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2-Tuple Linguistic Hesitant Fuzzy Aggregation Operators and Its Application to Multi-Attribute Decision Making

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Abstract. In this paper, a new class of uncertain linguistic variables called 2-tuple linguistic hesitant fuzzy sets (2-TLHFSs) is defined, which can express complex multi-attribute decision-making problems as well as reflect decision makers' hesitancy, uncertainty and inconsistency. Besides, it can avoid information and precision losing in aggregation process. Firstly, several new closed operational laws based on Einstein t-norm and t-conorm are defined over 2-TLHFSs, which can overcome granularity and logical problems of existing operational laws. Based on the new operational laws, 2-tuple linguistic hesitant fuzzy Einstein weighted averaging (2-TLHFEWA) operator and 2-tuple linguistic hesitant fuzzy Einstein weighted geometric (2-TLHFEWG) operator are proposed, and some of their properties are investigated. Then, a new model method based on similarity to ideal solution is proposed to determine weights of attribute, which takes both subjective and objective factors into consideration. Finally, a linguistic hesitant fuzzy multi-attribute decision making procedure is developed by means of 2-TLHFEWA and 2-TLHFEWG operators. An example is given to illustrate the practicality and efficiency of the proposed approach.

Key words: multi-attribute decision making, aggregation operator, 2-tuple linguistic hesitant fuzzy sets, 2-tuple linguistic hesitant fuzzy Einstein weighted averaging operators, 2-tuple linguistic hesitant fuzzy Einstein weighted geometric operator.

1. Introduction

Multi-attribute decision making (MADM) has been deeply studied and widely applied in many fields, such as education (Bryson and Mobolurin, 1997), medical care (James and Dolan, 2010), military (Robert and Swezey, 1979), engineering (Lennon *et al.*, 2013), social sciences (Cavus, 2011) and economics (Vaidogas and Sakenaite, 2011; Zeng and Chen, 2015; Zeng *et al.*, 2015). Because of the complexity and uncertainty of multi-attribute decision-making problems, it is impractical to use exact number to evaluate the attribute, so Zadeh (1965, 1973) proposed (type-2) fuzzy set, Atanassov (1986), Atanassov and Gargov (1989) introduced (interval-valued) intuitionistic fuzzy sets, and

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(interval-valued) hesitant fuzzy sets (Rodriguez *et al.*, 2014a, 2014b; Mu *et al.*, 2015; Chen *et al.*, 2013) are defined. However, there are some too complex or ill-defined problems by means of above quantitative expressions. For example, experts evaluate the refrigeration or quietness of an air condition; they prefer to apply linguistic expressions "extremely good", "good" or "little good". So it might be more appropriate to evaluate the attribute by linguistic information than by numerical values for too complex or ill-defined problems.

Zadeh (1975) first introduced the linguistic variables. Since then, linguistic variables have been deeply studied. And how to process the linguistic information have been paid more attention. There are four main methods for computing with words (CW): a model based on the extension principle, which transforms linguistic information into the fuzzy numbers, such as triangular fuzzy number, trapezoidal fuzzy number, and type-2 fuzzy sets (Chou, 2012; Deng *et al.*, 2011); a model based on the symbol, which aggregates linguistic variables on the indexes of linguistic terms (Chen and Lee, 2010; Xu, 2006); a method based on the cloud model (Liu *et al.*, 2014a; Wang, 2008), which can achieve uncertain transformation between a qualitative concept and its quantitative values; a model based on 2-tuple representation model (Herrera and Martinez, 2001; Dong *et al.*, 2013; Merigo and Gil, 2013), which transforms linguistic information into consecutive 2-tuple linguistic term. For the former three methods, the calculated results maybe do not match the initial linguistic term, and a transformation procedure should be introduced to transform the calculated result into the initial expression, which can cause information loss. However, 2-tuple linguistic representation model does not need this transformation procedure and can avoid the information loss and distortion (Herrera and Martinez, 2001). So 2-tuple linguistic representation model has been widely applied for computing with words (Marti and Herrera, 2012).

2-tuple linguistic representation model introduced by Herrera and Martinez (2001) is made up of a linguistic term and a numeric value expressed in $[-0.5, 0.5)$. Although 2-tuple linguistic representation model can avoid information loss, it cannot reflect the membership and non-membership degrees of an element to a certain concept. Inspired by intuitionistic fuzzy sets, Liu *et al.* (2014b) introduced intuitionistic linguistic sets (ILSs). After Wang and Li, some experts give a few extensions of ILSs, such as intuitionistic uncertain linguistic sets (IULSs) (Liu and Jin, 2012) and interval-valued intuitionistic uncertain linguistic sets (IVIULSs) (Xu and Shen, 2014). Torra (2010) pointed out the fact that decision-makers may hesitate between several values for evaluating an alternative with respect to attribute. So Rodriguez *et al.* (2014a, 2014b) developed hesitant fuzzy linguistic term sets (HFLSs), which keep several linguistic terms to express decision makers' hesitancy. Based on the HFLSs, interval-valued hesitant fuzzy linguistic sets (Wang and Wu, 2014) have been developed. Recently, some researches based on HFLSs and 2-tuple linguistic representation model are developed. Zhang and Guo (2015) proposed new operations of hesitant fuzzy linguistic term sets based on 2-tuple linguistic terms. Beg and Rashid (2016) introduced hesitant 2-tuple linguistic information. The hesitant 2-tuple linguistic information is denoted by (s_i, β_{ij}), the symbolic translation β_{ij} is a hesitant fuzzy set, which expresses the hesitancy by presenting its possible linguistic translations.

Wang *et al.* (2016) developed 2-tuple linguistic aggregation operators with multi-hesitant fuzzy linguistic information, the multi-hesitant fuzzy linguistic information contains inconsecutive and repeatable linguistic terms. For the above three researches, they can reflect decision makers' hesitancy by providing several possible linguistic terms of a linguistic variable as well as avoid information loss, but they cannot reflect the possible membership degree of each linguistic term. In reality, because of the fuzziness and uncertainty of MADM problems and the vagueness of human preferences, many MADM problems are highly complex. Sometimes, in order to express the decision makers' hesitancy, we have to give the membership of linguistic term. For the same example, experts evaluate the quietness of an air condition using linguistic expressions "extremely good", "good" or "little good"; expert may give the value 0.2 for "extremely good", the value 0.7 for "good" and the value 0.1 for "little good".

Therefore, in order to express and deal with complex and uncertain linguistic assessment, inspired by 2-tuple linguistic representation model and linguistic hesitant fuzzy sets (LHFSs) proposed by Meng *et al.* (2014), we define a new class of linguistic variables, 2-tuple linguistic hesitant fuzzy sets (2-TLHFSs), which can reflect decision makers' uncertainty and hesitancy by providing the information about several possible linguistic terms of a linguistic variable and several possible membership degrees of each linguistic term, besides, it can avoid information loss and the lack of precision in MADM aggregation process.

In MADM problems, aggregation operators are important tools to aggregate information. For intuitionistic fuzzy sets, Zhou *et al.* (2014a, 2014b, 2016), Zhou and Chen (2013) developed some new aggregation operators. For hesitant fuzzy linguistic sets, Zhang and Guo (2015) proposed HFLWA and HFLOWA operators, Wang *et al.* (2016) developed G2TLWA and G2TLOWA operators. For linguistic hesitant fuzzy sets, Meng *et al.* (2014) proposed GLHFHWA and GLHFHSWA operators. It is worthy of note that the above aggregation operators are based on Algebra t-norm and t-conorm, which is one of triangular norms and conorms (briefly t-norms and t-conorms) (Beliakov *et al.*, 2007). There are numerous basic t-norms and t-conorms, such as, Algebra t-norm and t-conorm, Einstein t-norm and t-conorm, Hammer t-norm and t-conorm, Frank t-norm and t-conorm. In fuzzy set theory, triangular norms and conorms are very useful to address "and" and "or" operations for decision-making problems and different aggregation operators based on different t-conorms and t-norms can provide more choices for the decision makers. So with respect to 2-TLHFSs, inspired by references (Wan, 2013; Tao *et al.*, 2014; Yu, 2014; Zhao *et al.*, 2015), we propose some new operational laws based on Einstein t-norm and t-conorm in this paper. The new operational laws based on Einstein t-norm and t-conorm are closed, and they can overcome granularity and logical problems of existing operational laws. Based on the new operational laws, 2-tuple linguistic hesitant fuzzy Einstein weighted averaging (2-TLHFEWA) operator and 2-tuple linguistic hesitant fuzzy Einstein weighted geometric (2-TLHFEWG) operator are developed.

However, in MADM problems under 2-TLHFSs environment, the weight information of attribute may be unknown. In this situation, we should firstly determine the weights of attribute. There are many methods for obtaining attribute's weight (Poyhonen and

Hamalainen, 2001), which can mainly be divided into two categories (Zardari *et al.*, 2015): objective weighting methods (Deng *et al.*, 2000; Diakoulaki *et al.*, 1995; Jahan *et al.*, 2012; Zavadskas and Podvezko, 2016) (such as Entropy method, Criteria Importance Through Inter-criteria Correlation (CRITIC), Standard Deviation) and subjective weighting methods (Rybarczyk and Wu, 2010; Edwards and Barron, 1994; Figueira and Roy, 2002; Krylovas *et al.*, 2014; Kersuliene and Turskis, 2010) (such as Direct Rating method, Ranking Method, Ratio Method, Swing Method, SIMOS Method). Objective weighting methods are based on some mathematical models where decision-makers play no role in determining the relative importance of attribute, which may not reflect the actual importance of attributes. Subjective weighting methods determine the weight vector based on preferences of decision-makers, which are not objective. In this paper, we propose a new model method based on the similarity to ideal solution to obtain weights of attribute, which takes both subjective and objective factors into consideration and can be effective and reliable. Finally, based on the 2-TLHFEWA, 2-TLHFEWG operator and new model for the optimal weight vectors, we develop an approach for 2-tuple linguistic hesitant fuzzy multi-attribute decision making.

This paper is organized as follows: In Section 2, some basic concepts about 2-tuple linguistic term sets, HFLSs and LHFSs are briefly introduced. In Section 3, the 2-TLHFSs, order relationship between 2-TLHFSs and some new operational laws based on Einstein t-norm and t-conorm are proposed. In Section 4, two new classes of aggregation operators, 2-tuple linguistic hesitant fuzzy Einstein weighted averaging operator and 2-tuple linguistic hesitant fuzzy Einstein weighted geometric operator, are developed. Some properties are investigated. In section 5, an approach to 2-tuple linguistic hesitant fuzzy multi-attribute decision making is developed, where a new model for the optimal weight vectors is proposed. In Section 6, a real example is provided to demonstrate the application of proposed aggregation operators. A detailed comparison between the proposed method and existing methods are given. Conclusions are provided in the last section.

2. Preliminaries

In the following, some related concepts are briefly reviewed, such as 2-tuple linguistic term sets, hesitant fuzzy linguistic term sets (HFLSs) and the linguistic hesitant fuzzy term sets (LHFSs). These concepts can help us better understand 2-tuple linguistic hesitant fuzzy sets.

Let $S = \{s_i \mid i = 0, 1, 2, \dots, g\}$ be a linguistic term set with odd cardinality. Any label s_i stands for a possible value of a linguistic variable and the label s_i should have the following properties (Herrera and Herrera-Viedma, 2000):

(1) The set is ordered: $s_i > s_j$, if $i > j$;
(2) Max operator: $\max(s_i, s_j) = s_i$, if $s_i \geqslant s_j$;
(3) Min operator: $\min(s_i, s_j) = s_i$, if $s_i \leqslant s_j$.

In order to avoid the information loss and distortion in aggregation process, Herrera and Martinez (2001) introduced the 2-tuple linguistic term sets, which are defined as follows.

Definition 1. (See Herrera and Martinez, 2001.) Let $S = \{s_i \mid i = 0, 1, 2, \dots, g\}$ be a linguistic term set and β be a number value representing the aggregation result of a linguistic symbolic aggregation operation. 2-tuple fuzzy linguistic representation model is made up of a linguistic term $s_i \in S$ and a numeric value $\alpha \in [-0.5, 0.5)$, denoted by (s_i, α), where

s_i represents the linguistic label of the information;

α is a numerical value expressing the value of the translation from the original aggregation result β to the closest index label i in the linguistic term set ($s_i \in S$), i.e. the symbolic translation.

Definition 2. (See Herrera and Martinez, 2001.) Let (s_i, α_i) be a 2-tuple linguistic term, the function Δ used to obtain the 2-tuple linguistic information equivalent to β is defined as:

$$\Delta : [1, g] \to S \times [-0.5, 0.5), \tag{1}$$

$$\Delta : (\beta) = \begin{cases} s_i, & i = round(\beta), \\ \alpha = \beta - i, & \alpha \in [-0.5, 0.5), \end{cases} \tag{2}$$

where round (.) is the usual round operation.

And, the function Δ^{-1} from a 2-tuple linguistic term (s_i, α_i) to its equivalent numerical value $\beta \in [1, g] \subset R$ can be defined as follows:

$$\Delta^{-1} : S \times [-0.5, 0.5) \to [1, g], \tag{3}$$

$$\Delta^{-1}(s_i, \alpha_i) = i + \alpha = \beta. \tag{4}$$

From the above definitions, when a linguistic term is transformed into a linguistic 2-tuple, we can add a value 0 as symbolic translation:

$$\Delta(s_i) = (s_i, 0). \tag{5}$$

Definition 3. (See Herrera and Martinez, 2001.) Let (s_i, α_i) and (s_j, α_j) be two linguistic 2-tuples, they have the following properties:

(1) If $i < j$, then (s_i, α_i) is smaller than (s_j, α_j),
(2) If $i = j$, and
  (a) if $\alpha_i = \alpha_j$, then (s_i, α_i) and (s_j, α_j) represent the same information;
  (b) if $\alpha_i < \alpha_j$, then (s_i, α_i) is smaller than (s_j, α_j);
  (c) if $\alpha_i > \alpha_j$, then (s_i, α_i) is bigger than (s_j, α_j).

In some situations, experts maybe hesitate between several values when evaluating an alternative with respect to an attribute. Rodriguez *et al.* (2014a) proposed hesitant fuzzy linguistic term sets (HFLSs) which allow linguistic variable to keep several linguistic terms to reflect decision makers' hesitancy.

Definition 4. (See Rodriguez *et al.*, 2014a.) Let S be a linguistic term set, $S = \{s_0, s_1, \dots, s_g\}$, a hesitant fuzzy linguistic term set (HFLS), H_s is an ordered finite subset of the consecutive linguistic terms of S.

For example let $S = \{s_0$: very good, s_1: good, s_2: above average, s_3: average, s_4: below average, s_5: bad, s_6: very bad$\}$ be a linguistic term set and let P be a qualitative reference; an HFLS could be $H_s(P) = \{s_2, s_3, s_4\}$.

Although HFLSs have taken decision makers' hesitancy into consideration, they cannot express the membership degree of each linguistic term. In order to reflect decision makers' hesitancy and possible membership degree of each linguistic term, Meng *et al.* (2014) introduced linguistic hesitant fuzzy sets (LHFSs) as follows.

Definition 5. (See Meng *et al.*, 2014.) Let S be a linguistic term set, $S = \{s_0, s_1, \dots, s_g\}$, a linguistic hesitant fuzzy set, LHF in S can be expressed by:

$$LHF = \left\{\left(s_{\theta_i}, lhf(s_{\theta_i})\right) \middle| s_{\theta_i} \in S\right\}$$

where $lhf(s_{\theta_i}) = \{r_1, r_2, r_3, \dots, r_{m_i}\}$ is a set with m_i values in [0, 1], denoting the possible membership degrees of the elements $x \in X$ to the set LHF.

3. 2-Tuple Linguistic Hesitant Fuzzy Set

In this section, a new uncertain linguistic variable called 2-tuple linguistic hesitant fuzzy set (2-TLHFS) is defined. Based on Einstein t-norm and t-conorm, some new operational laws over 2-TLHFSs are proposed. And order relation between 2-TLHFSs is given.

3.1. *2-Tuple Linguistic Hesitant Fuzzy Set*

Because of the fuzziness and uncertainty of MADM problems and the vagueness of human preferences, many MADM problems are rather complicated, while the existing linguistic variables, such as 2-tuple linguistic term sets, intuitionistic linguistic term sets, hesitant fuzzy linguistic term sets, are unable to express the complexity. To deal with the situation and ensure information integrity in aggregation process, this section gives a new uncertain linguistic variable, 2-tuple linguistic hesitant fuzzy sets (2-TLHFSs), that not only reflect decision makers' uncertainty and hesitancy by providing several possible linguistic terms of a linguistic variable and several possible membership degrees of each linguistic term but also avoid the information loss and distortion.

Definition 6. Let $S = \{(s_0, \alpha_0), (s_1, \alpha_1), \dots, (s_g, \alpha_g)\}$ be a 2-tuple linguistic term set. 2-tuple linguistic hesitant fuzzy set (2-TLHFS) LH in S can be expressed as follows:

$$LH = \left\{\left((s_{\theta_i}, \alpha_{\theta_i}), lh(s_{\theta_i}, \alpha_{\theta_i})\right) \middle| (s_{\theta_i}, \alpha_{\theta_i}) \in S\right\}$$

where $lh((s_{\theta_i}, \alpha_{\theta_i})) = \{r_1, r_2, r_3, \dots, r_{m_i}\}$ is a set with m_i values in [0, 1], denoting the possible membership degrees of the elements $x \in X$ to the set LH.

For example, some decision makers evaluate the weight of the goods, let $S = \{s_0$: very heavy, s_1: heavy, s_2: above average, s_3: average, s_4: below average, s_5: light, s_6: very light$\}$, one may give the value 0.2 for "very heavy", the value 0.7 for "heavy" and the value 0.1 for "above average", some may give the value 0.4 for "very heavy" and the value 0.6 for "heavy", the other may give the value 0.9 for "heavy". In this situation, a 2-TLHFS, $LH = ((s_0, 0), 0.2, 0.4), ((s_1, 0), 0.7, 0.6, 0.9), ((s_2, 0), 0.1)$ may be the best appropriate expression.

Compared with some new linguistic variables based on HFLSs and 2-tuple linguistic representation model, such as hesitant fuzzy linguistic term sets based on 2-tuple linguistic representation model (Zhang and Guo, 2015), hesitant 2-tuple linguistic information (Beg and Rashid, 2016), multi-hesitant fuzzy linguistic information (Wang *et al.*, 2016), the proposed 2-TLHFSs have a wider range of application. The 2-TLHFSs can express and deal with more complex linguistic assessment, such as decision makers' hesitancy and membership degree of a linguistic term. Compared with the LHFSs introduced by Meng *et al.* (2014), 2-TLHFSs can ensure information integrity in aggregation process. For LHFSs, the calculated results maybe do not match the initial linguistic terms, so an approximation procedure should be introduced to express the result in the initial expression domain. During the approximation procedure information might be lost. However, 2-TLHFSs do not need the approximation procedure and can avoid information loss and distortion.

3.2. *New Operational Laws over 2-TLHFSs Based on Einstein t-Norm and t-Conorm*

Definition 7. (See Figueira and Roy, 2002.) If a function $T : [0, 1]^2 \to [0, 1]$, for all $x, y, z \in [0, 1]$, satisfies the following four axioms, function T is called a triangular norm (briefly t-norm):

(1) $T(1, x) = x$ (boundary condition);
(2) $T(x, y) = T(y, x)$ (commutativity);
(3) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity);
(4) $T(x, y) \leqslant T(x, z)$ whenever $y \leqslant z$ (monotonicity).

The corresponding triangular conorm (briefly called t-conorm or s-norm) of T is the function $S : [0, 1]^2 \to [0, 1]$ denoted by $S(x, y) = 1 - T(1 - x, 1 - y)$.

There are numerous basic t-norms and t-conorms, such as, minimum T_M and maximum S_M, product T_P and probabilistic sum S_P, Lukasiewicz t-norm T_L and Lukasiewicz t-conorm S_L, Einstein product T_E and Einstein product S_E, and drastic product T_D and drastic sum S_D. Especially, Einstein product T_E and Einstein product S_E are defined by: $T_E(x, y) = \frac{xy}{1+(1-x)(1-y)}$ $S_E(x, y) = \frac{x+y}{1+xy}$.

Based on the above Einstein t-norm and t-conorm, new operational laws are defined as follows.

Definition 8. Let LH_1 and LH_2 be any two 2-TLHFSs, where g is the upper limit of the 2-tuple linguistic term set. Some operations on these sets are defined by (where $\lambda > 0$):

(1) $LH_1 \oplus_E LH_2$
$$= \bigcup_{((s_{\theta_i},\alpha_{\theta_i}),lh(s_{\theta_i},\alpha_{\theta_i}))\in LH_1,((s_{\theta_j},\alpha_{\theta_j}),lh(s_{\theta_j},\alpha_{\theta_j}))\in LH_2} \left\{\left(\Delta\left(g\cdot\frac{\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g}+\frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g}}{1+\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g}\frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g}}\right), \bigcup_{r_i\in lh(s_{\theta_i},\alpha_{\theta_i}),r_j\in lh(s_{\theta_j},\alpha_{\theta_j})}\frac{r_i+r_j}{1+r_ir_j}\right)\right\};$$
(2) $LH_1 \otimes_E LH_2$
$$= \bigcup_{((s_{\theta_i},\alpha_{\theta_i}),lh(s_{\theta_i},\alpha_{\theta_i}))\in LH_1,((s_{\theta_j},\alpha_{\theta_j}),lh(s_{\theta_j},\alpha_{\theta_j}))\in LH_2} \left\{\left(\Delta\left(g\cdot\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})\times\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g^2+(g-\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i}))(g-\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j}))}\right), \bigcup_{r_i\in lh(s_{\theta_i},\alpha_{\theta_i}),r_j\in lh(s_{\theta_j},\alpha_{\theta_j})}\frac{r_ir_j}{1+(1-r_i)(1-r_j)}\right)\right\};$$
(3) $$\lambda LH_1 = \bigcup_{((s_{\theta_i},\alpha_{\theta_i}),lh(s_{\theta_i},\alpha_{\theta_i}))\in LH_1} \left\{\left(\Delta\left(g\cdot\frac{(g+\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i}))^\lambda-(g-\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i}))^\lambda}{(g+\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i}))^\lambda+(g-\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i}))^\lambda}\right), \bigcup_{r\in lh(s_{\theta_i},\alpha_{\theta_i})}\frac{(1+r)^\lambda-(1-r)^\lambda}{(1+r)^\lambda+(1-r)^\lambda}\right)\right\};$$
(4) $$LH_1^\lambda = \bigcup_{((s_{\theta_i},\alpha_{\theta_i}),lh(s_{\theta_i},\alpha_{\theta_i}))\in LH_1} \left\{\left(\Delta\left(g\cdot\frac{2(\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i}))^\lambda}{(2g-\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i}))^\lambda+(\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i}))^\lambda}\right), \bigcup_{r\in lh(s_{\theta_i},\alpha_{\theta_i})}\frac{2r^\lambda}{(2-r)^\lambda+r^\lambda}\right)\right\}.$$

Theorem 1. *Let $S = \{(s_0,\alpha_0),(s_1,\alpha_1),\dots,(s_g,\alpha_g)\}$ be a 2-tuple linguistic term set, where g is the upper limit of the 2-tuple linguistic term set. And $\Omega = \{((s_{\theta_i},\alpha_{\theta_i}), lh(s_{\theta_i},\alpha_{\theta_i}))|(s_{\theta_i},\alpha_{\theta_i})\in S\}$ be the set of all the 2-tuple linguistic hesitant fuzzy term based on S, $LH_1, LH_2 \in \Omega$ and $\lambda \geqslant 0$ is a scalar. The new operational laws of 2-TLHFSs based on Einstein t-norm and t-conorm are closed, i.e.*

(1) $LH_1 \oplus_E LH_2 \in \Omega$;
(2) $LH_1 \otimes_E LH_2 \in \Omega$;
(3) $\lambda LH_1 \in \Omega$;
(4) $LH_1^\lambda \in \Omega$.

Proof. It is known that $\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g}, \frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g}, r_i, r_j \in [0,1]$.

(1) It is easy to know that $1+r_ir_j-(r_i+r_j)=(1-r_i)(1-r_j)\geqslant 0$, thus $1+r_ir_j\geqslant (r_i+r_j)$, then $\frac{r_i+r_j}{1+r_ir_j}\in[0,1]$.

Similarly, $g(1+\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g}\frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g}-(\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g}+\frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g}))\geqslant 0$, thus $1+\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g}\frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g}\geqslant\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g}+\frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g}$, then $\Delta(g\frac{\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g}+\frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g}}{1+\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g}\frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g}})\in S$.

So, $LH_1 \oplus_E LH_2 \in \Omega$;

(2) It is easy to know $1+(1-r_i)(1-r_j)-r_ir_j=(1-r_i)+(1-r_j)\geqslant 0$, thus $1+(1-r_i)(1-r_j)\geqslant r_ir_j$, then $\frac{r_ir_j}{1+(1-r_i)(1-r_j)}\in[0,1]$.

Similarly, $1+(1-\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g})(g-\frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g})-\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g}\times\frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g}\geqslant 0$, thus $1+(1-\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g})(g-\frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g})\geqslant\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g}\times\frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g}$, then $\Delta(g\cdot\frac{\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g}\times\frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g}}{1+(1-\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g})(g-\frac{\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j})}{g})})\in S$.

So, $LH_1\otimes_E LH_2\in\Omega$;

(3) It is easy to know that $(1+r)^\lambda+(1-r)^\lambda-((1+r)^\lambda+(1-r)^\lambda)=2(1-r)^\lambda\geqslant 0$, thus $(1+r)^\lambda+(1-r)^\lambda\geqslant(1+r)^\lambda-(1-r)^\lambda$, then $\frac{(1+r)^\lambda-(1-r)^\lambda}{(1+r)^\lambda+(1-r)^\lambda}\in[0,1]$. Similarly, we have $\Delta(g\cdot\frac{(1+\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g})^\lambda-(1-\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g})^\lambda}{(1+\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g})^\lambda+(1-\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{g})^\lambda})\in S$.

So, $\lambda LH_1\in\Omega$;

(4) It is easy to know that $(2-r)^\lambda+r^\lambda-2r^\lambda=(2-r)^\lambda-r^\lambda\geqslant 0$, thus $(2-r)^\lambda+r^\lambda\geqslant 2r^\lambda$, then $\frac{2r^\lambda}{(2-r)^\lambda+r^\lambda}\in[0,1]$. Similarly, we have $\Delta(g\cdot\frac{2(\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i}))^\lambda}{(2g-\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i}))^\lambda+(\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i}))^\lambda})\in S$.

So, $LH_1{}^\lambda\in\Omega$. □

According to the above theorem, it is known that the proposed operational laws are closed. And they can overcome granularity and logical problems of existing operational laws as follows. Let's take the operational laws over LHFSs defined by Meng *et al.* (2014) for example.

Granularity problem: when we use the defined operational laws (Meng *et al.*, 2014), the calculated results do not exist in the set. For example, assume that $S=\{s_0,s_1,\dots,s_8\}$ is a linguistic term set and any two linguistic hesitant fuzzy sets LHF_1 and LHF_2 in S are given: $LHF_1=(s_5,0.1,0.2)$, $LHF_2=(s_7,0.2)$. According to the operational laws, we can obtain $(s_5,0.1,0.2)\oplus(s_7,0.2)=(s_{12},0.28,0.36)$ and $(s_5,0.1,0.2)\otimes(s_7,0.2)=(s_{35},0.72,0.64)$, and the linguistic variables of s_{12} and s_{35} exceed the range of S.

Logical problem: this problem comes from two aspects. On the one hand, for a linguistic term set $S=\{(s_0,\alpha_0),(s_1,\alpha_1),\dots,(s_g,\alpha_g)\}$, the addition operation identifies a new linguistic term set with $2g$ linguistic labels, while the product operation has g^2 linguistic labels. That is to say that we need to use different granularity standards to assess the linguistic labels. On the other hand, the linear operations cannot illustrate the non-linearity of logical thinking.

Proposition 1. *Let* LH_1, LH_2 *and* LH_3 *be any three 2-TLHFSs, then*

(1) $LH_1\oplus_E LH_2=LH_2\oplus_E LH_1$;
(2) $LH_1\otimes_E LH_2=LH_2\otimes_E LH_1$;
(3) $\lambda(LH_1\oplus_E LH_2)=\lambda LH_1\oplus_E\lambda LH_2$, $\lambda>0$;
(4) $(LH_1\otimes_E LH_2)^\lambda=LH_1^\lambda\otimes_E LH_2^\lambda$, $\lambda>0$;
(5) $(\lambda_1+\lambda_2)LH_1=\lambda_1 LH_1\oplus_E\lambda_2 LH_1$, $\lambda_1,\lambda_2>0$;
(6) $LH_1^{\lambda_1+\lambda_2}=LH_1^{\lambda_1}\otimes_E LH_1^{\lambda_2}$, $\lambda_1,\lambda_2>0$;

(7) $(LH_1 \oplus_E LH_2) \oplus_E LH_3 = LH_1 \oplus_E (LH_2 \oplus_E LH_3)$;
(8) $(LH_1 \otimes_E LH_2) \otimes_E LH_3 = LH_1 \otimes_E (LH_2 \otimes_E LH_3)$.

Proof. For the sake of simplicity, let $a_i = ((s_{\theta_i}, \alpha_{\theta_i}), lh(s_{\theta_i}, \alpha_{\theta_i}))$, $\beta_i = \Delta^{-1}(s_{\theta_i}, \alpha_{\theta_i})$, $c_i = lh(s_{\theta_i}, \alpha_{\theta_i})$ $(i = j, i = m)$.

(1) $LH_1 \oplus_E LH_2$

$$= \bigcup_{a_i \in LH_1, a_j \in LH_2} \left\{ \left(\Delta\left(g^2 \cdot \frac{\beta_i + \beta_j}{g^2 + \beta_i \beta_j} \right), \bigcup_{r_i \in c_i, r_j \in c_j} \frac{r_i + r_j}{1 + r_i r_j} \right) \right\}$$
$$= \bigcup_{a_j \in LH_2, a_i \in LH_1} \left\{ \left(\Delta\left(g^2 \cdot \frac{\beta_j + \beta_i}{g^2 + \beta_j \beta_i} \right), \bigcup_{r_i \in c_i, r_j \in c_j} \frac{r_j + r_i}{1 + r_j r_i} \right) \right\} = LH_2 \oplus_E LH_1;$$

(2) $LH_1 \otimes_E LH_2$

$$= \bigcup_{a_i \in LH_1, a_j \in LH_2} \left\{ \left(\Delta\left(g \cdot \frac{\beta_i \beta_j}{g^2 + (g - \beta_i)(g - \beta_j)} \right), \bigcup_{r_i \in c_i, r_j \in c_j} \frac{r_i r_j}{1 + (1 - r_i)(1 - r_j)} \right) \right\}$$
$$= \bigcup_{a_j \in LH_2, a_i \in LH_1} \left\{ \left(\Delta\left(g \cdot \frac{\beta_j \beta_i}{g^2 + (g - \beta_j)(g - \beta_i)} \right), \bigcup_{r_i \in c_i, r_j \in c_j} \frac{r_j r_i}{1 + (1 - r_j)(1 - r_i)} \right) \right\}$$
$$= LH_2 \otimes_E LH_1;$$

(3) $\lambda(LH_1 \oplus_E LH_2)$

$$= \lambda \bigcup_{a_i \in LH_1, a_j \in LH_2} \left\{ \left(\Delta\left(g^2 \frac{\beta_i + \beta_j}{g^2 + \beta_i \beta_j} \right), \bigcup_{r_i \in c_i, r_j \in c_j} \frac{r_i + r_j}{1 + r_i r_j} \right) \right\}$$
$$= \bigcup_{a_i \in LH_1, a_j \in LH_2} \left\{ \left(\Delta\left(g \frac{(g^2 + \beta_i \beta_j + g\beta_i + g\beta_j)^\lambda - (g^2 + \beta_i \beta_j - g\beta_i - g\beta_j)^\lambda}{(g^2 + \beta_i \beta_j + g\beta_i + g\beta_j)^\lambda + (g^2 + \beta_i \beta_j - g\beta_i - g\beta_j)^\lambda} \right), \right.\right.$$
$$\left.\left. \bigcup_{r_i \in c_i, r_j \in c_j} \frac{(1 + r_i r_j + r_i + r_j)^\lambda - (1 + r_i r_j - r_i - r_j)^\lambda}{(1 + r_i r_j + r_i + r_j)^\lambda + (1 + r_i r_j - r_i - r_j)^\lambda} \right) \right\};$$

$\lambda LH_1 \oplus_E \lambda LH_2$

$$= \bigcup_{a_i \in LH_1} \left\{ \left(\Delta\left(g \cdot \frac{(g + \beta_i)^\lambda - (g - \beta_i)^\lambda}{(g + \beta_i)^\lambda + (g - \beta_i)^\lambda} \right), \bigcup_{r_i \in c_i} \frac{(1 + r_i)^\lambda - (1 - r_i)^\lambda}{(1 + r_i)^\lambda + (1 - r_i)^\lambda} \right) \right\}$$
$$\oplus_E \bigcup_{a_j \in LH_2} \left\{ \left(\Delta\left(g \cdot \frac{(g + \beta_j)^\lambda - (g - \beta_j)^\lambda}{(g + \beta_j)^\lambda + (g - \beta_j)^\lambda} \right), \bigcup_{r_j \in c_j} \frac{(1 + r_j)^\lambda - (1 - r_j)^\lambda}{(1 + r_j)^\lambda + (1 - r_j)^\lambda} \right) \right\}$$
$$= \bigcup_{a_i \in LH_1, a_j \in LH_2} \left\{ \left(\Delta\left(g \frac{(g^2 + \beta_i \beta_j + g\beta_i + g\beta_j)^\lambda - (g^2 + \beta_i \beta_j - g\beta_i - g\beta_j)^\lambda}{(g^2 + \beta_i \beta_j + g\beta_i + g\beta_j)^\lambda + (g^2 + \beta_i \beta_j - g\beta_i - g\beta_j)^\lambda} \right), \right.\right.$$
$$\left.\left. \bigcup_{r_i \in c_i, r_j \in c_j} \frac{(1 + r_i r_j + r_i + r_j)^\lambda - (1 + r_i r_j - r_i - r_j)^\lambda}{(1 + r_i r_j + r_i + r_j)^\lambda + (1 + r_i r_j - r_i - r_j)^\lambda} \right) \right\} \quad = \lambda(LH_1 \oplus_E LH_2);$$

(4) $(LH_1 \otimes_E LH_2)^\lambda$

$$= \left(\bigcup_{a_i \in LH_1, a_j \in LH_2} \left\{ \left(\Delta\left(\frac{\beta_i \beta_j}{1 + (1 - \beta_i)(1 - \beta_j)} \beta_i \beta_j \right), \bigcup_{r_i \in c_i, r_j \in c_j} \frac{r_i r_j}{1 + (1 - r_i)(1 - r_j)} \right) \right\} \right)^\lambda$$
$$= \bigcup_{a_i \in LH_1, a_j \in LH_2} \left\{ \left(\Delta\left(\frac{2g(\beta_i \beta_j)^\lambda}{(4g^2 + \beta_i \beta_j - 2g\beta_i - 2g\beta_j)^\lambda + (\beta_i \beta_j)^\lambda} \right), \right.\right.$$
$$\left.\left. \bigcup_{r_i \in c_i, r_j \in c_j} \frac{2(r_i r_j)^\lambda}{(4 - 2r_i - 2r_j + r_i r_j)^\lambda + (r_i r_j)^\lambda} \right) \right\};$$

$LH_1^\lambda \otimes_E LH_2^\lambda$

$$= \bigcup_{a_i \in LH_1} \left\{ \left(\Delta\left(g \cdot \frac{2\beta_i{}^\lambda}{(2g - \beta_i)^\lambda + \beta_i{}^\lambda} \right), \bigcup_{r_i \in c_i} \frac{2r_i{}^\lambda}{(2 - r_i)^\lambda + r_i{}^\lambda} \right) \right\}$$
$$\otimes_E \bigcup_{a_j \in LH_2} \left\{ \left(\Delta\left(g \cdot \frac{2\beta_j{}^\lambda}{(2g - \beta_j)^\lambda + \beta_j{}^\lambda} \right), \bigcup_{r_j \in c_j} \frac{2r_j{}^\lambda}{(2 - r_j)^\lambda + r_j{}^\lambda} \right) \right\}$$
$$= \bigcup_{a_i \in LH_1, a_j \in LH_2} \left\{ \left(\Delta\left(\frac{2g(\beta_i \beta_j)^\lambda}{(4g^2 + \beta_i \beta_j - 2g\beta_i - 2g\beta_j)^\lambda + (\beta_i \beta_j)^\lambda} \right), \right.\right.$$
$$\left.\left. \bigcup_{r_i \in c_i, r_j \in c_j} \frac{2(r_i r_j)^\lambda}{(4 - 2r_i - 2r_j + r_i r_j)^\lambda + (r_i r_j)^\lambda} \right) \right\} = (LH_1 \otimes_E LH_2)^\lambda;$$

(5) $\lambda_1 LH_1 \oplus_E \lambda_2 LH_2$

$$= \bigcup_{a_i \in LH_1} \left\{ \left(\Delta\left(g \cdot \frac{(g+\beta_i)^{\lambda_1} - (g+\beta_i)^{\lambda_1}}{(g+\beta_i)^{\lambda_1} + (g+\beta_i)^{\lambda_1}} \right), \bigcup_{r_i \in c_i} \frac{(1+r_i)^{\lambda_1} - (1+r_i)^{\lambda_1}}{(1+r_i)^{\lambda_1} + (1+r_i)^{\lambda_1}} \right) \right\}$$

$$\oplus_E \bigcup_{a_i \in LH_1} \left\{ \left(\Delta\left(g \cdot \frac{(g+\beta_i)^{\lambda_2} - (g-\beta_i)^{\lambda_2}}{(g+\beta_i)^{\lambda_2} + (g-\beta_i)^{\lambda_2}} \right), \bigcup_{r_i \in c_i} \frac{(1+r_i)^{\lambda_2} - (1-r_i)^{\lambda_2}}{(1+r_i)^{\lambda_2} + (1-r_i)^{\lambda_2}} \right) \right\}$$

$$= \bigcup_{a_i \in LH_1} \left\{ \left(\Delta\left(g \cdot \frac{(g+\beta_i)^{\lambda_1+\lambda_2} - (g-\beta_i)^{\lambda_1+\lambda_2}}{(g+\beta_i)^{\lambda_1+\lambda_2} + (g-\beta_i)^{\lambda_1+\lambda_2}} \right), \right.\right.$$

$$\left.\left. \bigcup_{r_i \in c_i} \frac{(1+r_i)^{\lambda_1+\lambda_2} - (1-r_i)^{\lambda_1+\lambda_2}}{(1+r_i)^{\lambda_1+\lambda_2} + (1-r_i)^{\lambda_1+\lambda_2}} \right) \right\} = (\lambda_1 + \lambda_2) LH_1;$$

(6) $LH_1^{\lambda_1} \otimes_E LH_1^{\lambda_2}$

$$= \bigcup_{a_i \in LH_1} \left\{ \left(\Delta\left(g \cdot \frac{2\beta_i{}^{\lambda_1}}{(2g-\beta_i)^{\lambda_1} + \beta_i{}^{\lambda_1}} \right), \bigcup_{r_i \in c_i} \frac{2r_i{}^{\lambda_1}}{(2-r_i)^{\lambda_1} + r_i{}^{\lambda_1}} \right) \right\}$$

$$\otimes_E \bigcup_{a_i \in LH_1} \left\{ \left(\Delta\left(g \cdot \frac{2\beta_i{}^{\lambda_2}}{(2g-\beta_i)^{\lambda_2} + \beta_i{}^{\lambda_2}} \right), \bigcup_{r_i \in c_i} \frac{2r_i{}^{\lambda_2}}{(2-r_i)^{\lambda_2} + r_i{}^{\lambda_2}} \right) \right\}$$

$$= \bigcup_{a_i \in LH_1} \left\{ \left(\Delta\left(g \cdot \frac{2\beta_i{}^{\lambda_1+\lambda_2}}{(2g-\beta_i)^{\lambda_1+\lambda_2} + \beta_i{}^{\lambda_1+\lambda_2}} \right), \bigcup_{r_i \in c_i} \frac{2r_i{}^{\lambda_1+\lambda_2}}{(2-r_i)^{\lambda_1+\lambda_2} + r_i{}^{\lambda_1+\lambda_2}} \right) \right\}$$

$$= LH_1^{\lambda_1+\lambda_2};$$

(7) $(LH_1 \oplus_E LH_2) \oplus_E LH_3$

$$= \bigcup_{a_i \in LH_1, a_j \in LH_2} \left\{ \left(\Delta\left(g^2 \cdot \frac{\beta_i + \beta_j}{g^2 + \beta_i \beta_j} \right), \bigcup_{r_i \in c_i, r_j \in c_j} \frac{r_i + r_j}{1 + r_i r_j} \right) \right\} \oplus_E LH_3$$

$$= \bigcup_{a_i \in LH_1, a_j \in LH_2, a_m \in LH_3} \left\{ \left(\Delta\left(\frac{g^2(\beta_i + \beta_j + \beta_m) + \beta_i \beta_j \beta_m}{g^2 + \beta_i \beta_j + \beta_j \beta_m + \beta_i \beta_m} \right), \right.\right.$$

$$\left.\left. \bigcup_{r_i \in c_i, r_j \in c_j, r_m \in c_m} \frac{r_i + r_j + r_m + r_i r_j r_m}{1 + r_i r_j + r_m r_i + r_j r_m} \right) \right\};$$

$LH_1 \oplus_E (LH_2 \oplus_E LH_3)$

$$= LH_1 \oplus_E \bigcup_{a_j \in LH_2, a_m \in LH_3} \left\{ \left(\Delta\left(g^2 \cdot \frac{\beta_j + \beta_m}{g^2 + \beta_j \beta_m} \right), \bigcup_{r_j \in c_j, r_m \in c_m} \frac{r_j + r_m}{1 + r_j r_m} \right) \right\}$$

$$= \bigcup_{a_i \in LH_1, a_j \in LH_2, a_m \in LH_3} \left\{ \left(\Delta\left(\frac{g^2(\beta_i + \beta_j + \beta_m) + \beta_i \beta_j \beta_m}{g^2 + \beta_i \beta_j + \beta_j \beta_m + \beta_i \beta_m} \right), \right.\right.$$

$$\left.\left. \bigcup_{r_i \in c_i, r_j \in c_j, r_m \in c_m} \frac{r_i + r_j + r_m + r_i r_j r_m}{1 + r_i r_j + r_m r_i + r_j r_m} \right) \right\} = (LH_1 \oplus_E LH_2) \oplus_E LH_3;$$

(8) $(LH_1 \otimes_E LH_2) \otimes_E LH_3$

$$= \bigcup_{a_i \in LH_1, a_j \in LH_2} \left\{ \left(\Delta(g \cdot \frac{\beta_i \beta_j}{g^2 + (g-\beta_i)(g-\beta_j)}), \bigcup_{r_i \in c_i, r_j \in c_j} \frac{r_i r_j}{1 + (1-r_i)(1-r_j)} \right) \right\}$$

$$\otimes_E LH_3$$

$$= \bigcup_{a_i \in LH_1, a_j \in LH_2, a_m \in LH_3} \left\{ \left(\Delta\left(\frac{\beta_i \beta_j \beta_m}{4g^2 - 2g(\beta_i + \beta_j + \beta_m) + \beta_i \beta_j + \beta_i \beta_m + \beta_j \beta_m} \right), \right.\right.$$

$$\left.\left. \bigcup_{r_i \in c_i, r_j \in c_j, r_m \in c_m} \frac{r_i r_j r_m}{4 - 2(r_i + r_j + r_m) + r_i r_m + r_i r_j + r_j r_m} \right) \right\};$$

$LH_1 \otimes_E (LH_2 \otimes_E LH_3)$

$$= LH_1 \otimes_E \bigcup_{a_j \in LH_2, a_m \in LH_3} \left\{ \left(\Delta\left(g \cdot \frac{\beta_j \beta_m}{g^2 + (g-\beta_j)(g-\beta_m)} \right), \right.\right.$$

$$\left.\left. \bigcup_{r_j \in c_j, r_m \in c_m} \frac{r_j r_m}{1 + (1-r_j)(1-r_m)} \right) \right\}$$

$$= \bigcup_{a_i \in LH_1, a_j \in LH_2, a_m \in LH_3} \left\{ \left(\Delta\left(\frac{\beta_i \beta_j \beta_m}{4g^2 - 2g(\beta_i + \beta_j + \beta_m) + \beta_i \beta_j + \beta_i \beta_m + \beta_j \beta_m} \right), \right.\right.$$

$$\left.\left. \bigcup_{r_i \in c_i, r_j \in c_j, r_m \in c_m} \frac{r_i r_j r_m}{4 - 2(r_i + r_j + r_m) + r_i r_m + r_i r_j + r_j r_m} \right) \right\}$$

$$= (LH_1 \otimes_E LH_2) \otimes_E LH_3. \qquad \square$$

3.3. *Order Relations Between 2-TLHFSs*

Definition 9. For any 2-TLHFS *LH*, the expectation function of *LH* is denoted by $E(LH) = (s_{\theta_i}, \alpha_{\theta_i}) = \Delta^{-1}(e(LH))$, and

$$e(LH) = \frac{1}{|index(LH)|}\left(\sum_{(s_{\theta_i},\alpha_{\theta_i}):(s_{\theta_i},\alpha_{\theta_i})\in index(LH)} \frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{|lh(s_{\theta_i},\alpha_{\theta_i})|}\left(\sum_{r\in lh(s_{\theta_i},\alpha_{\theta_i})} r\right)\right)$$

where $|lh(s_{\theta_i}, \alpha_{\theta_i})|$ is the count of real numbers in $lh(s_{\theta_i}, \alpha_{\theta_i})$, and $|index(LH)|$ is the cardinality of $index(LH) := \{((s_{\theta_i}, \alpha_{\theta_i})|((s_{\theta_i}, \alpha_{\theta_i}), lh(s_{\theta_i}, \alpha_{\theta_i})) \in LH),\ lh(s_{\theta_i}, \alpha_{\theta_i}) \neq \{0\}\}$, with $(s_{\theta_i}, \alpha_{\theta_i}) \in S$.

Definition 10. For any 2-TLHFS *LH*, the variance function of *LH* is denoted by $V(LH) = (s_{\theta_i}, \alpha_{\theta_i}) = \Delta^{-1}(v(LH))$, and

$$\begin{aligned} v(LH) &= \frac{1}{|index(LH)|} \\ &\quad\times\left(\sum_{(s_{\theta_i},\alpha_{\theta_i}):(s_{\theta_i},\alpha_{\theta_i})\in index(LH)} \left(\frac{\Delta^{-1}(s_{\theta_i},\alpha_{\theta_i})}{|lh(s_{\theta_i},\alpha_{\theta_i})|}\left(\sum_{r\in lh(s_{\theta_i},\alpha_{\theta_i})} r\right) - e(LH)\right)^2\right) \end{aligned}$$

where $|lh(s_{\theta_i}, \alpha_{\theta_i})|$ is the count of real numbers in $lh(s_{\theta_i}, \alpha_{\theta_i})$, and $|index(LH)|$ is the cardinality of $index(LH) := \{((s_{\theta_i}, \alpha_{\theta_i})|((s_{\theta_i}, \alpha_{\theta_i}), lh(s_{\theta_i}, \alpha_{\theta_i})) \in LH),\ lh(s_{\theta_i}, \alpha_{\theta_i}) \neq \{0\}\}$, with $(s_{\theta_i}, \alpha_{\theta_i}) \in S$.

Definition 11. The order relationship for any two 2-TLHFSs LH_1 and LH_2 is defined by:

If $E(LH_1) < E(LH_2)$, then $LH_1 < LH_2$.

If $E(LH_1) = E(LH_2)$, then $\begin{cases} V(LH_1) > V(LH_2), & LH_1 < LH_2 \\ V(LH_1) < V(LH_2), & LH_1 > LH_2. \end{cases}$

Example 1. Let LH_1 and LH_2 be two 2-TLHFSs, $LH_1 = \{((s_0, 0.4), 0.2, 0.3), ((s_1, -0.1), 0.7, 0.8, 0.9), ((s_1, 0.1), 0.7)\}$, $LH_2 = \{((s_1, 0.3), 0.5, 0.6, 0.7), ((s_1, -0.3), 0.8)\}$.

$$\begin{aligned} e(LH_1) &= \frac{1}{3}\times\left[\frac{0.4}{2}(0.2+0.3)+\frac{0.9}{3}(0.7+0.8+0.9)+1.1\times 0.7\right]\approx 0.52, \\ v(LH_1) &= \frac{1}{3}\left[\frac{0.4}{2}(0.2+0.3)-0.52\right]^2+\frac{1}{3}\left[\frac{0.9}{3}(0.7+0.8+0.9)-0.52\right]^2 \\ &\quad+\frac{1}{3}(1.1\times 0.7-0.52)^2\}\approx 0.093. \end{aligned}$$

Then $E(LH_1) = (s_1, -0.48)$, $V(LH_1) = (s_0, 0.093)$.

$$e(LH_2) = \frac{1}{2}\left[\frac{1.3}{3}(0.5+0.6+0.7)+0.7\times 0.8\right]\approx 0.52,$$

$$(LH_2) = \frac{1}{2}\left[\left[\frac{1.3}{3}(0.5+0.6+0.7)-0.52\right]^2 + (0.7\times 0.8-0.52)^2\right] \approx 0.0017.$$

Then $E(LH_2) = (s_1, -0.48)$, $V(LH_2) = (s_0, 0.0017)$.

Thus, $LH_1 < LH_2$ for $E(LH_1) = E(LH_2)$, $V(LH_1) > V(LH_2)$.

4. The 2-Tuple Linguistic Hesitant Fuzzy Einstein Aggregation Operators

This section introduces 2-tuple linguistic hesitant fuzzy Einstein weighted averaging (2-TLHFEWA) operator and 2-tuple linguistic hesitant fuzzy Einstein weighted geometric (2-TLHFEWG) operator. In addition, some properties are studied.

4.1. *Order Relations Between 2-TLHFSs*

Definition 12. Let LH_i $(i = 1, 2, \dots, n)$ be a collection of n 2-TLHFSs, A 2-tuple linguistic hesitant fuzzy Einstein weighted averaging (2-TLHFEWA) operator is a function $LH^n \to LH$, defined by

$$\begin{aligned}&\text{2-TLHFEWA}(LH_1, LH_2, LH_3, \dots, LH_n)\\&= w_1 LH_1 \oplus_E w_2 LH_2 \oplus_E \cdots \oplus_E w_n LH_n = \bigoplus_{j=1}^{n}{}_E\, w_j LH_j\end{aligned}$$

where $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of LH_j $(j = 1, 2, \dots, n)$, satisfying $w_j \in [0, 1]$ $(j = 1, 2, \dots, n)$ and $\sum_{j=1}^{n} w_j = 1$.

Theorem 2. *Let LH_i $(i = 1, 2, \dots, n)$ be a collection of n 2-TLHFSs, then the aggregated value by 2-TLHFEWA operator is also a 2-TLHFS, furthermore,*

$$\begin{aligned}&\textit{2-TLHFEWA}(LH_1, LH_2, LH_3, \dots, LH_n)\\&= \bigcup_{((s_{\theta_1},\alpha_{\theta_1}),lh(s_{\theta_1},\alpha_{\theta_1}))\in LH_1,\dots,((s_{\theta_n},\alpha_{\theta_n}),lh(s_{\theta_n},\alpha_{\theta_n}))\in LH_n}\\&\quad\left\{\left(\Delta\left(g\cdot\frac{\prod_{j=1}^{n}(g+\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j}))^{w_j}-\prod_{j=1}^{n}(g-\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j}))^{w_j}}{\prod_{j=1}^{n}(g+\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j}))^{w_j}+\prod_{j=1}^{n}(g-\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j}))^{w_j}}\right),\right.\right.\\&\quad\left.\left.\bigcup_{r_1\in lh(s_{\theta_1},\alpha_{\theta_1}),\dots,r_n\in lh(s_{\theta_n},\alpha_{\theta_n})}\frac{\prod_{j=1}^{n}(1+r_j)^{w_j}-\prod_{j=1}^{n}(1-r_j)^{w_j}}{\prod_{j=1}^{n}(1+r_j)^{w_j}+\prod_{j=1}^{n}(1-r_j)^{w_j}}\right)\right\}\end{aligned}\tag{6}$$

where g is the upper limit of the 2-tuple linguistic term set.

Proof. Eq. (6) can be proven by using a mathematical induction on n.

(1) For $n = 1$,

$$\begin{aligned}
&\text{2-TLHFEWA}\ (LH_1, LH_2, LH_3, \ldots, LH_n)\\
&= \text{2-TLHFEWA}(LH_1) = w_1 LH_1\\
&= \left\{\Delta\left(g \cdot \frac{(g + \Delta^{-1}(s_{\theta_1}, \alpha_{\theta_1}))^{w_1} - (g - \Delta^{-1}(s_{\theta_1}, \alpha_{\theta_1}))^{w_1}}{(g + \Delta^{-1}(s_{\theta_1}, \alpha_{\theta_1}))^{w_1} + (g - \Delta^{-1}(s_{\theta_1}, \alpha_{\theta_1}))^{w_1}}\right),\right.\\
&\left.\frac{(1 + r_1)^{w_1} - (1 - r_1)^{w_1}}{(1 + r_1)^{w_1} + (1 - r_1)^{w_1}}\right\}.
\end{aligned}$$

Therefore, the result of Eq. (6) is sure.

(2) Suppose Eq. (6) holds for $n = k$, that is

$$\begin{aligned}
&\text{2-TLHFEWA}(LH_1, LH_2, LH_3, \ldots, LH_k)\\
&= \bigcup_{((s_{\theta_1}, \alpha_{\theta_1}), lh(s_{\theta_1}, \alpha_{\theta_1})) \in LH_1, \ldots, ((s_{\theta_k}, \alpha_{\theta_k}), lh(s_{\theta_k}, \alpha_{\theta_k})) \in LH_k}\\
&\left\{\left(\Delta\left(g \cdot \frac{\prod_{j=1}^{k} (g + \Delta^{-1}(s_{\theta_j}, \alpha_{\theta_j}))^{w_j} - \prod_{j=1}^{k} (g - \Delta^{-1}(s_{\theta_j}, \alpha_{\theta_j}))^{w_j}}{\prod_{j=1}^{k} (g + \Delta^{-1}(s_{\theta_j}, \alpha_{\theta_j}))^{w_j} + \prod_{j=1}^{k} (g - \Delta^{-1}(s_{\theta_j}, \alpha_{\theta_j}))^{w_j}}\right),\right.\right.\\
&\left.\left.\bigcup_{r_1 \in lh(s_{\theta_1}, \alpha_{\theta_1}), \ldots, r_k \in lh(s_{\theta_k}, \alpha_{\theta_k})} \frac{\prod_{j=1}^{k} (1 + r_j)^{w_j} - \prod_{j=1}^{k} (1 - r_j)^{w_j}}{\prod_{j=1}^{k} (1 + r_j)^{w_j} + \prod_{j=1}^{k} (1 - r_j)^{w_j}}\right)\right\}.
\end{aligned}$$

When $n = k + 1$, by Definition 12. We have

$$\begin{aligned}
&\text{2-TLHFEWA}(LH_1, LH_2, \ldots, LH_{k+1})\\
&= \text{2-TLHFEWA}(LH_1, LH_2, \ldots, LH_k) \oplus_E w_{k+1} LH_{k+1}\\
&= \bigcup_{((s_{\theta_1}, \alpha_{\theta_1}), lh(s_{\theta_1}, \alpha_{\theta_1})) \in LH_1, \ldots, ((s_{\theta_k}, \alpha_{\theta_k}), lh(s_{\theta_k}, \alpha_{\theta_k})) \in LH_k}\\
&\left\{\left(\Delta\left(g \cdot \frac{\prod_{j=1}^{k} (g + \Delta^{-1}(s_{\theta_j}, \alpha_{\theta_j}))^{w_j} - \prod_{j=1}^{k} (g - \Delta^{-1}(s_{\theta_j}, \alpha_{\theta_j}))^{w_j}}{\prod_{j=1}^{k} (g + \Delta^{-1}(s_{\theta_j}, \alpha_{\theta_j}))^{w_j} + \prod_{j=1}^{k} (g - \Delta^{-1}(s_{\theta_j}, \alpha_{\theta_j}))^{w_j}}\right),\right.\right.\\
&\left.\left.\bigcup_{r_1 \in lh(s_{\theta_1}, \alpha_{\theta_1}), \ldots, r_k \in lh(s_{\theta_k}, \alpha_{\theta_k})} \frac{\prod_{j=1}^{k} (1 + r_j)^{w_j} - \prod_{j=1}^{k} (1 - r_j)^{w_j}}{\prod_{j=1}^{k} (1 + r_j)^{w_j} + \prod_{j=1}^{k} (1 - r_j)^{w_j}}\right)\right\}\\
&\oplus_E \left\{\left(\Delta\left(g \cdot \frac{(g + \Delta^{-1}(s_{\theta_{k+1}}, \alpha_{\theta_{k+1}}))^{w_{k+1}} - (g - \Delta^{-1}(s_{\theta_{k+1}}, \alpha_{\theta_{k+1}}))^{w_{k+1}}}{(g + \Delta^{-1}(s_{\theta_{k+1}}, \alpha_{\theta_{k+1}}))^{w_{k+1}} + (g - \Delta^{-1}(s_{\theta_{k+1}}, \alpha_{\theta_{k+1}}))^{w_{k+1}}}\right),\right.\right.\\
&\left.\left.\frac{(1 + r_{k+1})^{w_{k+1}} - (1 - r_{k+1})^{w_{k+1}}}{(1 + r_{k+1})^{w_{k+1}} + (1 - r_{k+1})^{w_{k+1}}}\right)\right\}
\end{aligned}$$

$$= \bigcup_{((s_{\theta_1},\alpha_{\theta_1}),lh(s_{\theta_1},\alpha_{\theta_1}))\in LH_1,\dots,((s_{\theta_{k+1}},\alpha_{\theta_{k+1}}),lh(s_{\theta_{k+1}},\alpha_{\theta_{k+1}}))\in LH_{k+1}}$$

$$\left\{\left(\Delta\left(g \cdot \frac{\prod_{j=1}^{k+1}(g+\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j}))^{w_j} - \prod_{j=1}^{k+1}(g-\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j}))^{w_j}}{\prod_{j=1}^{k+1}(g+\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j}))^{w_j} + \prod_{j=1}^{k+1}(g-\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j}))^{w_j}}\right),\right.\right.$$

$$\left.\left.\bigcup_{r_1\in lh(s_{\theta_1},\alpha_{\theta_1}),\dots,r_{k+1}\in lh(s_{\theta_{k+1}},\alpha_{\theta_{k+1}})} \frac{\prod_{j=1}^{k+1}(1+r_j)^{w_j} - \prod_{j=1}^{k+1}(1-r_j)^{w_j}}{\prod_{j=1}^{k+1}(1+r_j)^{w_j} + \prod_{j=1}^{k+1}(1-r_j)^{w_j}}\right)\right\}$$

i.e. Eq. (6) holds for $n = k + 1$. Thus, Eq. (6) holds for all n. □

Proposition 2. *Let $LH = \{LH_1, LH_2, LH_3, \dots, LH_n\}$ be a set of n 2-TLHFSs. If all LH_j $(j = 1, 2, \dots, n)$ are equal, i.e. $LH_j = LH_b = ((s_{\theta_b}, \alpha_{\theta_b}), lh(s_{\theta_b}, \alpha_{\theta_b}))$, for all $j = (1, 2, \dots, n)$, then*

$$\textit{2-TLHFEWA}(LH_1, LH_2, LH_3, \dots, LH_n) = LH_b.$$

Proof.

$$\text{2-TLHFEWA}(LH_1, LH_2, \dots, LH_n) = \text{2-TLHFEWA}(LH_b, \dots, LH_b)$$

$$= \bigcup_{((s_{\theta_b},\alpha_{\theta_b}),lh(s_{\theta_b},\alpha_{\theta_b}))\in LH_b}$$

$$\left\{\left(\Delta\left(g \cdot \frac{(g+\Delta^{-1}(s_{\theta_b},\alpha_{\theta_b}))^{\sum_{j=1}^n w_j} - (g-\Delta^{-1}(s_{\theta_b},\alpha_{\theta_b}))^{\sum_{j=1}^n w_j}}{(g+\Delta^{-1}(s_{\theta_b},\alpha_{\theta_b}))^{\sum_{j=1}^n w_j} + (g-\Delta^{-1}(s_{\theta_b},\alpha_{\theta_b}))^{\sum_{j=1}^n w_j}}\right),\right.\right.$$

$$\left.\left.\bigcup_{r_b\in lh(s_{\theta_b},\alpha_{\theta_b})} \frac{(1+r_b)^{\sum_{j=1}^n w_j} - (1-r_b)^{\sum_{j=1}^n w_j}}{(1+r_b)^{\sum_{j=1}^n w_j} + (1-r_b)^{\sum_{j=1}^n w_j}}\right)\right\}$$

$$= ((s_{\theta_b}, \alpha_{\theta_b}), r_b) = LH_b. \qquad \square$$

Proposition 3. *Let $LH = \{LH_1, LH_2, LH_3, \dots, LH_n\}$ be a set of n 2-TLHFSs. If $\gamma > 0$, then*

$$\text{2-TLHFEWA}(\gamma LH_1, \gamma LH_2, \gamma LH_3, \dots, \gamma LH_n)$$
$$= \gamma\,\text{2-TLHFEWA}(LH_1, LH_2, LH_3, \dots, LH_n).$$

Proof. For the sake of simplicity, let $\alpha_j = \Delta^{-1}(s_{\theta_j}, \beta_{\theta_j})$,

$$\text{2-TLHFEWA}(\gamma LH_1, \gamma LH_2, \gamma LH_3, \dots, \gamma LH_n)$$

$$= \bigcup_{((s_{\theta_1},\alpha_{\theta_1}),lh(s_{\theta_1},\alpha_{\theta_1}))\in LH_1,\dots,((s_{\theta_n},\alpha_{\theta_n}),lh(s_{\theta_n},\alpha_{\theta_n}))\in LH_n}$$

$$\left\{\left(\Delta\left(g\cdot\frac{\prod_{j=1}^{n}(g+g\frac{(g+\alpha_j)^{\gamma}-(g-\alpha_j)^{\gamma}}{(g+\alpha_j)^{\gamma}+(g-\alpha_j)^{\gamma}})^{w_j}-\prod_{j=1}^{n}(g-g\frac{(g+\alpha_j)^{\gamma}-(g-\alpha_j)^{\gamma}}{(g+\alpha_j)^{\gamma}+(g-\alpha_j)^{\gamma}})^{w_j}}{\prod_{j=1}^{n}(g+g\frac{(g+\alpha_j)^{\gamma}-(g-\alpha_j)^{\gamma}}{(g+\alpha_j)^{\gamma}+(g-\alpha_j)^{\gamma}})^{w_j}+\prod_{j=1}^{n}(g-g\frac{(g+\alpha_j)^{\gamma}-(g-\alpha_j)^{\gamma}}{(g+\alpha_j)^{\gamma}+(g-\alpha_j)^{\gamma}})^{w_j}}\right),\right.\right.$$

$$\left.\left.\bigcup_{r_n\in lh(s_{\theta_n},\alpha_{\theta_n})}\frac{\prod_{j=1}^{n}(1+\frac{(1+r_j)^{\gamma}-(1-r_j)^{\gamma}}{(1+r_j)^{\gamma}+(1-r_j)^{\gamma}})^{w_j}-\prod_{j=1}^{n}(1-\frac{(1+r_j)^{\gamma}-(1-r_j)^{\gamma}}{(1+r_j)^{\gamma}+(1-r_j)^{\gamma}})^{w_j}}{\prod_{j=1}^{n}(1+\frac{(1+r_j)^{\gamma}-(1-r_j)^{\gamma}}{(1+r_j)^{\gamma}+(1-r_j)^{\gamma}})^{w_j}+\prod_{j=1}^{n}(1-\frac{(1+r_j)^{\gamma}-(1-r_j)^{\gamma}}{(1+r_j)^{\gamma}+(1-r_j)^{\gamma}})^{w_j}}\right)\right\}$$

$$=\bigcup_{((s_{\theta_1},\alpha_{\theta_1}),lh(s_{\theta_1},\alpha_{\theta_1}))\in LH_1,\ldots,((s_{\theta_n},\alpha_{\theta_n}),lh(s_{\theta_n},\alpha_{\theta_n}))\in LH_n}$$

$$\left\{\left(\Delta\left(g\cdot\frac{\prod_{j=1}^{n}(g+\alpha_j)^{\gamma w_j}-\prod_{j=1}^{n}(g-\alpha_j)^{\gamma w_j}}{\prod_{j=1}^{n}(g+\alpha_j)^{\gamma w_j}+\prod_{j=1}^{n}(g-\alpha_j)^{\gamma w_j}}\right),\right.\right.$$

$$\left.\left.\bigcup_{r_n\in lh(s_{\theta_n},\alpha_{\theta_n})}\frac{\prod_{j=1}^{n}(1+r_j)^{\gamma w_j}-\prod_{j=1}^{n}(1-r_j)^{\gamma w_j}}{\prod_{j=1}^{n}(1+r_j)^{\gamma w_j}+\prod_{j=1}^{n}(1-r_j)^{\gamma w_j}}\right)\right\}.$$

$$\gamma\,2\text{-TLHFEWA}(LH_1,LH_2,LH_3,\ldots,LH_n)$$

$$=\bigcup_{((s_{\theta_1},\alpha_{\theta_1}),lh(s_{\theta_1},\alpha_{\theta_1}))\in LH_1,\ldots,((s_{\theta_n},\alpha_{\theta_n}),lh(s_{\theta_n},\alpha_{\theta_n}))\in LH_n}$$

$$\left\{\left(g\cdot\frac{(g+g\cdot\frac{\prod_{j=1}^{n}(g+\alpha_j)^{w_j}-\prod_{j=1}^{n}(g-\alpha_j)^{w_j}}{\prod_{j=1}^{n}(g+\alpha_j)^{w_j}+\prod_{j=1}^{n}(g-\alpha_j)^{w_j}})^{\gamma}-(1-g\cdot\frac{\prod_{j=1}^{n}(g+\alpha_j)^{w_j}-\prod_{j=1}^{n}(g-\alpha_j)^{w_j}}{\prod_{j=1}^{n}(g+\alpha_j)^{w_j}+\prod_{j=1}^{n}(g-\alpha_j)^{w_j}})^{\gamma}}{(g+g\cdot\frac{\prod_{j=1}^{n}(g+\alpha_j)^{w_j}-\prod_{j=1}^{n}(g-\alpha_j)^{w_j}}{\prod_{j=1}^{n}(g+\alpha_j)^{w_j}+\prod_{j=1}^{n}(g-\alpha_j)^{w_j}})^{\gamma}+(1-g\cdot\frac{\prod_{j=1}^{n}(g+\alpha_j)^{w_j}-\prod_{j=1}^{n}(g-\alpha_j)^{w_j}}{\prod_{j=1}^{n}(g+\alpha_j)^{w_j}+\prod_{j=1}^{n}(g-\alpha_j)^{w_j}})^{\gamma}},\right.\right.$$

$$\left.\left.\bigcup_{r_n\in lh(s_{\theta_n},\alpha_{\theta_n})}\frac{(1+\frac{\prod_{j=1}^{n}(1+r_j)^{w_j}-\prod_{j=1}^{n}(1-r_j)^{w_j}}{\prod_{j=1}^{n}(1+r_j)^{w_j}+\prod_{j=1}^{n}(1-r_j)^{w_j}})^{\gamma}-(1-\frac{\prod_{j=1}^{n}(1+r_j)^{w_j}-\prod_{j=1}^{n}(1-r_j)^{w_j}}{\prod_{j=1}^{n}(1+r_j)^{w_j}+\prod_{j=1}^{n}(1-r_j)^{w_j}})^{\gamma}}{(1+\frac{\prod_{j=1}^{n}(1+r_j)^{w_j}-\prod_{j=1}^{n}(1-r_j)^{w_j}}{\prod_{j=1}^{n}(1+r_j)^{w_j}+\prod_{j=1}^{n}(1-r_j)^{w_j}})^{\gamma}+(1-\frac{\prod_{j=1}^{n}(1+r_j)^{w_j}-\prod_{j=1}^{n}(1-r_j)^{w_j}}{\prod_{j=1}^{n}(1+r_j)^{w_j}+\prod_{j=1}^{n}(1-r_j)^{w_j}})^{\gamma}}\right)\right\}$$

$$=\bigcup_{((s_{\theta_1},\alpha_{\theta_1}),lh(s_{\theta_1},\alpha_{\theta_1}))\in LH_1,\ldots,((s_{\theta_n},\alpha_{\theta_n}),lh(s_{\theta_n},\alpha_{\theta_n}))\in LH_n}$$

$$\left\{(\Delta(g\cdot\frac{\prod_{j=1}^{n}(g+\alpha_j)^{\gamma w_j}-\prod_{j=1}^{n}(g-\alpha_j)^{\gamma w_j}}{\prod_{j=1}^{n}(g+\alpha_j)^{\gamma w_j}+\prod_{j=1}^{n}(g-\alpha_j)^{\gamma w_j}}),\right.$$

$$\left.\bigcup_{r_n\in lh(s_{\theta_n},\alpha_{\theta_n})}\frac{\prod_{j=1}^{n}(1+r_j)^{\gamma w_j}-\prod_{j=1}^{n}(1-r_j)^{\gamma w_j}}{\prod_{j=1}^{n}(1+r_j)^{\gamma w_j}+\prod_{j=1}^{n}(1-r_j)^{\gamma w_j}})\right\}.$$

Thus

$$2\text{-TLHFEWA}(\gamma LH_1,\gamma LH_2,\gamma LH_3,\ldots,\gamma LH_n)$$
$$=\gamma\,2\text{-TLHFEWA}(LH_1,LH_2,LH_3,\ldots,LH_n).\qquad\square$$

Proposition 4. *Let $LH = \{LH_1, LH_2, LH_3, \ldots, LH_n\}$ and $LH' = \{LH'_1, LH'_2, LH'_3, \ldots, LH'_n\}$ be a set of n 2-TLHFSs, then*

$$\begin{aligned}&\text{2-TLHFEWA}\big(LH_1 \oplus_E LH'_1, LH_2 \oplus_E LH'_2, \ldots, LH_n \oplus_E LH'_n\big)\\ &\quad = \text{2-TLHFEWA}(LH_1, LH_2, \ldots, LH_n) \oplus_E \text{2-TLHFEWA}\big(LH'_1, LH'_2, \ldots, LH'_n\big).\end{aligned}$$

Proof. For the sake of simplicity, let $\alpha_j = \Delta^{-1}(s_{\theta_j}, \beta_{\theta_j})$, $\alpha_j{}' = \Delta^{-1}(s_{\theta_j}, \beta_{\theta_j})'$,

$$\begin{aligned}&\text{2-TLHFEWA}(LH_1 \oplus_E LH'_1, LH_2 \oplus_E LH'_2, \ldots, LH_n \oplus_E LH'_n)\\ &\quad = \bigcup_{((s_{\theta_1},\alpha_{\theta_1}),lh(s_{\theta_1},\alpha_{\theta_1}))\in LH_1,\ldots,((s_{\theta_n},\alpha_{\theta_n})',lh'(s_{\theta_n},\alpha_{\theta_n}))\in LH'_n}\\ &\qquad \left\{\left(\Delta\left(g \cdot \frac{\prod_{j=1}^n (g + g^2 \frac{\alpha_j+\alpha_j{}'}{g^2+\alpha_j\alpha_j{}'})^{w_j} - \prod_{j=1}^n (g - g^2 \frac{\alpha_j+\alpha_j{}'}{g^2+\alpha_j\alpha_j{}'})^{w_j}}{\prod_{j=1}^n (g + g^2 \frac{\alpha_j+\alpha_j{}'}{g^2+\alpha_j\alpha_j{}'})^{w_j} + \prod_{j=1}^n (g - g^2 \frac{\alpha_j+\alpha_j{}'}{g^2+\alpha_j\alpha_j{}'})^{w_j}}\right),\right.\right.\\ &\qquad \left.\left.\bigcup_{r_1\in lh(s_{\theta_1},\alpha_{\theta_1}),\ldots,r_n{}'\in lh'(s_{\theta_n},\alpha_{\theta_n})} \frac{\prod_{j=1}^n (1 + \frac{r_j+r'_j}{1+r_jr'_j})^{w_j} - \prod_{j=1}^n (1 - \frac{r_j+r'_j}{1+r_jr'_j})^{w_j}}{\prod_{j=1}^n (1 + \frac{r_j+r'_j}{1+r_jr'_j})^{w_j} + \prod_{j=1}^n (1 - \frac{r_j+r'_j}{1+r_jr'_j})^{w_j}}\right)\right\}\\ &\quad = \bigcup_{((s_{\theta_1},\alpha_{\theta_1}),lh(s_{\theta_1},\alpha_{\theta_1}))\in LH_1,\ldots,((s_{\theta_n},\alpha_{\theta_n})',lh'(s_{\theta_n},\alpha_{\theta_n}))\in LH'_n}\\ &\qquad \left\{\left(\Delta\left(g \cdot \frac{\prod_{j=1}^n (1 + \alpha_j + \alpha_j{}' + \alpha_j\alpha_j{}')^{w_j} - \prod_{j=1}^n (1 - \alpha_j - \alpha_j{}' + \alpha_j\alpha_j{}')^{w_j}}{\prod_{j=1}^n (1 + \alpha_j + \alpha_j{}' + \alpha_j\alpha_j{}')^{w_j} + \prod_{j=1}^n (1 - \alpha_j - \alpha_j{}' + \alpha_j\alpha_j{}')^{w_j}}\right),\right.\right.\\ &\qquad \left.\left.\bigcup_{r_n{}'\in lh'(s_{\theta_n},\alpha_{\theta_n})} \frac{\prod_{j=1}^n (1 + r_j + r'_j + r_jr'_j)^{w_j} - \prod_{j=1}^n (1 - r_j - r'_j + r_jr'_j)^{w_j}}{\prod_{j=1}^n (1 + r_j + r'_j + r_jr'_j)^{w_j} + \prod_{j=1}^n (1 - r_j - r'_j + r_jr'_j)^{w_j}}\right)\right\},\end{aligned}$$

$$\begin{aligned}&\text{2-TLHFEWA}(LH_1, LH_2, \ldots, LH_n) \oplus_E \text{2-TLHFEWA}\big(LH'_1, LH'_2, \ldots, LH'_n\big)\\ &\quad = \bigcup_{((s_{\theta_1},\alpha_{\theta_1}),lh(s_{\theta_1},\alpha_{\theta_1}))\in LH_1,\ldots,((s_{\theta_n},\alpha_{\theta_n})',lh'(s_{\theta_n},\alpha_{\theta_n}))\in LH'_n}\\ &\qquad \left\{\left(\Delta\left(g^2 \cdot \frac{g \cdot \frac{\prod_{j=1}^n (g+\alpha_j)^{w_j} - \prod_{j=1}^n (g-\alpha_j)^{w_j}}{\prod_{j=1}^n (g+\alpha_j)^{w_j} + \prod_{j=1}^n (g-\alpha_j)^{w_j}} + g \cdot \frac{\prod_{j=1}^n (g+\alpha_j{}')^{w_j} - \prod_{j=1}^n (g-\alpha_j{}')^{w_j}}{\prod_{j=1}^n (g+\alpha_j{}')^{w_j} + \prod_{j=1}^n (g-\alpha_j{}')^{w_j}}}{g^2 + g \cdot \frac{\prod_{j=1}^n (g+\alpha_j)^{w_j} - \prod_{j=1}^n (g-\alpha_j)^{w_j}}{\prod_{j=1}^n (g+\alpha_j)^{w_j} + \prod_{j=1}^n (g-\alpha_j)^{w_j}} g \cdot \frac{\prod_{j=1}^n (g+\alpha_j{}')^{w_j} - \prod_{j=1}^n (g-\alpha_j{}')^{w_j}}{\prod_{j=1}^n (g+\alpha_j)')^{w_j} + \prod_{j=1}^n (g-\alpha_j{}')^{w_j}}}\right),\right.\right.\\ &\qquad \left.\left.\bigcup_{r_n{}'\in lh'(s_{\theta_n},\alpha_{\theta_n})} \frac{\frac{\prod_{j=1}^n (1+r_j)^{w_j} - \prod_{j=1}^n (1-r_j)^{w_j}}{\prod_{j=1}^n (1+r_j)^{w_j} + \prod_{j=1}^n (1-r_j)^{w_j}} + \frac{\prod_{j=1}^n (1+r'_j)^{w_j} - \prod_{j=1}^n (1-r'_j)^{w_j}}{\prod_{j=1}^n (1+r'_j)^{w_j} + \prod_{j=1}^n (1-r'_j)^{w_j}}}{1 + \frac{\prod_{j=1}^n (1+r_j)^{w_j} - \prod_{j=1}^n (1-r_j)^{w_j}}{\prod_{j=1}^n (1+r_j)^{w_j} + \prod_{j=1}^n (1-r_j)^{w_j}} \frac{\prod_{j=1}^n (1+r'_j)^{w_j} - \prod_{j=1}^n (1-r'_j)^{w_j}}{\prod_{j=1}^n (1+r'_j)^{w_j} + \prod_{j=1}^n (1-r'_j)^{w_j}}}\right)\right\}\\ &\quad = \bigcup_{((s_{\theta_1},\alpha_{\theta_1}),lh(s_{\theta_1},\alpha_{\theta_1}))\in LH_1,\ldots,((s_{\theta_n},\alpha_{\theta_n})',lh'(s_{\theta_n},\alpha_{\theta_n}))\in LH'_n}\end{aligned}$$

$$\left\{\left(\Delta\left(g\cdot\frac{\prod_{j=1}^{n}(1+\alpha_j+\alpha_j{}'+\alpha_j\alpha_j{}')^{w_j}-\prod_{j=1}^{n}(1-\alpha_j-\alpha_j{}'+\alpha_j\alpha_j{}')^{w_j}}{\prod_{j=1}^{n}(1+\alpha_j+\alpha_j{}'+\alpha_j\alpha_j{}')^{w_j}+\prod_{j=1}^{n}(1-\alpha_j-\alpha_j{}'+\alpha_j\alpha_j{}')^{w_j}}\right),\right.\right.$$
$$\left.\left.\bigcup_{r_n{}'\in lh'(s_{\theta_n},\alpha_{\theta_n})}\frac{\prod_{j=1}^{n}(1+r_j+r'_j+r_jr'_j)^{w_j}-\prod_{j=1}^{n}(1-r_j-r'_j+r_jr'_j)^{w_j}}{\prod_{j=1}^{n}(1+r_j+r'_j+r_jr'_j)^{w_j}+\prod_{j=1}^{n}(1-r_j-r'_j+r_jr'_j)^{w_j}}\right)\right\}.$$

Thus we have that

$$\begin{aligned}&\text{2-TLHFEWA}\big(LH_1\oplus_E LH'_1, LH_2\oplus_E LH'_2,\ldots,LH_n\oplus_E LH'_n\big)\\&\quad=\text{2-TLHFEWA}(LH_1,LH_2,\ldots,LH_n)\oplus_E \text{2-TLHFEWA}\big(LH'_1,LH'_2,\ldots,LH'_n\big).\quad\square\end{aligned}$$

4.2. *2-Tuple Linguistic Hesitant Fuzzy Einstein Weighted Geometric Operator*

Definition 13. Let LH_i $(i=1,2,\ldots,n)$ be a collection of 2-TLHFSs. A 2-tuple linguistic hesitant fuzzy Einstein weighted geometric (2-TLHFEWG) operator is a function $LH^n\to LH$, defined by

$$\begin{aligned}&\text{2-TLHFEWG}(LH_1,LH_2,LH_3,\ldots,LH_n)\\&\quad=LH_1^{w_1}\otimes_E LH_2^{w_2}\otimes_E\cdots\otimes_E LH_n{}^{w_n}=\bigotimes_{j=1}^{n}{}_E\,LH_j{}^{w_j}\end{aligned}$$

where $w=(w_1,w_2,\ldots,w_n)^T$ is the weight vector of LH_j $(j=1,2,\ldots,n)$, satisfying $w_j\in[0,1]$ $(j=1,2,\ldots,n)$ and $\sum_{j=1}^{n}w_j=1$.

The 2-TLHFEWG operator has some similar properties with the 2-TLHFEWA operator, which is given as follows.

Theorem 3. *Let LH_i $(i=1,2,\ldots,n)$ be a collection of n 2-TLHFSs, then the aggregated value by 2-TLHFEWG operator is also a 2-TLHFS, furthermore,*

$$\begin{aligned}&\text{2-TLHFEWG}(LH_1,LH_2,LH_3,\ldots,LH_n)\\&\quad=\bigcup_{((s_{\theta_1},\alpha_{\theta_1}),lh(s_{\theta_1},\alpha_{\theta_1}))\in LH_1,\ldots,((s_{\theta_n},\alpha_{\theta_n}),lh(s_{\theta_n},\alpha_{\theta_n}))\in LH_n}\\&\quad\left\{\left(\Delta\left(g\cdot\frac{2\prod_{j=1}^{n}(\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j}))^{w_j}}{\prod_{j=1}^{n}(2g-\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j}))^{w_j}+\prod_{j=1}^{n}(\Delta^{-1}(s_{\theta_j},\alpha_{\theta_j}))^{w_j}}\right),\right.\right.\\&\quad\left.\left.\bigcup_{r_1\in lh(s_{\theta_1},\alpha_{\theta_1}),\ldots,r_n\in lh(s_{\theta_n},\alpha_{\theta_n})}\frac{2\prod_{j=1}^{n}(r_j)^{w_j}}{\prod_{j=1}^{n}(2-r_j)^{w_j}+\prod_{j=1}^{n}(r_j)^{w_j}}\right)\right\}\end{aligned}\tag{7}$$

where g is the upper limit of the 2-tuple linguistic term set.

Proposition 5. *Let* $LH = \{LH_1, LH_2, LH_3, \dots, LH_n\}$ *be a set of* n *2-TLHFSs. If all* LH_j $(j = 1, 2, \dots, n)$ *are equal, i.e.* $LH_j = LH_b = ((s_{\theta_b}, \alpha_{\theta_b}), lh(s_{\theta_b}, \alpha_{\theta_b}))$*, for all* $j = (1, 2, \dots, n)$*, then*

$$\text{2-TLHFEWG}(LH_1, LH_2, LH_3, \dots, LH_n) = LH_b.$$

Proposition 6. *Let* $LH = \{LH_1, LH_2, LH_3, \dots, LH_n\}$ *be a set of* n *2-TLHFSs. If* $\gamma > 0$*, then*

$$\text{2-TLHFEWG}\big(LH_1^{\gamma}, LH_2^{\gamma}, \dots, LH_n^{\gamma}\big) = \text{2-TLHFEWG}(LH_1, LH_2, LH_3, \dots, LH_n)^{\gamma}.$$

Proposition 7. *Let* $LH = \{LH_1, LH_2, LH_3, \dots, LH_n\}$ *and* $LH' = \{LH'_1, LH'_2, LH'_3, \dots, LH'_n\}$ *be a set of* n *2-TLHFSs, then*

$$\begin{aligned}&\text{2-TLHFEWA}\big(LH_1 \otimes_E LH'_1, LH_2 \otimes_E LH'_2, \dots, LH_n \otimes_E LH'_n\big)\\&\quad = \text{2-TLHFEWA}(LH_1, LH_2, \dots, LH_n) \otimes_E \text{2-TLHFEWA}\big(LH'_1, LH'_2, \dots, LH'_n\big).\end{aligned}$$

5. An Approach to 2-Tuple Linguistic Hesitant Fuzzy Multi-Attribute Decision Making

In this section, the 2-TLHFEWA and 2-TLHFEWG operators are used to develop an approach to multi-attribute decision making with qualitative attributes, and a new model by similarity to ideal solution for optimal weight vectors is developed. The multi-attribute decision making approach can deal with the situation where there are complex linguistic assessment and the unknown weight information.

For a multi-attribute decision-making problem with 2-tuple linguistic hesitant fuzzy information, there are m alternatives $A = \{A_1, A_2, \dots, A_m\}$ and n attributes $C = \{C_1, C_2, \dots, C_n\}$. Assume that the experts give their personal preferences for alternatives $A_i \in A$ $(i = 1, 2, \dots, m)$ with respect to each attribute $C_j \in C$ $(j = 1, 2, \dots, n)$ on the predefined linguistic term set $S = \{s_0, s_1, \dots, s_g\}$. If the experts have different assessments for one alternative and they can't come to consensus, then there can be several linguistic terms with some possible membership degrees for the alternative A_i with respect to the attribute C_j, which can be expressed by a 2-TLHFS LH_{ij}. Then we can obtain a 2-tuple linguistic hesitant fuzzy decision matrix $H = (LH_{ij})_{m\times n}$ $(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$.

If the weight information is completely known, then we can use the aggregation operators to obtain the alternative comprehensive values; otherwise, we must firstly obtain the weight vectors, where attributes' weight vectors $w = (w_1, w_2, \dots, w_n)^T$ satisfy $w_j > 0$ $(j = 1, 2, \dots, n)$ and $\sum_{j=1}^{n} w_j = 1$.

5.1. *Models for the Optimal Weight Vectors*

The weights of an attribute could be different for different problems and they could be affected by decision makers' preference, so we should take objective factors and people's subjective preference into consideration when determining the weight vectors. However, existing methods for obtaining weight vectors do not consider the two factors simultaneously. In this subsection, we propose a new model to obtain weight vectors. In the model, we firstly select the biggest value to be the positive ideal solution (PIS) and the smallest value to be the negative ideal solution (NIS) with respect to each attribute. Then we make alternatives satisfy the shortest distance from the PIS and the farthest distance from the NIS with respect to each attribute, which can construct a linear programming model; finally we can obtain weight vectors by solving linear programming model. The model considers decision makers' preference by selecting the PIS and NIS and approaching to ideal solution, and it is objective and effective. The specific process of proposed model for the optimal weight vectors is shown as follows:

(1) Selecting the PIS and NIS with respect to each attribute.

Under the 2-tuple linguistic hesitant fuzzy environment, we use Eqs. (8) and (9) to select the positive ideal solution (PIS) A_j^+ and the negative ideal solution (NIS) A_j^-,

$$A_j{}^+ = \max_{1 \leqslant i \leqslant m} e(LH_{ij}), \tag{8}$$

$$A_j{}^- = \min_{1 \leqslant i \leqslant m} e(LH_{ij}). \tag{9}$$

(2) Measuring the distances between the every alternative and A_j^+, A_j^-, with respect to each attribute, respectively.

For each attribute, we use Eq. (10) and (11) to measure the distances between every alternative and A_j^+, A_j^-. And the distances are denoted by d_j^+, d_j^-, respectively.

$$d_j{}^+ = \sqrt{\sum_{i=1}^{m} \big(e\big(LH_{ij} - A_j{}^+\big)\big)^2}, \tag{10}$$

$$d_j{}^- = \sqrt{\sum_{i=1}^{m} \big(e\big(LH_{ij} - A_j{}^-\big)\big)^2}. \tag{11}$$

(3) Constructing a model to obtain the optimal weight vector.

We construct a model as Eq. (12) to obtain the optimal weight vector, which can minimize the distance from every alternative to the positive ideal solution (PIS) and maximize the distance to the negative ideal solution (NIS).

$$\begin{aligned} &\min \sum_{j=1}^{n} \frac{d_j{}^+}{d_j{}^+ + d_j{}^-} w_j \\ &\quad = \min \sum_{j=1}^{n} \frac{\sqrt{\sum_{i=1}^{m} (e(LH_{ij} - A_j{}^+))^2}}{\sqrt{\sum_{i=1}^{m} (e(LH_{ij} - A_j{}^+))^2} + \sqrt{\sum_{i=1}^{m} (e(LH_{ij} - A_j{}^-))^2}} w_j, \end{aligned}$$

$$\text{s.t.} \begin{cases} \sum_{i=1}^{n} w_i = 1, \\ w_i \in W_j, \\ w_i \geqslant 0, \end{cases} \tag{12}$$

where W_j is the partially known weight information.

5.2. *A Multi-Attribute Decision Making Approach under 2-Tuple Linguistic Hesitant Fuzzy Environment*

Based on the above model for optimal weight vectors and proposed aggregation operators, an approach to 2-tuple linguistic hesitant fuzzy sets is developed. The main decision steps are shown as follows:

Step 1. Obtain decision matrix H.

The decision makers give their evaluated values of alternative A_i with respect to attribute C_j, which are expressed by the 2-TLHFSs on the predefined linguistic term set $S = \{s_0, s_1, \dots, s_g\}$. Then we can obtain a 2-tuple linguistic hesitant fuzzy decision matrix $H = (LH_{ij})_{m\times n}$ $(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$.

Step 2. Normalize decision matrix H.

As we all known, there are two usual kinds of attributes in MADM problems, such as benefit attributes (the bigger the better) and cost attributes (the smaller the better). If every attribute C_j $(j = 1, 2, \dots, n)$ is a benefit attribute, it does not normalize decision matrix H. Otherwise, 2-TLHFSs decision matrix $H = (LH_{ij})_{m\times n}$ should be transformed into $\overline{H} = (\overline{LH_{ij}})_{m\times n}$, where

$$\overline{LH_{ij}} = \begin{cases} LH_{ij}, & \text{benefit,} \\ LH_{ij}^{c}, & \text{cost} \end{cases}$$

with $LH_{ij}{}^{c} = \{(\Delta[g - \Delta^{-1}(s_i{}^{j}, 0)], \bigcup_{r_{ij}\in lh^{j}(s_i)} (1 - r_{ij}))|lh^{j}(s_i) \in s\}$.

Step 3. Determine the weight vectors.

If the weight information is completely known, then we can use the aggregation operators to obtain the alternative comprehensive values; otherwise, we must obtain the weight vectors by using Eq. (12).

Step 4. Aggregate the LH_{ij} $(j = 1, 2, \dots, n)$ for each alternative.

For alternative A_j, one can aggregate all 2-tuple linguistic hesitant fuzzy values LH_{ij} $(j = 1, 2, \dots, n)$ into a global value LH_i by means of 2-tuple linguistic hesitant fuzzy Einstein weighted averaging (2-TLHFEWA) operator or 2-tuple linguistic hesitant fuzzy Einstein weighted geometric (2-TLHFEWG) operator.

Step 5. Calculate the expectation value $E(LH_i)$ and the variance value $V(LH_i)$.

According to the comprehensive 2-TLHFSs LH_i $(i = 1, 2, \dots, m)$, we calculate the expectation value $E(LH_i)$ and the variance value $V(LH_i)$ in accordance with Definitions 9 and 10.

Step 6. Rank all alternatives according to the comprehensive 2-TLHFSs LH_i $(i = 1, 2, \dots, m)$, and select the best one(s).

Step 7. End.

Table 1
2-TLHFSs decision matrix.

	C_1	C_2	C_3
A_1	$\{((s_5,0),0.1,0.2),((s_6,0),0.7),((s_7,0),0.8)\}$	$\{((s_6,0),0.9)\}$	$\{((s_6,0),0.5,0.4),((s_7,0),0.7)\}$
A_2	$\{((s_5,0),0.5,0.6),((s_6,0),0.6,0.5)\}$	$\{((s_7,0),0.6),((s_8,0),0.7)\}$	$\{((s_6,0),0.3,0.5,0.8)\}$
A_3	$\{((s_5,0),0.2),((s_6,0),0.7,0.8)\}$	$\{((s_6,0),0.8,0.9)\}$	$\{((s_7,0),0.4,0.5),((s_8,0),0.1)\}$

6. Illustrative Example and Comparison Analysis

In this section, a real example is applied to illustrate the efficiency and practicality of the above approach. Then, we give a detailed comparison between the approaches based on 2-TLHFEWA operator and HFLWA operator (Lin *et al.*, 2014), GLHFHSWA operator (Meng *et al.*, 2014).

6.1. *Illustrative Example*

College plans to choose the most appropriate college faculty for tenure and promotion (Martinez, 2007). Three evaluation criteria, including teaching, research and service, are given. An expert team is invited to assess three faculty candidates. A linguistic term set $S = \{s_0$: extremely poor, s_1: very poor, s_2: poor, s_3: slightly poor, s_4: fair, s_5: slightly good, s_6: good, s_7: very good, s_8: extremely good$\}$ is given. Expert team assesses three faculty candidates (alternatives) $A = \{A_1, A_2, A_3\}$ by the above linguistic term set S with respect to three evaluation criteria: C_1: teaching, C_2: research and C_3: service.

Expert team may have different assessments for one alternative. For example, evaluating candidate A_1 with respect to C_1 teaching; one give the value 0.2 for slightly good and the value 0.8 for very good, others give the value 0.1 for slightly good and the value 0.7 for good. In this case, LH_{11} can be denoted by a 2-TLHFS, $\{((s_5,0),0.1,0.2),((s_6,0),0.7),((s_7,0),0.8)\}$. Therefore, in order to reflect experts inconsistency and hesitancy, experts assessment values to every attribute are expressed by 2-TLHFS shown in Table 1.

For the weight of attribute, weight vector is unknown, but experts can provide partial information: $w_1 \in [0.2, 0.35]$, $w_2 \in [0.3, 0.45]$, $w_3 \in [0.3, 0.4]$.

Using the proposed MADM approach based on 2-TLHFEWA operator, the following decision procedure is involved.

Step 1: The decision makers give their evaluated values of the alternative A_i with respect to the attribute C_j. Then we can obtain a 2-tuple linguistic hesitant fuzzy decision matrix $H = (LH_{ij})_{m\times n}$ $(i = 1,2,\dots,m;\ j = 1,2,\dots,n)$ shown as Table 1. As each attribute C_j $(j = 1,2,\dots,n)$ is benefit attribute, there is no need to normalize the decision matrix.

Step 2: The weight of attribute is unknown, so we should firstly determine the weight vectors. The procedure for the optimal weight vectors is as follows: Using Eqs. (8) and (9) to obtain the 2-tuple linguistic hesitant fuzzy PIS and NIS: ${A_1}^+ = 3.517$, ${A_1}^- = 2.75$; ${A_2}^+ = 5.4$, ${A_2}^- = 4.9$; ${A_3}^+ = 3.8$, ${A_3}^- = 1.975$; Using Eqs. (10) and (11) to obtain the distances: ${d_1}^+ = 0.911$, ${d_1}^- = 0.815$; ${d_2}^+ = 0.583$, ${d_2}^- = 0.5385$; ${d_3}^+ = 1.921$, ${d_3}^- =$

2.198; finally, we get the following linear programming model by minimizing the distance from every alternative to the positive ideal solution (PIS) and maximizing the distance to the negative ideal solution (NIS) with respect to each attribute.

$$\min \sum_{j=1}^{n} \frac{{d_j}^+}{{d_j}^+ + {d_j}^-} w_j = \min 0.5278 w_1 + 0.5198 w_2 + 0.4664 w_3,$$

$$\text{s.t.} \begin{cases} \sum_{i=1}^{3} w_i = 1, \\ w_1 \in [0.2, 0.35], \\ w_2 \in [0.3, 0.45], \\ w_3 \in [0.3, 0.4]. \end{cases}$$

By solving the above model, we obtain the optimal weight vector $w = (0.2, 0.4, 0.4)$.

Step 3: Based on the 2-TLHFEWA operator, we have (where $g = 8$):

$LH_1 = \text{2-TLHFEWA}(LH_{11}, LH_{12}, LH_{13}) = \{((s_6, -0.174), 0.679, 0.357, 0.691, 0.663), ((s_6, 0.338), 0.742, 0.751), ((s_6, 0), 0.754, 0.731), ((s_6, 0.475), 0.804), ((s_6, 0.252), 0.773, 0.752), ((s_7, -0.327), 0.820)\}$.

$LH_2 = \text{2-TLHFEWA}(LH_{21}, LH_{22}, LH_{23}) = \{((s_6, 0.338), 0.471, 0.542, 0.679, 0.493, 0.562, 0.694), ((s_8, 0), 0.523, 0.589, 0.714, 0.544, 0.608, 0.728), ((s_6, 0.475), 0.493, 0.562, 0.694, 0.471, 0.542, 0.679), ((s_8, 0), 0.544, 0.608, 0.728, 0.523, 0.589, 0.714)\}$.

$LH_3 = \text{2-TLHFEWA}(LH_{31}, LH_{32}, LH_{33}) = \{((s_6, 0.338), 0.571, 0.604, 0.663, 0.691), ((s_8, 0), 0.478, 0.585), ((s_6, 0.475), 0.654, 0.682, 0.731, 0.754, 0.678, 0.706, 0.752, 0.773), ((s_8, 0), 0.574, 0.665, 0.604, 0.690)\}$.

Step 4: By Definition 9, the expectation function $E(LH_i)$ $(i = 1, 2, 3)$ is acquired as follows:

$$E(LH_1) = (s_5, -0.315), \qquad E(LH_2) = (s_4, 0.3089), \qquad E(LH_3) = (s_4, 0.491).$$

Step 5: According to Definition 11, we have the following ranking: $A_1 > A_3 > A_2$.

Using the proposed MADM approach based on 2-TLHFEWG operator, the following decision procedure is involved.

Step 1: We can obtain a 2-tuple linguistic hesitant fuzzy decision matrix $H = (LH_{ij})_{m\times n}$ $(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$ as Table 1. Because every attribute C_j $(j = 1, 2, \dots, n)$ is benefit attribute, there is no need to normalize the decision matrix.

Step 2: The weight of attribute is unknown, so we should firstly determine the weight vectors. By applying the model by similarity to ideal solution, we obtain the optimal weight vector $w = (0.2, 0.4, 0.4)$.

Step 3: Based on the 2-TLHFEWG operator, we have (where $g = 8$):

$LH_1 = \text{2-TLHFEWA}(LH_{11}, LH_{12}, LH_{13}) = \{((s_6, -0.207), 0.496, 0.455, 0.554, 0.509), ((s_6, 0.182), 0.571, 0.634), ((s_6, 0), 0.689, 0.638), ((s_6, 0.394), 0.778), ((s_6, 0.196), 0.708, 0.657), ((s_7, -0.406), 0.798)\}$.

$LH_2 = \text{2-TLHFEWA}(LH_{21}, LH_{22}, LH_{23}) = \{((s_6, 0.182), 0.445, 0.539, 0.654, 0.462, 0.559, 0.677), ((s_7, -0.432), 0.477, 0.575, 0.695, 0.495, 0.596, 0.718), ((s_6, 0.394),$

0.462, 0.559, 0.677, 0.445, 0.539, 0.654), $((s_7, -0.216)$, 0.495, 0.596, 0.718, 0.477, 0.575, 0.695)}.

$LH_3 =$ 2-TLHFEWA$(LH_{31}, LH_{32}, LH_{33}) = \{((s_6, 0.182)$, 0.479, 0.522, 0.509, 0.554), $((s_7, -0.432)$, 0.289, 0.310), $((s_6, 0.394)$, 0.603, 0.652, 0.638, 0.689, 0.621, 0.671, 0.657, 0.708), $((s_7, -0.216)$, 0.376, 0.401, 0.389, 0.415)}.

Step 4: By Definition 9, the expectation function $E(LH_i)$ $(i = 1, 2, 3)$ is acquired as follows:

$$E(LH_1) = (s_4, 0.181), \qquad E(LH_2) = (s_5, -0.274), \qquad E(LH_3) = (s_3, 0.006).$$

Step 5: According to Definition 11, we have the following ranking: $A_1 > A_2 > A_3$.

From the above analysis, the best choice in both cases is A_1. So we should choose college faculty A_1 for tenure and promotion.

6.2. *Comparison Analysis*

(1) A comparison with the approach based on HFLWA operator (Lin *et al.*, 2014).

For the above numerical example, if the 2-TLHFEWA operator is replaced by the HFLWA operator, the decision making result is as follows. It is known for the weight vector of the attribute $w = (0.2, 0.4, 0.4)$.

$$\begin{aligned}
&\mathrm{HFLWA}(LH_1, LH_2, LH_3, \ldots, LH_n)\\
&\quad = w_1 LH_1 \oplus w_2 LH_2 \oplus \cdots \oplus w_n LH_n = \bigoplus_{j=1}^{n} w_j LH_j\\
&\quad = \bigcup_{((s_{\theta_1},\alpha_{\theta_1}), lh(s_{\theta_1},\alpha_{\theta_1}))\in LH_1,\ldots,((s_{\theta_n},\alpha_{\theta_n}), lh(s_{\theta_n},\alpha_{\theta_n}))\in LH_n}\\
&\qquad \left\{\left(s_{(\sum_{j=1}^{n} w_j\theta_j)}, \bigcup_{r_1\in lh(s_{\theta_1},\alpha_{\theta_1}),\ldots,r_n\in lh(s_{\theta_n},\alpha_{\theta_n})} \left(1 - \prod_{j=1}^{n}(1-r_j)^{w_j}\right)\right)\right\}.
\end{aligned}$$

Based on the LHFWA operator, we have:

$LH_1 = \{(s_{5.8}$, 0.705, 0.682, 0.711, 0.690), $(s_{6.2}$, 0.0.759, 0.765), $(s_6$, 0.763, 0.745), $(s_{6.4}$, 0.807), $(s_{6.2}$, 0.781, 0.765), $(s_{6.6}$, 0.822)}.

$LH_2 = \{(s_{6.2}$, 0.477, 0.543, 0.683, 0.500, 0.563, 0.700), $(s_{6.6}$, 0.534, 0.592, 0.7170, 0.554, 0.610, 0.730), $(s_{6.4}$, 0.500, 0.563, 0.700, 0.477, 0.543, 0.683), $(s_{6.8}$, 0.554, 0.610, 0.730, 0.534, 0.592, 0.7170)}.

$LH_3 = \{(s_{6.2}$, 0.590, 0.619, 0.690, 0.711), $(s_{6.6}$, 0.518, 0.635), $(s_{6.4}$, 0.663, 0.687, 0.745, 0.763, 0.690, 0.711, 0.765, 0.781), $(s_{6.8}$, 0.604, 0.700, 0.635, 0.723)}.

The expectation function $E(LH_i)$ $(i = 1, 2, 3)$ is acquired as follows:

$$E(LH_1) = (s_{4.779}), \qquad E(LH_2) = (s_{3.906}), \qquad E(LH_3) = (s_{4.255}).$$

So we have ranking: $A_1 > A_3 > A_2$.

Table 2
2-TLHFSs of alternatives.

	C_1	C_2	C_3
A_1	$\{((s_5,0),0.1,0.2),((s_6,0),0.4),((s_7,0),0.3)\}$	$\{((s_6,0),0.4),((s_7,0),0.2,0.3)\}$	$\{((s_6,0),0.2,0.4),((s_7,0),0.3)\}$
A_2	$\{((s_5,0),0.2,0.4),((s_6,0),0.3,0.5)\}$	$\{((s_7,0),0.3,0.6),((s_8,0),0.2)\}$	$\{((s_6,0),0.3,0.5,0.8)\}$
A_3	$\{((s_5,0),0.2),((s_6,0),0.3,0.5)\}$	$\{((s_5,0),0.3,0.5),((s_6,0),0.2,0.3),((s_7,0),0.1)\}$	$\{((s_7,0),0.3,0.5),((s_8,0),0.1,0.3)\}$

According to the results, the LHFWA and the 2-TLHFEWA operators have the same ranking results and the most desirable alternative A_1.

(2) A comparison with the approach based on GLHFHSWA operator (Meng *et al.*, 2014).

We apply proposed approach based on 2-TLHFEWA operator to the Example 1 (Lin *et al.*, 2014), and the partially known weight vector is given by: $w_1 \in [0.2, 0.35]$, $w_2 \in [0.3, 0.45]$, $w_3 \in [0.3, 0.4]$. So decision procedure is involved as follows.

Step 1: A LHFSs decision matrix *HF* as Table 1 (Lin *et al.*, 2014).

Step 2: Translate LHFSs into 2-TLHFSs and obtain a 2-TLHFSs decision matrix H as Table 2. And because every attribute C_j $(j = 1, 2, \dots, n)$ is benefit attribute, it does not need to normalize decision matrix H.

Step 3: Weight vectors are unknown, so we firstly determine the weight vectors. Using the proposed model, we can obtain the following linear programming model to determine the best weight vector.

$$\min \sum_{j=1}^{n} \frac{{d_j}^+}{{d_j}^+ + {d_j}^-} w_j = \min 0.4767 w_1 + 0.5316 w_2 + 0.5248 w_3$$

$$\text{s.t.} \begin{cases} \sum_{i=1}^{3} w_i = 1, \\ w_1 \in [0.2, 0.35], \\ w_2 \in [0.3, 0.45], \\ w_3 \in [0.3, 0.4]. \end{cases}$$

By solving the above model, we get the optimal weight vector $w = (0.35, 0.3, 0.35)$.

Step 4: Based on 2-TLHFEWA operator, we have (where $g = 8$):

$LH_1 = \text{2-TLHFEWA}(LH_{11}, LH_{12}, LH_{13}) = \{((s_6, -0.313), 0.229, 0.301, 0.263, 0.333), ((s_6, 0.167), 0.264, 0.297), ((s_6, 0.104), 0.165, 0.239, 0.196, 0.269, 0.2, 0.273, 0.231, 0.302), ((s_7, -0.494), 0.201, 0.232, 0.236, 0.266), ((s_6, 0), 0.333, 0.4), ((s_6, 0.422), 0.366), ((s_6, 0.367), 0.273, 0.343, 0.302, 0.371), ((s_7, -0.281), 0.307, 0.336), ((s_6, 0.422), 0.297, 0.366), ((s_7, -0.237), 0.331), ((s_7, -0.281), 0.236, 0.307, 0.266, 0.336), ((s_7, 0), 0.271, 0.3)\}$.

$LH_2 = \text{2-TLHFEWA}(LH_{21}, LH_{22}, LH_{23}) = \{((s_6, 0.104), 0.266, 0.342, 0.499, 0.369, 0.439, 0.581, 0.336, 0.408, 0.555, 0.434, 0.499, 0.630), ((s_8, 0), 0.236, 0.313, 0.745, 0.307, 0.381, 0.532), ((s_6, 0.367), 0.3, 0.374, 0.527, 0.401, 0.469, 0.605, 0.374, 0.444, 0.585, 0.469, 0.532, 0.655), ((s_8, 0), 0.271, 0.346, 0.503, 0.346, 0.418, 0.563)\}$.

$LH_3 = \text{2-TLHFEWA}(LH_{31}, LH_{32}, LH_{33}) = \{((s_6, -0.08), 0.266, 0.342, 0.331, 0.404), ((s_8, 0), 0.196, 0.266, 0.264, 0.331), ((s_6, 0.167), 0.236, 0.313, 0.266, 0.342), ((s_8, 0), 0.165, 0.236, 0.196, 0.266), ((s_7, -0.494), 0.206, 0.285), ((s_8, 0), 0.135, 0.206),$

$((s_6, 0.205), 0.3, 0.375, 0.364, 0.434, 0.374, 0.444, 0.434, 0.5), ((s_8, 0), 0.232, 0.3, 0.299, 0.364, 0.310, 0.374, 0.373, 0.434), ((s_6, 0.422), 0.271, 0.346, 0.3, 0.374, 0.346, 0.418, 0.374, 0.444), ((s_8, 0), 0.201, 0.271, 0.232, 0.3, 0.280, 0.346, 0.374, 0.310), ((s_7, -0.281), 0.242, 0.319, 0.319, 0.392), ((s_8, 0), 0.172, 0.242, 0.252, 0.319)\}$.

Step 5: By Definition 9, the expectation function $E(LH_i)$ $(i = 1, 2, 3)$ is acquired as follows:

$$E(LH_1) = (s_2, -0.056), \qquad E(LH_2) = (s_3, 0.006), \qquad E(LH_3) = (s_2, 0.041).$$

Step 6: The ranking is $A_2 > A_3 > A_1$.

It is obvious that there are different orders, but the best choice in both cases is A_2.

By the analysis of the two above comparisons, the validity of new proposed aggregation operators is tested. And for the proposed MADM approach in this paper, there are four main differences from existing approaches.

Firstly, we proposed a new uncertain linguistic variable, 2-tuple linguistic hesitant fuzzy sets (2-TLHFSs), which can reflect decision makers' uncertainty and hesitancy by providing the information about several possible linguistic terms of a linguistic variable and several possible membership degrees of each linguistic term. 2-TLHFSs have a wider range of application and can express and address rather complex multi-attribute decision-making problems that existing linguistic variables cannot address.

Secondly, 2-TLHFEWA and 2-TLHFEWG operators are based on 2-tuple linguistic representation model, 2-tuple linguistic representation model can make linguistic variable continuous and prevent information from losing in aggregation process. So 2-TLHFEWA and 2-TLHFEWG operators are also efficient and can avoid information loss and the lack of precision.

Thirdly, based on Einstein t-norm and t-conorm, we propose the new operational laws and 2-TLHFEWA and 2-TLHFEWG operators. The new operational laws are closed and can overcome granularity and logical problems. Compared with most aggregation operators based on Algebraic t-conorm and t-norm, the aggregation operators based on Einstein t-norm and t-conorm can provide another choice for decision makers.

Finally, we propose a model to deal with the situation where the weights information is unknown. The proposed model for optimal weight vector is advantaged and effective, which takes both subjective and objective weights information into consideration.

7. Conclusion

In order to deal with rather complex linguistic assessment and express membership degrees of linguistic term, this paper proposed a new class of uncertain linguistic variables, 2-tuple linguistic hesitant fuzzy sets (2-TLHFSs). It can reflect sufficiently decision makers' hesitancy and ensure information integrity in aggregation process. Then based on Einstein t-norm and t-conorm, the new closed operational laws are defined, which can overcome the granularity and logical problems. Based on the new closed operational laws, 2-tuple linguistic hesitant fuzzy aggregation operators (2-TLHFEWA operator and

2-TLHFEWG operator) and relative basic properties are defined. Additionally, to deal with the situation where the weight information of attribute is unknown, a new model is build to obtain optimal weight vectors, which take both subjective and objective factors into consideration. Then a multi-attribute decision making approach under 2-tuple linguistic hesitant fuzzy environment is developed and an example is presented to demonstrate the application of the proposed approach. It should be noted that the proposed approach to deal with 2-tuple linguistic hesitant fuzzy information needs to be further extended, and other possible approaches addressing complex MADM linguistic problems will be studied in the future.

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