i_k}}{|LH_1||lh(s_{\theta(i)})|}\Bigg)^2 \\
&\quad = \Bigg(\sum_{lh_i=(s_{\theta(i)}, lh(s_{\theta(i)}))\in LH_1} \sum_{\substack{r_{i_k}\in lh(s_{\theta(i)}):r_{j_p}^{i_k}\in lh(s_{\theta^i(j)}),\\ (s_{\theta^i(j)}, lh(s_{\theta^i(j)}))\in LH_2}} \frac{\theta(i)r_{i_k}}{\sqrt{|LH_1||lh(s_{\theta(i)})|}} \frac{\theta^i(j)r_{j_p}^{i_k}}{\sqrt{|LH_1||lh(s_{\theta(i)})|}}\Bigg)^2 \\
&\quad \leqslant \Bigg(\sum_{lh_i=(s_{\theta(i)}, lh(s_{\theta(i)}))\in LH_1} \sum_{\substack{r_{i_k}\in lh(s_{\theta(i)}):r_{j_p}^{i_k}\in lh(s_{\theta^i(j)}),\\ (s_{\theta^i(j)}, lh(s_{\theta^i(j)}))\in LH_2}} \frac{(\theta(i)r_{i_k})^2}{|LH_1||lh(s_{\theta(i)})|}\Bigg) \\
&\qquad \times \Bigg(\sum_{lh_i=(s_{\theta(i)}, lh(s_{\theta(i)}))\in LH_1} \sum_{\substack{r_{i_k}\in lh(s_{\theta(i)}):r_{j_p}^{i_k}\in lh(s_{\theta^i(j)}),\\ (s_{\theta^i(j)}, lh(s_{\theta^i(j)}))\in LH_2}} \frac{(\theta^i(j)r_{j_p}^{i_k})^2}{|LH_1||lh(s_{\theta(i)})|}\Bigg) \\
&\quad = \Bigg(\sum_{lh_i=(s_{\theta(i)}, lh(s_{\theta(i)}))\in LH_1} \sum_{r_{i_k}\in lh(s_{\theta(i)})} \frac{(\theta(i)r_{i_k})^2}{|LH_1||lh(s_{\theta(i)})|}\Bigg)
\end{aligned}$$

$$\times\Bigg(\sum_{\substack{lh^i_j=(s_{\theta^i(j)},lh(s_{\theta^i(j)}))\in LH_2:\\ lh_i=(s_{\theta(i)},lh(s_{\theta(i)}))\in LH_1}}\ \sum_{r^{i_k}_{jp}\in lh(s_{\theta^i(j)}):r_{i_k}\in lh(s_{\theta(i)})}\frac{(\theta^i(j)r^{i_k}_{jp})^2}{|LH_1||lh(s_{\theta(i)})|}\Bigg)$$
$$= D(LH_1)D\big(LH_2^{LH_1}\big).$$

From $\sqrt{D(LH_1)D(LH_2^{LH_1})} \leqslant \frac{D(LH_1)+D(LH_2^{LH_1})}{2} \leqslant \max\{D(LH_1), D(LH_2^{LH_1})\}$, we derive

$$\overrightarrow{C(LH_1, LH_2)} \leqslant \max\big\{D(LH_1), D\big(LH_2^{LH_1}\big)\big\}.$$

Similarly, we have $\overrightarrow{C(LH_2, LH_1)} \leqslant \max\{D(LH_2), D(LH_1^{LH_2})\}$. Thus,

$$CC(LH_1, LH_2) \leqslant 1. \quad \square$$

Remark 3. Similar to the correlation coefficient defined in the equation (7), we can define other correlation coefficients of LHFSs in a similar way to Meng and Chen (2015). When the membership degree of each linguistic term in LHFSs is one, then LHFSs degenerate to HFLTSs. In this situation, the correlation coefficient (7) reduces to the correlation coefficient for HFLTSs. Furthermore, when all LHFSs only have the same one linguistic term, then the correlation coefficient (7) reduces to the correlation coefficient for HFSs given by Meng and Chen (2015).

For example, let LH_1 and LH_2 be two LHFSs for the predefined linguistic term set $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$, where LH_1 and LH_2 as shown above, namely, $LH_1 = \{(s_2, 0.2, 0.3), (s_3, 0.3, 0.5, 0.7), (s_4, 0.1)\}$ and $LH_2 = \{(s_4, 0.5, 0.6), (s_5, 0.1, 0.2)\}$. Then, the correlation coefficient between LH_1 and LH_2 is $CC(LH_1, LH_2) = 0.8747$.

4. Hybrid Weighted Distance Measure and Correlation Coefficient of LHFSs

This section studies two types of the hybrid weighted distance measures and the hybrid weighted correlation coefficients of LHFSs. One is based on additive measures, which does not consider the interactions between elements in a set; the other adopts 2-additive measures to reflect their interactive characteristics. It is worth noting that the former can be seen as a special case of the latter.

4.1. *Distance Measure and Correlation Coefficient Based on Additive Measures*

To both consider the importance of the elements and their ordered positions, this subsection defines the hybrid weighted distance measure and the hybrid weighted correlation coefficient based on additive measures.

Definition 9. Let $S = \{s_1, \dots, s_t\}$ be a linguistic term set, and let $A = \{LH_1, LH_2, \dots, LH_n\}$ and $B = \{LH'_1, LH'_2, \dots, LH'_n\}$ be any two collections of LHFSs for the predefined linguistic term set S. Then, the hybrid weighted distance measure (HWDM) between A and B is defined as follows:

$$HWDM(A, B) = \sum_{i=1}^{n} \frac{w_i \omega_{(i)}}{\sum_{i=1}^{n} w_i \omega_{(i)}} d\big(LH_{(i)}, LH'_{(i)}\big),$$

and the geometric hybrid weighted distance measure (GHWDM) between A and B is defined as follows:

$$GHWDM(A, B) = \prod_{i=1}^{n} d\big(LH_{(i)}, LH'_{(i)}\big)^{\frac{w_i \omega_{(i)}}{\sum_{i=1}^{n} w_i \omega_{(i)}}},$$

where $(\cdot)$ is a permutation on the distance measures $d(LH_j, LH'_j)$, $j = 1, 2, \dots, n$, with $\omega_{(j)} d(LH_{(j)}, LH'_{(j)})$ being the jth largest value of $\omega_j d(LH_j, LH'_j)$ and $d(LH_{(j)}, LH'_{(j)})^{\omega_{(j)}}$ being the jth largest value of $d(LH_j, LH'_j)^{\omega_j}$, $w = \{w_1, w_2, \dots, w_n\}$ is the weighting vector on the ordered position set $N = \{1, 2, \dots, n\}$ such that $\sum_{i=1}^{n} w_i = 1$ and $w_i \geqslant 0$ for all $i = 1, 2, \dots, n$, and $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is the weighting vector on $D = \{d(LH_i, LH'_i)\}_{i \in N}$ such that $\sum_{i=1}^{n} \omega_i = 1$ and $\omega_i \geqslant 0$ for all $i = 1, 2, \dots, n$.

Property 3. Let $A = \{LH_1, LH_2, \dots, LH_n\}$ and $B = \{LH'_1, LH'_2, \dots, LH'_n\}$ be any two collections of LHFSs for the predefined linguistic term set $S = \{s_1, \dots, s_t\}$. Then, $GHWDM(A, B) \leqslant HWDM(A, B)$.

Proof. According to $\sum_{i=1}^{n} \omega_i a_i \leqslant \prod_{i=1}^{n} a_i^{\omega_i}$ with $a_i > 0$ for all $i = 1, 2, \dots, n$, where ω is a weighting vector as shown in Definition 9, we have $GHWDM(A, B) \leqslant HWDM(A, B)$ from their expressions. □

Property 4. Let $A = \{LH_1, LH_2, \dots, LH_n\}$ and $B = \{LH'_1, LH'_2, \dots, LH'_n\}$ be any two collections of LHFSs for the predefined linguistic term set $S = \{s_1, \dots, s_t\}$.

(i) *Commutativity*. Let σ be a permutation of A and B, respectively, where $\dot{A} = \{LH_{\sigma(1)}, LH_{\sigma(2)}, \dots, LH_{\sigma(n)}\}$ and $\dot{B} = \{LH'_{\sigma(1)}, LH'_{\sigma(2)}, \dots, LH'_{\sigma(n)}\}$. Then,

$$HWDM(A, B) = HWDM(\dot{A}, \dot{B}) \text{ and } \mathrm{GHWDM}(A, B) = \mathrm{GHWDM}(\dot{A}, \dot{B}).$$

(ii) *Monotonicity*. Let $C = \{Cauchy\text{–}Schwarz''_1, Cauchy\text{–}Schwarz''_2, \dots, LH''_n\}$ be another collection of LHFSs for the predefined linguistic term set S, where $d(LH_i, LH'_i) \leqslant d(LH_i, LH''_i)$ for all $i = 1, 2, \dots, n$. Then,

$$HWDM(A, B) \leqslant HWDM(A, C) \text{ and } GHWDM(A, B) \leqslant GHWDM(A, C).$$

(iii) *Idempotency*. If $d(LH_i, LH'_i) = c$ for all $i = 1, 2, \dots, n$, then

$$HWDM(A, B) = c \quad \text{and} \quad GHWDM(A, B) = c.$$

(iv) *Boundary*.

$$\min_{1\leqslant i\leqslant n} d\big(LH_i, LH_i'\big) \leqslant HWDM(A,B) \leqslant \max_{1\leqslant i\leqslant n} d\big(LH_i, LH_i'\big),$$

$$\min_{1\leqslant i\leqslant n} d\big(LH_i, LH_i'\big) \leqslant GHWDM(A,B) \leqslant \max_{1\leqslant i\leqslant n} d\big(LH_i, LH_i'\big).$$

Proof. For (i): we have

$$\begin{aligned} HWDM(A,B) &= \sum_{i=1}^{n} \frac{w_i\omega_{(i)}}{\sum_{i=1}^{n} w_i\omega_{(i)}} d\big(LH_{(i)}, LH_{(i)}'\big) \\ &= \sum_{i=1}^{n} \frac{w_{\sigma(i)}\omega_{(\sigma(i))}}{\sum_{i=1}^{n} w_{\sigma(i)}\omega_{(\sigma(i))}} d\big(LH_{(\sigma(i))}, LH_{(\sigma(i))}'\big) \\ &= HWDM(\dot{A}, \dot{B}). \end{aligned}$$

Similarly, we get $GHWDM(A,B) = GHWDM(\dot{A}, \dot{B})$.

For (ii): from Definition 9 and $d(LH_i, LH_i') \leqslant d(LH_i, LH_i'')$ for all $i = 1, 2, \dots, n$, one can easily derive the conclusion.

For (iii): from $d(LH_i, LH_i') = c$ for all $i = 1, 2, \dots, n$, we get

$$\begin{aligned} HWDM(A,B) &= \sum_{i=1}^{n} \frac{w_i\omega_{(i)}}{\sum_{i=1}^{n} w_i\omega_{(i)}} d\big(LH_{(i)}, LH_{(i)}'\big) = \sum_{i=1}^{n} \frac{w_i\omega_{(i)}}{\sum_{i=1}^{n} w_i\omega_{(i)}} c \\ &= c\sum_{i=1}^{n} \frac{w_i\omega_{(i)}}{\sum_{i=1}^{n} w_i\omega_{(i)}} = c. \end{aligned}$$

In a similar way, we have $GHWDM(A,B) = c$.

For (iv): because $\min_{1\leqslant i\leqslant n} d(LH_i, LH_i') \leqslant d(LH_{(i)}, LH_{(i)}') \leqslant \max_{1\leqslant i\leqslant n} d(LH_i, LH_i')$ for all $i = 1, 2, \dots, n$, we derive

$$\begin{aligned} \sum_{i=1}^{n} \frac{w_i\omega_{(i)}}{\sum_{i=1}^{n} w_i\omega_{(i)}} \min_{1\leqslant i\leqslant n} d\big(LH_i, LH_i'\big) &\leqslant \sum_{i=1}^{n} \frac{w_i\omega_{(i)}}{\sum_{i=1}^{n} w_i\omega_{(i)}} d\big(LH_{(i)}, LH_{(i)}'\big) \\ &\leqslant \sum_{i=1}^{n} \frac{w_i\omega_{(i)}}{\sum_{i=1}^{n} w_i\omega_{(i)}} \max_{1\leqslant i\leqslant n} d\big(LH_i, LH_i'\big) \end{aligned}$$

and

$$\min_{1\leqslant i\leqslant n} d\big(LH_i, LH_i'\big) \leqslant \sum_{i=1}^{n} \frac{w_i\omega_{(i)}}{\sum_{i=1}^{n} w_i\omega_{(i)}} d(LH_{(i)}, LH_{(i)}') \leqslant \max_{1\leqslant i\leqslant n} d\big(LH_i, LH_i'\big).$$

Thus,

$$\min_{1\leqslant i\leqslant n} d\big(LH_i, LH_i'\big) \leqslant HWDM(A,B) \leqslant \max_{1\leqslant i\leqslant n} d\big(LH_i, LH_i'\big).$$

In a similar way, one can derive $\min_{1\leqslant i\leqslant n} d(LH_i, LH'_i) \leqslant GHWDM(A, B) \leqslant \max_{1\leqslant i\leqslant n} d(LH_i, LH'_i)$. □

Remark 4. When $w_i = 1/n$ for all $i = 1, 2, \ldots, n$, then the HWDM degenerates to the weighted distance measure (WDM):

$$WDM(A, B) = \sum_{i=1}^{n} \omega_i d\big(LH_i, LH'_i\big),$$

and the GHWDM degenerates to the geometric weighted distance measure (GWDM):

$$GWDM(A, B) = \prod_{i=1}^{n} d\big(LH_i, LH'_i\big)^{\omega_i}.$$

When $\omega_i = 1/n$ for all $i = 1, 2, \ldots, n$, then the HWDM degenerates to the ordered weighted distance measure (OWDM):

$$OWDM(A, B) = \sum_{i=1}^{n} w_i d\big(LH_{(i)}, LH'_{(i)}\big),$$

and the GHWDM degenerates to the geometric ordered weighted distance measure (GOWDM):

$$GOWDM(A, B) = \prod_{i=1}^{n} d\big(LH_{(i)}, LH'_{(i)}\big)^{w_i}.$$

Definition 10. Let $S = \{s_1, \ldots, s_t\}$ be a linguistic term set, and let $A = \{LH_1, LH_2, \ldots, LH_n\}$ and $B = \{LH'_1, LH'_2, \ldots, LH'_n\}$ be any two collections of LHFSs for the predefined linguistic term set S. Then, the hybrid weighted correlation coefficient (HWCC) between A and B is defined as follows:

$$HWCC(A, B) = \sum_{i=1}^{n} \frac{w_i \omega_{(i)}}{\sum_{i=1}^{n} w_i \omega_{(i)}} CC\big(LH_{(i)}, LH'_{(i)}\big),$$

and the geometric hybrid weighted correlation coefficient (GHWCC) between A and B is defined as follows:

$$GHWCC(A, B) = \prod_{i=1}^{n} CC\big(LH_{(i)}, LH'_{(i)}\big)^{\frac{w_i \omega_{(i)}}{\sum_{i=1}^{n} w_i \omega_{(i)}}}$$

where $(\cdot)$ is a permutation on the correlation coefficients $CC(LH_j, LH'_j)$, $j = 1, 2, \ldots, n$, with $\omega_{(j)} CC(LH_{(j)}, LH'_{(j)})$ being the jth largest value of $\omega_j CC(LH_j, LH'_j)$ and $CC(LH_{(j)}, LH'_{(j)})^{\omega_{(j)}}$ being the jth largest value of $CC(LH_j, LH'_j)^{\omega_j}$, $w = \{w_1, w_2, \ldots,$

$w_n\}$ is the weighting vector on the ordered position set $N = \{1, 2, \dots, n\}$ such that $\sum_{i=1}^{n} w_i = 1$ and $w_i \geqslant 0$ for all $i = 1, 2, \dots, n$, and $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ is the weighting vector on $D = \{d(LH_i, LH_i')\}_{i \in N}$ such that $\sum_{i=1}^{n} \omega_i = 1$ and $\omega_i \geqslant 0$ for all $i = 1, 2, \dots, n$.

Remark 5. When $w_i = 1/n$ for all $i = 1, 2, \dots, n$, then the HWCC degenerates to the weighted correlation coefficient (WCC):

$$WCC(A, B) = \sum_{i=1}^{n} \omega_i CC\big(LH_i, LH_i'\big),$$

and the GHWCC degenerates to the geometric weighted correlation coefficient (GWCC):

$$GWCC(A, B) = \prod_{i=1}^{n} CC\big(LH_i, LH_i'\big)^{\omega_i}.$$

When $\omega_i = 1/n$ for all $i = 1, 2, \dots, n$, then the HWCC degenerates to the ordered weighted correlation coefficient (OWCC):

$$OWCC(A, B) = \sum_{i=1}^{n} w_i CC\big(LH_{(i)}, LH_{(i)}'\big),$$

and the GHWCC degenerates to the geometric ordered weighted correlation coefficient (GOWCC):

$$GOWCC(A, B) = \prod_{i=1}^{n} CC\big(LH_{(i)}, LH_{(i)}'\big)^{w_i}.$$

Property 5. Let $A = \{LH_1, LH_2, \dots, LH_n\}$ and $B = \{LH_1', LH_2', \dots, LH_n'\}$ be any two collections of LHFSs for the predefined linguistic term set $S = \{s_1, \dots, s_t\}$. Then, $GHWCC(A, B) \leqslant HWCC(A, B)$.

Remark 6. Similar to the HWDM and the GHWDM, one can check that the HWCC and the GHWCC satisfy the properties listed in Property 4.

4.2. *Distance Measure and Correlation Coefficient Based on 2-Additive Measures*

The hybrid weighted distance measure and correlation coefficient defined in the subsection 4.1 are all based on the assumption that the importance of elements in a set is independent. However, in some situations, this assumption is incorrect (Grabisch, 1997; Meng *et al.*, 2016b, 2016f; Meng and Chen, 2017b; Meng and Chen, 2016a; Meng and Chen, 2016c; Tan, 2011; Tan and Chen, 2011; Xu, 2010a). Considering the interactions between elements, this subsection uses the Shapley function (Shapley, 1953) with respect to 2-additive measures (Grabisch, 1997) to define the hybrid 2-additive Shapley distance measure and the hybrid 2-additive Shapley correlation coefficient. Sugeno (1974) introduced the following concept of fuzzy measures:

Definition 11. (See Sugeno, 1974.) A fuzzy measure on finite set $N = \{1, 2, \dots, n\}$ is a set function $\mu : P(N) \to [0, 1]$ satisfying:

(1) $\mu(\emptyset) = 0, \mu(N) = 1$;
(2) For all $A, B \in P(N)$ with $A \subseteq B$, $\mu(A) \leqslant \mu(B)$,

where $P(N)$ is the power set of N.

From Definition 11, we know that fuzzy measures do not only give the importance of elements separately but also consider the importance of all their combinations. For any two coalitions $A, B \in P(N)$ with $A \cap B = \emptyset$, when $\mu(A) + \mu(B) < \mu(A \cup B)$, then there exists complementary interaction between A and B; when $\mu(A) + \mu(B) > \mu(A \cup B)$, then there exists redundant interaction between A and B; otherwise, there is no interaction between A and B, namely, $\mu(A) + \mu(B) = \mu(A \cup B)$. Although fuzzy measures well reflect the interactions between elements, they define on the power set. This means that it is not an easy thing to determine a fuzzy measure. To cope with this problem, 2-additive measures (Grabisch, 1997) are a good choice.

Definition 12. (See Grabisch, 1997.) The fuzzy measure μ on $N = \{1, 2, \dots, n\}$ is called a 2-additive measure, if, for any $S \subseteq N$ with $s \geqslant 2$, we have

$$\mu(S) = \sum_{\{i,j\} \subseteq S} \mu(i, j) - (s - 2) \sum_{i \in S} \mu(i), \tag{14}$$

where s is the cardinality of S.

According to Definition 12, when we know the importance of each element and all of their combined in pairs, then we can derive an associated 2-additive measure. Following the work of Meng and Tang (2013), we apply the Shapley function (Shapley, 1953) with respect to 2-additive measures to give the weights of the elements.

Theorem 1. *(See* Meng and Tang, 2013.*) Let v be a 2-additive measure defined on $N = \{1, 2, \dots, n\}$, then the Shapley function for v can be expressed as follows:*

$$\phi_i(N, v) = \frac{3 - n}{2} v(i) + \frac{1}{2} \sum_{j \in N \setminus i} \big(v(i, j) - v(j)\big). \tag{15}$$

Now, let us define the following distance measure and correlation coefficient using the Shapley function with respect to 2-additive measures.

Definition 13. Let $S = \{s_1, \dots, s_t\}$ be a linguistic term set, and let $A = \{LH_1, LH_2, \dots, LH_n\}$ and $B = \{LH'_1, LH'_2, \dots, LH'_n\}$ be any two collections of LHFSs for the predefined linguistic term set S. Then, the hybrid 2-additive Shapley distance measure (H2SDM) between A and B is defined as follows:

$$H2SDM(A,B)=\sum_{i=1}^{n}\frac{\phi_i(N,v)\varphi_{(i)}(D,\mu)}{\sum_{i=1}^{n}\phi_i(N,v)\varphi_{(i)}(D,\mu)}d(LH_{(i)},LH'_{(i)}),$$

and the geometric hybrid 2-additive Shapley distance measure (GH2SDM) between A and B is defined as follows:

$$GH2SDM(A,B)=\prod_{i=1}^{n}d\big(LH_{(i)},LH'_{(i)}\big)^{\frac{\phi_i(N,v)\varphi_{(i)}(D,\mu)}{\sum_{i=1}^{n}\phi_i(N,v)\varphi_{(i)}(D,\mu)}},$$

where $(\cdot)$ is a permutation on the distance measures $d(LH_j,LH'_j)$, $j=1,2,\dots,n$, with $\varphi_{(j)}(D,\mu)d(LH_{(j)},LH'_{(j)})$ being the jth largest value of $\varphi_j(D,\mu)d(LH_j,LH'_j)$ and $d(LH_{(j)},LH'_{(j)})^{\varphi_{(j)}(D,\mu)}$ being the jth largest value of $d(LH_j,LH'_j)^{\varphi_j(D,\mu)}$, $\phi_i(N,v)$ is the Shapley value of the ith position with respect to the 2-additive measure v on the ordered position set $N=\{1,2,\dots,n\}$, and $\varphi_i(D,\mu)$ is the Shapley value of the distance measure $d(LH_i,LH'_i)$ with respect to the 2-additive measure μ on $D=\{d(LH_i,LH'_i)\}_{i\in N}$.

Remark 7. When there are no interactions between elements as well as between the ordered positions, then the H2SDM reduces to the HWDM, and the GH2SDM reduces to the GHWDM. Furthermore, when $\phi_i(N,v)=1/n$ for all $i=1,2,\dots,n$, then the H2SDM degenerates to the 2-additive Shapley distance measure (2SDM):

$$2SDM(A,B)=\sum_{i=1}^{n}\varphi_i(D,\mu)d\big(LH_i,LH'_i\big),$$

and the GH2SDM degenerates to the geometric 2-additive Shapley distance measure (G2SDM):

$$G2SDM(A,B)=\prod_{i=1}^{n}d\big(LH_i,LH'_i\big)^{\varphi_i(D,\mu)}.$$

When $\varphi_i(D,\mu)=1/n$ for all $i=1,2,\dots,n$, then the H2SDM degenerates to the ordered 2-additive Shapley distance measure (O2SDM):

$$O2SDM(A,B)=\sum_{i=1}^{n}\phi_i(N,v)d\big(LH_{(i)},LH'_{(i)}\big),$$

and the GH2SDM degenerates to the geometric ordered 2-additive Shapley distance measure (GO2SDM):

$$GO2SDM(A, B) = \prod_{i=1}^{n} d\big(LH_{(i)}, LH'_{(i)}\big)^{\phi_i(N,v)}.$$

Definition 14. Let $S = \{s_1, \ldots, s_t\}$ be a linguistic term set, and let $A = \{LH_1, LH_2, \ldots, LH_n\}$ and $B = \{LH'_1, LH'_2, \ldots, LH'_n\}$ be any two collections of LHFSs for the predefined linguistic term set S. Then, the hybrid 2-additive Shapley correlation coefficient (H2SCC) between A and B is defined as follows:

$$H2SCC(A, B) = \sum_{i=1}^{n} \frac{\phi_i(N, v)\varphi_{(i)}(E, \mu)}{\sum_{i=1}^{n} \phi_i(N, v)\varphi_{(i)}(E, \mu)} CC\big(LH_{(i)}, LH'_{(i)}\big),$$

and the geometric hybrid 2-additive Shapley correlation coefficient (GH2SCC) between A and B is defined as follows:

$$GH2SCC(A, B) = \prod_{i=1}^{n} CC\big(LH_{(i)}, LH'_{(i)}\big)^{\frac{\phi_i(N,v)\varphi_{(i)}(E,\mu)}{\sum_{i=1}^{n} \phi_i(N,v)\varphi_{(i)}(E,\mu)}},$$

where $(\cdot)$ is a permutation on the correlation coefficients $CC(LH_j, LH'_j)$, $j = 1, 2, \ldots, n$, with $\varphi_{(j)}(E, \mu)CC(LH_{(j)}, LH'_{(j)})$ being the jth largest value of $\varphi_j(E, \mu)CC(LH_j, LH'_j)$ and $CC(LH_{(j)}, LH'_{(j)})^{\varphi_{(j)}(E,\mu)}$ being the jth largest value of $CC(LH_j, LH'_j)^{\varphi_j(E,\mu)}$, $\phi_i(N, v)$ is the Shapley value of the ith position with respect to the 2-additive measures on the ordered position set $N = \{1, 2, \ldots, n\}$, and $\varphi_i(E, \mu)$ is the Shapley value of the distance measure $CC(LH_i, LH'_i)$ with respect to the 2-additive measure μ on $E = \{CC(LH_i, LH'_i)\}_{i \in N}$.

Remark 8. When there are no interactions between elements as well as between the ordered positions, then the H2SCC reduces to the HWCC, and the GH2SCC reduces to the GHWCC. Furthermore, when $\phi_i(N, v) = 1/n$ for all $i = 1, 2, \ldots, n$, then the H2SCC degenerates to the 2-additive Shapley correlation coefficient (2SCC):

$$2SCC(A, B) = \sum_{i=1}^{n} \varphi_i(E, \mu) CC\big(LH_i, LH'_i\big),$$

and the GH2SDM degenerates to the geometric 2-additive Shapley correlation coefficient (G2SCC):

$$G2SCC(A, B) = \prod_{i=1}^{n} CC\big(LH_i, LH'_i\big)^{\varphi_i(E,\mu)}.$$

When $\varphi_i(D,\mu) = 1/n$ for all $i = 1, 2, \dots, n$, then the H2SCC degenerates to the ordered 2-additive Shapley correlation coefficient (O2SCC):

$$O2SCC(A,B) = \sum_{i=1}^{n} \phi_i(N,v) CC\big(LH_{(i)}, LH'_{(i)}\big),$$

and the GH2SCC degenerates to the geometric ordered 2-additive Shapley correlation coefficient (GO2SCC):

$$GO2SCC(A,B) = \prod_{i=1}^{n} CC\big(LH_{(i)}, LH'_{(i)}\big)^{\phi_i(N,v)}.$$

Furthermore, one can easily show that all distance measures and correlation coefficients defined in this subsection satisfy the properties given in Property 4.

5. Application to Pattern Recognition and to Multi-Attribute Decision Making

This section researches the application of the distance measure and the correlation coefficient to pattern recognition and to multi-attribute decision making. Additionally, a comparison analysis is made to validate the effectiveness of the proposed approach.

5.1. *The Application to Pattern Recognition*

This subsection presents an approach to pattern recognition with LHFSs using the distance measure and the correlation coefficient. The main decision procedure can be described as follows:

Step 1: Suppose that there are m patterns $A = \{A_1, A_2, \dots, A_m\}$ and n features $C = \{c_1, c_2, \dots, c_n\}$. Let $S = \{s_1, \dots, s_t\}$ be the predefined linguistic term set. The evaluation of each pattern A_i with respect to the feature c_j is a LHFS, denoted by

$$A_i = \big\{\big\langle c_j, LH_{ij} = \big\{(s_{\theta(ij)}, lh(s_{\theta(ij)})\,\big|\, s_{\theta(ij)} \in S)\big\}\big\rangle \,\big|\, j = 1, 2, \dots, n\big\}$$

for each $i = 1, 2, \dots, m$. Furthermore, assume there is a sample ε to be recognized, which is represented by

$$\varepsilon = \big\{\big\langle c_j, LH_j = \big\{(s_{\theta(j)}, lh(s_{\theta(j)})\,\big|\, s_{\theta(j)} \in S)\big\}\big\rangle \,\big|\, j = 1, 2, \dots, n\big\}.$$

Step 2: Use the HWDM or GHWDM to calculate the distance measure between A_i $(i = 1, 2, \dots, m)$ and ε, or adopt the H2SDM or GH2SDM to calculate the distance measure between A_i $(i = 1, 2, \dots, m)$ and ε.

Step 3: According to the distance measure, identify the most likely pattern.

Step 3: End.

Similar to the distance measure, we can apply the correlation coefficient to give a method to pattern recognition.

Example 1. (See Zhang and Jiang, 2008.) Let us consider a set of diagnoses $A = \{A_1(\text{Viral fever}), A_2(\text{Malaria}), A_3(\text{Typhoid})\}$, and a set of symptoms $C = \{c_1$: temperature; c_2: headache; c_3: cough$\}$. Let $S_1 = \{s_1$: fair, s_2: a little high, s_3: high, s_4: very high, s_5: extremely high$\}$ be the predefined linguistic term set for the feature c_1, and let $S_2 = \{s_1$: extremely slight, s_2: very slight, s_3: slight, s_4: a little slight, s_5: fair, s_6: a little heavy, s_7: heavy; s_8: very heavy s_9: extremely heavy$\}$ be the predefined linguistic term set for the features c_2 and c_3. Suppose that a patient, with respect to all the symptoms, can be represented by the following LHFS:

$$\begin{aligned}\varepsilon(\text{patient}) = \big\{&\big\langle c_1, \big\{(s_4, 0.5, 0.7), (s_5, 0.4)\big\}\big\rangle, \big\langle c_2, \big\{(s_6, 0.6), (s_7, 0.5, 0.8)\big\}\big\rangle,\\ &\big\langle c_3, \big\{(s_4, 0.6)\big\}\big\rangle\big\}\end{aligned}$$

and assume that each diagnosis A_i $(i = 1, 2, 3)$ is viewed as a LHFS with respect to all the symptoms, where

$$\begin{aligned}A_1 &= \big\{\big\langle c_1, \big\{(s_2, 0.3), (s_3, 0.6, 0.7)\big\}\big\rangle, \big\langle c_2, \big\{(s_5, 0.7, 0.9), (s_6, 0.4)\big\}\big\rangle,\\ &\quad \big\langle c_3, \big\{(s_6, 0.4, 0.6), (s_7, 0.7)\big\}\big\rangle\big\},\\ A_2 &= \big\{\big\langle c_1, \big\{(s_4, 0.6, 0.8)\big\}\big\rangle, \big\langle c_2, \big\{(s_4, 0.5, 0.7), (s_5, 0.6, 0.8)\big\}\big\rangle, \big\langle c_3, \big\{(s_6, 0.5, 0.7)\big\}\big\rangle\big\},\\ A_3 &= \big\{\big\langle c_1, \big\{(s_4, 0.3, 0.5), (s_5, 0.7)\big\}\big\rangle, \big\langle c_2, \big\{(s_6, 0.6, 0.8)\big\}\big\rangle, \big\langle c_3, \big\{(s_4, 0.3, 0.6),\\ &\quad (s_5, 0.7, 0.8)\big\}\big\rangle\big\}.\end{aligned}$$

Our aim is to classify the patient to one of the diagnoses A_1, A_2 and A_3. Assume that the importance of features is given by 0.5, 0.2 and 0.3, and the weights of the ordered positions are defined by 0.3, 0.4 and 0.3.

Using the HWDM, we derive:

$$\begin{cases} HWDM(A_1, \varepsilon) = 0.6525,\\ HWDM(A_2, \varepsilon) = 0.5907,\\ HWDM(A_3, \varepsilon) = 0.4166.\end{cases}$$

From $HWDM(A_1, \varepsilon) \geqslant HWDM(A_2, \varepsilon) \geqslant HWDM(A_3, \varepsilon)$, we know that the patient suffers from typhoid.

When the GHWDM is applied to calculate the distance measure, we obtain:

$$\begin{cases} GHWDM(A_1, \varepsilon) = 0.6286,\\ GHWDM(A_2, \varepsilon) = 0.5904,\\ GHWDM(A_3, \varepsilon) = 0.3932,\end{cases}$$

which also shows that the patient has typhoid.

Furthermore, when the HWCC and GHWCC are used to compute the correlation coefficients, we get:

$$\begin{cases} HWCC(A_1, \varepsilon) = 0.7747, \\ HWCC(A_2, \varepsilon) = 0.8169, \\ HWCC(A_3, \varepsilon) = 0.8744 \end{cases} \quad \text{and} \quad \begin{cases} GHWCC(A_1, \varepsilon) = 0.7705, \\ GHWCC(A_2, \varepsilon) = 0.8321, \\ GHWCC(A_3, \varepsilon) = 0.889 \end{cases}$$

which means that the patient has typhoid too.

All of the above distance measures and correlation coefficients only consider the importance of elements separately. However, the importance of their combinations is not given. Considering the following facts, the symptoms of temperature and cough give more information than that of headache and temperature to diagnose the patient, and the symptoms of headache and temperature give more information than that of headache and cough. Suppose that the importance of temperature and cough, headache and temperature, and headache and cough is respectively defined by 0.9, 0.7, and 0.4. Furthermore, the importance of the ordered positions 1 and 2 is equal to that of the ordered positions 2 and 3, which both equal 0.75. However, the importance of the ordered positions 1 and 3 is less than that of the ordered positions 1 and 2 or 2 and 3, which is defined by 0.5. In this case, the distance measures and the correlation coefficients based on additive measures are helpless. However, the distance measures and the correlation coefficients using the Shapley function with respect to 2-additive measures are good choices to deal with this situation.

From the above depiction, we know that the 2-additive measure μ on the feature set C is

$$\mu(\emptyset) = 0, \quad \mu(c_1) = 0.5, \quad \mu(c_2) = 0.2, \quad \mu(c_3) = 0.3, \quad \mu(c_1, c_2) = 0.9,$$
$$\mu(c_1, c_3) = 0.7, \quad \mu(c_2, c_3) = 0.4.$$

Furthermore, the 2-additive measure v on the ordered position set N is $v(\emptyset) = 0$, $v(1) = 0.3$, $v(2) = 0.4$, $v(3) = 0.3$, $v(1, 2) = v(2, 3) = 0.75$, $v(1, 3) = 0.5$. Using the equation (15), the Shapley values of the features are

$$\phi_{c_1}(C, \mu) = 0.55, \quad \phi_{c_2}(C, \mu) = 0.25, \quad \phi_{c_3}(C, \mu) = 0.2$$

and the Shapley values of the ordered positions are

$$\phi_1(N, v) = \phi_3(N, v) = 0.275, \quad \phi_2(N, v) = 0.45.$$

Using the H2SDM and the GH2SDM, we obtain:

$$\begin{cases} H2SDM(A_1, \varepsilon) = 0.6196, \\ H2SDM(A_2, \varepsilon) = 0.6142, \\ H2SDM(A_3, \varepsilon) = 0.3993 \end{cases} \quad \text{and} \quad \begin{cases} GH2SDM(A_1, \varepsilon) = 0.6105, \\ GH2SDM(A_2, \varepsilon) = 0.5777, \\ GH2SDM(A_3, \varepsilon) = 0.3780 \end{cases}$$

which both show that the patient suffers from typhoid.

Using the H2SCC and the GH2SCC, we get:

$$\begin{cases} H2SCC(A_1,\varepsilon)=0.8154, \\ H2SCC(A_2,\varepsilon)=0.8469, \\ H2SCC(A_3,\varepsilon)=0.8872 \end{cases} \quad \text{and} \quad \begin{cases} GH2SCC(A_1,\varepsilon)=0.7895, \\ GH2SCC(A_2,\varepsilon)=0.8414, \\ GH2SCC(A_3,\varepsilon)=0.8655 \end{cases}$$

which still show that the patient has typhoid.

It is interesting that all distance measures and correlation coefficients show the patient having typhoid. However, their ranking values are different. In practical application, when we only need to consider the importance of each element, then the decision makers can apply the distance measures and correlation coefficients based on additive measures; otherwise, we suggest that the decision makers use the 2-additive measure based distance measures and correlation coefficients.

5.2. *An Approach to Multi-Attribute Decision Making*

This subsection introduces a new decision-making method with linguistic hesitant fuzzy information. Considering a multi-attribute decision making problem, let $S=\{s_1,\dots,s_t\}$ be the predefined linguistic term set. Suppose that there are m alternatives $A=\{A_1,A_2,\dots,A_m\}$ to be evaluated according to n attributes $C=\{c_1,c_2,\dots,c_n\}$. The main steps are given as follows:

Step 1: Assume that the evaluation of the alternative A_i with respect to the attribute c_j is a LHFS $LH_{ij}=\{(s_{\theta(ij)},lh(s_{\theta(ij)})\mid s_{\theta(ij)}\in S)\}$ $(i=1,2,\dots,m;\ j=1,2,\dots,n)$. Let $LH=(LH_{ij})_{m\times n}$ be the evaluation LHFS matrix.

Step 2: If all attributes c_j $(j=1,2,\dots,n)$ are benefits (i.e. the larger, the greater preference), then the attribute values need not to be normalized. Otherwise, we normalize the decision matrix $LH=(LH_{ij})_{m\times n}$ into $LH'=(LH'_{ij})_{m\times n}$, where $LH'_{ij}=\begin{cases} LH_{ij} & \text{for benefit criterion } c_j \\ (LH'_{ij})^C & \text{for cost criterion } c_j \end{cases}$ $(i=1,2,\dots,m;\ j=1,2,\dots,n)$. $(LH'_{ij})^C$ is the complement of LH_{ij} such that $(LH'_{ij})^C=\{(s_{t-\theta_{(ij)}},\bigcup_{r_{ij}\in lh(s_{\theta(ij)})}(1-r_{ij})\mid s_{\theta(ij)}\in S)\}$.

Step 3: Give the positive-ideal LHFS $LH_j^+=(s_t,1)$ and the negative-ideal LHFS $LH_j^-=(s_1,0)$ for each $j=1,2,\dots,n$. Let

$$LH^+=\{LH_1'^+,LH_2'^+,\dots,LH_n'^+\} \quad \text{and} \quad LH^-=\{LH_1'^-,LH_2'^-,\dots,LH_n'^-\}.$$

Step 4: Apply the HWDM or GHWDM to calculate the distance measure between $LH_i=\{LH_{i1},LH_{i2},\dots,LH_{in}\}$ $(i=1,2,\dots,m)$ and LH^+ as well as the distance measure between LH_i $(i=1,2,\dots,m)$ and LH^-, or we adopt the H2SDM or GH2SDM to calculate the distance measure between LH_i $(i=1,2,\dots,m)$ and LH^+ as well as the distance measure between LH_i $(i=1,2,\dots,m)$ and LH^-.

Table 1
LHFSs of alternatives.

	c_1	c_2	c_3
A_1	$\{(s_5, 0.1, 0.2), (s_6, 0.4), s_7, 0.3)\}$	$\{(s_6, 0.4), (s_7, 0.2, 0.3)\}$	$\{(s_6, 0.2, 0.4), (s_7, 0.3)\}$
A_2	$\{(s_5, 0.2, 0.4), (s_6, 0.3, 0.5)\}$	$\{(s_7, 0.3, 0.6), (s_8, 0.2)\}$	$\{(s_6, 0.3, 0.5, 0.8)\}$
A_3	$\{(s_5, 0.2), (s_6, 0.3, 0.5)\}$	$\{(s_5, 0.3, 0.5), (s_6, 0.2, 0.3), (s_7, 0.1)\}$	$\{(s_7, 0.3, 0.5), (s_8, 0.1, 0.3)\}$

Step 5: Calculate the similarity measure D_i for the alternative A_i $(i = 1, 2, \dots, m)$, where

$$D_i = 1 - \frac{HWDM(LH_i, M^+)}{HWDM(LH_i, M^+) + HWDM(LH_i, M^-)},$$

$$D_i = 1 - \frac{GHWDM(LH_i, M^+)}{GHWDM(LH_i, M^+) + GHWDM(LH_i, M^-)},$$

$$D_i = 1 - \frac{H2SDM(LH_i, M^+)}{H2SDM(LH_i, M^+) + H2SDM(LH_i, M^-)},$$

$$D_i = 1 - \frac{GH2SDM(LH_i, M^+)}{GH2SDM(LH_i, M^+) + GH2SDM(LH_i, M^-)}.$$

According to D_i $(i = 1, 2, \dots, m)$, select the best choice.
Step 6: End.

Similar to the distance measure, we can apply the correlation coefficient to give a method to multi-attribute decision making.

Example 2. (See Bryson and Mobolurin, 1995.) Let us consider the decision-making problem of evaluating university faculty for tenure and promotion. There are three faculty candidates (alternatives) $A = \{A_1, A_2, A_3\}$ to be evaluated using the linguistic term set $S = \{s_1$: extremely poor, s_2: very poor, s_3: poor, s_4: slightly poor, s_5: fair, s_6: slightly good, s_7: good, s_8: very good, s_9: extremely good$\}$ by an expert with respect to three attributes: $C = \{c_1$: teaching, c_2: research, c_3: service$\}$. The associated assessment values are shown as listed in Table 1.

These three faculty candidates are from one research-based university, which gives more importance to c_2 than to c_1 and c_3, but, on the other hand, the committee gives some advantages to the candidates that are both good in c_2 and in either c_1 or c_3. Their importance is defined as follows:

$$\mu(c_1) = \mu(c_3) = 0.4, \quad \mu(c_2) = 0.6, \quad \mu(c_1, c_2) = \mu(c_2, c_3) = 0.85,$$
$$\mu(c_1, c_3) = 0.7.$$

Furthermore, the ordered position values' importance is defined as follows:

$$v(1) = 0.5, \quad v(2) = 0.6, \quad v(3) = 0.5, \quad v(1, 2) = v(2, 3) = 0.9, \quad v(1, 3) = 0.8,$$

where $N = \{1, 2, 3\}$.

One can easily check that μ and v are both a 2-additive measure. Using the equation (15), the Shapley values of the attributes are $\phi_{c_1}(C,\mu) = 0.275$, $\phi_{c_2}(C,\mu) = 0.45$, $\phi_{c_3}(C,\mu) = 0.275$; and the Shapley values of the ordered positions are $\phi_1(N,v) = \phi_3(N,v) = 0.3$, $\phi_2(N,v) = 0.4$. To derive the optimal candidate, the following procedure is followed:

Step 1: Because all attributes are benefit, we derive $LH = LH'$. According to the predefined linguistic term set S, we have $M^+ = \{(s_9,1),(s_9,1),(s_9,1)\}$ and $M^- = \{(s_1,0),(s_1,0),(s_1,0)\}$.

Step 2: Using the H2SDM, we obtain:

$$\begin{cases} H2SDM(LH_1, M^+) = 6.8328, \\ H2SDM(LH_2, M^+) = 5.8059, \\ H2SDM(LH_3, M^+) = 6.4055 \end{cases} \quad \text{and} \quad \begin{cases} H2SDM(LH_1, M^-) = 1.5286, \\ H2SDM(LH_2, M^-) = 2.0305, \\ H2SDM(LH_3, M^-) = 1.2767. \end{cases}$$

Step 3: From $H2SDM(LH_i, M^+)$ and $H2SDM(LH_i, M^-)$, we derive $D_1 = 0.1828$, $D_2 = 0.2591$, $D_3 = 0.1662$. Thus, the best candidate is A_2.

In this example, when the GH2SWDM is applied, we have:

$$\begin{cases} GH2SDM(LH_1, M^+) = 6.8325, \\ GH2SDM(LH_2, M^+) = 5.7763, \\ GH2SDM(LH_3, M^+) = 6.3478 \end{cases} \quad \text{and} \quad \begin{cases} GH2SDM(LH_1, M^-) = 1.5068, \\ GH2SDM(LH_2, M^-) = 1.9913, \\ GH2SDM(LH_3, M^-) = 1.2499 \end{cases}$$

by which we get $D_1 = 0.1807$, $D_2 = 0.2564$, $D_3 = 0.1645$. It also shows that the candidate A_2 is the best choice.

Furthermore, when the H2SCC is used to select the best candidate, we obtain:

$$\begin{cases} H2SCC(LH_1, M^+) = 0.2417, \\ H2SCC(LH_2, M^+) = 0.3691, \\ H2SCC(LH_3, M^+) = 0.2615. \end{cases}$$

From $H2SCC(LH_2, M^+) > H2SCC(LH_3, M^+) > H2SCC(LH_1, M^+)$, we know that the candidate A_2 is the best choice. When we adopt the GH2SCC to calculate the correlation coefficient, we derive:

$$\begin{cases} GH2SCC(LH_1, M^+) = 0.2407, \\ GH2SCC(LH_2, M^+) = 0.3489, \\ GH2SCC(LH_3, M^+) = 0.2538 \end{cases}$$

which still shows that the candidate A_2 is the best choice.

Because the correlations between the attribute LHFSs of alternatives and the negative-ideal LHFS are zero, we only calculate the correlations between the attribute LHFSs of alternatives and the positive-ideal LHFS. Then, we give the ranking order of alternatives according to them.

Table 2
Ranking results based on the GLHFHSWA operator to different values of γ.

	$E(LH_1)$	$E(LH_2)$	$E(LH_3)$	Ranking orders
$\gamma = 0.1$	$s_{1.91}$	$s_{2.68}$	$s_{1.78}$	$A_2 > A_1 > A_3$
$\gamma = 0.2$	$s_{1.91}$	$s_{2.69}$	$s_{1.78}$	$A_2 > A_1 > A_3$
$\gamma = 0.5$	$s_{1.93}$	$s_{2.73}$	$s_{1.79}$	$A_2 > A_1 > A_3$
$\gamma = 1.0$	$s_{1.97}$	$s_{2.79}$	$s_{1.82}$	$A_2 > A_1 > A_3$
$\gamma = 2.0$	$s_{2.04}$	$s_{2.92}$	$s_{1.87}$	$A_2 > A_1 > A_3$
$\gamma = 5.0$	$s_{2.14}$	$s_{3.14}$	$s_{1.90}$	$A_2 > A_1 > A_3$
$\gamma = 10.0$	$s_{2.13}$	$s_{3.14}$	$s_{1.83}$	$A_2 > A_1 > A_3$
$\gamma = 30.0$	$s_{1.73}$	$s_{2.87}$	$s_{1.43}$	$A_2 > A_1 > A_3$

Table 3
Ranking results based on the GLHFHSGM operator to different values of γ.

	$E(LH_1)$	$E(LH_2)$	$E(LH_3)$	Ranking orders
$\gamma = 0.1$	$s_{1.89}$	$s_{2.45}$	$s_{1.62}$	$A_2 > A_1 > A_3$
$\gamma = 0.2$	$s_{1.89}$	$s_{2.44}$	$s_{1.61}$	$A_2 > A_1 > A_3$
$\gamma = 0.5$	$s_{1.88}$	$s_{2.41}$	$s_{1.61}$	$A_2 > A_1 > A_3$
$\gamma = 1.0$	$s_{1.88}$	$s_{2.37}$	$s_{1.59}$	$A_2 > A_1 > A_3$
$\gamma = 2.0$	$s_{1.86}$	$s_{2.29}$	$s_{1.56}$	$A_2 > A_1 > A_3$
$\gamma = 5.0$	$s_{1.82}$	$s_{2.11}$	$s_{1.48}$	$A_2 > A_1 > A_3$
$\gamma = 10.0$	$s_{1.74}$	$s_{1.94}$	$s_{1.38}$	$A_2 > A_1 > A_3$
$\gamma = 30.0$	$s_{1.58}$	$s_{1.74}$	$s_{1.20}$	$A_2 > A_1 > A_3$

5.3. *Comparison Analysis*

Because most of previous studies applied multi-attribute decision making problem to linguistic hesitant fuzzy environment, several methods are applied to Example 2 to verify the effectiveness of the proposed approach.

Considering there are interactive characteristics between the attributes and their orders, Meng *et al.* (2014) proposed the GLHFHSWA and GLHFHSGM operators, by which the related results and ranking orders are obtained as shown in Tables 2 and 3.

Tables 2 and 3 show that the same ranking order and the same best choice are derived in this example, which is the same as that derived by using the H2SDM and GH2SWDM.

Most of previous studies assume that no interactive characteristics exist between the attributes and between the orders. To further show the advantage of the proposed approach under a linguistic hesitant fuzzy environment, we compare it with the methods proposed by Zhu *et al.* (2016) and Zhou *et al.* (2015) that are based on independent assumption. For the Zhu *et al.*'s method, the weights on the attributes are $\omega_1 = 0.275$, $\omega_2 = 0.45$ and $\omega_3 = 0.275$, and the nine clouds are $A_1(0, 3.01, 0.107)$, $A_2(1.93, 2.75, 0.193)$, $A_3(3.31, 2.27, 0.353)$, $A_4(4.30, 1.932, 0.466)$, $A_5(5, 1.667, 0.554)$, $A_6(5.70, 1.932, 0.466)$, $A_7(6.69, 2.27, 0.353)$, $A_8(8.07, 2.75, 0.193)$, $A_9(10, 3.01, 0.107)$ provided by Wang *et al.* (2014). The related results and ranking orders are shown in Table 4.

With respect to the Zhou *et al.*'s extended evidential reasoning (ER) method, the related results and ranking orders are shown in Table 5.

Table 4
Ranking results based on the LHFPWA operator and the LHFPWG operator.

	$E(LH_1)$	$E(LH_2)$	$E(LH_3)$	Ranking orders
LHFPWA	$s_{1.97}$	$s_{2.72}$	$s_{1.78}$	$A_2 > A_1 > A_3$
LHFPWG	$s_{1.88}$	$s_{2.32}$	$s_{1.56}$	$A_2 > A_1 > A_3$

Table 5
Ranking results based on the ER method.

	$E(LH_1)$	$E(LH_2)$	$E(LH_3)$	Ranking orders
Using LSF(1)	$s_{0.46}$	$s_{0.44}$	$s_{0.42}$	$A_1 > A_2 > A_3$
Using LSF(2)	$s_{0.43}$	$s_{0.41}$	$s_{0.40}$	$A_1 > A_2 > A_3$
Using LSF(3)	$s_{0.48}$	$s_{0.45}$	$s_{0.42}$	$A_1 > A_2 > A_3$

Using three kinds of linguistic scale functions (LSFs) defined by Zhou *et al.* (2015), the ranking result is $A_1 > A_2 > A_3$, and the best alternative is A_1.

From the above ranking results and ranking orders, one can find the difference between methods based on the independent and interactive analysis. This example shows that the new method is effective, and it is simpler than some previous methods.

6. Conclusions

As we know, distance measure and correlation coefficient are two important tools to decision making. Considering the application of linguistic hesitant fuzzy sets, this paper defines a distance measure, and then introduces a correlation coefficient. After that, we develop two types of the hybrid weighted distance measures and the hybrid weighted correlation coefficients for linguistic hesitant fuzzy sets, by which we can derive the comprehensive evaluated values of the objects. Furthermore, we study their applications to pattern recognition and to multi-attribute decision making.

Comparing with the previous researches about decision making with LHFSs, there are several contributions of our method:

(i) It is simpler than the previous method (Meng *et al.*, 2014);
(ii) It addresses the issues in the previous distance measure (Zhou *et al.*, 2015);
(iii) It considers the interaction between elements in a set and the complexity of determining a fuzzy measure;
(iv) It extends the application of LHFSs.

However, we only present one distance measure and one correlation coefficient, and it will be interesting to study other distance measures and correlation coefficients for linguistic hesitant fuzzy sets. Furthermore, we shall discuss their application in other fields, such as expert systems, digital image processing, and clustering analysis. Moreover, we can extend the developed theoretical results to other types of fuzzy sets, such as hesitant interval neutrosophic linguistic sets (Ye, 2013).

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