An Approach to Hesitant Fuzzy Group Decision Making with Multi-Granularity Linguistic Information

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Abstract. The 2-tuple linguistic computational model is an important tool to deal with linguistic information. To extend the application of hesitant fuzzy linguistic term sets and avoid information loss, this paper introduces hesitant fuzzy 2-tuple linguistic term sets that are expressed by using several symbolic numbers in [0, 1]. Considering the order relationship between hesitant fuzzy 2-tuple linguistic term sets, measures of expected value and variance are defined. Meanwhile, several induced generalized hesitant fuzzy 2-tuple linguistic aggregation operators are defined, by which the comprehensive attribute values of alternatives can be obtained. Then, models for the optimal weight vector on a decision maker set, on an attribute set and on their ordered sets are constructed, respectively. Furthermore, an approach to multi-granularity group decision making with hesitant fuzzy linguistic information is developed. Finally, an example is selected to illustrate the feasibility and practicality of the proposed procedure.

Key words: group decision making; 2-tuple linguistic computational model; hesitant fuzzy linguistic term set; aggregation operator.

1. Introduction

Since Zadeh (1975a, 1975b, 1975c) first introduced linguistic variables to cope with qualitative information, linguistic variables have received considerable attention (Herrera *et al.*, 1995, 2000; Herrera and Herrera-Viedma, 1997; Herrera and Martínez, 2000; Liu, 2009; Xu, 2009) and have been applied in many fields, especially in decision making (Alonso *et al.*, 2013; Agell *et al.*, 2012; Chen and Ben-Arieh, 2006; Delgado *et al.*, 1993; Degani and Bortolan, 1988; Herrera and Verdegay, 1993; Merigó *et al.*, 2010; Tan *et al.*, 2011; Wang, 2013; Wang *et al.*, 2015; Wei and Zhao, 2014). Later, researchers noted that the linguistic variable only permits the decision maker to express his/her qualitative individual information by using one linguistic term from the predefined linguistic term set. This seems to be inadequate in some situations. For example, a decision maker may think that the quality of a product is between 'good' and 'very good' rather than

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'good' or 'very good'. Therefore, many group decision-making methods with uncertain linguistic information are proposed (Jin *et al.*, 2013; Liu *et al.*, 2011; Park *et al.*, 2011; Xu, 2004, 2006a, 2006b; Zhang, 2013; Zhang and Guo, 2012). Furthermore, Wang and Hao (2006) introduced the proportional linguistic 2-tuple variable that is expressed by two consecutive linguistic terms and the possible proportions of these two linguistic terms. Ma *et al.* (2007) developed a model to join different single terms in a new synthesized term by using a fuzzy model and measures of consistency and determinacy. To reflect the membership and non-membership degrees of a given linguistic variable, Wang and Li (2009) defined the concept of intuitionistic linguistic sets, which are expressed by a linguistic term, a membership degree and a non-membership degree. Later, Liu (2013a, 2013b) introduced several intuitionistic linguistic aggregation operators. In a similar way to intuitionistic linguistic sets (Wang and Li, 2009), Liu (2013a, 2013b) and Liu and Jin (2012) proposed interval-valued intuitionistic uncertain linguistic sets that are denoted by an uncertain linguistic variable, an interval membership and non-membership degrees. The authors further defined several interval-valued intuitionistic uncertain linguistic aggregation operators and researched their application in group decision making.

Very recently, Rodríguez *et al.* (2012) presented the concept of hesitant fuzzy linguistic term sets (HFLTSs) that are denoted by several linguistic terms from the predefined linguistic term set. Such a generalization further addresses the hesitancy and inconsistency of the decision maker. Meanwhile, some properties of HFLTSs are discussed, and the envelope of HFLTSs is defined. Later, Rodríguez *et al.* (2013) further developed an approach to group decision making with hesitant fuzzy linguistic preference relations, which is based on the envelope of HFLTSs and the 2-tuple arithmetic mean operator (Herrera and Martínez, 2000). After that, according to the preference relation on intervals (Wang *et al.*, 2005) and the defined non-dominance degree, the authors developed an approach to rank the comprehensive attribute values. Liao *et al.* (2015) studied the correlation coefficients of HFLTSs and discussed their application in decision making. Later, Wei *et al.* (2014) analysed the issues of the method in Rodríguez *et al.* (2013) and studied the hesitant fuzzy linguistic multi-criteria group decision-making problem. To compare HFLTSs, the authors defined a possibility degree formula, which is based on the comparison of HFLTSs of the same length. However, this method in fact results in a distinct HFLTS that contains more linguistic terms. Furthermore, the comparison method is not in accordance with common sense. For example, let $H_1 = \{s_3, s_4, s_5, s_6\}$ and $H_2 = \{s_2, s_3, s_4\}$ be two HFLTSs on the predefined linguistic term set $S = \{s_i, i = 1, 2, \ldots, 6\}$. To compare H_1 and H_2, the authors added one linguistic term $\bar{s}_2$ into H_1 and added two linguistic terms $\bar{s}_5$ and $\bar{s}_6$ into H_2, then it derives $H_1^* = \{\bar{s}_2, s_3, s_4, s_5, s_6\}$ and $H_2^* = \{s_2, s_3, s_4, \bar{s}_5, \bar{s}_6\}$, where $\bar{s}_2 \in H_1$ and $\bar{s}_5, \bar{s}_6 \in H_2$. After that, the authors compared H_1^* and H_2^* to represent the relationship between H_1 and H_2. According to the defined possibility degree formula (Wei *et al.*, 2014), we have $p(H_1 > H_2) = 0.8$. However, when H_1 and H_2 are directly compared according to the principle in Wei *et al.* (2014), we get $p(H_1 > H_2) = 1/2 + 1/4 = 0.75$. Furthermore, let $H_1 = \{s_1, s_2, s_3, s_4\}$ and $H_2 = \{s_2, s_3\}$, then $p(H_1 > H_2) = p(H_2 > H_1) = 0.5$. However, we usually conclude that H_2 is better than H_1 for the former has a smaller hesitancy degree. Moreover, Wei *et al.* (2014) defined two hesitant fuzzy linguistic operators

based on the convex combination of two linguistic terms (Delgado *et al.*, 1993), which may cause a loss of information by the use of the round operator (Herrera and Martínez, 2000). Liu and Rodríguez (2014) presented a method to transform a HFLTS into an associated trapezoidal fuzzy number called the fuzzy envelope and researched its application in multi-attribute decision making. As Herrera and Martínez (2000) noted, linguistic computational model based on the associated membership function may also be loss of information. Furthermore, Zhu and Xu (2014) discussed the hesitant fuzzy linguistic preference relation by using the defined distant consistency index; Beg and Rashid (2013) developed an approach to hesitant fuzzy linguistic multi-attribute decision making based on TOPSIS method, which assumes that all attributes have the same importance. Meng and Chen (2015) defined a new distance measure on HFLTSs, which needn't consider the number of elements in a HFLTS. Then, the authors developed an approach to hesitant fuzzy linguistic multi-granularity decision making based on distance measures.

At present, there are three main methods to process linguistic information: the membership function (Degani and Bortolan, 1988), the ordinal scale (Yager, 1981), and the discrete fuzzy number (Massanet *et al.*, 2014). It is worth noting that the linguistic symbolic computational model based on the ordinal scale has received considerable attention for its adaptation and simplicity (Agell *et al.*, 2012; Chen and Ben-Arieh, 2006; Delgado *et al.*, 1993; Yager, 1981; Zhu and Hipel, 2012). The 2-tuple linguistic computational model (Herrera and Martínez, 2000) is one of the most important and popular methods to express linguistic variables on the ordinal scale that contains a linguistic term and a symbolic translation value. This model can avoid the loss of information. Since it was first introduced by Herrera and Martínez (2000), several 2-tuple linguistic aggregation operators are defined such as the induced 2-tuple linguistic generalized aggregation operator (Merigó and Gil-Lafuente, 2013), the proportional 2-tuple geometric aggregation operator (Xu *et al.*, 2013) and the 2-tuple linguistic power aggregation operator (Xu and Wang, 2011). Furthermore, Martínez and Herrera (2012) reviewed the current researches for the 2-tuple linguistic computational model in detail.

To make HFLTSs (Rodríguez *et al.*, 2012) more easy to use and to avoid the information loss (Herrera and Martínez, 2000), this paper develops an approach to hesitant fuzzy linguistic multi-granularity group decision making by using the 2-tuple linguistic representation model, which can eliminate the problem in Wei *et al.* (2014). To do this, the concept of hesitant fuzzy 2-tuple linguistic term sets (HFTLTSs) is introduced. Based on measures of expected value and variance, an order relationship between HFTLTSs is offered. Then, several induced generalized aggregation operators are defined, by which the comprehensive attribute values can be obtained. Based on the defined similarity degree of HFTLTSs, models for the optimal weight vector are built. Finally, an approach to hesitant fuzzy linguistic multi-granularity group decision making with incomplete weight formation and interactive characteristics is developed.

This paper is organized as follows: Section 2 introduces some basic concepts such as 2-tuple linguistic representation models, hesitant fuzzy linguistic term sets and hesitant fuzzy 2-tuple linguistic term sets. Section 3 defines several induced generalized hesitant fuzzy 2-tuple linguistic operators such as the induced generalized hesitant fuzzy 2-tuple

linguistic hybrid weighted averaging (IG-HFTLHWA) operator and the induced general-ized hesitant fuzzy 2-tuple linguistic hybrid Shapley averaging (IG-HFTLHSA) operator. Meanwhile, several special cases are discussed. Section 4 first introduces a similarity de-gree of HFTLTSs. Then, models for the optimal fuzzy vector on a decision maker set, on an attribute set and on their ordered sets are built, respectively. Section 5 develops an approach to multi-granularity hesitant fuzzy linguistic group decision making. Section 6 offers an illustrative example to show the concrete application of the developed procedure. Conclusions are made in the last section.

2. Basic Concepts

2.1. *2-Tuple Linguistic Variables and Hesitant Fuzzy Linguistic Term Sets*

As Zadeh (1975a, 1975b, 1975c) noted, in some situations, it is insufficient to express fuzzy information by using quantitative variables. To deal with this issue, we usually use qualitative variables: linguistic variables such as "unimportant", "fair", and "important".

Let $S = \{s_i | i = 0, 1, \ldots, t\}$ be a linguistic term set with odd cardinality. Any label s_i represents a possible value for a linguistic variable and it should satisfy the following characteristics (Herrera and Martínez, 2000): (i) The set is ordered: $s_i > s_j$, if $i > j$; (ii) Max operator: $\max(s_i, s_j) = s_i$, if $s_i \geqslant s_j$; (iii) Min operator: $\min(s_i, s_j) = s_i$, if $s_i \leqslant s_j$; (iv) A negation operator: $\text{neg}(s_i) = s_j$ such that $j = t - i$.

For example, the linguistic term set S can be expressed by $S = \{s_0$: worst, s_1: worse, s_2: bad, s_3: fair, s_4: good, s_5: better, s_6: best$\}$.

DEFINITION 1. (See Herrera and Martínez, 2000.) The symbolic translation is a numerical value assessed in $[0.5, 0.5)$ that supports the difference of information between a counting of information β assessed in the interval of granularity $[0, t]$ of the predefined linguistic term set S and the closest value in $\{0, 1, \ldots, t\}$, which indicates the index of the closest linguistic term in S.

To improve the accuracy and facilitate the process of computing with words (CW), Herrera and Martínez (2000) introduced the 2-tuple linguistic representation model that consists of a pair of values, namely, $(s_i, \alpha_i) \in S \times [0.5, 0.5)$ with s_i being a linguistic term from the predefined linguistic term set S and α_i being a symbolic translation value in $[0.5, 0.5)$.

DEFINITION 2. (See Herrera and Martínez, 2000.) Let $S = \{s_0, s_1, \ldots, s_t\}$ be a linguis-tic term set, and $\beta \in [0, t]$ be a real number representing the aggregation result of lin-guistic symbolic, then the 2-tuple linguistic variable that expresses the equivalent infor-mation to β is obtained by using the following function $\Delta: \Delta : [0, t] \rightarrow S \times [0.5, 0.5)$, $\Delta(\beta) = (s_i, \alpha_i)$ with $\begin{cases} s_i, & i = round(\beta) \\ \alpha_i = \beta - i, & \alpha_i \in [-0.5, 0.5), \end{cases}$ where $round(\cdot)$ is the usual round operation, s_i has the closest index label to β and α_i is the value of the symbolic translation.

DEFINITION 3. (See Herrera and Martínez, 2000.) Let $S = \{s_0, s_1, \ldots, s_t\}$ be a linguistic term set, and (s_i, α_i) be a 2-tuple linguistic variable. There is always a function Δ^{-1}:

$$\Delta^{-1} : S \times [0.5, 0.5) \rightarrow [0, t],$$

$$\Delta^{-1}(s_i, \alpha_i) = i + \alpha_i = \beta.$$

In a similar way to Herrera and Martínez (2000), Chen and Tai (2005) introduced another form of the 2-tuple linguistic representation model.

DEFINITION 4. (See Chen and Tai, 2005.) Let $S = \{s_i | i = 0, 1, \ldots, t\}$ be a linguistic term set with odd cardinality, then any $\beta \in [0, 1]$ can be transformed into a 2-tuple linguistic variable, denoted by $\Delta(\beta) = (s_i, \alpha)$ with $\begin{cases} s_i, & i = round(\beta \cdot t), \\ \alpha = \beta - i/t, & \alpha \in [-0.5/t, 0.5/t). \end{cases}$

REMARK 1. From Definition 4, one can conclude that any 2-tuple linguistic variable (s_i, α) can be converted into a crisp value $\beta \in [0, 1]$, denoted by $\Delta^{-1}(s_i, \alpha) = i/t + \alpha$. Thus, the model presented in Definition 4 restricts the value of β from $[0, t]$ into $[0, 1]$. This representation model eliminates the cardinal influence of the linguistic term set and can deal with multi-granularity linguistic group decision-making problems. For this reason, this paper adopts the 2-tuple linguistic representation model shown in Definition 4.

Because of various reasons such as time pressure, the decision makers' limited decision expertise about the problem domain, and the inconsistency and uncertainty of the decision makers' subjective judgements; it may be more suitable to express qualitative information by using several linguistic terms. For this purpose, Rodríguez *et al.* (2012) defined hesitant fuzzy linguistic term sets (HFLTSs) that permit the decision makers to use several linguistic terms to represent qualitative information.

DEFINITION 5. (See Rodríguez *et al.*, 2012.) A HFLTS H is an ordered finite subset of consecutive linguistic terms of S with $S = \{s_0, s_1, \ldots, s_t\}$ being a linguistic term set.

For example, let $S = \{s_0:$ worst, $s_1:$ worse, $s_2:$ bad, $s_3:$ fair, $s_4:$ good, $s_5:$ better, $s_6:$ best$\}$ be a linguistic term set, then the qualitative information Q could be expressed by $H(Q) = \{s_0, s_1, s_2, s_3\}$.

2.2. *Hesitant Fuzzy 2-Tuple Linguistic Term Sets*

To avoid information loss during the calculation of HFLTSs, this section introduces the concept of hesitant fuzzy 2-tuple linguistic term sets (HFTLTSs). It is worth noting that the relationship between HFLTSs and HFTLTSs is similar to that between 2-tuple linguistic variables and linguistic variables (Chen and Tai, 2005).

DEFINITION 6. Let $S = \{s_i | i = 0, 1, \ldots, t\}$ be a linguistic term set with odd cardinality. A HFTLTS is composed of several linguistic terms and several numbers in $[0.5/t, 0.5/t)$,

denoted by $\{(s_l, \alpha_l)\}_{l=i,i+1,\ldots,j}$, where $j \leqslant t$, s_l represents the linguistic label in S and α_l is the value of the symbolic translation. Any HFTLTS $\{(s_l, \alpha_l)\}_{l=i,i+1,\ldots,j}$ can be converted into a real set $\{\beta_i, \beta_{i+1}, \ldots, \beta_j\}$ with $\beta_l \in [0, 1]$, $l = i, i + 1, \ldots, j$, and $\beta_k \leqslant \beta_{k+1}$, $k = i, i + 1, \ldots, j - 1$, denoted by

$$\Delta^{-1}\big(\{(s_l, \alpha_l)\}_{l=i,i+1,\ldots,j}\big) = \{l/t + \alpha_l\}_{l=i,i+1,\ldots,j} = \{\beta_l\}_{l=i,i+1,\ldots,j}. \tag{1}$$

Equivalently, any real set $A = \{\beta_1, \beta_2, \ldots, \beta_p\}$ with $\beta_r \in [0, 1]$, $r = 1, 2, \ldots, p$, and $\beta_k \leqslant \beta_{k+1}$, $k = 1, 2, \ldots, p - 1$, can be converted into a HFTLTS, expressed by

$$\Delta(A) = \big\{(s_r, \alpha_r)\big\}_{r=1,2,\ldots,p} \tag{2}$$

with $\begin{cases} s_r, & r = round(\beta_r \cdot t), \ r = 1, 2, \ldots, p, \\ \alpha_r = \beta_r - r/t, & \alpha_r \in [-0.5/t, 0.5/t), \ r = 1, 2, \ldots, p. \end{cases}$

For example, let $S = \{s_i | i = 0, 1, \ldots, 6\}$ be the predefined linguistic term set. Let $\{(s_1, 0.033), (s_2, 0.042), (s_3, 0.005), (s_4, 0.021)\}$ be a HFTLTS for S. According to the equation (1), we derive

$$\Delta^{-1}\big(\{(s_1, 0.033), (s_2, 0.042), (s_3, 0.005), (s_4, 0.021)\}\big)$$
$$= \{0.1997, 0.3753, 0.505, 0.6877\}.$$

On the other hand, for the real number set $A = \{0.2, 0.25, 0.3, 0.36\}$, using the equation (2), we have $\Delta(A) = \{(s_1, 0.033), (s_2, 0.083), (s_2, 0.027), (s_2, 0.067)\}$.

REMARK 2. HFTLTSs are not new linguistic fuzzy variables. It is a linguistic computational model for HFLTSs. Because the decision maker usually applies the linguistic term from the predefined linguistic term set to express his/her qualitative information, the value of the symbolic translation is equal to zero. The situation that the symbolic translation is not equal to zero only appears in the process of calculation.

To compare HFTLTSs, let us consider the concepts of expected value and variance on HFLTSs.

DEFINITION 7. Measure of expected value E on HFTLTSs, for any HFTLTS $H = \{(s_l, \alpha_l)\}_{l=i,i+1,\ldots,j}$ on the predefined linguistic term set S, is defined by $E(H) = \sum_{l=i}^{j} \frac{l/t + \alpha_l}{j - i + 1}$, and measure of variance V on HFTLTSs, for the HFTLTS H, is defined by $V(H) = \sum_{l=i}^{j} \left(\frac{l/t + \alpha_l}{j - i + 1} - E(H) \right)^2$ with $(s_l, \alpha_l) = l/t + \alpha_l$ for each $l = i, i + 1, \ldots, j$.

The order relationship, for any two HFTLTSs H and K on the predefined linguistic term set S, is defined as follows:

If $E(H) < E(K)$, then $H < K$.

If $E(H) = E(K)$, then $\begin{cases} V(H) > V(K), & H < K, \\ V(H) = V(K), & H = K. \end{cases}$

For example, let $S = \{s_i | i = 0, 1, \ldots, 6\}$ be the predefined linguistic term set. Let $H_1 = \{(s_2, 0.04), (s_3, 0.05), (s_4, 0.03)\}$ and $H_2 = \{(s_3, 0.02), (s_4, 0.02)\}$ be two HFTLTSs for S. Then, their expected values are $E(H_1) = 0.54$ and $E(H_2) = 0.603$. According to the above order relationship, we have $H_1 < H_2$. When $H_1 = \{(s_2, 0.02), (s_3, 0.04), (s_4, 0.02), (s_5, 0.00)\}$, we get $E(H_1) = 0.603$. From $V(H_1) = 0.8269$ and $V(H_2) = 0.1855$, we derive $H_1 < H_2$.

3. Several Hesitant Fuzzy 2-Tuple Linguistic Aggregation Operators

To obtain the comprehensive hesitant fuzzy linguistic attribute values, this section defines several hesitant fuzzy 2-tuple linguistic aggregation operators.

3.1. *Aggregation Operators based on Additive Measures*

DEFINITION 8. Let H_i $(i = 1, 2, \ldots, n)$ be a collection of HFTLTSs on the predefined linguistic term set S. The generalized hesitant fuzzy 2-tuple linguistic weighted averaging (GHFTLWA) operator of dimension n is a mapping GHFTLWA: HFTLTSs$^n \to$ HFTLTSs, defined by

$$\text{GHFTLWA}(H_1, H_2, \ldots, H_n)$$

$$= \left(\bigoplus_{i=1}^{n} \omega_{H_i} H_i^{\gamma} \right)^{\frac{1}{\gamma}} = \bigcup_{(s_{\theta_1}, \alpha_{\theta_1}) \in H_1, \ldots, (s_{\theta_n}, \alpha_{\theta_n}) \in H_n} \Delta \left(\left(\sum_{i=1}^{n} \omega_{H_i} \Delta^{-1}(s_{\theta_i}, \alpha_{\theta_i})^{\gamma} \right)^{\frac{1}{\gamma}} \right),$$

where $\gamma \in R^+$, and ω_{H_i} is the weight of the HFTLTS H_i with $\omega_{H_i} \geqslant 0$ and $\sum_{i=1}^{n} \omega_{H_i} = 1$.

Next, let us consider several special cases of the GHFTLWA operator.

REMARK 3. Let $\gamma \to 0^+$, then the GHFTLWA operator reduces to the hesitant fuzzy 2-tuple linguistic geometric mean (HFTLGM) operator

$$\text{HFTLGM}(H_1, H_2, \ldots, H_n)$$

$$= \prod_{i=1}^{n} H_i^{\omega_{H_i}} = \bigcup_{(s_{\theta_1}, \alpha_{\theta_1}) \in H_1, \ldots, (s_{\theta_n}, \alpha_{\theta_n}) \in H_n} \Delta \left(\prod_{i=1}^{n} \Delta^{-1}(s_{\theta_i}, \alpha_{\theta_i})^{\omega_{H_i}} \right),$$

Let $\gamma = 1$, then the GHFTLWA operator reduces to the hesitant fuzzy 2-tuple linguistic weighted averaging (HFTLWA) operator

$$\text{HFTLWA}(H_1, H_2, \ldots, H_n)$$

$$= \bigoplus_{i=1}^{n} \omega_{H_i} H_i = \bigcup_{(s_{\theta_1}, \alpha_{\theta_1}) \in H_1, \ldots, (s_{\theta_n}, \alpha_{\theta_n}) \in H_n} \Delta \left(\sum_{i=1}^{n} \omega_{H_i} \Delta^{-1}(s_{\theta_i}, \alpha_{\theta_i}) \right).$$

Let $\gamma = 2$, then the GHFTLWA operator reduces to the hesitant fuzzy 2-tuple linguistic quadratic weighted averaging (HFTLQWA) operator

$$\text{HFTLQWA}(H_1, H_2, \ldots, H_n)$$

$$= \left(\bigoplus_{i=1}^{n} \omega_{H_i} H_i^2 \right)^{\frac{1}{2}} = \bigcup_{(s_{\theta_1}, \alpha_{\theta_1}) \in H_1, \ldots, (s_{\theta_n}, \alpha_{\theta_n}) \in H_n} \Delta \left(\left(\sum_{i=1}^{n} \omega_{H_i} \Delta^{-1}(s_{\theta_i}, \alpha_{\theta_i})^2 \right)^{\frac{1}{2}} \right).$$

Let $\gamma \to +\infty$, then the GHFTLWA operator reduces to the Max operator

$$\text{Max}(H_1, H_2, \ldots, H_n) = \bigcup_{(s_{\theta_1}, \alpha_{\theta_1}) \in H_1, \ldots, (s_{\theta_n}, \alpha_{\theta_n}) \in H_n} \left(\max_{i=1}^{n}(s_{\theta_i}, \alpha_{\theta_i}) \right),$$

and let $\gamma \to -\infty$, then the GHFTLWA operator reduces to the Min operator

$$\text{Min}(H_1, H_2, \ldots, H_n) = \bigcup_{(s_{\theta_1}, \alpha_{\theta_1}) \in H_1, \ldots, (s_{\theta_n}, \alpha_{\theta_n}) \in H_n} \left(\min_{i=1}^{n}(s_{\theta_i}, \alpha_{\theta_i}) \right).$$

Let $\gamma = -1$, then the GHFTLWA operator reduces to the hesitant fuzzy 2-tuple linguistic harmonic mean (HFTLHM) operator

$$\text{HFTLHM}(H_1, H_2, \ldots, H_n)$$

$$= \left(\bigoplus_{i=1}^{n} \frac{\omega_{H_i}}{H_i} \right)^{-1} = \bigcup_{(s_{\theta_1}, \alpha_{\theta_1}) \in H_1, \ldots, (s_{\theta_n}, \alpha_{\theta_n}) \in H_n} \Delta \left(\sum_{i=1}^{n} \frac{\omega_{H_i}}{\Delta^{-1}(s_{\theta_i}, \alpha_{\theta_i})} \right)^{-1}.$$

In a similar way to the GHFTLWA operator, the induced generalized hesitant fuzzy 2-tuple linguistic ordered weighted averaging (IG-HFTLOWA) operator is defined as follows:

DEFINITION 9. Let $H_i(i = 1, 2, \ldots, n)$ be a collection of HFTLTSs on the predefined linguistic term set S. The IG-HFTLOWA operator of dimension n is a mapping IG-HFTLOWA: HFTLTSs^n → HFTLTSs defined on the set of second arguments of two tuples $\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \ldots, \langle u_n, H_n \rangle$ with a set of order-inducing variables u_i ($i = 1, 2, \ldots, n$), denoted by

$$\text{IG-HFTLOWA}\left(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \ldots, \langle u_n, H_n \rangle \right) = \left(\bigoplus_{j=1}^{n} w_j H_{(j)}^{\gamma} \right)^{\frac{1}{\gamma}}$$

$$= \bigcup_{(s_{\theta_{(1)}}, \alpha_{\theta_{(1)}}) \in H_{(1)}, \ldots, (s_{\theta_{(n)}}, \alpha_{\theta_{(n)}}) \in H_{(n)}} \Delta \left(\left(\sum_{j=1}^{n} w_j \Delta^{-1}(s_{\theta_{(j)}}, \alpha_{\theta_{(j)}})^{\gamma} \right)^{\frac{1}{\gamma}} \right),$$

where $\gamma \in R^+$, $(\cdot)$ is a permutation on u_i $(i = 1, 2, \ldots, n)$ such that $u_{(j)}$ is the jth largest value of u_i, and w_j is the weight of the jth position with $w_j \geqslant 0$ and $\sum_{i=1}^{n} w_j = 1$.

Similar to the GHFTLWA operator, there are several special cases of the IG-HFTLOWA operator.

REMARK 4. Let $\gamma \to 0^+$, then the IG-HFTLOWA operator reduces to the induced hesitant fuzzy 2-tuple linguistic ordered geometric mean (I-HFTLOGM) operator

$$\text{I-HFTLOGM}\big(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \ldots, \langle u_n, H_n \rangle\big)$$

$$= \prod_{j=1}^{n} H_{(j)}^{w_j} = \bigcup_{(s_{\theta_{(1)}}, \alpha_{\theta_{(1)}}) \in H_{(1)}, \ldots, (s_{\theta_{(n)}}, \alpha_{\theta_{(n)}}) \in H_{(n)}} \Delta \left(\prod_{j=1}^{n} \Delta^{-1}(s_{\theta_{(j)}}, \alpha_{\theta_{(j)}})^{w_j} \right).$$

Let $\gamma = 1$, then the IG-HFTLOWA operator reduces to the induced hesitant fuzzy 2-tuple linguistic ordered weighted averaging (I-HFTLOWA) operator

$$\text{I-HFTLOWA}\big(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \ldots, \langle u_n, H_n \rangle\big)$$

$$= \bigoplus_{j=1}^{n} w_j H_{(j)} = \bigcup_{(s_{\theta_{(1)}}, \alpha_{\theta_{(1)}}) \in H_{(1)}, \ldots, (s_{\theta_{(n)}}, \alpha_{\theta_{(n)}}) \in H_{(n)}} \Delta \left(\sum_{j=1}^{n} w_j \Delta^{-1}(s_{\theta_{(j)}}, \alpha_{\theta_{(j)}}) \right).$$

Let $\gamma = 2$, then the IG-HFTLOWA operator reduces to the induced hesitant fuzzy 2-tuple linguistic quadratic ordered weighted averaging (I-HFTLQOWA) operator

$$\text{I-HFTLQOWA}\big(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \ldots, \langle u_n, H_n \rangle\big) = \left(\bigoplus_{j=1}^{n} w_j H_{(j)}^2 \right)^{\frac{1}{2}}$$

$$= \bigcup_{(s_{\theta_{(1)}}, \alpha_{\theta_{(1)}}) \in H_{(1)}, \ldots, (s_{\theta_{(n)}}, \alpha_{\theta_{(n)}}) \in H_{(n)}} \Delta \left(\left(\sum_{j=1}^{n} w_j \Delta^{-1}(s_{\theta_{(j)}}, \alpha_{\theta_{(j)}})^2 \right)^{\frac{1}{2}} \right).$$

Let $\gamma = -1$, then the IG-HFTLOWA operator reduces to the induced hesitant fuzzy 2-tuple linguistic ordered harmonic mean (I-HFTLOHM) operator

$$\text{I-HFTLOHM}\big(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \ldots, \langle u_n, H_n \rangle\big) = \left(\bigoplus_{j=1}^{n} \frac{w_j}{H_{(j)}} \right)^{-1}$$

$$= \bigcup_{(s_{\theta_{(1)}}, \alpha_{\theta_{(1)}}) \in H_{(1)}, \ldots, (s_{\theta_{(n)}}, \alpha_{\theta_{(n)}}) \in H_{(n)}} \Delta \left(\sum_{j=1}^{n} \frac{w_j}{\Delta^{-1}(s_{\theta_{(j)}}, \alpha_{\theta_{(j)}})} \right)^{-1}.$$

From Definitions 8 and 9, we know that the GHFTLWA operator only considers the importance of the attributes, while the IG-HFTLOWA operator gives the importance of the

ordered positions. To reflect these two aspects, we further introduce the induced generalized hesitant fuzzy 2-tuple linguistic hybrid weighted averaging (IG-HFTLHWA) operator as follows:

DEFINITION 10. Let H_i $(i = 1, 2, \ldots, n)$ be a collection of HFTLTSs on the predefined linguistic term set S. The IG-HFTLHWA operator of dimension n is a mapping IG-HFTLHWA: $\text{HFTLTSs}^n \to \text{HFTLTSs}$ defined on the set of second arguments of two tuples $\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \ldots, \langle u_n, H_n \rangle$ with a set of order-inducing variables u_i $(i = 1, 2, \ldots, n)$, denoted by

$$\text{IG-HFTLHWA}\big(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \ldots, \langle u_n, H_n \rangle\big) = \left(\bigoplus_{j=1}^{n} \frac{w_j \dot{H}_{(j)}^{\gamma}}{\sum_{j=1}^{n} w_j \omega_{H_{(j)}}^{\gamma}} \right)^{\frac{1}{\gamma}}$$

$$= \bigcup_{(s_{\theta_{(1)}}, \alpha_{\theta_{(1)}}) \in H_{(1)}, \ldots, (s_{\theta_{(n)}}, \alpha_{\theta_{(n)}}) \in H_{(n)}} \Delta \left(\left(\sum_{j=1}^{n} \frac{w_j (\omega_{H_{(j)}} \Delta^{-1}(s_{\theta_{(j)}}, \alpha_{\theta_{(j)}}))^{\gamma}}{\sum_{j=1}^{n} w_j \omega_{H_{(j)}}^{\gamma}} \right)^{\frac{1}{\gamma}} \right),$$

where $\gamma \in R^+$, $(\cdot)$ is a permutation on u_i $(i = 1, 2, \ldots, n)$ such that $u_{(j)}$ is the jth largest value of u_i, and w_j is the weight of the jth position with $w_j \geqslant 0$ and $\sum_{i=1}^{n} w_j = 1$, and ω_{H_i} is the weight of H_i the HFTLTS Hi with $omega_{H_i} \geqslant 0$ and $\sum_{i=1}^{n} \omega_{H_i} = 1$.

From Definition 10, it is easy to obtain the following special cases.

REMARK 5. Let $\omega_{H_i} = 1/n$, for each $i = 1, 2, \ldots, n$, then the IG-HFTLHWA operator reduces to the IG-HFTLOWA operator; Let $w_j = 1/n$ for each $j = 1, 2, \ldots, n$, and $u_i = u_j$ for all $i, j = 1, 2, \ldots, n$ with $i \neq j$, then the IG-HFTLHWA operator reduces to the GHFTLWA operator.

Let $\gamma = 1$, then the IG-HFTLHWA operator reduces to the induced hesitant fuzzy 2-tuple linguistic hybrid weighted averaging (I-HFTLHWA) operator

$$\text{I-HFTLHWA}\big(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \ldots, \langle u_n, H_n \rangle\big) = \bigoplus_{j=1}^{n} \frac{w_j \omega_{H_{(j)}} H_{(j)}}{\sum_{j=1}^{n} w_j \omega_{H_{(j)}}}$$

$$= \bigcup_{(s_{\theta_{(1)}}, \alpha_{\theta_{(1)}}) \in H_{(1)}, \ldots, (s_{\theta_{(n)}}, \alpha_{\theta_{(n)}}) \in H_{(n)}} \Delta \left(\sum_{j=1}^{n} \frac{w_j \omega_{H_{(j)}} \Delta^{-1}(s_{\theta_{(j)}}, \alpha_{\theta_{(j)}})}{\sum_{j=1}^{n} w_j \omega_{H_{(j)}}} \right).$$

Let $\gamma = 2$, then the IG-HFTLHWA operator reduces to the induced hesitant fuzzy 2-tuple linguistic quadratic hybrid weighted averaging (I-HFTLQHWA) operator

$$\text{I-HFTLQHWA}\big(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \ldots, \langle u_n, H_n \rangle\big) = \left(\bigoplus_{j=1}^{n} \frac{w_j \dot{H}_{(j)}^{2}}{\sum_{j=1}^{n} w_j \omega_{H_{(j)}}^{2}} \right)^{\frac{1}{2}}$$

$$= \bigcup_{(s_{\theta_{(1)}}, \alpha_{\theta_{(1)}}) \in H_{(1)}, \dots, (s_{\theta_{(n)}}, \alpha_{\theta_{(n)}}) \in H_{(n)}} \Delta\left(\left(\sum_{j=1}^{n} \frac{w_j(\omega_{H_{(j)}} \Delta^{-1}(s_{\theta_{(j)}}, \alpha_{\theta_{(j)}}))^2}{\sum_{j=1}^{n} w_j \omega_{H_{(j)}}^2}\right)^{\frac{1}{2}}\right).$$

Let $\gamma = -1$, then the IG-HFTLHWA operator reduces to the induced hesitant fuzzy 2-tuple linguistic hybrid harmonic mean (I-HFTLHHM) operator

$$\text{I-HFTLHHM}(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \dots, \langle u_n, H_n \rangle) = \left(\bigoplus_{j=1}^{n} \frac{w_j/\omega_{H_{(j)}}}{\sum_{j=1}^{n}(w_j/\omega_{H_{(j)}})H_{(j)}}\right)^{-1}$$

$$= \bigcup_{(s_{\theta_{(1)}}, \alpha_{\theta_{(1)}}) \in H_{(1)}, \dots, (s_{\theta_{(n)}}, \alpha_{\theta_{(n)}}) \in H_{(n)}} \Delta\left(\left(\sum_{j=1}^{n} \frac{w_j/\omega_{H_{(j)}}}{\sum_{j=1}^{n}(w_j/\omega_{H_{(j)}})\Delta^{-1}(s_{\theta_{(j)}}, \alpha_{\theta_{(j)}})}\right)^{-1}\right).$$

Similar to the quasi aggregation operator, we can also define the Quasi IG-HFTLHWA (QIG-HFTLHWA) operator as follows:

DEFINITION 11. Let H_i $(i = 1, 2, \dots, n)$ be a collection of HFTLTSs on the predefined linguistic term set S. The QIG-HFTLHWA operator of dimension n is a mapping QIG-HFTLHWA: HFTLTSs$^n \rightarrow$ HFTLTSs defined on the set of second arguments of two tuples $\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \dots, \langle u_n, H_n \rangle$ with a set of order-inducing variables u_i $(i = 1, 2, \dots, n)$, denoted by

$$\text{QIG-HFTLHWA}(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \dots, \langle u_n, H_n \rangle)$$

$$= \bigcup_{(s_{\theta_{(1)}}, \alpha_{\theta_{(1)}}) \in H_{(1)}, \dots, (s_{\theta_{(n)}}, \alpha_{\theta_{(n)}}) \in H_{(n)}} \Delta\left(g^{-1}\left(\sum_{j=1}^{n} \frac{w_j g(\omega_{H_{(j)}} \Delta^{-1}(s_{\theta_{(j)}}, \alpha_{\theta_{(j)}}))}{\sum_{j=1}^{n} w_j g(\omega_{H_{(j)}})}\right)\right),$$

where g is a strictly continuous monotonic function such that $g : [0, 1] arrow R, \gamma \in R^+$, $(\cdot)$ is a permutation on u_i $(i = 1, 2, \dots, n)$ such that $u_{(j)}$ is the jth largest value of u_i, and w_j is the weight of the jth position with $w_j \geqslant 0$ and $\sum_{i=1}^{n} w_j = 1$, and ω_{H_i} is the weight of H_i the HFTLTS Hi with $\omega_{H_i} \geqslant 0$ and $\sum_{i=1}^{n} \omega_{H_i} = 1$.

Let $g = x^\gamma$, $x \in [0, 1]$, then the QIG-HFTLHWA operator is the IG-HFTLHWA operator.

3.2. *Aggregation Operators Based on Fuzzy Measures*

In Section 3.1, all defined generalized hesitant fuzzy 2-tuple linguistic aggregation operators are based on the assumption that the elements in a set are independent. However, in some situations, there usually exist some degrees of correlations. To cope with this issue, researchers usually adopt the correlated aggregation operators to compute the

comprehensive attribute values. At present, there are two types of the correlated aggregation operators. One type is the Choquet aggregation operator (Meng *et al.*, 2014a, 2014b; Meng and Zhang, 2014; Xu, 2010), and the other type is the Shapley aggregation operator (Meng and Chen, 2014a, 2014b, 2014c; Meng *et al.*, 2014c, 2014d). Because the Shapley function globally considers the interdependence between elements (Meng and Chen, 2014a, 2014b, 2014c; Meng *et al.*, 2014c, 2014d), we define the induced generalized hesitant fuzzy 2-tuple linguistic hybrid Shapley averaging (IG-HFTLHSA) operator. First, let us consider the following expression of the Shapley function (Shapley, 1953):

$$Sh_i(\mu, N) = \sum_{T \subseteq N \setminus i} \frac{(n - t - 1)!t!}{n!} \big(\mu(T \cup i) - \mu(T) \big) \quad \forall i \in N, \tag{3}$$

where μ is a fuzzy measure on $N = \{1, 2, \ldots, n\}$ (Sugeno, 1974), s and n denote the cardinalities of T and N, respectively.

DEFINITION 12. Let H_i $(i = 1, 2, \ldots, n)$ be a collection of HFTLTSs on the predefined linguistic term set S. The IG-HFTLHSA operator of dimension n is a mapping IG-HFTLHSA: HFTLTSs^n → HFTLTSs defined on the set of second arguments of two tuples $\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \ldots, \langle u_n, H_n \rangle$ with a set of order-inducing variables u_i $(i = 1, 2, \ldots, n)$, denoted by

$$\text{IG-HFTLHSA}\big(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \ldots, \langle u_n, H_n \rangle \big)$$

$$= \left(\bigoplus_{j=1}^{n} \frac{Sh_j(\mu, N) \dot{H}_{(j)}^{\gamma}}{\sum_{j=1}^{n} Sh_j(\mu, N) Sh_{H_{(j)}}^{\gamma}(v, H)} \right)^{\frac{1}{\gamma}}$$

$$= \bigcup_{(s_{\theta(1)}, \alpha_{\theta(1)}) \in H_{(1)}, \ldots, (s_{\theta(n)}, \alpha_{\theta(n)}) \in H_{(n)}} \Delta \left(\left(\sum_{j=1}^{n} \frac{Sh_j(\mu, N)(Sh_{H_{(j)}}(v, H) \Delta^{-1}(s_{\theta(j)}, \alpha_{\theta(j)}))^{\gamma}}{\sum_{j=1}^{n} Sh_j(\mu, N) Sh_{H_{(j)}}^{\gamma}(v, H)} \right)^{\frac{1}{\gamma}} \right),$$

where $\gamma \in R^+$, $(\cdot)$ is a permutation on u_i $(i = 1, 2, \ldots, n)$ such that $u_{(j)}$ is the jth largest value of u_i, $Sh(v, H)$ is the Shapley function for the fuzzy measure v on $H = \{H_i\}_{i \in N}$, and $Sh(\mu, H)$ is the Shapley function for the fuzzy measure μ on the ordered set $N = \{1, 2, \ldots, n\}$.

From the expression of the Shapley function, it is easy to check that when v and μ are two additive measures, then the IG-GHFTLHSA operator is the IG-HFTLHWA operator.

REMARK 6. Let $Sh_{H_i}(v, H) = 1/n$ for each $i = 1, 2, \ldots, n$, then the IG-HFTLHSA operator reduces to the induced generalized hesitant fuzzy 2-tuple linguistic ordered Shapley averaging (IG-HFTLOSA) operator

$$\text{IG-HFTLOSA}\big(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \ldots, \langle u_n, H_n \rangle \big) = \left(\bigoplus_{j=1}^{n} Sh_j(\mu, N) H_{(j)}^{\gamma} \right)^{\frac{1}{\gamma}}$$

$$= \bigcup_{(s_{\theta_{(1)}}, \alpha_{\theta_{(1)}}) \in H_{(1)}, \dots, (s_{\theta_{(n)}}, \alpha_{\theta_{(n)}}) \in H_{(n)}} \Delta \left(\left(\sum_{j=1}^{n} Sh_j(\mu, N) \Delta^{-1}(s_{\theta_{(j)}}, \alpha_{\theta_{(j)}})^{\gamma} \right)^{\frac{1}{\gamma}} \right).$$

Let $\gamma = 1$, then the IG-HFTLHSA operator reduces to the induced hesitant fuzzy 2-tuple linguistic hybrid Shapley averaging (I-HFTLHSA) operator

$$\text{I-HFTLHSA}\big(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \dots, \langle u_n, H_n \rangle\big) = \bigoplus_{j=1}^{n} \frac{Sh_j(\mu, N) Sh_{H_{(j)}}(v, H) H_{(j)}}{\sum_{j=1}^{n} Sh_j(\mu, N) Sh_{H_{(j)}}(v, H)}$$

$$= \bigcup_{(s_{\theta_{(1)}}, \alpha_{\theta_{(1)}}) \in H_{(1)}, \dots, (s_{\theta_{(n)}}, \alpha_{\theta_{(n)}}) \in H_{(n)}} \Delta \left(\sum_{j=1}^{n} \frac{Sh_j(\mu, N) Sh_{H_{(j)}}(v, H) \Delta^{-1}(s_{\theta_{(j)}}, \alpha_{\theta_{(j)}})}{\sum_{j=1}^{n} Sh_j(\mu, N) Sh_{H_{(j)}}(v, H)} \right).$$

Let $\gamma = 2$, then the IG-HFTLHSA operator reduces to the induced hesitant fuzzy 2-tuple linguistic quadratic hybrid Shapley averaging (I-HFTLQHSA) operator

$$\text{I-HFTLQHSA}\big(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \dots, \langle u_n, H_n \rangle\big)$$

$$= \left(\bigoplus_{j=1}^{n} \frac{Sh_j(\mu, N) \dot{H}_{(j)}^2}{\sum_{j=1}^{n} Sh_j(\mu, N) Sh_{H_{(j)}}^{\gamma}(v, H)} \right)^{\frac{1}{2}}$$

$$= \bigcup_{(s_{\theta_{(1)}}, \alpha_{\theta_{(1)}}) \in H_{(1)}, \dots, (s_{\theta_{(n)}}, \alpha_{\theta_{(n)}}) \in H_{(n)}} \Delta \left(\left(\sum_{j=1}^{n} \frac{Sh_j(\mu, N)(Sh_{H_{(j)}}(v, H) \Delta^{-1}(s_{\theta_{(j)}}, \alpha_{\theta_{(j)}}))^2}{\sum_{j=1}^{n} Sh_j(\mu, N) Sh_{H_{(j)}}^{\gamma}(v, H)} \right)^{\frac{1}{2}} \right).$$

Let $\gamma = -1$, then the IG-HFTLHSA operator reduces to the induced hesitant fuzzy 2-tuple linguistic hybrid harmonic Shapley mean (I-HFTLHHSM) operator

$$\text{I-HFTLHHSM}\big(\langle u_1, H_1 \rangle, \langle u_2, H_2 \rangle, \dots, \langle u_n, H_n \rangle\big)$$

$$= \left(\bigoplus_{j=1}^{n} \frac{Sh_j(\mu, N)/Sh_{H_{(j)}}(v, H)}{\sum_{j=1}^{n} (Sh_j(\mu, N)/Sh_{H_{(j)}}(v, H)) H_{(j)}} \right)^{-1}$$

$$= \bigcup_{(s_{\theta_{(1)}}, \alpha_{\theta_{(1)}}) \in H_{(1)}, \dots, (s_{\theta_{(n)}}, \alpha_{\theta_{(n)}}) \in H_{(n)}} \Delta \left(\left(\sum_{j=1}^{n} \frac{Sh_j(\mu, N)/Sh_{H_{(j)}}(v, H)}{\sum_{j=1}^{n} (Sh_j(\mu, N)/Sh_{H_{(j)}}(v, H)) \Delta^{-1}(s_{\theta_{(j)}}, \alpha_{\theta_{(j)}})} \right)^{-1} \right).$$

4. Models for the Optimal Weight Vector

Because of various reasons such as the complexity of the decision-making problems, the time pressure, and the decision makers' limited decision expertise, the weight information may be not exactly known. As a hot research topic in decision-making theory, models for the weight vector have been researched by many researchers (Ma *et al.*, 2007; Massanet *et al.*, 2014; Martínez and Herrera, 2012; Merigó and Gil-Lafuente, 2013; Meng *et al.*, 2014a, 2014b; Meng and Zhang, 2014). This section continues to study models for the weight vector. First, let us consider a similarity degree of HFTLTSs.

4.1. *A Similarity Degree of HFTLTSs*

Let H_1 and H_2 be any two HFTLTSs on the predefined linguistic term set S. For any $(l_i, \alpha_i) \in H_1$, the distance between (l_i, α_i) and H_2 is defined by $d((l_i, \alpha_i), H_2) = \min_{(l_j, \alpha_j) \in H_2} |\Delta^{-1}(l_i, \alpha_i) - \Delta^{-1}(l_j, \alpha_j)|$.

DEFINITION 13. Let H_1 and H_2 be any two HFTLTSs on the predefined linguistic term set S. The distance from H_1 to H_2 is defined by

$$d(H_1, H_2) = \sum_{(l_i, \alpha_i) \in H_1} \frac{1}{h_1} d((l_i, \alpha_i), H_2)$$

and the distance from H_2 to H_1 is defined by

$$d(H_2, H_1) = \sum_{(l_j, \alpha_j) \in H_2} \frac{1}{h_2} d(H_1, (l_j, \alpha_j))$$

where h_1 and h_2 are the counts of H_1 and H_2, respectively.

From Definition 13, one can easily check that the distance between H_2 and H_1 can be denoted by $D(H_1, H_2) = \frac{d(H_1, H_2) + d(H_2, H_1)}{2}$. The similarity degree between HFTLTSs is defined as follows:

DEFINITION 14. Let H_1 and H_2 be any two HFTLTSs on the predefined linguistic term set S. The similarity degree between H_1 and H_2 is defined by

$$CC(H_1, H_2) = 1 - D(H_1, H_2). \tag{4}$$

Proposition 1. *The similarity degree CC, for any two HFTLTSs H_1 and H_2 on the predefined linguistic term set S, satisfies*

   (i) $CC(H_1, H_1) = 1$;
  (ii) $CC(H_1, H_2) = CC(H_2, H_1)$;
 (iii) $0 \leqslant CC(H_1, H_2) \leqslant 1$.

Corollary 1. *The distance D, for any two HFTLTSs H_1 and H_2 on the predefined linguistic term set S, satisfies*

   (i) $D(H_1, H_1) = 0$;
  (ii) $D(H_1, H_2) = D(H_2, H_1)$;
 (iii) $0 \leqslant D(H_1, H_2) \leqslant 1$.

EXAMPLE 1. Let $H_1 = \{(s_2, 0.05), (s_3, 0.07)\}$ and $H_2 = \{(s_3, 0.04), (s_4, 0.07), (s_5, 0.02)\}$ be two HFTLTSs on the predefined linguistic term set $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$. We have

$$d(H_1, H_2) = 0.0938$$

and

$$d(H_2, H_1) = 0.1601.$$

Thus, $D(H_1, H_2) = 0.117$. The similarity degree between H_1 and H_2 is $CC(H_1, H_2) = 0.873$.

For a given multi-granularity hesitant fuzzy linguistic group decision-making problem, without loss of generality, suppose there are m alternatives $A = \{a_1, a_2, \ldots, a_m\}$ and n attributes $C = \{c_1, c_2, \ldots, c_n\}$, which are judged by q decision makers $E = \{e_1, e_2, \ldots, e_q\}$. Let $S_j = \{s_i | i = 0, 1, \ldots, t_j\}$ be the predefined linguistic term set for the attribute $c_j \in C$, $j = 1, 2, \ldots, n$. Assume that $G^k = (G^k_{ij})_{m \times n}$ is the hesitant fuzzy linguistic decision matrix given by the decision maker e_k, where G^k_{ij} is the hesitant fuzzy linguistic judgement value of the alternative $a_i \in A$ for the attribute $c_j \in C$ on the predefined linguistic term set S_j. For brevity, let $M = \{1, 2, \ldots, m\}$, $N = \{1, 2, \ldots, n\}$ and $Q = \{1, 2, \ldots, q\}$.

4.2. *Models for the Optimal Weight Vectors on the Expert Set and on the Associated Ordered Set*

For each hesitant fuzzy linguistic decision matrix $G^k = (G^k_{ij})_{m \times n}$, $k \in Q$, we transform it into the hesitant fuzzy 2-tuple linguistic decision matrix $H^k = (H^k_{ij})_{m \times n}$ with $H^k_{ij} = \cup_{s^k_{i_j} \in H^k_{ij}} (s^k_{i_j}, 0)$ for each pair of (i, j). Calculate the similarity degree $CC(H^k_{ij}, H^l_{ij})$ between H^k_{ij} and H^l_{ij} for each pair of (i, j), where $k, l \in Q$ with $k \neq l$. When the weight information on the decision maker set is not exactly known, we build the following model for the optimal weight vector ω:

$$\max \sum_{k \in Q} \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{l \in Q, l \neq k} \omega_{e_k} CC(H^k_{ij}, H^l_{ij})$$

$$\text{s.t.} \begin{cases} \sum_{k \in Q} \omega_{e_k} = 1 \, A\omega \leqslant b \\ F\omega = d \\ \omega_{e_k} \geqslant 0, \ \omega_{e_k} \in W_{e_k}, \ k \in Q, \end{cases} \tag{5}$$

where W_{e_k} is the known weight information, and $A\omega \leqslant b$ and $F\omega = d$ are the known inequality and equality constraints, respectively.

When there are interactions between the decision makers, then the following model for the optimal fuzzy measure v^E on the decision maker set E is constructed:

$$\max \sum_{k \in Q} \sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{l \in Q, l \neq k} Sh_{e_k}(v^E, E) CC(H^k_{ij}, H^l_{ij})$$

$$\text{s.t.} \begin{cases} v^E(E) = 1, \\ \tilde{A}(v^E(S_1), v^E(S_1), \ldots, v^E(S_{p_1})) \leqslant \tilde{b}, \ S_r \subseteq E, \ r = 1, \ldots, p_1, \\ \tilde{F}(v^E(T_1), v^E(T_1), \ldots, v^E(T_{p_2})) = \tilde{d}, \ T_r \subseteq E, \ r = 1, \ldots, p_2, \\ v^E(S) \leqslant v^E(T) \ \forall S, T \subseteq E \text{ s.t. } S \subseteq T, \\ v^E(e_k) \in W_{e_k}, \ v^E(e_k) \geqslant 0, \ k \in Q, \end{cases} \tag{6}$$

where $Sh(v^E, E)$ is the Shapley function for the fuzzy measure v^E on the decision maker set E, and $\tilde{A}(v^E(S_1), v^E(S_1), \ldots, v^E(S_{p_1})) \leqslant \tilde{b}$ and $\tilde{F}(v^E(T_1), v^E(T_1), \ldots, v^E(T_{p_2})) = \tilde{d}$ are the known inequality and equality constraints, respectively.

Let $CC_{ij}^k = \sum_{l \in Q, l \neq k} CC(H_{ij}^k, H_{ij}^l)$, reorder CC_{ij}^k, $k = 1, 2, \ldots, q$, such that $CC_{ij}^{(1)} \geqslant CC_{ij}^{(2)} \geqslant \cdots \geqslant CC_{ij}^{(q)}$ for each pair of (i, j), where $(\cdot)$ is a permutation on Q. When the weight information on the ordered set Q is incompletely known, then we build the following model for the optimal weight vector w:

$$\max \sum_{k \in Q} \sum_{j=1}^{n} \sum_{i=1}^{m} w_k CC_{ij}^{(k)}$$

$$\text{s.t.} \begin{cases} \sum_{k \in Q} w_k = 1, \\ A'w \leqslant b', \\ F'w = d', \\ w_k \geqslant 0, \ w_k \in W_k, \ k \in Q, \end{cases} \tag{7}$$

where W_k is the known weight information, and $A'w \leqslant b'$ and $F'w = d'$ are the known inequality and equality constraints, respectively.

Considering interactions between the ordered positions, model for the optimal fuzzy measure μ^Q on the ordered set Q is constructed as follows:

$$\max \sum_{k \in Q} \sum_{j=1}^{n} \sum_{i=1}^{m} Sh_k(\mu^Q, Q) CC_{ij}^{(k)}$$

$$\text{s.t.} \begin{cases} \mu^Q(Q) = 1, \\ \tilde{A}'(\mu^Q(S_1), \mu^Q(S_2), \ldots, \mu^Q(S_{t_1})) \leqslant \tilde{b}', \ S_r \subseteq Q, \ r = 1, \ldots, t_1, \\ \tilde{F}'(\mu^Q(T_1), \mu^Q(T_2), \ldots, \mu^Q(T_{t_2})) = \tilde{d}', \ T_r \subseteq Q, \ r = 1, \ldots, t_2, \\ \mu^Q(S) \leqslant \mu^Q(T) \ \forall S, T \subseteq Q \text{ s.t. } S \subseteq T, \\ \mu^Q(k) \in W_k, \ \mu^Q(k) \geqslant 0, \ k \in Q, \end{cases} \tag{8}$$

where $Sh(\mu^Q, Q)$ is the Shapley function for the fuzzy measure μ^Q on the ordered set Q and $\tilde{A}'(\mu^Q(S_1), \mu^Q(S_2), \ldots, \mu^Q(S_{t_1})) \leqslant \tilde{b}'$ and $\tilde{F}'(\mu^Q(T_1), \mu^Q(T_2), \ldots, \mu^Q(T_{t_2})) = \tilde{d}'$ are the known inequality and equality constraints, respectively.

4.3. *Models for the Optimal Weight Vectors on the Attribute Set and on the Associated Ordered Set*

Suppose that $H = (H_{ij})_{m \times n}$ is the comprehensive hesitant fuzzy 2-tuple linguistic decision matrix. Let $H^+ = (H_1^+, H_2^+, \ldots, H_n^+)$ and $H^+ = (H_1^-, H_2^-, \ldots, H_n^-)$ be the positive and negative hesitant fuzzy 2-tuple linguistic vectors, respectively, where $H_j^+ = \max_{i=1}^n H_{ij}$ and $H_j^- = \min_{i=1}^n H_{ij}$ for each $j \in N$. Calculate the similarity degrees $CC(H_j^+, H_{ij})$ and $CC(H_j^-, H_{ij})$ for each pair of (i, j).

When the weight information of the attributes is not exactly known, then we build the following model for the optimal weight vector ω:

$$\max \sum_{i=1}^m \sum_{j=1}^n \omega_{c_j} \frac{CC(H_j^+, H_{ij})}{CC(H_j^+, H_{ij}) + CC(H_j^-, H_{ij})}$$

$$\text{s.t.} \begin{cases} \sum_{j \in N} \omega_{c_j} = 1, \\ R\omega \leqslant \alpha, \\ P\omega = \beta, \\ \omega_{c_j} \geqslant 0, \ \omega_{c_j} \in W_{c_j}, \ j \in N, \end{cases} \quad (9)$$

where W_{c_j} is the known weight information, and $R\omega \leqslant \alpha$ and $P\omega = \beta$ are the known inequality and equality constraints, respectively.

When there are correlations between the attributes, the following model for the optimal fuzzy measure v^C on the attribute set C is constructed:

$$\max \sum_{i=1}^m \sum_{j=1}^n Sh_{c_j}(v^C, C) \frac{CC(H_j^+, H_{ij})}{CC(H_j^+, H_{ij}) + CC(H_j^-, H_{ij})}$$

$$\text{s.t.} \begin{cases} v^C(C) = 1, \\ \tilde{R}(v^C(S_1), v^C(S_1), \ldots, v^C(S_{d_1})) \leqslant \tilde{\alpha}, \ S_r \subseteq C, \ r = 1, \ldots, d_1, \\ \tilde{P}(v^C(T_1), v^C(T_1), \ldots, v^C(T_{d_2})) = \tilde{\beta}, \ T_r \subseteq C, \ r = 1, \ldots, d_2, \\ v^C(S) \leqslant v^C(T) \ \forall S, T \subseteq C \ \text{s.t.} \ S \subseteq T, \\ v^C(c_j) \in W_{c_j}, \ v^C(c_j) \geqslant 0, \ j \in N, \end{cases} \quad (10)$$

where $Sh(v^C, C)$ is the Shapley function for the fuzzy measure v^C on the attribute set C, and $\tilde{R}(v^C(S_1), v^C(S_1), \ldots, v^C(S_{d_1})) \leqslant \tilde{\alpha}$ and $\tilde{P}(v^C(T_1), v^C(T_1), \ldots, v^C(T_{d_2})) = \tilde{\beta}$ are the known inequality and equality constraints, respectively.

Let $H_\omega = (\dot{H}_{ij})_{m \times n}$ be the weighted comprehensive matrix for $H = (H_{ij})_{m \times n}$ with

$$\dot{H}_{ij} = \bigcup_{(s_{ij}, \alpha_{ij}) \in H_{ij}} \Delta \big(\omega_{c_j} \Delta^{-1}(s_{ij}, \alpha_{ij}) \big).$$

Let $\dot{H}^+ = (\dot{H}_1^+, \dot{H}_2^+, \ldots, \dot{H}_n^+)$ and $\dot{H}^- = (\dot{H}_1^-, \dot{H}_2^-, \ldots, \dot{H}_n^-)$ be the positive and negative hesitant fuzzy 2-tuple linguistic vectors, respectively, where $\dot{H}_j^+ = \max_{i=1}^n \dot{H}_{ij}$

and $\dot{H}_j^- = \min_{i=1}^n \dot{H}_{ij}$ for each $j \in N$. Calculate the similarity degrees $CC(\dot{H}_j^+, \dot{H}_{ij})$ and $CC(\dot{H}_j^-, \dot{H}_{ij})$ for each pair of (i, j).

Let $CC_{ij} = \frac{CC(\dot{H}_j^+, \dot{H}_{ij})}{CC(\dot{H}_j^+, \dot{H}_{ij}) + CC(\dot{H}_j^-, \dot{H}_{ij})}$. For each $i \in M$, reorder CC_{ij}, $j = 1, 2, \ldots, n$, such that $CC_{i(1)} \geqslant CC_{i(2)} \geqslant \cdots \geqslant CC_{i(n)}$, where $(\cdot)$ is a permutation on N. When the weight information on the ordered set N is incompletely known, then we build the following model for the optimal weight vector w:

$$\max \sum_{i=1}^m \sum_{j=1}^n w_j CC_{i(j)}$$

$$\text{s.t.} \begin{cases} \sum_{j \in N} w_j = 1, \\ R'\omega \leqslant \alpha', \\ P'\omega = \beta', \\ w_j \geqslant 0, w_j \in W_j, j \in N, \end{cases} \tag{11}$$

where W_j is the known weight information, and $R'\omega \leqslant \alpha'$ and $P'\omega = \beta'$ are the known inequality and equality constraints, respectively.

Considering correlations between the ordered positions in N, calculate the Shapley weighted matrix $H_{Sh} = (\ddot{H}_{ij})_{m \times n}$ with $\ddot{H}_{ij} = \cup_{(s_{ij}, \alpha_{ij}) \in H_{ij}} \Delta(Sh_{c_j}(v^C, C) \Delta^{-1}(s_{ij}, \alpha_{ij}))$. Let $\ddot{H}^+ = (\ddot{H}_1^+, \ddot{H}_2^+, \ldots, \ddot{H}_n^+)$ and $\ddot{H}^- = (\ddot{H}_1^-, \ddot{H}_2^-, \ldots, \ddot{H}_n^-)$ be the positive and negative hesitant fuzzy 2-tuple linguistic vectors, respectively, where $\ddot{H}_j^+ = \max_{i=1}^n \ddot{H}_{ij}$ and $\ddot{H}_j^- = \min_{i=1}^n \ddot{H}_{ij}$ for each $j \in N$. Calculate the similarity degrees $CC(\ddot{H}_j^+, \ddot{H}_{ij})$ and $CC(\ddot{H}_j^-, \ddot{H}_{ij})$ for each pair (i, j). For each $i \in M$, reorder $\dot{C}\dot{C}_{ij}$, $j = 1, 2, \ldots, n$, such that $\dot{C}\dot{C}_{i(1)} \geqslant \dot{C}\dot{C}_{i(2)} \geqslant \cdots \geqslant \dot{C}\dot{C}_{i(n)}$, where $\dot{C}\dot{C}_{ij} = \frac{CC(\dot{H}_j^+, \dot{H}_{ij})}{CC(\dot{H}_j^+, \dot{H}_{ij}) + CC(\dot{H}_j^-, \dot{H}_{ij})}$ and $(\cdot)$ is a permutation on N. Then, we build the following model for the optimal fuzzy measure μ^N on the ordered set N:

$$\max \sum_{i=1}^m \sum_{j=1}^n Sh_j(\mu^N, N) \dot{C}\dot{C}_{i(j)}$$

$$\text{s.t.} \begin{cases} \mu^N(N) = 1, \\ \tilde{R}'(\mu^N(S_1), \mu^N(S_2), \ldots, \mu^N(S_{h_1})) \leqslant \tilde{\alpha}', \ S_r \subseteq N, \ r = 1, \ldots, h_1, \\ \tilde{P}'(\mu^N(T_1), \mu^N(T_2), \ldots, \mu^N(T_{h_2})) = \tilde{\beta}', \ T_r \subseteq N, \ r = 1, \ldots, h_2, \\ \mu^N(S) \leqslant \mu^N(T) \ \forall S, T \subseteq N \text{ s.t. } S \subseteq T, \\ \mu^N(j) \in W_j, \ \mu^N(j) \geqslant 0, \ j \in N, \end{cases} \tag{12}$$

where $Sh(\mu^N, N)$ is the Shapley function for the fuzzy measure μ^N on the ordered set N, and $\tilde{R}'(\mu^N(S_1), \mu^N(S_2), \ldots, \mu^N(S_{h_1})) \leqslant \tilde{\alpha}'$ and $\tilde{P}'(\mu^N(T_1), \mu^N(T_2), \ldots, \mu^N(T_{h_2})) = \tilde{\beta}'$ are the known inequality and equality constraints, respectively.

5. An Approach to Multi-Granularity Hesitant Fuzzy Linguistic Group Decision Making

This section considers a decision-making method to multi-granularity hesitant fuzzy linguistic group decision making by using the defined aggregation operators and the built models. The main decision steps are involved as follows:

Step 1: Transform the hesitant fuzzy linguistic decision matrix $G^k = (G_{ij}^k)_{m \times n}$ into
$$R^k = (R_{ij}^k)_{m \times n}, \text{ where } R_{ij}^k = \begin{cases} G_{ij}^k & \text{for benefit attribute } c_j \\ (G_{ij}^k)^c & \text{for cost attribute } c_j \end{cases} \text{ with } (G_{ij}^k)^c = \bigcup_{s_{ij}^k \in G_{ij}^k} \{s_{t_j - i_j}^k\}, i \in M, j \in N.$$

Step 2: Convert the hesitant fuzzy linguistic decision matrix $R^k = (R_{ij}^k)_{m \times n}$ $(k \in Q)$ into the hesitant fuzzy 2-tuple linguistic decision matrix $H^k = (H_{ij}^k)_{m \times n}$ with $H_{ij}^k = \bigcup_{s_{ij}^k \in H_{ij}^k} (s_{ij}^k, 0)$ for each pair (i, j), where s_{ij}^k belongs to the predefined linguistic term set S_j with respect to the attribute c_j.

Step 3: When the weight information on the decision maker set is not exactly known, we utilize model (7) (or (6)) to calculate the optimal weight vector.

Step 4: When the weight information on the ordered position set is not exactly known, we utilize model (9) (or (8)) to calculate the optimal weight vector.

Step 5: Use the IG-HFTLHSA (or IG-HFTLHWA) operator to calculate the comprehensive hesitant fuzzy 2-tuple linguistic decision matrix $H = (H_{ij})_{m \times n}$.

Step 6: When the weight information on the attribute set is not exactly known, we apply model (11) (or (10)) to calculate the optimal weight vector.

Step 7: When the weight information on the ordered position set is not exactly known, we apply model (13) (or (12)) to calculate the optimal weight vector.

Step 8: Again use the IG-HFTLHSA (or IG-HFTLHWA) operator to calculate the comprehensive hesitant fuzzy 2-tuple linguistic term set H_i of the alternative $a_i, i \in M$.

Step 9: According to the comprehensive hesitant fuzzy 2-tuple linguistic term set H_i, calculate the expected value $E(H_i)$ and the variance $V(H_i)$. Then, rank the value H_i, $i \in M$, and select the best choice.

Step 10: End.

EXAMPLE 2. Let us consider the multi-granularity hesitant fuzzy linguistic decision-making problem of evaluating investment. Suppose that there is an investment company, which intends to invest a sum of money in the best option (Tan, 2011). There is a panel with four possible alternatives to invest the money: a car company a_1, a food company a_2, a computer company a_3, and a TV company a_4. The investment company must take a decision according to the following four attributes: the risk factor c_1, the growth factor c_2, the social-political impact c_3, and the environmental impact c_4. The four possible alternatives are evaluated by three decision makers $E = \{e_1, e_2, e_3\}$ under the above attributes. With respect to these four attributes, their evaluation on alternatives by using the different linguistic term sets is as follows:

$S_1 = S_4 = \{s_0$: very small; s_1: small; s_2: fair; s_3: big; s_4: very big$\}$;

$S_2 = \{s_0$: extremely slow; s_1: very slow; s_2: slow; s_3: fair; s_4: fast; s_5: very fast; s_6: extremely fast$\}$;

$S_3 = \{s_0$: little; s_1: fair; s_2: much$\}$.

The evaluation information given by these three decision makers is given in the following matrices:

$$G^1 = \begin{pmatrix} \{s_2, s_3\} & \{s_5\} & \{s_1, s_2\} & \{s_3, s_4\} \\ \{s_3\} & \{s_2, s_3\} & \{s_0\} & \{s_1\} \\ \{s_1, s_2\} & \{s_2, s_3, s_4\} & \{s_1\} & \{s_2, s_3\} \\ \{s_2\} & \{s_3\} & \{s_1\} & \{s_2\} \end{pmatrix},$$

$$G^2 = \begin{pmatrix} \{s_3\} & \{s_4, s_5\} & \{s_2\} & \{s_4\} \\ \{s_2, s_3\} & \{s_4\} & \{s_1\} & \{s_0, s_1\} \\ \{s_3\} & \{s_4\} & \{s_0\} & \{s_3\} \\ \{s_3, s_4\} & \{s_2, s_3\} & \{s_1\} & \{s_2, s_3\} \end{pmatrix},$$

$$G^3 = \begin{pmatrix} \{s_3, s_4\} & \{s_5, s_6\} & \{s_2\} & \{s_3\} \\ \{s_0, s_1, s_2\} & \{s_3, s_4\} & \{s_0\} & \{s_2\} \\ \{s_2, s_3\} & \{s_4\} & \{s_2\} & \{s_2\} \\ \{s_3, s_4\} & \{s_2\} & \{s_1\} & \{s_1\} \end{pmatrix}.$$

Based on the decision makers' reputation, experience and decision expertise, the weight information on the decision maker set E is defined by

$$\omega_{e_1} \geqslant \omega_{e_2}, \ \omega_{e_1} \geqslant \omega_{e_3}, \ \omega_{e_1} \leqslant 0.5, \ 0.2 \leqslant \omega_{e_2} \leqslant 0.4, \ 0.2 \leqslant \omega_{e_3} \leqslant 0.3. \tag{13}$$

Namely, the importance of the decision maker e_1 is no smaller than that of the decision maker e_2 or e_3. The importance of the decision maker e_1 is no bigger than the sum of the other two decision makers'. Compared with the other two decision makers, the percentage of the importance of the decision maker e_2 is given between 20% and 40%, and the percentage of the importance of the decision maker e_3 is given between 20% and 30%.

Based on the principle of the larger similarity degree the bigger weight, the weight information on the ordered set $Q = \{1, 2, 3\}$ is defined by

$$0.2 \geqslant w_1 - w_2 \geqslant 0.1, \ 0.2 \geqslant w_2 - w_3 \geqslant 0.1, \ 0.2 \leqslant w_3 \leqslant 0.3, \ w_1 \leqslant 0.5. \tag{14}$$

Namely, the difference between any two adjacent positions belongs to $[0.1, 0.2]$, and the importance of the 3th position belongs to $[0.2, 0.3]$. Furthermore, the importance of the 1st position is no bigger than the sum of the other two positions.

These four companies belong to one country, whose government always attaches a greater importance to environmental protection than any other factor. However, the importance of environment is no bigger than the sum of the other three attributes' importance. Furthermore, this country has a stable social-political environment, which means that the influence of the social-political factor is smaller than that of the risk or growth

factor. With respect to the other three attributes, the percentage of the importance of the social-political factor is given between 10% and 20%. For the risk and growth factors, since it is difficult to decide which is more important, it assumes that their importance is equal and defined between 20% and 40%. Based on these facts, the weight information of the attributes is given as follows:

$$0 \leqslant \omega_{c_4} - \omega_{c_1} \leqslant 0.1, \ 0 \leqslant \omega_{c_1} - \omega_{c_3} \leqslant 0.1, \ 0.2 \leqslant \omega_{c_1} = \omega_{c_2} \leqslant 0.4,$$
$$0.1 \leqslant \omega_{c_3} \leqslant 0.2, \ \omega_{c_4} \leqslant 0.5. \tag{15}$$

Similar to the weight information on the ordered set Q, the weight information on the ordered set $N = \{1, 2, 3, 4\}$ is given by

$$0.2 \geqslant w_1 - w_2 \geqslant 0.1, \ 0.2 \geqslant w_2 - w_3 \geqslant 0.1, \ 0.2 \geqslant w_3 - w_4 \geqslant 0.1,$$
$$0.1 \leqslant w_4 \leqslant 0.2, \ w_1 \leqslant 0.5. \tag{16}$$

To obtain the most desirable alternative(s), the following procedure is involved.

Step 1: Transform the hesitant fuzzy linguistic decision matrix $G^k = (G_{ij}^k)_{4\times4}$ into $R^k = (R_{ij}^k)_{4\times4}$ ($k \in Q$), denoted by

$$R^1 = \begin{pmatrix} \{s_1, s_2\} & \{s_5\} & \{s_0, s_1\} & \{s_0, s_1\} \\ \{s_1\} & \{s_2, s_3\} & \{s_2\} & \{s_3\} \\ \{s_2, s_3\} & \{s_2, s_3, s_4\} & \{s_1\} & \{s_1, s_2\} \\ \{s_2\} & \{s_3\} & \{s_1\} & \{s_2\} \end{pmatrix},$$

$$R^2 = \begin{pmatrix} \{s_1\} & \{s_4, s_5\} & \{s_0\} & \{s_0\} \\ \{s_1, s_2\} & \{s_4\} & \{s_1\} & \{s_3, s_4\} \\ \{s_1\} & \{s_4\} & \{s_2\} & \{s_1\} \\ \{s_0, s_1\} & \{s_2, s_3\} & \{s_1\} & \{s_1, s_2\} \end{pmatrix},$$

$$R^3 = \begin{pmatrix} \{s_0, s_1\} & \{s_5, s_6\} & \{s_0\} & \{s_1\} \\ \{s_2, s_3, s_4\} & \{s_3, s_4\} & \{s_2\} & \{s_2\} \\ \{s_1, s_2\} & \{s_4\} & \{s_0\} & \{s_2\} \\ \{s_0, s_1\} & \{s_2\} & \{s_1\} & \{s_3\} \end{pmatrix}.$$

Step 2: Convert the hesitant fuzzy linguistic decision matrix $R^k = (R_{ij}^k)_{4\times4}$ ($k \in Q$) into the hesitant fuzzy 2-tuple linguistic decision matrix, take $k = 1$ for example,

$$H^1 = \begin{pmatrix} \{(s_1, 0), (s_2, 0)\} & \{(s_5, 0)\} & \{(s_0, 0), (s_1, 0)\} & \{(s_0, 0), (s_1, 0)\} \\ \{(s_1, 0)\} & \{(s_2, 0), (s_3, 0)\} & \{(s_2, 0)\} & \{(s_3, 0)\} \\ \{(s_2, 0), (s_3, 0)\} & \{(s_2, 0), (s_3, 0), (s_4, 0)\} & \{(s_1, 0)\} & \{(s_1, 0), (s_2, 0)\} \\ \{(s_2, 0)\} & \{(s_3, 0)\} & \{(s_1, 0)\} & \{(s_2, 0)\} \end{pmatrix}.$$

Step 3: According to the model (7) and the condition (14), the following linear programming model for the optimal fuzzy measure v^E on the decision maker set E is built:

$$\max 0.191\big(v^E(e_1) - v^E(e_2, e_3)\big) - 0.08\big(v^E(e_2) - v^E(e_1, e_3)\big)$$
$$- 0.11\big(v^E(e_3) - v^E(e_1, e_2)\big) + 26.56$$

$$\text{s.t.} \begin{cases} v^E(e_1) \geqslant v^E(e_2), \\ v^E(e_1) \geqslant v^E(e_3), \\ v^E(e_1) \leqslant 0.5, \\ v^E(S) - v^E(T) \leqslant 0, \ S, T \subseteq E \text{ s.t. } S \subseteq T, \\ v^E(e_2) \in [0.2, 0.4], \ v^E(e_3) \in [0.2, 0.3]. \end{cases}$$

Solving the above model, we derive

$$v^E(e_1) = 0.5, \quad v^E(e_2) = v^E(e_3) = v^E(e_2, e_3) = 0.2,$$
$$v^E(e_1, e_2) = v^E(e_1, e_3) = v^E(e_1, e_2, e_3) = 1.$$

According to the fuzzy measure v^E, formula yields $Sh_{e_1}(v^E, E) = 0.7$, $Sh_{e_2}(v^E, E) = Sh_{e_3}(v^E, E) = 0.15$.

Step 4: From model (9) and the condition (15), the following linear programming model for the optimal fuzzy measure μ^Q on the ordered set Q is built:

$$\max 0.847\big(\mu^Q(1) - \mu^Q(2, 3)\big) - 0.017\big(\mu^Q(2) - \mu^Q(1, 3)\big)$$
$$- 0.83\big(\mu^Q(3) - \mu^Q(1, 2)\big) + 26.56$$

$$\text{s.t.} \begin{cases} \mu^Q(1 + j) - \mu^Q(2 + j) \geqslant 0.1, j = 0, 1, \\ \mu^Q(1 + j) - \mu^Q(2 + j) \leqslant 0.2, j = 0, 1, \\ \mu^Q(1) \leqslant 0.5, \\ \mu^Q(S) - \mu^Q(T) \leqslant 0, \ S, T \subseteq Q \text{ s.t. } S \subseteq T, \\ \mu^Q(3) \in [0.2, 0.3]. \end{cases}$$

Solving the above model, we have

$$\mu^Q(1) = 0.5, \quad \mu^Q(2) = \mu^Q(2, 3) = 0.3, \quad \mu^Q(3) = 0.2,$$
$$\mu^Q(1, 2) = \mu^Q(1, 3) = \mu^Q(1, 2, 3) = 1.$$

According to the fuzzy measure μ^Q, formula yields $Sh_1(\mu^Q, Q) = 0.65$, $Sh_2(\mu^Q, Q) = 0.2$, $Sh_3(\mu^Q, Q) = 0.15$.

Step 5: For each pair of (i, j), let $u_k = CC_{ij}^k$, $k = 1, 2, 3$. Let $\gamma = 2$, using the IG-HFTLHSA operator, the comprehensive hesitant fuzzy 2-tuple linguistic decision matrix

H is obtained as follows:

$$H = \begin{pmatrix} \{(s_1, -0.015), (s_1, 0), (s_2, -0.12), (s_2, -0.13)\} \\ \{(s_1, 0.043), (s_1, 0.11), (s_1, 0.199), (s_1, 0.21), (s_1, 0.269), (s_1, 0.347)\} \\ \{(s_2, -0.161), (s_2, -0.037), (s_2, 0.711), (s_3, -0.203)\} \\ \{(s_2, -0.219), (s_2, -0.175), (s_2, -0.205), (s_2, -0.161)\} \\ \{(s_5, -0.012), (s_5, -0.001), (s_5, 0), (s_5, 0.011)\} \\ \{(s_2, 0.318), (s_3, -0.455), (s_3, 0.056), (s_3, 0.232)\} \\ \{(s_3, -0.455), (s_3, 0.232), (s_4, 0)\} \\ \{(s_3, -0.177), (s_3, -0.135)\} \\ \{(s_0, 0), (s_1, -0.109)\} \quad \{(s_0, 0.102), (s_1, -0.007)\} \\ \{(s_2, -0.008)\} \qquad \{(s_3, -0.009), (s_3, 0.007)\} \\ \{(s_1, 0.015)\} \qquad \{(s_1, 0.015), (s_2, -0.01)\} \\ \{(s_1, 0)\} \qquad \{(s_2, 0.003), (s_2, 0.013)\} \end{pmatrix}.$$

Step 6: Because the risk factor c_1 and the growth factor c_2 are considered to have the same importance, we have $v^C(c_1, c_j) = v^C(c_2, c_j)$, $j = 3, 4$, and $C(c_1, c_3, c_4) = v^C(c_2, c_3, c_4)$. From the comprehensive decision matrix H, model (11) and the condition (16), the following linear programming model for the optimal fuzzy measure v^C on the attribute set C is built:

$$\max 0.037\big(v^C(c_1) - v^C(c_2, c_3, c_4)\big) - 0.016\big(v^C(c_2) - v^C(c_1, c_3, c_4)\big)$$
$$- 0.045\big(v^C(c_3) - v^C(c_1, c_2, c_4)\big) - 0.023\big(v^C(c_4) - v^C(c_1, c_2, c_3)\big)$$
$$+ 0.011\big(v^C(c_1, c_2) - v^C(c_3, c_4)\big) - 0.004\big(v^C(c_1, c_3) - v^C(c_2, c_4)\big)$$
$$+ 0.03\big(v^C(c_1, c_4) - v^C(c_2, c_3)\big) + 1.892$$

$$\text{s.t.} \begin{cases} v^C(c_4) - v^C(c_1) \geqslant 0, \\ v^C(c_4) - v^C(c_1) \leqslant 0.1, \\ v^C(c_1) - v^C(c_3) \geqslant 0, \\ v^C(c_1) - v^C(c_3) \leqslant 0.1, \\ v^C(c_1) - v^C(c_2) = 0, \\ v^C(c_1, c_j) - v^C(c_2, c_j) = 0, \ j = 3, 4, \\ v^C(c_1, c_3, c_4) - v^C(c_2, c_3, c_4) = 0, \\ v^C(c_4) \leqslant 0.5, \\ v^C(S) - v^C(T) \leqslant 0, \ S, T \subseteq C \text{ s.t. } S \subseteq T, \\ v^C(c_1), v^C(c_2) \in [0.2, 0.4], \ v^C(c_3) \in [0.1, 0.2]. \end{cases}$$

Solving the above model, we derive

$$v^C(c_1) = v^C(c_2) = v^C(c_1, c_2) = v^C(c_1, c_3) = v^C(c_2, c_3) = v^C(c_1, c_2, c_3) = 0.2,$$
$$v^C(c_3) = 0.1, \ v^C(c_4) = v^C(c_3, c_4) = 0.3,$$
$$v^C(c_1, c_4) = v^C(c_2, c_4) = v^C(c_1, c_2, c_4) = v^C(c_1, c_3, c_4) = v^C(c_2, c_3, c_4)$$
$$= v^C(C) = 1.$$

According to the fuzzy measure v^C, formula yields

$$Sh_{c_1}(v^C, C) = Sh_{c_2}(v^C, C) = 0.1, \quad Sh_{c_3}(v^C, C) = 0.025, \quad Sh_{c_4}(v^C, C) = 0.625.$$

Step 7: From model (13) and the condition (17), the following linear programming model for the optimal fuzzy measure μ^N on the ordered set N is built:

$$\max 0.112(\mu^N(1) - \mu^N(2, 3, 4)) - 0.019(\mu^N(2) - \mu^N(1, 3, 4))$$
$$- 0.025(\mu^N(3) - \mu^N(1, 2, 4)) - 0.069(\mu^N(4) - \mu^N(1, 2, 3))$$
$$+ 0.047(\mu^N(1, 2) - \mu^N(3, 4)) + 0.044(\mu^N(1, 3) - \mu^N(2, 4))$$
$$+ 0.022(\mu^N(1, 4) - \mu^N(2, 3)) + 1.434$$

$$\text{s.t.} \begin{cases} \mu^N(1 + j) - \mu^N(2 + j) \geqslant 0.1, \ j = 0, 1, 2, \\ \mu^N(1 + j) - \mu^N(2 + j) \leqslant 0.2, \ j = 0, 1, 2, \\ \mu^N(1) \leqslant 0.5, \\ \mu^N(S) - \mu^N(T) \leqslant 0, \ S, T \subseteq N \text{ s.t. } S \subseteq T, \\ \mu^Q(4) \in [0.1, 0.2]. \end{cases}$$

Solving the above model, we derive

$$\mu^N(1) = 0.5, \quad \mu^N(2) = \mu^N(2, 3) = \mu^N(2, 4) = \mu^N(2, 3, 4) = 0.3,$$
$$\mu^N(3) = \mu^N(3, 4) = 0.2, \quad \mu^N(4) = 0.1,$$
$$\mu^N(1, 2) = \mu^N(1, 3) = \mu^N(1, 4) = \mu^N(1, 2, 3) = \mu^N(1, 2, 4) = \mu^N(1, 3, 4)$$
$$= \mu^N(N) = 1.$$

According to the fuzzy measure μ^N, formula yields

$$Sh_1(\mu^N, N) = 0.683, \quad Sh_2(\mu^N, N) = 0.15,$$
$$Sh_3(\mu^N, N) = 0.1, \quad Sh_4(\mu^N, N) = 0.07.$$

Step 8: Without loss of generality, let $S = \{s_0, s_1, \ldots, s_6\}$. Furthermore, for each i, let $u_j = \frac{CC(H_j^+, H_{ij})}{CC(H_j^+, H_{ij}) + CC(H_j^-, H_{ij})}$, $j = 1, 2, 3, 4$. Let $\gamma = 2$, using the IG-HFTLHSA operator, the comprehensive HFTLTS H_i of the alternative a_i ($i = 1, 2, 3, 4$) is obtained. Take H_4 for example,

$$H_4 = \{(s_3, 0.11), (s_3, 0.1), (s_3, 0.109), (s_3, 0.098), (s_3, 0.089), (s_3, 0.079),$$
$$(s_3, 0.088), (s_3, 0.078), (s_3, 0.103), (s_3, 0.093), (s_3, 0.102), (s_3, 0.092),$$
$$(s_3, 0.083), (s_3, 0.073), (s_3, 0.081), (s_2, 0.071)\}.$$

Step 9: According to the comprehensive hesitant fuzzy 2-tuple linguistic term sets H_i ($i = 1, 2, 3, 4$), the expected values are

$$E(H_1) = 0.503, \quad E(H_2) = 0.729, \quad E(H_3) = 0.469, \quad E(H_4) = 0.485.$$

Table 1
Ranking order with respect to the IG-HFTLHSA operator.

	$E(H_1)$	$E(H_2)$	$E(H_3)$	$E(H_4)$	Ranking order
$\gamma = -1$	0.613	0.935	0.525	0.480	$H_2 > H_1 > H_3 > H_4$
$\gamma \to 0^+$	0.278	0.804	0.538	0.466	$H_2 > H_3 > H_4 > H_1$
$\gamma = 0.1$	0.443	0.786	0.537	0.465	$H_2 > H_3 > H_4 > H_1$
$\gamma = 0.2$	0.510	0.769	0.536	0.465	$H_2 > H_3 > H_1 > H_4$
$\gamma = 0.5$	0.560	0.725	0.529	0.466	$H_2 > H_1 > H_3 > H_4$
$\gamma = 1.0$	0.562	0.700	0.513	0.470	$H_2 > H_1 > H_3 > H_4$
$\gamma = 2.0$	0.503	0.729	0.469	0.485	$H_2 > H_1 > H_4 > H_3$
$\gamma = 5.0$	0.348	0.750	0.395	0.501	$H_2 > H_4 > H_3 > H_1$
$\gamma = 10$	0.286	0.750	0.372	0.502	$H_2 > H_4 > H_3 > H_1$

According to $H_2 > H_1 > H_4 > H_3$, we know that the food company a_2 is the best choice.

With respect to the comprehensive hesitant fuzzy 2-tuple linguistic decision matrix H, when the different values of γ are used to calculate the comprehensive HFTLTSs of the alternatives, ranking order is obtained as shown in Table 1.

From Table 1, we know that ranking orders may be different with respect to the different values of γ. However, all ranking results show that the food company a_2 is the best choice. In this example, when we do not consider the interactions between the elements in the corresponding sets, using the IG-HFTLHWA operator, the following procedure is involved.

Step 1': From Step 2 and model (6), the following linear programming model for the optimal weight vector ω on the decision maker set E is constructed:

$$\max 26.937\omega_{e_1} + 26.396\omega_{e_2} + 26.333\omega_{e_3}$$

$$\text{s.t.} \begin{cases} \omega_{e_1} + \omega_{e_2} + \omega_{e_3} = 1, \\ \omega_{e_2} - \omega_{e_1} \leqslant 0, \\ \omega_{e_3} - \omega_{e_1} \leqslant 0, \\ \omega_{e_1} \leqslant 0.5, \\ \omega_{e_2} \in [0.2, 0.4], \ \omega_{e_3} \in [0.2, 0.3]. \end{cases}$$

Solving the above model, we have $\omega_{e_1} = 0.5$, $\omega_{e_2} = 0.3$, $\omega_{e_3} = 0.2$.

Step 2': From Step 2 and model (8), the following linear programming model for the optimal weight vector w on the ordered set Q is constructed:

$$\max 28.25w_1 + 26.52w_2 + 24.89w_3$$

$$\text{s.t.} \begin{cases} w_1 + w_2 + w_3 = 1, \\ w_{1+j} - w_{2+j} \geqslant 0.1, \ j = 0, 1, \\ w_{1+j} - w_{2+j} \leqslant 0.2, \ j = 0, 1, \\ w_1 \leqslant 0.5, \\ w_3 \in [0.2, 0.3]. \end{cases}$$

Solving the above model, we derive $w_1 = 0.5$, $w_2 = 0.3$, $w_3 = 0.2$.

Step 3': For each pair of (i, j), let $u_k = CC_{ij}^k$ $(k = 1, 2, 3)$. Let $\gamma = 2$, using the IG-HFTLHWA operator, the comprehensive hesitant fuzzy 2-tuple linguistic decision matrix H' is obtained as follows:

$$
H' = \begin{pmatrix}
\{(s_1, -0.032), (s_1, 0), (s_2, -0.34), (s_2, -0.39)\} \\
\{(s_1, 0.09), (s_1, 0.22), (s_1, 0.392), (s_1, 0.497), (s_2, -0.402), (s_2, 0.27)\} \\
\{(s_2, -0.404), (s_2, -0.221), (s_2, 0.264), (s_3, 0.396)\} \\
\{(s_1, 0.436), (s_2, -0.494), (s_2, -0.47), (s_2, -0.404)\} \\
\{(s_5, -0.154), (s_5, -0.098), (s_5, 0), (s_5, 0.055)\} \\
\{(s_3, 0.107), (s_3, 0.133), (s_3, 0.309), (s_4, 0.48)\} \\
\{(s_3, 0.133), (s_4, 0.48), (s_4, 0)\} \\
\{(s_3, -0.435), (s_3, -0.177)\} \\
\begin{array}{ll} \{(s_0, 0), (s_1, -0.282)\} & \{(s_0, 0.224), (s_1, -0.088)\} \\ \{(s_2, -0.09)\} & \{(s_3, -0.042), (s_3, 0.151)\} \\ \{(s_1, 0.207)\} & \{(s_1, 0.072), (s_2, -0.131)\} \\ \{(s_1, 0)\} & \{(s_2, -0.065), (s_2, 0.062)\} \end{array}
\end{pmatrix}.
$$

Step 4': From the comprehensive matrix H' and model (10), the following linear programming model for the optimal weight vector ω on the attribute set C is constructed:

$$\max 1.982\omega_{c_1} + 1.913\omega_{c_2} + 1.908\omega_{c_3} + 1.956\omega_{c_4}$$

$$
\text{s.t.} \begin{cases}
\omega_{c_1} + \omega_{c_2} + \omega_{c_3} + \omega_{c_4} = 1, \\
\omega_{c_1} - \omega_{c_2} = 0, \ \omega_{c_4} - \omega_{c_1} \geqslant 0, \\
\omega_{c_4} - \omega_{c_1} \leqslant 0.1, \ \omega_{c_1} - \omega_{c_3} \geqslant 0, \\
\omega_{c_1} - \omega_{c_3} \leqslant 0.1, \ \omega_{c_4} \leqslant 0.5, \\
\omega_{c_1}, \omega_{c_2} \in [0.2, 0.4], \\
\omega_{c_3} \in [0.1, 0.2].
\end{cases}
$$

Solving the above model, we have $\omega_{c_1} = \omega_{c_2} = 0.25$, $\omega_{c_3} = 0.15$, $\omega_{c_4} = 0.35$.

Step 5': From model (12), the following linear programming model for the optimal weight vector w on the ordered set N is constructed:

$$\max 2.111w_1 + 1.998w_2 + 1.956w_3 + 1.889w_4$$

$$
\text{s.t.} \begin{cases}
w_1 + w_2 + w_3 + w_4 = 1, \\
w_{1+j} - w_{2+j} \geqslant 0.1, \ j = 0, 1, 2, \\
w_{1+j} - w_{2+j} \leqslant 0.2, \ j = 0, 1, 2, \\
w_1 \leqslant 0.5, \\
w_4 \in [0.1, 0.2],
\end{cases}
$$

Solving the above model, we derive $w_1 = 0.4$, $w_2 = 0.3$, $w_3 = 0.2$, $w_4 = 0.1$.

Step 6': Without loss of generality, let $S = \{s_0, s_1, \ldots, s_6\}$. Furthermore, for each i, let $u_j = \dfrac{CC(H_j^+, H_{ij})}{CC(H_j^+, H_{ij}) + CC(H_j^-, H_{ij})}$, $j = 1, 2, 3, 4$. Let $\gamma = 2$, using the IG-HFTLHWA operator, the comprehensive HFTLTS H_i of the alternative a_i $(i = 1, 2, 3, 4)$ is obtained. Take

Table 2
Ranking order with respect to the IG-HFTLHWA operator.

	$E(H_1)$	$E(H_2)$	$E(H_3)$	$E(H_4)$	Ranking order
$\gamma = -1$	0.404	0.612	0.514	0.483	$H_2 > H_3 > H_4 > H_1$
$\gamma \to 0^+$	0.225	0.647	0.521	0.483	$H_2 > H_3 > H_4 > H_1$
$\gamma = 0.1$	0.258	0.650	0.522	0.484	$H_2 > H_3 > H_4 > H_1$
$\gamma = 0.2$	0.531	0.654	0.522	0.484	$H_2 > H_1 > H_3 > H_4$
$\gamma = 0.5$	0.427	0.663	0.524	0.484	$H_2 > H_3 > H_4 > H_1$
$\gamma = 1.0$	0.507	0.678	0.528	0.485	$H_2 > H_3 > H_1 > H_4$
$\gamma = 2.0$	0.586	0.702	0.532	0.488	$H_2 > H_1 > H_3 > H_4$
$\gamma = 5.0$	0.666	0.736	0.535	0.496	$H_2 > H_1 > H_3 > H_4$
$\gamma = 10$	0.669	0.748	0.521	0.501	$H_2 > H_1 > H_3 > H_4$

H_4 for example,

$$H_4 = \big\{ (s_3, 0.09), (s_3, 0.08), (s_3, 0.083), (s_3, 0.073), (s_3, 0.073), (s_3, 0.062),$$

$$(s_3, 0.069), (s_3, 0.059), (s_3, 0.082), (s_3, 0.072), (s_3, 0.079), (s_3, 0.069),$$

$$(s_3, 0.068), (s_3, 0.058), (s_3, 0.064), (s_2, 0.054) \big\}.$$

Step 7′: According to the comprehensive hesitant fuzzy 2-tuple linguistic term sets H_i ($i = 1, 2, 3, 4$), the expected values are

$$E(H_1) = 0.586, \quad E(H_2) = 0.702, \quad E(H_3) = 0.532, \quad E(H_4) = 0.488.$$

From $H_2 > H_1 > H_3 > H_4$, we know that the food company a_2 is the best choice.

With respect to the comprehensive hesitant fuzzy 2-tuple linguistic decision matrix H', when the different values of γ are used to calculate the comprehensive HFTLTSs, ranking order is obtained as shown in Table 2.

From Table 2, we also derive different ranking results with respect to the different values of γ, and all of them show that the food company a_2 is the best choice. In the practical decision-making problems, when it is sufficient to only consider the importance of elements separately, the decision maker can use the IG-HFTLHWA operator; otherwise, we suggest the decision maker to adopt the IG-HFTLHSA operator. Furthermore, the pessimistic decision maker could use the smaller value of γ, the optimistic decision maker may apply the larger value of γ, while the neutral decision maker could use the middle value of γ, for example, $\gamma = 1$.

REMARK 7. Because all existing methods cannot cope with group decision making with multi-granularity hesitant fuzzy linguistic information, they cannot be applied in this example. This also shows that the new method expands the application of HFLTSs.

6. Conclusion

Different to existing researches about HFLTSs, we introduce the concept of hesitant fuzzy 2-tuple linguistic term sets to express HFLTSs, which avoids the information loss and

distortion during the calculation of language information. To research the application of HFTLTSs, an order relationship is introduced. Meanwhile, several aggregation operators are defined, by which the comprehensive attribute values of the alternatives can be obtained. To deal with the situation where the weight information is incompletely known, models for the optimal weight vector by using the similarity degree are established. Then, we develop a method to multi-granularity hesitant fuzzy linguistic group decision making.

It is worth noting that we only discuss the application of HFTLTSs in decision making, and we will continue to study the application of HFTLTSs in some other fields such as industrial engineering, expert systems, neural networks, digital image processing, and uncertain systems and controls. Furthermore, we will continue to study HFLTSs including the computational model, the order relationship, the aggregation operator and model for the optimal weight vector.

All abovementioned researches can be classified into decision making with qualitative fuzzy information, and there are many studies (Hajiagha *et al.*, 2013a, 2013b; Kiris, 2013; Liao *et al.*, 2014; Meng *et al.*, 2014d; Singh, 2014; Tan *et al.*, 2015a, 2015b, Wang and Liu, 2014; Zhu and Xu, 2013; Zhang and Xu, 2015; Zhao *et al.*, 2014; Zhang and Wu, 2014; Zeng *et al.*, 2013) for decision-making based on quantitative fuzzy variables, which is another very important topic of multi-attribute decision making. Therefore, we shall study decision making in quantitative fuzzy environment in our future works.

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Būdas svyruojantiesiems neraiškiesiems grupiniams sprendimams priimti, grįstas daugiareikšme lingvistine informacija

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Kortežinis lingvistinis skaičiavimo modelis yra svarbi priemonė apdoroti lingvistinę informaciją. Straipsnis pristato svyruojančiąsias neraiškiąsias kortežines lingvistinių apibrėžčių aibes, kurios yra išreikštos keletu simbolinių skaičių iš intervalo [0, 1] norint praplėsti svyruojančiųjų neraiškiųjų lingvistinių apibrėžčių aibių taikymą ir išvengti informacijos netekčių. Atsižvelgiant į sąryšio tarp svyruojančiųjų lingvistinių apibrėžčių aibių tvarką, apibūdintas tikėtinų verčių ir dispersijos nustatymas. Taip pat keletas išplaukiančių iš apibendrintų svyruojančiųjų neraiškiųjų kortežų sujungimo operatorių, kuriais galima gauti išsamias alternatyvų rodiklių vertes, yra apibrėžti. Tuomet sudaryti sprendimų priėmėjo, rodiklių aibės optimalus svorių vektorius ir jų sutvarkytų aibių modeliai. Be to, būdas svyruojantiesiems neraiškiesiems grupiniams sprendimams priimti, grįstas daugiareikšme lingvistine informacija, yra pateiktas. Galiausiai pasirinktas pavyzdys parodyti siūlomos procedūros galimybes ir praktiškumą.