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A Hesitant Fuzzy Programming Method for Hybrid MADM with Incomplete Attribute Weight Information

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Abstract. This paper investigates a kind of hybrid multiple attribute decision making (MADM) problems with incomplete attribute weight information and develops a hesitant fuzzy programming method based on the linear programming technique for multidimensional analysis of preference (LINMAP). In this method, decision maker (DM) gives preferences over alternatives by the pair-wise comparison with hesitant fuzzy truth degrees and the evaluation values are expressed as crisp numbers, intervals, intuitionistic fuzzy sets (IFSs), linguistic variables and hesitant fuzzy sets (HFSs). First, by calculating the relative projections of alternatives on the positive ideal solution (PIS) and negative ideal solution (NIS), the overall relative closeness degrees of alternatives associated with attribute weights are derived. Then, the hesitant fuzzy consistency and inconsistency measures are defined. Through minimizing the inconsistency measure and maximizing the consistency measure simultaneously, a new bi-objective hesitant fuzzy programming model is constructed and a novel solution method is developed. Thereby, the weights of attributes are determined objectively. Subsequently, the ranking order of alternatives is generated based on the overall relative closeness degrees of alternatives. Finally, a supplier selection example is provided to show the validity and applicability of the proposed method.

Key words: multi-attribute decision making, hesitant fuzzy set, relative projection, hesitant fuzzy programming.

1. Intoduction

Hybrid multiple attribute decision making (MADM) is a type of MADM with multiple different types of assessment information. Due to the knowledge or preference of decision

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makers (DMs) and the nature of attributes, DMs may provide attribute values with different formats in decision making. Therefore, hybrid MADM often occurs in many fields, such as supply chain management (Wan and Li, 2015), risk investment (Sun *et al.*, 2015; Wan and Dong, 2014) and so on. For example, while selecting an appropriate supplier for a car manufacturer, quality, price and delivery time are usually considered. Generally, DMs express the quality as linguistic variables, describe price with crisp numbers and represent the delivery time by intervals. In recent years, the hybrid MADM has received more and more attention and many results about it have appeared. Roughly, these results can be divided into two categories: those which do not consider the pair-wise comparison between alternatives and those which consider these comparisons.

Aimed at the first category, two types of methods are usually employed, including transforming different types of attribute values into the same type of attribute values (Herrera *et al.*, 2001, 2005; Martinez *et al.*, 2007) and extending classical decision making methods, such as TODIM (an acronym in Portuguese for Interative Multi-criteria Decision Making), TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) and VIKOR (Visekriterijumska Optimizacija i Kompromisno Resenje), to fuzzy environment (Fan *et al.*, 2013; Zeng and Chen, 2015; Zeng and Xiao, 2016). As for the second category, the truth degrees on the pairwise comparisons between alternatives are divided into the crisp truth degree and the fuzzy truth degree.

For the crisp truth degree (i.e. the truth degree is crisp number 0 or 1), Srinivasan and Shocker (1973) proposed a linear programming technique for multidimensional analysis of preference (LINMAP) to solve MADM problems. In this method, the DM can not only provide the attribute values but give the incomplete preference relations on pairwise comparisons of alternatives. The idea of LINMAP is to define consistency and inconsistency measures based on pairwise comparisons of alternatives. According to the consistency and inconsistency measures, a crisp linear programming model is constructed to derive the ideal solution and attribute weights. Thus, the best compromise alternative that has the shortest distance to the ideal solution is obtained. Though the LINMAP method is simple and feasible, it is suitable only when the attribute values are crisp numbers and the truth degree on the pairwise comparison between alternatives is 0 or 1. However, due to the uncertainty and imprecision or the pressure of time often existing, the decision information is vague, imprecise and uncertain by nature. The crisp number is not adequate to model real-life decision problems. Thus, the LINMAP method has been extended to suit different situations where the attribute values of alternatives are fuzzy variables and the truth degree is still 0 or 1. For example, Xia *et al.* (2006) proposed the fuzzy LINMNAP method with linguistic variables. Li *et al.* (2010) presented an intuitionistic fuzzy multi-attribute group decision making method in the framework of LINMAP. Wang and Li (2012) extended the LINMAP method under interval-valued intuitionistic fuzzy environment (Jin *et al.*, 2014; Wan *et al.*, 2015a).

When the truth degrees on the pairwise comparisons between alternatives are fuzzy numbers or intuitionistic fuzzy sets (IFSs) (Wan *et al.*, 2015b, 2016a; Xu *et al.*, 2016; Zeng *et al.*, 2016b), different methods were proposed. For example, Zhang and Xu (2014) developed an interval programming approach in which the fuzzy truth degrees are intervals and the attribute values of alternatives are hesitant fuzzy sets (HFSs) (Wu *et al.*, 2013).

Representing the fuzzy truth degrees as trapezoidal fuzzy numbers (TrFNs), Li and Wan (2013) and Li and Wan (2014a) gave two different methods for solving hybrid MADM with real numbers, intervals and TrFNs. The difference between them is that only PIS is considered and unknown in the method of Li and Wan (2013), while PIS and NIS are considered simultaneously and given a priori in the method of Li and Wan (2014a). Li and Wan (2014b) generalized the method (Li and Wan, 2014a) by adding the attribute values with IFSs. Later, considering the alternative comparisons with IFSs and supposing that the PIS is given, Wan and Li (2013) constructed an intuitionistic fuzzy programming model and proposed a new heterogeneous MADM method. In this method, the fuzzy degrees on alternative comparisons are expressed as IFSs and the heterogeneous information of attribute values are represented as IFS, intervals, TrFNs and crisp numbers, respectively. Further, Wan and Li (2014) proposed the other intuitionistic fuzzy programming method in the situation that the PIS is not given and needed to be determined. Recently, Wan and Dong (2015) developed an interval-valued intuitionistic fuzzy mathematical programming method in the environment that the preference relations between alternatives are expressed as interval-valued intuitionistic fuzzy sets (IVIFSs) and the attribute values are in the form of IVIFSs, IFSs, TrFNs, linguistic variables, intervals and real numbers.

The aforementioned methods seem to be very effective for solving hybrid MADM problems. However, there are following drawbacks:

(i) Methods (Herrera *et al.*, 2001, 2005; Martinez *et al.*, 2007) transformed different types of information into the single one in the process of decision making. Therefore, some decision information may be lost or distorted in transforming process.

(ii) The classical TOPSIS method requires that attribute weights are completely given a priori, but the attribute weights are usually incomplete (Li *et al.*, 2010; Wan and Li, 2013, 2014). To determine the attribute weights, some existing LINMAP methods (Wan and Li, 2013, 2014) only minimized the inconsistency measure and did not consider the consistency measure. However, only minimizing the inconsistency cannot ensure that the consistency measure achieves the maximum. Therefore, it is not perfect to only consider the inconsistency while determining the attribute weights.

(iii) Existing LINMAP methods (Li *et al.*, 2010; Wan and Li, 2013, 2014) only considered the PIS and ignored the NIS. Moreover, methods (Li and Wan, 2013, 2014a; Wan and Li, 2013, 2014) did not consider the attribute values or fuzzy truth degrees represented with HFSs. Since the HFS can describe the uncertainty which cannot be described by intervals, fuzzy sets or IFSs, HFSs are more useful in real-life MADM problems.

As an example, in a supplier selection, three DMs evaluate the technology ability of a candidate supplier. The first DM assigns 0.8, the second one assigns 0.5, and the last one assigns 0.2. No consistency is reached among these DMs. In this case, the satisfactory degrees can be represented by a hesitant fuzzy element (HFE), i.e. $\{0.8, 0.5, 0.2\}$, which is obviously different from fuzzy number 0.5 (or 0.2), the interval $[0.2, 0.8]$ and an IFS $\langle 0.8, 0.2\rangle$.

To overcome above drawbacks, we propose a new hesitant fuzzy programming method for hybrid MADM problems and apply it to supplier selection problems. In this method, the truth degrees on the pairwise comparison between alternatives are expressed as HFSs, and the types of attribute values of alternatives include real numbers, intervals, IFSs, HFSs and linguistic variables. First, given the fuzzy positive and negative ideal solutions, the relative projection is utilized to define the overall relative closeness degrees of alternatives to the fuzzy PIS. Then, HFS-type fuzzy consistency and inconsistency measures are defined employing the relative closeness degree and the alternative comparisons with hesitant fuzzy truth degrees. By maximizing the consistency measure and minimizing the inconsistency measure simultaneously, a new bi-objective hesitant fuzzy mathematical programming model is constructed to derive attribute weights. Using the score functions of HFSs, the constructed bi-objective programming model is transformed into a single objective crisp programming model to be solved. Thus, the attribute weights can be objectively determined. Subsequently, the overall relative closeness degrees of alternatives are calculated and used to rank alternatives. Finally, an example of a supplier selection is provided to illustrate the proposed method.

Compared with existing research, the highlights of this method include the following points:

(1) Considering the alternative comparisons with hesitant fuzzy truth degrees, we firstly adopt HFSs to capture the fuzzy alternative comparisons. Since HFS generalizes fuzzy sets and all IFSs are HFSs, it is more suitable to express the fuzzy truth degrees with HFSs.
(2) A bi-objective hesitant fuzzy programming model is constructed to determine the weights of attributes. A notable characteristic of this model is that it can take the inconsistency and consistency into account simultaneously. However, methods (Li and Wan, 2013; Wan and Li, 2013) only minimized the inconsistency and ignored to maximize the consistency.
(3) An effective method is technically developed to solve the bi-objective hesitant fuzzy programming model. Thereby, the attribute weights are derived objectively.

The rest of this paper is organized as follows. In Section 2, some preliminaries for IFSs and HFSs are reviewed and the relative projection is defined. In Section 3, the hybrid MADM problems with hesitant fuzzy truth degrees and incomplete weight information are described and the normalization methods are given. A novel hesitant fuzzy programming method for such hybrid MADM problems is developed in Section 4. The proposed method is illustrated with a real supplier selection example and comparative analysis is conducted in Section 5. Section 6 shows the main conclusions.

2. Preliminaries

As a preparation for introducing our new method, some related concepts and operations are illustrated in this section.

2.1. *Intuitionistic Fuzzy Sets and Hesitant Fuzzy Sets*

Definition 1. (See Atanassov, 1986.) Let X be a finite universe of discourse, an IFS A in X is defined as $A = \{\langle x, \mu_A(x), \upsilon_A(x)\rangle | x \in X\}$, where the function $\mu_A(x) : X \to [0, 1]$ and $\upsilon_A(x) : X \to [0, 1]$ are the degrees of membership and nonmembership of an element $x \in X$, respectively, satisfying $0 \leqslant \mu_A(x) + \upsilon_A(x) \leqslant 1$, $\forall x \in X$. $\pi_A(x) = 1 - \mu_A(x) - \upsilon_A(x)$ is called the intuitionistic fuzzy index of $x \in A$. It represents the hesitation degree of $x \in A$. For each $x \in X$, $0 \leqslant \pi_A(x) \leqslant 1$. The pair $(\mu_A(x), \upsilon_A(x))$ is called an intuitionisitic fuzzy value (IFV) and simply demoted by $\alpha = (\mu_\alpha, \upsilon_\alpha)$.

Definition 2. (See Wan *et al.*, 2016b.) Let $A = \langle \mu_A, \upsilon_A\rangle$ and $B = \langle \mu_B, \upsilon_B\rangle$ be two IFVs. We stipulate:

(1) $A + B = \langle \mu_A + \mu_B - \mu_A\mu_B, \upsilon_A\upsilon_B\rangle$;
(2) $A \subseteq B$ if and only if $\mu_A \leqslant \mu_B$ and $\upsilon_A \geqslant \upsilon_B$;
(3) The complementary of an IFV A is $A^c = \langle \upsilon_A, \mu_A\rangle$.

Definition 3. (See Torra, 2010.) Let X be a finite universe of discourse, a HFS on X is in terms of a function that when applied to X returns a subset of [0, 1], which can be expressed as the following mathematical symbol:

$$E = \{\langle x, h_E(x)\rangle | x \in X\},$$

where $h_E(x)$ is a set of some values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to the set E. For convenience, we call $h = h_E(x)$ a hesitant fuzzy element (HFE).

Based on the relationship between HFSs and IFSs, Xia and Xu (2011) defined the following new operations on HFSs. Let h, h_1 and h_2 be three HFEs, then

(1) $h^\lambda = \bigcup_{\gamma\in h}\{\gamma^\lambda\}$;
(2) $\lambda h = \bigcup_{\gamma\in h}\{1 - (1-\gamma)^\lambda\}$;
(3) The complementary of a HFE h is $h^c = \bigcup_{\gamma\in h}\{1-\gamma\}$;
(4) $h_1 \cup h_2 = \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2} \max\{\gamma_1, \gamma_2\}$;
(5) $h_1 \cap h_2 = \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2} \min\{\gamma_1, \gamma_2\}$;
(6) $h_1 \oplus h_2 = \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$.

Definition 4. (See Xia and Xu, 2011.) For a HFE h, $s(h) = \frac{1}{l_h}\sum_{\gamma\in h}\gamma$ is called a score function of h, where l_h is the number of the elements in h. Moreover, for two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

Given a HFE $h(x) = \{\gamma_1, \gamma_2, \dots, \gamma_l\}$, where $\gamma_1, \gamma_2, \dots, \gamma_l$ are listed in decending order, Torra and Narukawa (2009) gave a method for transforming h into an IFS, that is

$$\mu(x) = h^-(x), \qquad v(x) = 1 - h^+(x), \tag{1}$$

where $h^-(x) = \min(h(x)) = \gamma_l$ and $h^+(x) = \max(h(x)) = \gamma_1$.

Table 1
The relations between linguistic variables and TFNs.

Linguistic variables	TFNs
Very strong (s_4)	(0.8, 0.9, 1.0)
Strong (s_3)	(0.6, 0.7, 0.8)
Medium (s_2)	(0.4, 0.5, 0.6)
Poor (s_1)	(0.2, 0.3, 0.4)
Very poor (s_0)	(0.0, 0.1, 0.3)

2.2. *Linguistic Variables*

For traditional MADM problems, DMs often express their preferences on alternatives with numerical values. However, due to the fuzziness and uncertainty, DMs may be unable to use numerical values for providing their assessment values of alternatives with respect to some attributes, especially some qualitative ones. In this case, it is more suitable for DMs to provide their assessment values by linguistic variables whose values are linguistic terms (Merigó *et al.*, 2016). For instance, while evaluating the technology ability of the suppliers, it is more suitable and easier to use terms like "strong (or good)", "medium", "poor" (Ju and Wang, 2012).

Suppose that $S = \{s_0, s_1, s_2, \ldots, s_l\}$ is a linguistic term set, where s_i represents a possible linguistic term for a linguistic variable, and $l + 1$ is called the granularity of the set S. For example, a set S with five terms could be given as $S = \{s_0, s_1, s_2, s_3, s_4\} =$ {very poor, poor, medium, strong, very strong}. In these cases, the following characteristics should be satisfied (Merigó *et al.*, 2016):

(i) A negation operator: $\mathrm{Neg}(s_i) = s_j$ such that $j = l - i$;
(ii) The set is ordered: $s_i \leqslant s_j$ if and only if $i \leqslant j$;
(iii) Max operator: $\mathrm{Max}(s_i, s_j) = s_i$ if $s_i \geqslant s_j$;
(iv) Min operator: $\mathrm{Min}(s_i, s_j) = s_i$ if $s_i \leqslant s_j$.

Usually, linguistic values are represented using positive triangular fuzzy numbers (TFNs) (Wan and Li, 2015; Wan and Dong, 2015). For example, "poor" and "strong" can be represented by TFNs (0.2, 0.3, 0.4) and (0.4, 0.5, 0.6), respectively. In this paper, the transformed relations between linguistic variables and TFNs are listed in Table 1.

2.3. *Relative Projection*

Definition 5. (See Xu and Liu, 2013). Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ be two n dimensions vectors, then

$$\Pr j_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) = |\boldsymbol{\alpha}| \cos(\boldsymbol{\alpha}, \boldsymbol{\beta}) = |\boldsymbol{\alpha}| \frac{\boldsymbol{\alpha}\boldsymbol{\beta}}{|\boldsymbol{\alpha}||\boldsymbol{\beta}|} = \frac{\boldsymbol{\alpha}\boldsymbol{\beta}}{|\boldsymbol{\beta}|} = \frac{\sum_{j=1}^{n} \alpha_j \beta_j}{\sqrt{\sum_{j=1}^{n} \beta_j^2}} \tag{2}$$

is called the projection of the vector $\boldsymbol{\alpha}$ on the vector $\boldsymbol{\beta}$, where $|\boldsymbol{\alpha}|$ and $|\boldsymbol{\beta}|$ are the modules of vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, respectively. It is shown in Fig. 1.

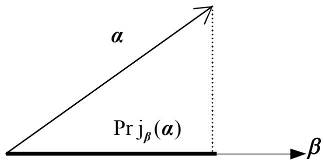

Fig. 1. Projection of vector $\boldsymbol{\alpha}$ on the $\boldsymbol{\beta}$.

From Definition 5, the larger the value of $\Pr j_{\boldsymbol{\beta}}(\boldsymbol{\alpha})$, the closer the degree of vector $\boldsymbol{\alpha}$ is to vector $\boldsymbol{\beta}$. When the module of vector $\boldsymbol{\alpha}$ is less than or equal to that of vector $\boldsymbol{\beta}$, the conclusion is right. However, when the module of vector $\boldsymbol{\alpha}$ is more than that of vector $\boldsymbol{\beta}$, the conclusion is wrong. For example, let $\boldsymbol{\alpha} = \boldsymbol{\beta}$ and $\boldsymbol{\gamma} = 2\boldsymbol{\beta}$, then $\Pr j_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) = |\boldsymbol{\beta}|$ and $\mathrm{Pr\,j}_{\boldsymbol{\beta}}(\boldsymbol{\gamma}) = 2|\boldsymbol{\beta}|$. Obviously, $\mathrm{Pr\,j}_{\boldsymbol{\beta}}(\boldsymbol{\gamma})$ is larger than $\Pr j_{\boldsymbol{\beta}}(\boldsymbol{\alpha})$. In fact, $\boldsymbol{\alpha}$ is closer to $\boldsymbol{\beta}$ than $\boldsymbol{\gamma}$. Therefore, the projection cannot accurately describe the degree of how close vector $\boldsymbol{\alpha}$ is to vector $\boldsymbol{\beta}$. It is necessary to seek new tools to measure the degree of how close vector $\boldsymbol{\alpha}$ is to vector $\boldsymbol{\beta}$.

Fusing Eq. (2) and the expression $|\boldsymbol{\beta}| = \sqrt{\sum_{j=1}^{n} \beta_j^2}$, a relative projection definition is given below.

Definition 6. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ be two n dimensional vectors. Then

$$\mathrm{R\,Pr\,j}_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) = \frac{\mathrm{Pr\,j}_{\boldsymbol{\beta}}(\boldsymbol{\alpha})}{|\beta|} = \frac{\sum_{j=1}^{n} \alpha_j \beta_j}{\sum_{j=1}^{n} \beta_j^2} \tag{3}$$

is called the relative projection of vector $\boldsymbol{\alpha}$ on vector $\boldsymbol{\beta}$.

Obviously, if $\boldsymbol{\alpha} = \boldsymbol{\beta}$, then $\mathrm{Pr\,j}_{\boldsymbol{\beta}}(\boldsymbol{\alpha}) = |\boldsymbol{\beta}|$ should hold. Thus, we get $\frac{\mathrm{Pr\,j}_{\boldsymbol{\beta}}(\boldsymbol{\alpha})}{|\boldsymbol{\beta}|} = 1$. Therefore, the closer $\mathrm{R\,Pr\,j}_{\boldsymbol{\beta}}(\boldsymbol{\alpha})$ is to 1, the closer vector $\boldsymbol{\alpha}$ is to vector $\boldsymbol{\beta}$. Accordingly, the distance between $\mathrm{R\,Pr\,j}_{\boldsymbol{\beta}}(\boldsymbol{\alpha})$ and 1 can be used to characterize the closeness degree of vector $\boldsymbol{\alpha}$ to vector $\boldsymbol{\beta}$.

Let a and b be two positive real numbers which can be considered as two one dimensional vectors. Then the relative projection a on b can be defined as

$$\mathrm{R\,Pr\,j}_b(a) = \frac{ab}{b^2} = \frac{a}{b}. \tag{4}$$

Let $\tilde{a} = [o_{\tilde{a}}, q_{\tilde{a}}]$ and $\tilde{b} = [o_{\tilde{b}}, q_{\tilde{b}}]$ be two interval numbers, where $0 < o_{\tilde{a}} \leqslant q_{\tilde{a}}$ and $0 < o_{\tilde{b}} \leqslant q_{\tilde{b}}$. then the relative projection $\tilde{a}$ on $\tilde{b}$ is defined as

$$\mathrm{R\,Pr\,j}_{\tilde{b}}(\tilde{a}) = \frac{o_{\tilde{a}} o_{\tilde{b}} + q_{\tilde{a}} q_{\tilde{b}}}{(o_{\tilde{b}})^2 + (q_{\tilde{b}})^2}. \tag{5}$$

Similarly, when $\tilde{a}_1 = (a_1, b_1, c_1)$ and $\tilde{a}_2 = (a_2, b_2, c_2)$ are two TFNs, where $0 < a_i \leqslant b_i \leqslant c_i$ $(i = 1, 2)$, the relative projection $\tilde{a}_1$ on $\tilde{a}_2$ is represented as

$$\mathrm{R\,Prj}_{\tilde{a}_2}(\tilde{a}_1) = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{(a_2)^2 + (b_2)^2 + (c_2)^2}. \tag{6}$$

If $\tilde{e}_1 = \langle \mu_1, \upsilon_1 \rangle$ and $\tilde{e}_2 = \langle \mu_2, \upsilon_2 \rangle$ are two IFSs, then

$$\mathrm{R\,Prj}_{\tilde{e}_2}(\tilde{e}_1) = \frac{\mu_1 \mu_2 + \upsilon_1 \upsilon_2}{(\mu_2)^2 + (\upsilon_2)^2}. \tag{7}$$

If $\tilde{h}_1 = \{h_{11}, h_{12}, \dots, h_{1l_1}\}$ and $\tilde{h}_2 = \{h_{21}, h_{22}, \dots, h_{2l_2}\}$ are two HFSs, then

$$\mathrm{R\,Prj}_{\tilde{h}_2}(\tilde{h}_1) = \frac{1}{l} \sum_{k=1}^{l} \frac{h_{1k}}{h_{2k}}, \tag{8}$$

where $l = \max\{l_1, l_2\}$.

In most cases, $l_1 \neq l_2$. Without loss of generality, let $l_1 < l_2$. Xu and Zhang (2013) pointed that $\tilde{h}_1$ can be extended by adding any value in it, such as adding the minimum or maximum value, until the number of the possible values in $\tilde{h}_1$ is equal to l_2. The pessimists may choose the minimum value, while the optimists may choose the maximum value. For example, let $\tilde{h}_1 = \{0.5, 0.4\}$ and $\tilde{h}_2 = \{0.6, 0.5, 0.3\}$, where $l_1 < l_2$. A pessimist can extend $\tilde{h}_1$ to $\tilde{h}_1 = \{0.5, 0.4, 0.4\}$, and an optimist can extend $\tilde{h}_1$ as $\tilde{h}_1 = \{0.5, 0.5, 0.4\}$. Although the results are different, they are reasonable because the decision makers' risk preferences can directly influence their final decisions. In this paper, DMs are considered to be pessimistic (other situations can be researched similarly).

3. Hybrid MADM Problems with Hesitant Fuzzy Alternative Comparisons

In this section, the hybrid MADM problems considered in this paper are described and the normalization methods are provided.

3.1. *The Description of Hybrid MADM Problems*

For hybrid MADM problems, let $X = \{x_1, x_2, \dots, x_n\}$ be the set of n feasible alternatives, $U = \{u_1, u_2, \dots, u_m\}$ be the set of m attributes, and $w = (w_1, w_2, \dots, w_m)^{\mathrm{T}}$ be the weight vector of attributes. Usually, the attribute weights are required to satisfy the normalization conditions: $\sum_{j=1}^{m} w_j = 1$ and $w_j \geqslant \varepsilon$ $(j = 1, 2, \dots, m)$. For convenience, denote $D_0 = \{\boldsymbol{w} | \sum_{j=1}^{m} w_j = 1,\ w_j \geqslant \varepsilon,\ \text{for } j = 1, 2, \dots, m\}$, where ε is a sufficiently small positive number to ensure the weights obtained are not zeros. The incomplete information structures of attribute weights are often given in the following five basic relations among attributes (Li, 2011):

(1) A ranking with times: $w_k \geqslant \varsigma_{kl} w_l$, $0 \leqslant \varsigma_{kl} \leqslant 1$;
(2) A weak ranking: $w_k \geqslant w_l$;

(3) A strict ranking: $0 < a_{kl} \leqslant w_k - w_l \leqslant b_{kl}$, $0 \leqslant a_{kl}, b_{kl} \leqslant 1$;
(4) An interval-valued form: $\xi_k \leqslant w_k \leqslant \chi_k$, $0 \leqslant \xi_k, \chi_k \leqslant 1$;
(5) A ranking of differences: $w_k - w_l \leqslant w_p - w_q$.

According to the characteristics of decision problems themselves or the capacities of DMs, DMs may give partial information about attribute weights. The incomplete information of attribute weights given by DMs, denoted by D, may consist of several or all of five basic relations in D_0.

Let $\boldsymbol{P}' = (p'_{ij})_{n\times m}$ be a decision matrix given by DMs, where p'_{ij} $(i = 1, 2, \dots, n;$ $j = 1, 2, \dots, m)$ be the ratings of alternative x_i on the attribute u_j. Assume that:

(1) for $j = 1, 2, \dots, j_1$, p'_{ij} are real numbers denoted by f'_{ij};
(2) for $j = j_1 + 1, 2, \dots, j_2$, p'_{ij} are linguistic variables denoted by s_{ij};
(3) for $j = j_2 + 1, 2, \dots, j_3$, p'_{ij} are IFSs denoted by $\langle \mu_{ij}, v_{ij} \rangle$;
(4) for $j = j_3 + 1, \dots, j_4$, p'_{ij} are HFSs denoted by $\tilde{h}_{ij} = \{h_{ij1}, h_{ij2}, \dots, h_{ijl_{ij}}\}$, where l_{ij} is the number of possible values in h_{ij};
(5) for $j = j_4 + 1, \dots, m$, p'_{ij} are intervals denoted by $[o'_{ij}, q'_{ij}]$.

3.2. *Normalization Methods*

Generally, there are benefit attributes and cost attributes in MADM problems, the higher the benefit attribute value, the better it will be. As for the cost attribute, it is opposite. Let J_1 and J_2 be the sets of benefit attributes and cost attributes, respectively. In order to measure all attributes in dimensionless units and to facilitate inter-attribute comparisons, we need to normalize above attribute values. Denote the normalized values by p_{ij}, and the normalized decision matrix by $\boldsymbol{P} = (p_{ij})_{n\times m}$. The normalizing formulas are as follows:

$$p_{ij} = \begin{cases} f'_{ij}/\max_i f'_{ij}, & \text{if } j \in J_1,\ j = 1, 2, \dots, j_1, \\ \min_i f'_{ij}/f'_{ij}, & \text{if } j \in J_2,\ j = 1, 2, \dots, j_1, \end{cases} \tag{9}$$

$$p_{ij} = \begin{cases} s_{ij}, & \text{if } j \in J_1,\ j = j_1 + 1, j_1 + 2, \dots, j_2, \\ \mathrm{Neg}(s_{ij}), & \text{if } j \in J_2,\ j = j_1 + 1, j_1 + 2, \dots, j_2, \end{cases} \tag{10}$$

$$p_{ij} = \begin{cases} \langle \mu_{ij}, v_{ij} \rangle, & \text{if } j \in J_1,\ j = j_2 + 1, j_2 + 2, \dots, j_3, \\ \langle v_{ij}, \mu_{ij} \rangle, & \text{if } j \in J_2,\ j = j_2 + 1, j_2 + 2, \dots, j_3, \end{cases} \tag{11}$$

$$p_{ij} = \begin{cases} \tilde{h}_{ij}, & \text{if } j \in J_1,\ j = j_3 + 1, j_3 + 2, \dots, j_4, \\ (\tilde{h}_{ij})^c, & \text{if } j \in J_2,\ j = j_3 + 1, j_3 + 2, \dots, j_4, \end{cases} \tag{12}$$

$$p_{ij} = \begin{cases} [o'_{ij}/\max_i q'_{ij}, q'_{ij}/\max_i q'_{ij}], & \text{if } j \in J_1,\ j = j_4 + 1, j_4 + 2, \dots, m, \\ [1 - q'_{ij}/\max_i q'_{ij}, 1 - o'_{ij}/\max_i q'_{ij}], & \text{if } j \in J_2,\ j = j_4 + 1, j_4 + 2, \dots, m. \end{cases} \tag{13}$$

4. A Novel Method for Hybrid MADM Problems with Hesitant Fuzzy Alternative Comparisons

In this section, a new hesitant fuzzy programming method is developed for solving the above hybrid MADM problems.

4.1. *Computing the Relative Closeness of Alternatives Based on the Relative Projection*

Denote the PIS and NIS by $r^+ = (r_1^+, r_2^+, \ldots, r_n^+)$ and $r^- = (r_1^-, r_2^-, \ldots, r_n^-)$, respectively, where

$$r_j^+ = \begin{cases} f_j^+ = \max_i f_{ij}, \quad \text{if } j = 1, 2, \ldots, j_1, \\ (a_j^+, b_j^+, c_j^+) = (\max_i a_{ij}, \max_i b_{ij}, \max_i c_{ij}), \quad \text{if } j = j_1+1, j_1+2, \ldots, j_2, \\ \langle \mu_j^+, v_j^+ \rangle = \langle \max_i \mu_{ij}, \min_i v_{ij} \rangle, \quad \text{if } j = j_2+1, j_2+2, \ldots, j_3, \\ (h_{j1}^+, h_{j2}^+, \ldots, h_{jl_j}^+) = (\max_i h_{ij1}, \max_i h_{ij2}, \ldots, \max_i h_{ijl_j}), \\ \quad \text{if } j = j_3+1, j_3+2, \ldots, j_4, \ \text{where } l_j = \max_i \{l_{ij}\}, \\ [o_j^+, q_j^+] = [\max_i o_{ij}, \max_i q_{ij}], \quad \text{if } j = j_4+1, j_4+2, \ldots, m \end{cases} \tag{14}$$

and

$$r_j^- = \begin{cases} f_j^- = \min_i f_{ij}, \quad \text{if } j = 1, 2, \ldots, j_1, \\ (a_j^-, b_j^-, c_j^-) = (\min_i a_{ij}, \min_i b_{ij}, \min_i c_{ij}), \quad \text{if } j = j_1+1, j_1+2, \ldots, j_2, \\ \langle \mu_j^-, v_j^- \rangle = \langle \min_i \mu_{ij}, \max_i v_{ij} \rangle, \quad \text{if } j = j_2+1, j_2+2, \ldots, j_3, \\ (h_{j1}^-, h_{j2}^-, \ldots, h_{jl_j}^-) = (\min_i h_{ij1}, \min_i h_{ij2}, \ldots, \min_i h_{ijl_j}), \\ \quad \text{if } j = j_3+1, j_3+2, \ldots, j_4, \ \text{where } l_j = \max_i \{l_{ij}\}, \\ [o_j^-, q_j^-] = [\min_i o_{ij}, \min_i q_{ij}], \quad \text{if } j = j_4+1, j_4+2, \ldots, m. \end{cases} \tag{15}$$

According to the TOPSIS method, the closer the alternative x_i is to the PIS r^+ and, at the same time, the farther is to the NIS r^-, the better the alternative x_i is. In this paper, the closeness degree between alternative x_i and the PIS or NIS is described by the distance between the relative projection referred in Definition 5 and crisp number 1. Using Eqs. (4)–(8), the relative projection between p_{ij} and r_j^+ as well as r_j^- is computed as follows:

$$\mathrm{RPr}_{r_j^+}(p_{ij}) = \begin{cases} \frac{f_{ij}}{f_j^+}, & \text{if } j = 1, 2, \ldots, j_1, \\ \frac{a_{ij}a_j^+ + b_{ij}b_j^+ + c_{ij}c_j^+}{(a_j^+)^2 + (b_j^+)^2 + (c_j^+)^2}, & \text{if } j = j_1+1, j_1+2, \ldots, j_2, \\ \frac{\mu_{ij}\mu_j^+ + v_{ij}v_j^+}{(\mu_j^+)^2 + (v_j^+)^2}, & \text{if } j = j_2+1, j_2+2, \ldots, j_3, \\ \frac{1}{l_j}\sum_{k=1}^{l_j} \frac{h_{ijk}}{h_{jk}^+}, & \text{if } j = j_3+1, j_3+2, \ldots, j_4; \ l_j = \max_i l_{ij}, \\ \frac{o_{ij}o_j^+ + q_{ij}q_j^+}{(o_j^+)^2 + (q_j^+)^2}, & \text{if } j = j_4+1, j_4+2, \ldots, m \end{cases} \tag{16}$$

and

$$\mathrm{RPr}_{r_j^-}(p_{ij}) = \begin{cases} \frac{f_{ij}}{f_j^-}, & \text{if } j = 1, 2, \dots, j_1, \\ \frac{a_{ij}a_j^- + b_{ij}b_j^- + c_{ij}c_j^-}{(a_j^-)^2 + (b_j^-)^2 + (b_j^-)^2}, & \text{if } j = j_1 + 1, j_1 + 2, \dots, j_2, \\ \frac{\mu_{ij}\mu_j^- + v_{ij}v_j^-}{(\mu_j^-)^2 + (v_j^-)^2}, & \text{if } j = j_2 + 1, j_2 + 2, \dots, j_3, \\ \frac{1}{l_j}\sum_{k=1}^{l_j} \frac{h_{ijk}}{h_{jk}^-}, & \text{if } j = j_3 + 1, j_3 + 2, \dots, j_4;\ l_j = \max_i l_{ij}, \\ \frac{o_{ij}o_j^- + q_{ij}q_j^-}{(o_j^-)^2 + (q_j^-)^2}, & \text{if } j = j_4 + 1, j_4 + 2, \dots, m. \end{cases} \tag{17}$$

Denote $\theta_{ij}^+ = (\mathrm{R\,Prj}_{r_j^+}(r_{ij}) - 1)^2$ and $\theta_{ij}^- = (\mathrm{R\,Prj}_{r_j^-}(r_{ij}) - 1)^2$. The relative closeness of p_{ij} with respect to r_j^+ is defined as

$$RC_{ij} = \frac{\theta_{ij}^-}{\theta_{ij}^+ + \theta_{ij}^-}. \tag{18}$$

Let $\boldsymbol{RC} = (RC_{ij})_{n\times m}$ be the relative closeness matrix. Therefore, the overall relative closeness of alternative x_i can be described as

$$T_i = \sum_{j=1}^{n} w_j RC_{ij}. \tag{19}$$

If attribute weights are known in advance, then the alternatives can be ranked according to the descending order of T_i, and the one with the maximum value of T_i is the best. To determine attribute weights, a new bi-objective hesitant fuzzy programming model is constructed in the sequel.

4.2. *A New Bi-Objective Hesitant Fuzzy Programming Model for Determining Attribute Weights*

To estimate attribute weights, a new bi-objective hesitant fuzzy programming method is developed in this subsection.

4.2.1. *Hesitant Fuzzy Consistency and Inconsistency Measures*

Under certain circumstances, DM may compare two alternatives directly without consideration of particular attributes. For example, a DM may prefer supplier A to supplier B without considering specific attributes of the suppliers. Assume that the DM gives the preference relations between alternatives by a HFS of ordered pairs $E = \{\langle (k, i), h_E(k, i)\rangle | x_k \succeq x_i \text{ with } h_E(k, i)\}$, where $\langle (k, i), h_E(k, i)\rangle$ represents an ordered pairs of alternatives x_k and x_i that the DM prefers x_k to x_i (denoted by $x_k \succeq x_i$) with the hesitant fuzzy truth degree $h_E(k, i)$, which is a HFE and denoted by $h_E(k, i) = \{\gamma_{ki}^1, \gamma_{ki}^2, \dots, \gamma_{ki}^{l_{ki}}\}$, satisfying $\gamma_{ki}^g \in [0, 1]$ for any $g = 1, 2, \dots, l_{ki}$.

For each pair of alternatives $(k,i) \in E$, alternative x_k is closer to the PIS than alternative x_i if $T_k \geqslant T_i$. Hence, it yields $x_k \succeq x_i$. This ranking order is consistent with the subjective preference relation given by DM. Conversely, if $T_k < T_i$, then $x_i \succeq x_k$. Thus, the chosen $\boldsymbol{w}$ is not proper since it results in that the ranking order of alternatives x_k and x_i determined by T_k and T_i associated with $\boldsymbol{w}$ is inconsistent with the subjective preference relation given by DM. Therefore, $\boldsymbol{w}$ should be chosen in order to make the ranking order determined by T_k and T_i consistent with the subjective preference relation $(k,i) \in E$ provided by DM.

Bearing this idea in mind, we introduce the inconsistency measure

$$Y_{ki} = \begin{cases} h_E(k,i)(T_i - T_k), & \text{if } T_k < T_i, \\ 0, & \text{if } T_k \geqslant T_i \end{cases} \tag{20}$$

to measure inconsistency between the ranking order of alternatives x_k and x_i determined by T_k and T_i and the preference relation $(k,i) \in E$.

In Eq. (20), $T_k \geqslant T_i$ demonstrates that alternative x_k is preferred to x_i, which is in accordance with the subjective preference relation $(k,i) \in E$. Hence, the inconsistency measure Y_{ki} is equal to 0. Otherwise, the inconsistency measure $Y_{ki} = h_E(k,i)(T_i - T_k)$ represents the expected value of the inconsistency degree between the ranking order of alternatives x_k and x_i determined by T_k and T_i and the preference relation $(k,i) \in E$. In order to unify the two expressions of consistency measure into one expression, we conduct the following analyses.

If $T_k < T_i$, then $T_i - T_k > 0$. Thus, it is followed that $\max\{0, T_i - T_k\} = T_i - T_k$. Consequently, $Y_{ki} = h_E(k,i)(T_i - T_k)$ in Eq. (20) can be expressed as $Y_{ki} = h_E(k,i)\max\{0, T_i - T_k\}$. If $T_k \geqslant T_i$, then we get $\max\{0, T_i - T_k\} = 0$. Thereby, $Y_{ki} = 0$ in Eq. (20) can also be written as $Y_{ki} = h_E(k,i)\max\{0, T_i - T_k\}$. Accordingly, Eq. (20) can be unified into the following equation:

$$Y_{ki} = h_E(k,i)\max\{0, T_i - T_k\}. \tag{21}$$

Denote the total inconsistency measure by ICI. We derive

$$\mathrm{ICI} = \sum_{(k,i)\in E} Y_{ki} = \sum_{(k,i)\in E} h_E(k,i)\max\{0, T_i - T_k\}. \tag{22}$$

Similar to the inconsistency measure, the consistent measure can be defined as

$$B_{ki} = \begin{cases} h_E(k,i)(T_k - T_i), & \text{if } T_k \geqslant T_i, \\ 0, & \text{if } T_k < T_i, \end{cases} \tag{23}$$

which can be rewritten as

$$B_{ki} = h_E(k,i)\max\{0, T_k - T_i\}. \tag{24}$$

Hence, the total consistency index CI can be defined as

$$\mathrm{CI} = \sum_{(k,i)\in E} B_{ki} = \sum_{(k,i)\in E} h_E(k,i)\max\{0, T_k - T_i\}. \tag{25}$$

4.2.2. *Construction of a Bi-Objective Hesitant Fuzzy Programming Model*

To determine the attribute weight vector $\boldsymbol{w}$, a new bi-objective hesitant fuzzy mathematical programming model is constructed as

$$\begin{array}{l} \max\ \ \mathrm{CI} \\ \min\ \ \mathrm{ICI} \\ \text{s.t.}\ \ w \in D \end{array} \tag{26}$$

where D is the incomplete information of the attribute importance given by the DM referred in Section 3.2. Eq. (26) intends to maximize the consistency measure CI and minimize the inconsistency measure ICI simutaneously.

According to Eqs. (22) and (25), Eq. (26) can be rewritten as

$$\begin{array}{l} \max\ \left\{ \sum\limits_{(k,i)\in E} h_E(k,i)\max\{0, T_k - T_i\} \right\}, \\ \min\ \left\{ \sum\limits_{(k,i)\in E} h_E(k,i)\max\{0, T_i - T_k\} \right\}, \\ \text{s.t.}\ \ \boldsymbol{w} \in D. \end{array} \tag{27}$$

For each pair of alternatives $(k,i) \in E$, let $\eta_{ki} = \max\{0, T_k - T_i\}$ and $\xi_{ki} = \max\{0, T_i - T_k\}$, then $\eta_{ki} \geqslant 0$, $\xi_{ki} \geqslant 0$, $\eta_{ki} \geqslant T_k - T_i$ and $\xi_{ki} \geqslant T_i - T_k$, i.e. $T_k - T_i - \eta_{ki} \leqslant 0$ and $T_k - T_i + \xi_{ki} \geqslant 0$. Furthermore, η_{ki} and ξ_{ki} satisfy the equation $\eta_{ki} - \xi_{ki} = T_k - T_i$.

Thus, Eq. (27) can be transformed into a bi-objective hesitant fuzzy programming, i.e.

$$\begin{array}{l} \max\ \left\{ \sum\limits_{(k,i)\in E} h_E(k,i)\eta_{ki} \right\}, \\ \min\ \left\{ \sum\limits_{(k,i)\in E} h_E(k,i)\xi_{ki} \right\}, \\ \text{s.t.}\ \begin{cases} \eta_{ki} - \xi_{ki} = T_k - T_i, & (k,i) \in E, \\ T_k - T_i - \eta_{ki} \leqslant 0, & (k,i) \in E, \\ T_k - T_i + \xi_{ki} \geqslant 0, & (k,i) \in E, \\ \eta_{ki} \geqslant 0, \quad \xi_{ki} \geqslant 0, & (k,i) \in E, \\ \boldsymbol{w} \in D. \end{cases} \end{array} \tag{28}$$

According to Eq. (19), one has

$$T_k - T_i = \sum_{j=1}^{n} w_j (RC_{kj} - RC_{ij}). \tag{29}$$

Putting Eq. (29) into (28), we have

$$\begin{aligned} &\max \left\{ \sum_{(k,i)\in E} h_E(k,i)\eta_{ki} \right\}, \\ &\min \left\{ \sum_{(k,i)\in E} h_E(k,i)\xi_{ki} \right\}. \\ &\text{s.t.} \begin{cases} \eta_{ki} - \xi_{ki} = \sum_{j=1}^{n} w_j (RC_{kj} - RC_{ij}), \quad (k,i) \in E, \\ \sum_{j=1}^{n} w_j (RC_{kj} - RC_{ij}) - \eta_{ki} \leqslant 0, \quad (k,i) \in E, \\ \sum_{j=1}^{n} w_j (RC_{kj} - RC_{ij}) + \xi_{ki} \geqslant 0, \quad (k,i) \in E, \\ \eta_{ki} \geqslant 0, \quad \xi_{ki} \geqslant 0, \quad (k,i) \in E, \\ \boldsymbol{w} \in D. \end{cases} \end{aligned} \tag{30}$$

4.2.3. *The Resolution Method of the Bi-Objecitve Hesitant Fuzzy Programming Model*
From operational rules of HFEs, the objective function $\sum_{(k,i)\in E} h_E(k,i)\eta_{ki}$ and $\sum_{(k,i)\in E} h_E(k,i)\xi_{ki}$ in Eq. (30) are HFEs, that is

$$\sum_{(k,i)\in E} h_E(k,i)\eta_{ki} = \bigoplus_{(k,i)\in E} h_E(k,i)\eta_{ki} = \bigcup_{\gamma_{ki}^{g} \in h_E(k,i)} \left\{ 1 - \prod_{(k,i)\in E} (1 - \gamma_{ki}^{g})^{\eta_{ki}} \right\} \tag{31}$$

and

$$\sum_{(k,i)\in E} h_E(k,i)\xi_{ki} = \bigoplus_{(k,i)\in E} h_E(k,i)\xi_{ki} = \bigcup_{\gamma_{ki}^{g} \in h_E(k,i)} \left\{ 1 - \prod_{(k,i)\in E} (1 - \gamma_{ki}^{g})^{\xi_{ki}} \right\}. \tag{32}$$

For the sake of convenience, suppose that

$$\tilde{q} = \bigcup_{\gamma_{ki}^{g} \in h_E(k,i)} \left\{ 1 - \prod_{(k,i)\in E} (1 - \gamma_{ki}^{g})^{\eta_{ki}} \right\} \triangleq \{\gamma_1^{\tilde{q}}, \gamma_2^{\tilde{q}}, \dots, \gamma_{N_{\tilde{q}}}^{\tilde{q}}\} \tag{33}$$

and

$$\tilde{\theta} = \bigcup_{\gamma_{ki}^{g} \in h_E(k,i)} \left\{ 1 - \prod_{(k,i)\in E} (1 - \gamma_{ki}^{g})^{\xi_{ki}} \right\} \triangleq \{\gamma_1^{\tilde{\theta}}, \gamma_2^{\tilde{\theta}}, \dots, \gamma_{N_{\tilde{\theta}}}^{\tilde{\theta}}\}, \tag{34}$$

where $N_{\tilde{q}}$ and $N_{\tilde{\theta}}$ are the numbers of all possible values of HFEs $\tilde{\theta}$ and $\tilde{q}$, respectively, and $\gamma_{t_1}^{\tilde{\theta}}$ and $\gamma_{t_2}^{\tilde{q}}$ are corresponding possible values, $t_1 = 1, 2, \dots, N_{\tilde{q}}$ and $t_2 = 1, 2, \dots, N_{\tilde{\theta}}$.

According to the ranking relation of HFEs (see Definition 4), minimizing $\sum_{(k,i)\in E} h_E(k,i)\xi_{ki}$ and maximizing $\sum_{(k,i)\in E} h_E(k,i)\eta_{ki}$ in Eq. (30) are equivalent to minimize the score function of the former and maximize that of the latter. Let s($\tilde{q}$) and s($\tilde{\theta}$) be respectively the score functions of the objective functions $\sum_{(k,i)\in E} h_E(k,i)\xi_{ki}$ and $\sum_{(k,i)\in E} h_E(k,i)\eta_{ki}$, then Eq. (30) can be converted by Eqs. (31)–(34) as follows:

$$\max\ \mathrm{s}(\tilde{q}),$$

$$\min\ \mathrm{s}(\tilde{\theta}),$$

$$\text{s.t.}\ \begin{cases} \eta_{ki} - \xi_{ki} = \sum_{j=1}^{n} w_j(RC_{kj} - RC_{ij}), & (k,i)\in E, \\ \sum_{j=1}^{n} w_j(RC_{kj} - RC_{ij}) - \eta_{ki} \leqslant 0, & (k,i)\in E, \\ \sum_{j=1}^{n} w_j(RC_{kj} - RC_{ij}) + \xi_{ki} \geqslant 0, & (k,i)\in E, \\ \eta_{ki} \geqslant 0, \quad \xi_{ki} \geqslant 0, \quad (k,i)\in E, \\ \boldsymbol{w}\in D. \end{cases} \tag{35}$$

To solve Eq. (35), it is needed to determine the score functions s($\tilde{q}$) and s($\tilde{\theta}$). By employing Definition 4, the two score functions are obtained as

$$\mathrm{s}(\tilde{q}) = \frac{1}{N_{\tilde{q}}}\sum_{t_1=1}^{N_{\tilde{q}}} \gamma_{t_1}^{\tilde{q}} \tag{36}$$

and

$$\mathrm{s}(\tilde{\theta}) = \frac{1}{N_{\tilde{\theta}}}\sum_{t_2=1}^{N_{\tilde{\theta}}} \gamma_{t_2}^{\tilde{\theta}}. \tag{37}$$

By the operations on HFSs in Section 2 and Eqs. (33)–(34), it yields that

$$\mathrm{s}(\tilde{q}) = 1 - \frac{1}{N_{\tilde{q}}}\sum_{\gamma_{ki}^{g}\in h_E(k,i)} \prod_{(k,i)\in E} (1-\gamma_{ki}^{g})^{\eta_{ki}} \tag{38}$$

and

$$\mathrm{s}(\tilde{\theta}) = 1 - \frac{1}{N_{\tilde{\theta}}}\sum_{\gamma_{ki}^{g}\in h_E(k,i)} \prod_{(k,i)\in E} (1-\gamma_{ki}^{g})^{\xi_{ki}}. \tag{39}$$

Plugging Eqs. (38)–(39) into Eq. (35), we derive

$$\max\ \left\{1 - \frac{1}{N_{\tilde{q}}}\sum_{\gamma_{ki}^{g}\in h_E(k,i)} \prod_{(k,i)\in E} (1-\gamma_{ki}^{g})^{\eta_{ki}}\right\},$$

$$\min\ \left\{1 - \frac{1}{N_{\tilde{\theta}}}\sum_{\gamma_{ki}^{g}\in h_E(k,i)} \prod_{(k,i)\in E} (1-\gamma_{ki}^{g})^{\xi_{ki}}\right\},$$

$$\text{s.t.} \begin{cases} \eta_{ki} - \xi_{ki} = \sum_{j=1}^{n} w_j(RC_{kj} - RC_{ij}), & (k,i) \in E, \\ \sum_{j=1}^{n} w_j(RC_{kj} - RC_{ij}) - \eta_{ki} \leqslant 0, & (k,i) \in E, \\ \sum_{j=1}^{n} w_j(RC_{kj} - RC_{ij}) + \xi_{ki} \geqslant 0, & (k,i) \in E, \\ \eta_{ki} \geqslant 0, \quad \xi_{ki} \geqslant 0, \quad (k,i) \in E, \\ \boldsymbol{w} \in D. \end{cases} \tag{40}$$

Clearly, Eq. (40) can be simplified as the following model:

$$\begin{aligned} &\min \left\{ z_1 = \frac{1}{N_{\tilde{q}}} \sum_{\gamma_{ki}^{g} \in h_E(k,i)} \prod_{(k,i)\in E} (1 - \gamma_{ki}^{g})^{\eta_{ki}} \right\}, \\ &\max \left\{ z_2 = \frac{1}{N_{\tilde{\theta}}} \sum_{\gamma_{ki}^{g} \in h_E(k,i)} \prod_{(k,i)\in E} (1 - \gamma_{ki}^{g})^{\xi_{ki}} \right\}, \\ &\text{s.t.} \begin{cases} \eta_{ki} - \xi_{ki} = \sum_{j=1}^{n} w_j(RC_{kj} - RC_{ij}), & (k,i) \in E, \\ \sum_{j=1}^{n} w_j(RC_{kj} - RC_{ij}) - \eta_{ki} \leqslant 0, & (k,i) \in E, \\ \sum_{j=1}^{n} w_j(RC_{kj} - RC_{ij}) + \xi_{ki} \geqslant 0, & (k,i) \in E, \\ \eta_{ki} \geqslant 0, \quad \xi_{ki} \geqslant 0, \quad (k,i) \in E, \\ \boldsymbol{w} \in D. \end{cases} \end{aligned} \tag{41}$$

By the linear weighted summation method, Eq. (41) can be transformed into a single objective crisp programming model:

$$\begin{aligned} &\min \left\{ z = \delta \frac{1}{N_{\tilde{q}}} \sum_{\gamma_{ki}^{g} \in h_E(k,i)} \prod_{(k,i)\in E} (1 - \gamma_{ki}^{g})^{\eta_{ki}} - (1-\delta) \frac{1}{N_{\tilde{\theta}}} \sum_{\gamma_{ki}^{g} \in h_E(k,i)} \prod_{(k,i)\in E} d(1 - \gamma_{ki}^{g})^{\xi_{ki}} \right\}, \\ &\text{s.t.} \begin{cases} \eta_{ki} - \xi_{ki} = \sum_{j=1}^{n} w_j(RC_{kj} - RC_{ij}), & (k,i) \in E, \\ \sum_{j=1}^{n} w_j(RC_{kj} - RC_{ij}) - \eta_{ki} \leqslant 0, & (k,i) \in E, \\ \sum_{j=1}^{n} w_j(RC_{kj} - RC_{ij}) + \xi_{ki} \geqslant 0, & (k,i) \in E, \\ \eta_{ki} \geqslant 0, \quad \xi_{ki} \geqslant 0, \quad (k,i) \in E, \\ \boldsymbol{w} \in D. \end{cases} \end{aligned} \tag{42}$$

where the weighted coefficient $0 \leqslant \delta \leqslant 1$.

Specially, $\delta = 0$ means that only minimizing the inconsistency measure is considered; $\delta = 0.5$ indicates that maximizing the consistency measure is as important as minimizing the inconsistency measure; $\delta = 1$ implies that only maximizing the consistency measure is considered.

Solving Eq. (42), the vector of attribute weights, $w = (w_1, w_2, \ldots, w_m)^{\mathrm{T}}$, can be determined.

4.3. *Decision Process and Algorithm for Hybrid MADM Problems*

Based on the above analysis, the algorithm and decision process for hybrid MADM problems are summarized as follows:

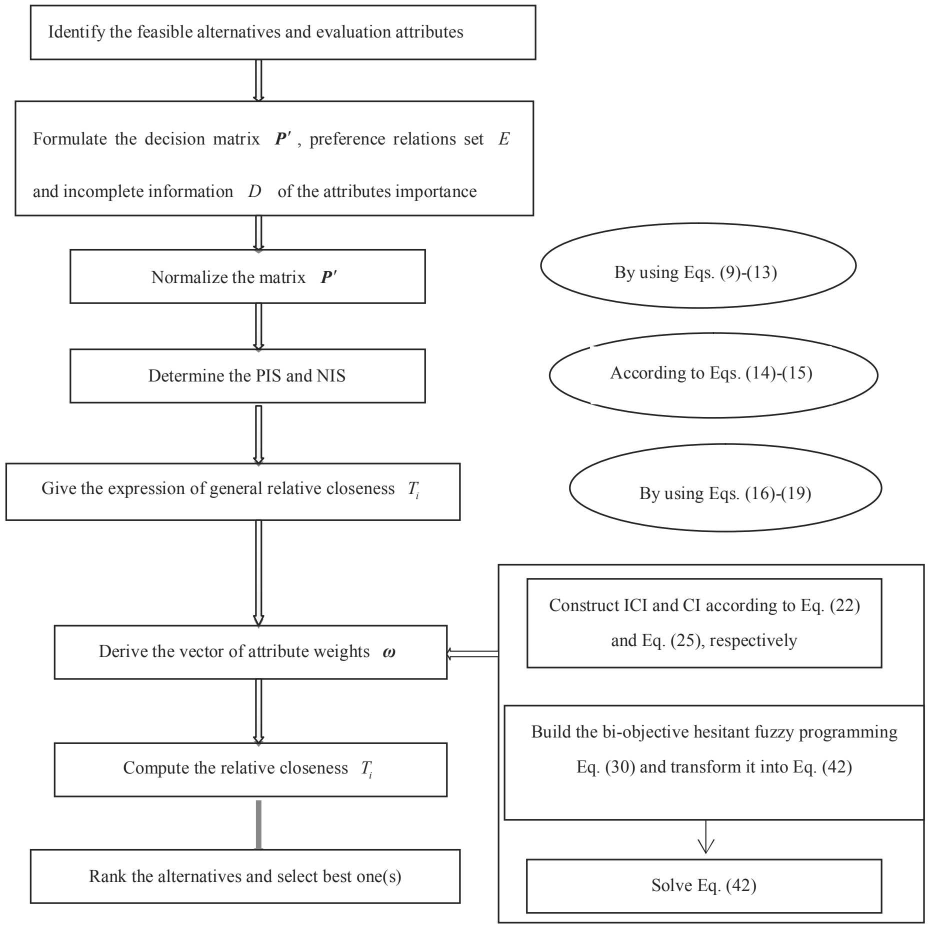

Fig. 2. The decision making process for the hybrid MADM with hesitant fuzzy alternative comparisons.

Step 1. Identify all the feasible alternatives and evaluation attributes.

Step 2. Elicit the fuzzy decision matrix $\boldsymbol{P}' = (p'_{ij})_{m\times n}$, formulate the preference relations between alternatives by a HFS of ordered pairs E, and acquire the incomplete information D of attribute weights.

Step 3. Normalize the matrix $\boldsymbol{P}' = (p'_{ij})_{m\times n}$ into $\boldsymbol{P} = (p_{ij})_{m\times n}$ via Eqs. (9)–(13).

Step 4. Determine the PIS and NIS by Eqs. (14)–(15).

Step 5. Give the expression of the overall relative closeness T_i using Eqs. (16)–(19).

Step 6. Derive the weight vector w by solving Eq. (42).

Step 7. Compute the overall relative closeness T_i by Eq. (19).

Step 8. Rank alternatives according to T_i and select the best one(s).

The above decision making process may be depicted by Fig. 2.

5. A Real Supplier Selection Example and Comparative Analysis

In this section, a real supplier selection example is given to illustrate the application of the proposed method. Meanwhile, the comparative analysis is also conducted to show the superiority of the proposed method.

5.1. *A Supplier Selection Problem and the Solving Process*

Yutong Bus Co., Ltd. (YBC for short) is one of the biggest companies in Chinese bus industry. In 1997, YBC became the first listed company among the bus industry in China. In 2013, bus sales in YBC reached 56068 units. To increase its core competencies, YBC needs to select a suitable supplier for its automotive upholstery. After preliminary screening, five candidate suppliers (alternatives) remain for further evaluation, denoted by x_1, x_2, x_3, x_4 and x_5. While evaluating these suppliers, eight attributes are considered, including the price (u_1), quality (u_2), reputation (u_3), technology ability (u_4), general management capability (u_5), risk (u_6), service performance (u_7) and the delivery time (u_8). The values for evaluating u_1 are certain and described by crisp numbers. Attributes u_2 and u_5 are qualitative attributes and the evaluations for them are expressed easily by linguistic variables. The values of attributes u_3 and u_7 are usually represented by IFSs. The assessments for attributes u_4 and u_6 are described in the form of HFSs. It is suitable to use interval number to describe attribute u_8. The DM evaluates candidate suppliers and provides the decision matrix as follows:

$$\begin{aligned}\boldsymbol{P}' &= (p'_{ij})_{6\times 8}\\ &= \begin{pmatrix} 200 & s_2 & \langle 0.60, 0.20\rangle & \{0.7, 0.5, 0.4\} & s_3 & \{0.9, 0.8, 0.6\} & \langle 0.10, 0.80\rangle & [6, 8] \\ 217 & s_1 & \langle 0.70, 0.15\rangle & \{0.8, 0.6\} & s_2 & \{0.9, 0.7\} & \langle 0.15, 0.70\rangle & [8, 10] \\ 212 & s_4 & \langle 0.80, 0.10\rangle & \{0.9, 0.8\} & s_2 & \{0.7, 0.5, 0.4\} & \langle 0.05, 0.75\rangle & [7, 9] \\ 232 & s_3 & \langle 0.50, 0.20\rangle & \{0.7, 0.6, 0.5\} & s_4 & \{0.8, 0.7, 0.6\} & \langle 0.05, 0.90\rangle & [5, 7] \\ 250 & s_0 & \langle 0.90, 0.05\rangle & \{0.8, 0.7, 0.5\} & s_3 & \{0.5, 0.4\} & \langle 0.20, 0.70\rangle & [3, 5] \\ 227 & s_3 & \langle 0.70, 0.20\rangle & \{0.9, 0.7, 0.6\} & s_1 & \{0.4, 0.3\} & \langle 0.10, 0.80\rangle & [8, 10] \end{pmatrix}.\end{aligned}$$

Combining the opinions of domain experts, general manager, financial manager and purchasing manager with DM's comprehensive judgements, the DM gives the following preference relations between candidate suppliers:

$$\begin{aligned}E = \big\{&\langle(1,2), h_E(1,2)\rangle, \langle(1,4), h_E(1,4)\rangle, \langle(3,2), h_E(3,2)\rangle, \langle(3,4), h_E(3,4)\rangle,\\ &\langle(5,6), h_E(5,6)\rangle, \langle(6,2), h_E(6,2)\rangle\big\},\end{aligned}$$

where $h_E(1,2) = \{0.4, 0.3, 0.1\}$, $h_E(1,4) = \{0.3, 0.2, 0.1\}$, $h_E(3,2) = \{0.8\}$, $h_E(3,4) = \{0.7, 0.6\}$, $h_E(6,2) = \{0.3\}$, $h_E(5,6) = \{0.2, 0.1\}$.

The attributes information supplied by the DM is

$$D = \left\{ w \in D_0 \,\middle|\, \begin{array}{l} w_2 - w_4 \geqslant 0.02;\ w_2 - w_3 \leqslant 0.05;\ 0.05 \leqslant w_2 \leqslant 0.15; \\ w_2 - w_3 < w_1 - w_7;\ w_4 \geqslant 2w_5; \\ w_5 \geqslant 0.02;\ w_6 - w_8 > 0.06w_7;\ w_8 < 2w_7;\ w_8 > 0.08; \\ w_1 + w_2 + w_6 + w_8 \geqslant 0.5. \end{array} \right\}$$

Step 1. Using Table 1 and Eqs. (9)–(13), the decision matrix $\boldsymbol{P}' = (p'_{ij})_{m\times n}$ can be normalized as

$$\boldsymbol{P} = \begin{pmatrix} 1.00 & (0.4, 0.5, 0.6) & \langle 0.60, 0.20\rangle & \{0.7, 0.5, 0.4\} & (0.6, 0.7, 0.8) & \{0.9, 0.8, 0.6\} & \langle 0.80, 0.10\rangle & [0.6, 0.8] \\ 0.92 & (0.2, 0.3, 0.4) & \langle 0.70, 0.10\rangle & \{0.8, 0.6\} & (0.4, 0.5, 0.6) & \{0.9, 0.7\} & \langle 0.70, 0.15\rangle & [0.8, 1.0] \\ 0.94 & (0.8, 0.9, 1.0) & \langle 0.80, 0.15\rangle & \{0.9, 0.8\} & (0.4, 0.5, 0.6) & \{0.7, 0.5, 0.4\} & \langle 0.75, 0.05\rangle & [0.7, 0.9] \\ 0.86 & (0.6, 0.7, 0.8) & \langle 0.50, 0.20\rangle & \{0.7, 0.6, 0.5\} & (0.8, 0.9, 1.0) & \{0.8, 0.7, 0.6\} & \langle 0.90, 0.05\rangle & [0.5, 0.7] \\ 0.80 & (0.0, 0.1, 0.2) & \langle 0.90, 0.05\rangle & \{0.8, 0.7, 0.5\} & (0.6, 0.7, 0.8) & \{0.5, 0.4\} & \langle 0.70, 0.20\rangle & [0.3, 0.5] \\ 0.88 & (0.6, 0.7, 0.8) & \langle 0.70, 0.20\rangle & \{0.9, 0.7, 0.6\} & (0.2, 0.3, 0.4) & \{0.4, 0.3\} & \langle 0.80, 0.10\rangle & [0.8, 1.0] \end{pmatrix}.$$

Step 2. The PIS r^+ and NIS $r^-$are obtained by Eqs. (14)–(15), i.e.

$$\begin{aligned} r^+ &= (1.0, (0.8, 0.9, 1.0), \langle 0.90, 0.05\rangle, \{0.9, 0.8, 0.8\}, (0.8, 0.9, 1), \{0.9, 0.8, 0.7\}, \\ &\quad \langle 0.90, 0.05\rangle, [0.8, 1.0]), \\ r^- &= (0.8, (0.0, 0.1, 0.2), \langle 0.5, 0.2\rangle, \{0.7, 0.5, 0.4\}, (0.2, 0.3, 0.4), \{0.4, 0.3, 0.3\}, \\ &\quad \langle 0.7, 0.2\rangle, [0.3, 0.5]). \end{aligned}$$

According to Eqs. (16)–(18), the relative closeness matrix $\boldsymbol{RC}$ of alternatives is derived as

$$\boldsymbol{RC} = \begin{pmatrix} 1.0000 & 0.9674 & 0.3582 & 0.0000 & 0.9694 & 0.9987 & 0.4342 & 0.9118 \\ 0.7785 & 0.7671 & 0.834 & 0.6554 & 0.6647 & 0.999 & 0.0076 & 1.0000 \\ 0.8948 & 1.0000 & 0.9997 & 1.0000 & 0.6647 & 0.7443 & 0.0032 & 0.9866 \\ 0.2571 & 0.9963 & 0.0000 & 0.2200 & 1.0000 & 0.9872 & 1.0000 & 0.6713 \\ 0.0000 & 0.0000 & 0.9627 & 0.6273 & 0.9694 & 0.3083 & 0.0000 & 0.0000 \\ 0.4098 & 0.9963 & 0.9111 & 0.9091 & 0.0000 & 0.0000 & 0.4342 & 1.0000 \end{pmatrix}.$$

Step 3. A bi-objective hesitant fuzzy programming model is constructed by Eq. (30) as follows:

$$\begin{aligned} \max\ \mathrm{CI} &= \{0.4, 0.3, 0.1\}\eta_{12} \oplus \{0.3, 0.2, 0.1\}\eta_{14} \oplus \{0.8\}\eta_{32} \oplus \{0.7, 0.6\}\eta_{34} \\ &\quad \oplus \{0.2, 0.1\}\eta_{56} \oplus \{0.4\}\eta_{62}, \\ \min\ \mathrm{ICI} &= \{0.4, 0.3, 0.1\}\xi_{12} \oplus \{0.3, 0.2, 0.1\}\xi_{14} \oplus \{0.8\}\xi_{32} \oplus \{0.7, 0.6\}\xi_{34} \\ &\quad \oplus \{0.2, 0.1\}\xi_{56} \oplus \{0.4\}\xi_{62}, \end{aligned}$$

$$
\text{s.t.}\begin{cases}
\eta_{12}-\xi_{12}=0.2215w_1+0.2003w_2-0.4758w_3-0.6554w_4+0.3047w_5\\
\qquad\qquad -0.0003w_6+0.4266w_7-0.0882w_8\\
\eta_{14}-\xi_{14}=0.7429w_1-0.0289w_2+0.3582w_3-0.22w_4-0.0306w_5\\
\qquad\qquad -0.0114w_6-0.5658w_7-0.2405w_8\\
\eta_{32}-\xi_{32}=0.1163w_1+0.2329w_2+0.1658w_3+0.3446w_4+0w_5\\
\qquad\qquad +0.2547w_6-0.0044w_7-0.0134w_8\\
\eta_{34}-\xi_{34}=0.6377w_1+0.0037w_2+0.9997w_3+0.78w_4-0.3353w_5\\
\qquad\qquad -0.243w_6-0.9968w_7+0.3152w_8\\
\eta_{56}-\xi_{56}=-0.4098w_1-0.9963w_2+0.0516w_3-0.2818w_4+0.9694w_5\\
\qquad\qquad +0.3083w_6-0.4342w_7-w_8\\
\eta_{62}-\xi_{62}=-0.3687w_1+0.2292w_2+0.0771w_3+0.2536w_4-0.6647w_5\\
\qquad\qquad -0.999w_6+0.4266w_7+0\times w_8\\
(0.2215w_1+0.2003w_2-0.4758w_3-0.6554w_4+0.3047w_5-0.0003w_6\\
\quad +0.4266w_7-0.0882w_8)-\eta_{12}\leqslant 0\\
(0.7429w_1-0.0289w_2+0.3582w_3-0.22w_4-0.0306w_5-0.0114w_6\\
\quad -0.5658w_7-0.2405w_8)-\eta_{14}\leqslant 0\\
(0.1163w_1+0.2329w_2+0.1658w_3+0.3446w_4+0w_5+0.2547w_6\\
\quad -0.0044w_7-0.0134w_8)-\eta_{32}\leqslant 0\\
(0.6377w_1+0.0037w_2+0.9997w_3+0.78w_4-0.3353w_5-0.243w_6\\
\quad -0.9968w_7+0.3152w_8)-\eta_{34}\leqslant 0\\
(-0.4098w_1-0.9963w_2+0.0516w_3-0.2818w_4+0.9694w_5+0.3083w_6\\
\quad -0.4342w_7-w_8)-\eta_{56}\leqslant 0\\
(-0.3687w_1+0.2292w_2+0.0771w_3+0.2536w_4-0.6647w_5-0.999w_6\\
\quad +0.4266w_7+0w_8)-\eta_{62}\leqslant 0\\
(0.2215w_1+0.2003w_2-0.4758w_3-0.6554w_4+0.3047w_5-0.0003w_6\\
\quad +0.4266w_7-0.0882w_8)+\xi_{12}\geqslant 0\\
(0.7429w_1-0.0289w_2+0.3582w_3-0.22w_4-0.0306w_5-0.0114w_6\\
\quad -0.5658w_7-0.2405w_8)+\xi_{14}\geqslant 0\\
(0.1163w_1+0.2329w_2+0.1658w_3+0.3446w_4+0\times w_5+0.2547w_6\\
\quad -0.0044w_7-0.0134w_8)+\xi_{32}\geqslant 0\\
(0.6377w_1+0.0037w_2+0.9997w_3+0.78w_4-0.3353w_5-0.243w_6\\
\quad -0.9968w_7+0.3152w_8)+\xi_{34}\geqslant 0\\
(-0.4098w_1-0.9963w_2+0.0516w_3-0.2818w_4+0.9694w_5+0.3083w_6\\
\quad -0.4342w_7-w_8)+\xi_{56}\geqslant 0\\
(-0.3687w_1+0.2292w_2+0.0771w_3+0.2536w_4-0.6647w_5-0.999w_6\\
\quad +0.4266w_7+0\times w_8)+\xi_{62}\geqslant 0\\
\xi_{12}\geqslant 0;\ \xi_{14}\geqslant 0;\ \xi_{32}\geqslant 0;\ \xi_{34}\geqslant 0;\ \xi_{56}\geqslant 0;\ \xi_{62}\geqslant 0;\\
\eta_{12}\geqslant 0;\ \eta_{14}\geqslant 0;\ \eta_{32}\geqslant 0;\ \eta_{34}\geqslant 0;\ \eta_{56}\geqslant 0;\ \eta_{62}\geqslant 0;\\
w_2-w_4\geqslant 0.02;\ w_2-w_3\leqslant 0.05;\ 0.05\leqslant w_2\leqslant 0.15;\ w_2-w_3<w_1-w_7;\\
w_4\geqslant 2w_5;\ w_5\geqslant 0.02;\ w_6-w_8>0.06w_7;\ w_8<2w_7;\ w_8>0.05w_6;\ w_8\geqslant 0.08;\\
w_1+w_2+w_6+w_8\geqslant 0.5,\\
w_1+w_2+w_3+w_4+w_5+w_6+w_7+w_8=1,\\
w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8\geqslant 0.
\end{cases}
\tag{43}
$$

Table 2
The vectors of attribute weights and ranking orders of candidates for different values of parameter δ.

δ	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	Ranking of candidates
0	0.1750	0.1500	0.2440	0.0040	0.0020	0.0950	0.2497	0.0800	$x_3 \succ x_1 \succ x_2 \succ x_6 \succ x_4 \succ x_5$
0.1	0.1784	0.1500	0.3053	0.0040	0.0020	0.0916	0.1934	0.0080	$x_3 \succ x_1 \succ x_2 \succ x_6 \succ x_4 \succ x_5$
0.2	0.1876	0.1500	0.4540	0.0040	0.0020	0.0824	0.0400	0.0080	$x_3 \succ x_2 \succ x_6 \succ x_1 \succ x_5 \succ x_4$
0.3	0.2876	0.0500	0.4540	0.0040	0.0020	0.0824	0.0400	0.0080	$x_3 \succ x_2 \succ x_6 \succ x_1 \succ x_5 \succ x_4$
0.4	0.2876	0.0500	0.4540	0.0040	0.0020	0.0824	0.0400	0.0080	$x_3 \succ x_2 \succ x_6 \succ x_1 \succ x_5 \succ x_4$
0.5	0.2876	0.0500	0.4540	0.0040	0.0020	0.0824	0.0400	0.0080	$x_3 \succ x_2 \succ x_6 \succ x_1 \succ x_5 \succ x_4$
0.6	0.2876	0.0500	0.4540	0.0040	0.0020	0.0824	0.0400	0.0080	$x_3 \succ x_2 \succ x_6 \succ x_1 \succ x_5 \succ x_4$
0.7	0.2876	0.0500	0.4540	0.0040	0.0020	0.0824	0.0400	0.0080	$x_3 \succ x_2 \succ x_6 \succ x_1 \succ x_5 \succ x_4$
0.8	0.2876	0.0500	0.4540	0.0040	0.0020	0.0824	0.0400	0.0080	$x_3 \succ x_2 \succ x_6 \succ x_1 \succ x_5 \succ x_4$
0.9	0.2876	0.0500	0.4540	0.0040	0.0020	0.0824	0.0400	0.0080	$x_3 \succ x_2 \succ x_6 \succ x_1 \succ x_5 \succ x_4$
1	0.1392	0.1259	0.1781	0.0836	0.0239	0.2150	0.1119	0.1221	$x_3 \succ x_2 \succ x_1 \succ x_4 \succ x_6 \succ x_5$

Step 4. Utilizing Eq. (42), Eq. (43) can be transformed into a single objective crisp programming model. We use Lingo Software Tool to solve it with $\delta = 0.5$. Main components for the optimal solution of the model are as follows:

$$\eta_{12} = 0.000, \quad \eta_{14} = 0.371, \quad \eta_{32} = 0.099, \quad \eta_{34} = 0.605, \quad \eta_{56} = 0.000,$$
$$\eta_{62} = 0.000, \quad \xi_{12} = 0.134, \quad \xi_{14} = \xi_{32} = \xi_{34} = 0.000, \quad \xi_{56} = 0.215,$$
$$\xi_{62} = 0.125, \quad w_1 = 0.2876, \quad w_2 = 0.0500, \quad w_3 = 0.4540, \quad w_4 = 0.0040,$$
$$w_5 = 0.0020, \quad w_6 = 0.0824, \quad w_7 = 0.0400, \quad w_8 = 0.0800.$$

Putting η_{ik} and ξ_{ik} into the objective functions of Eq. (43), the consistency and inconsistency measures are obtained as $\mathrm{CI} = 0.5861$ and $\mathrm{ICI} = 0.1159$, respectively.

Step 5. The overall relative closeness of alternatives to PIS can be calculated by Eq. (19) as follows:

$$T_1 = 0.6731, \quad T_2 = 0.8075, \quad T_3 = 0.9069, \quad T_4 = 0.3017,$$
$$T_5 = 0.4669, \quad T_6 = 0.6823.$$

Step 6. The ranking order of six candidate suppliers is $x_3 \succ x_2 \succ x_6 \succ x_1 \succ x_5 \succ x_4$. Therefore, supplier x_3 is the best one.

In the same way, the attribute weights can be calculated when the parameter δ takes different values between 0 and 1. The corresponding computation results and ranking orders are listed in Table 2. Fig. 3 intuitively reflects the changes of attribute weights.

As shown in Fig. 3, when the values of the parameter δ vary from 0 to 0.3, some attribute weights change apparently, whereas others vary slightly or remain unchanged. For example, w_1 and w_3 increase remarkably, while w_7 gradually decreases, w_6 varies slightly, and w_5 remains unchanged. When the values of δ changes from 0.4 to 0.9, all the weights of attributes are invariable when $\delta = 1$, the weight of each attribute varies again.

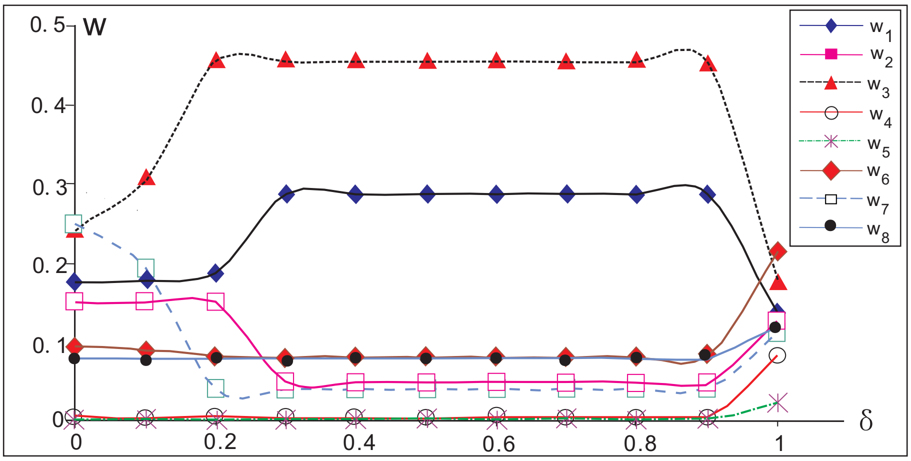

Fig. 3. The corresponding vectors of attribute weights according to different values of parameter δ.

Table 2 and Fig. 3 show that the weights of attributes and ranking orders of candidates may depend on the values of parameter δ. In real application, the DM can choose the appropriate value of parameter δ based on his/her preferences.

5.2. *Comparison with Intuitionistic Fuzzy LINMAP Method*

In this subsection, the comparision with fuzzy LINMAP method (Wan and Li, 2013) is given. Before comparing the both methods, we first transform HFSs into IFSs via Eq. (1) and use method (Wan and Li, 2013) to solve the above supplier selection problem again.

Transforming HFSs into IFSs by Eq. (1), the normalized decision matrix is transformed into the following matrix:

$$\boldsymbol{P} = \begin{pmatrix} 1.00 & (0.4, 0.5, 0.6) & \langle 0.60, 0.20\rangle & \langle 0.4, 0.3\rangle & (0.6, 0.7, 0.8) & \langle 0.60, 0.10\rangle & \langle 0.80, 0.10\rangle & [0.6, 0.8] \\ 0.92 & (0.2, 0.3, 0.4) & \langle 0.70, 0.10\rangle & \langle 0.6, 0.2\rangle & (0.4, 0.5, 0.6) & \langle 0.70, 0.10\rangle & \langle 0.70, 0.15\rangle & [0.8, 1.0] \\ 0.94 & (0.8, 0.9, 1) & \langle 0.80, 0.15\rangle & \langle 0.8, 0.1\rangle & (0.4, 0.5, 0.6) & \langle 0.40, 0.30\rangle & \langle 0.75, 0.05\rangle & [0.7, 0.9] \\ 0.86 & (0.6, 0.7, 0.8) & \langle 0.50, 0.20\rangle & \langle 0.5, 0.3\rangle & (0.8, 0.9, 1.0) & \langle 0.60, 0.20\rangle & \langle 0.90, 0.05\rangle & [0.5, 0.7] \\ 0.80 & (0, 0.1, 0.2) & \langle 0.90, 0.05\rangle & \langle 0.5, 0.2\rangle & (0.6, 0.7, 0.8) & \langle 0.40, 0.50\rangle & \langle 0.70, 0.20\rangle & [0.3, 0.5] \\ 0.88 & ((0.6, 0.7, 0.8) & \langle 0.7, 0.20\rangle & \langle 0.6, 0.1\rangle & (0.2, 0.3, 0.4) & \langle 0.30, 0.60\rangle & \langle 0.8, 0.10\rangle & [0.8, 1.0] \end{pmatrix}.$$

Meanwhile, the elements of the preference relation set E are transformed into IFSs from HFSs, i.e.

$$\begin{aligned} \tilde{C} = \{ & \langle (1,2), \tilde{C}(1,2)\rangle, \langle (1,4), \tilde{C}(1,4)\rangle, \langle (3,2), \tilde{C}(3,2)\rangle, \langle (3,4), \tilde{C}(3,4)\rangle, \\ & \langle (5,6), \tilde{C}(5,6)\rangle, \langle (6,2), \tilde{C}(6,2)\rangle\}, \end{aligned}$$

where $\tilde{C}(1,2) = \langle 0.1, 0.6\rangle$, $\tilde{C}(1,4) = \langle 0.1, 0.7\rangle$, $\tilde{C}(3,2) = \langle 0.8, 0.2\rangle$, $\tilde{C}(3,4) = \langle 0.6, 0.3\rangle$, $\tilde{C}(5,6) = \langle 0.1, 0.8\rangle$, $\tilde{C}(6,2) = \langle 0.3, 0.7\rangle$.

Solving a fuzzy LINMAP programming Eq. (28) in Wan and Li (2013), we get the following results:

$$\xi_{12} = \xi_{14} = \xi_{32} = \xi_{34} = 0.000, \quad \xi_{56} = 0.395, \quad \xi_{62} = 0.000,$$
$$w_1 = 0.1745, \quad w_2 = 0.1500, \quad w_3 = 0.2364, \quad w_4 = 0.0040,$$
$$w_5 = 0.0020, \quad w_6 = 0.0954, \quad w_7 = 0.2575, \quad w_8 = 0.0800, \quad T_1 = 0.6691,$$
$$T_2 = 0.6285, \quad T_3 = 0.6785, \quad T_4 = 0.5962, \quad T_5 = 0.2331, \quad T_6 = 0.6285.$$

Therefore, the ranking order of six candidate suppliers is $x_3 \succ x_1 \succ x_2 \sim x_6 \succ x_4 \succ x_5$.

From the expressions of decision information and the decision results, we compare method (Wan and Li, 2013) with the proposed method and get the following conclusions:

(1) Method (Wan and Li, 2013) only considered four types of attribute values, including crisp numbers, TrFNs, IFSs and intervals, while this paper adds the linguistic variables and HFSs. Since HFS can describe the uncertainty which cannot be described by interval, fuzzy sets, or IFS, adding HFS-type attribute values into consideration can make the decision making more flexible.

(2) Although the best suppliers are the same (i.e. supplier x_3), the proposed method has stronger distinguishing power than method (Wan and Li, 2013). For example, the overall relative closeness for alternatives x_2 and x_6 obtained by the proposed method are $T_2 = 0.8075$ and $T_6 = 0.6823$, respectively, which shows that alternative x_2 is obviously superior to alternative x_6, whereas the overall relative closeness of these two alternatives obtained by method (Wan and Li, 2013) are the same, i.e. $T_2 = T_6 = 0.6285$, which implies these two alternatives are considered indifference. In fact, as the DM prefers x_6 to x_2 with the hesitant fuzzy truth degree $h_E(6,2) = 0.3 < 0.5$, alternative x_6 is inferior to alternative x_2. In other words, it is more reasonable to interpret the alternative x_2 to be superior to alternative x_6, which verifies that the results obtained by the proposed method are more consistent with the subjective preferences given by the DMs.

5.3. *Comparison with the Method without Considering Hesitant Fuzzy Truth Degree*

In the above supplier selection example, if the hesitant fuzzy truth degrees are reduced into crisp truth degrees 0 or 1. i.e. $h_E(k,i) = 1$ for all $(k,i) \in E$. Then the above fuzzy programming model is simplified to the following linear programming model:

$$\max \ \mathrm{CI} = \eta_{12} + \eta_{14} + \eta_{32} + \eta_{34} + \eta_{56} + \eta_{62}, \tag{44}$$

$$\min \ \mathrm{ICI} = \xi_{12} + \xi_{14} + \xi_{32} + \xi_{34} + \xi_{56} + \xi_{62}, \tag{45}$$

$$
s.t.\left\{\begin{array}{l}
\eta_{12}-\xi_{12}=0.2215w_1+0.2003w_2-0.4758w_3-0.6554w_4+0.3047w_5\\
\qquad\qquad -0.0003w_6+0.4266w_7-0.0882w_8\\
\eta_{14}-\xi_{14}=0.7429w_1-0.0289w_2+0.3582w_3-0.22w_4-0.0306w_5\\
\qquad\qquad -0.0114w_6-0.5658w_7-0.2405w_8\\
\eta_{32}-\xi_{32}=0.1163w_1+0.2329w_2+0.1658w_3+0.3446w_4+0w_5\\
\qquad\qquad +0.2547w_6-0.0044w_7-0.0134w_8\\
\eta_{34}-\xi_{34}=0.6377w_1+0.0037w_2+0.9997w_3+0.78w_4-0.3353w_5\\
\qquad\qquad -0.243w_6-0.9968w_7+0.3152w_8\\
\eta_{56}-\xi_{56}=-0.4098w_1-0.9963w_2+0.0516w_3-0.2818w_4+0.9694w_5\\
\qquad\qquad +0.3083w_6-0.4342w_7-w_8\\
\eta_{62}-\xi_{62}=-0.3687w_1+0.2292w_2+0.0771w_3+0.2536w_4-0.6647w_5\\
\qquad\qquad -0.999w_6+0.4266w_7+0\times w_8\\
(0.2215w_1+0.2003w_2-0.4758w_3-0.6554w_4+0.3047w_5-0.0003w_6\\
\quad +0.4266w_7-0.0882w_8)-\eta_{12}\leqslant 0\\
(0.7429w_1-0.0289w_2+0.3582w_3-0.22w_4-0.0306w_5-0.0114w_6\\
\quad -0.5658w_7-0.2405w_8)-\eta_{14}\leqslant 0\\
(0.1163w_1+0.2329w_2+0.1658w_3+0.3446w_4+0w_5+0.2547w_6\\
\quad -0.0044w_7-0.0134w_8)-\eta_{32}\leqslant 0\\
(0.6377w_1+0.0037w_2+0.9997w_3+0.78w_4-0.3353w_5-0.243w_6\\
\quad -0.9968w_7+0.3152w_8)-\eta_{34}\leqslant 0\\
(-0.4098w_1-0.9963w_2+0.0516w_3-0.2818w_4+0.9694w_5+0.3083w_6\\
\quad -0.4342w_7-w_8)-\eta_{56}\leqslant 0\\
(-0.3687w_1+0.2292w_2+0.0771w_3+0.2536w_4-0.6647w_5-0.999w_6\\
\quad +0.4266w_7+0\times w_8)-\eta_{62}\leqslant 0\\
(0.2215w_1+0.2003w_2-0.4758w_3-0.6554w_4+0.3047w_5-0.0003w_6\\
\quad +0.4266w_7-0.0882w_8)+\xi_{12}\geqslant 0\\
(0.7429w_1-0.0289w_2+0.3582w_3-0.22w_4-0.0306w_5-0.0114w_6\\
\quad -0.5658w_7-0.2405w_8)+\xi_{14}\geqslant 0\\
(0.1163w_1+0.2329w_2+0.1658w_3+0.3446w_4+0\times w_5+0.2547w_6\\
\quad -0.0044w_7-0.0134w_8)+\xi_{32}\geqslant 0\\
(0.6377\omega_1+0.0037w_2+0.9997w_3+0.78w_4-0.3353w_5\\
\quad -0.243w_6-0.9968w_7+0.3152w_8)+\xi_{34}\geqslant 0\\
(-0.4098w_1-0.9963w_2+0.0516w_3-0.2818w_4+0.9694w_5+0.3083w_6\\
\quad -0.4342w_7-w_8)+\xi_{56}\geqslant 0\\
(-0.3687w_1+0.2292w_2+0.0771w_3+0.2536w_4-0.6647w_5-0.999w_6\\
\quad +0.4266w_7+0\times w_8)+\xi_{62}\geqslant 0\\
\xi_{12}\geqslant 0;\ \xi_{14}\geqslant 0;\ \xi_{32}\geqslant 0;\ \xi_{34}\geqslant 0;\ \xi_{56}\geqslant 0;\ \xi_{62}\geqslant 0;\\
\eta_{12}\geqslant 0;\ \eta_{14}\geqslant 0;\ \eta_{32}\geqslant 0;\ \eta_{34}\geqslant 0;\ \eta_{56}\geqslant 0;\ \eta_{62}\geqslant 0;\\
w_2-w_4\geqslant 0.02;\ w_2-w_3\leqslant 0.05;\ 0.05\leqslant w_2\leqslant 0.15;\ w_2-w_3<w_1-w_7;\\
w_4\geqslant 2w_5;\ w_5\geqslant 0.02;\ w_6-w_8>0.06w_7;\ w_8<2w_7;\ w_8>0.05w_6;\ w_8\geqslant 0.08;\\
w_1+w_2+w_6+w_8\geqslant 0.5,\\
w_1+w_2+w_3+w_4+w_5+w_6+w_7+w_8=1,\\
w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8\geqslant 0.
\end{array}\right. \tag{46}
$$

By the equal weighted summation method, Eq. (44) can be transformed into a single objective linear programming model. Solving the transformed model with Simplex

Method yields the main components of the optimal solution as follows:

$$\eta_{12} = 0.0000, \quad \eta_{14} = 0.0349, \quad \eta_{32} = \eta_{34} = 0.0000, \quad \eta_{56} = 0.0850,$$
$$\eta_{62} = 0.0000, \quad \xi_{12} = 0.025, \quad \xi_{14} = 0.000, \quad \xi_{32} = 0.160, \quad \xi_{34} = 0.100,$$
$$\xi_{56} = 0.000, \quad \xi_{62} = 0.698, \quad w_1 = 0.090, \quad w_2 = 0.050, \quad w_3 = 0.000,$$
$$w_4 = 0.004, \quad w_5 = 0.002, \quad w_6 = 0.734, \quad w_7 = 0.040, \quad w_8 = 0.080.$$

Thus, the overall relative closeness of alternative to PIS can be acquired by Eq. (19) as:

$$T_1 = 0.9636, \quad T_2 = 0.9259, \quad T_3 = 0.7612, \quad T_4 = 0.8942,$$
$$T_5 = 0.2307, \quad T_6 = 0.1877.$$

Therefore, the ranking order of six candidate suppliers is $x_1 \succ x_2 \succ x_4 \succ x_3 \succ x_5 \succ x_6$ and the best supplier is x_1 which is different from the results obtained by the above two methods. In fact, the ranking result is scarcely trust-worthy because the weight of attribute u_3 obtained by Eq. (44) is 0 which means the attribute u_3 completely does not work. In fact, reputation (u_3) plays a crucial role in the process of selecting suppliers for DMs in many industries because the decision information is incomplete or asymmetric in decision making. Therefore, the weight of attribute u_3 (0.454) obtained by the method proposed in this paper is closer to the reality. Since the ranking order of alternatives highly depends on the attribute weights, the decision results in this paper are more reasonable.

This analysis indicates that it is reasonable and necessary to introduce the hesitant fuzzy truth degrees to characterize the pairwise alternatives' comparisons. The HFS can flexibly reflect the fuzzy preference information of alternatives and the hesitant fuzzy truth degrees play an important role in the decision results indeed.

The comparisons of ranking orders between the method with crisp truth degree and the proposed method in this paper are also depicted in Fig. 4.

6. Conclusions

In this paper, a new hesitant fuzzy programming method is proposed to solve the hybrid MADM problems with hesitant fuzzy alternative comparisons and incomplete attribute weight information. In the proposed method, DM gave the preference relations between alternatives with hesitant fuzzy truth degrees represented by HFSs. Considering PIS and NIS simultaneously, the overall relative closeness is defined by the relative projection. According to the preference relations and the overall relative closeness degrees, the hesitant fuzzy consistency and inconsistency were measured and a new bi-objective hesitant fuzzy programming model was then constructed to determine attribute weights. Subsequently, a novel method for solving such a model is proposed and the vector of attribute weights was derived. Finally, the overall relative closeness degrees of alternatives were calculated and used to rank alternatives. The main contributions of this paper are outlined as follows:

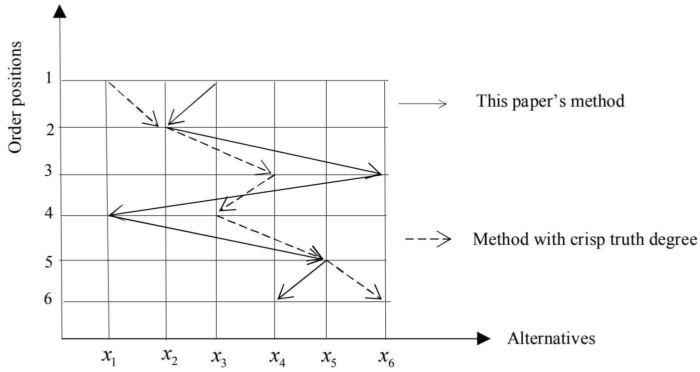

Fig. 4. Comparisons of ranking orders between the two methods given in the figure on the right.

(1) The fuzzy truth degrees of alternative comparisons are firstly represented by HFSs which can express the uncertain information being not described by fuzzy numbers or IFSs. The supplier selection example and comparison analysis show that using HFSs can make the decision results much closer to the real decision situation.
(2) Minimizing the inconsistency measure and maximizing the consistency measure simultaneously, a new bi-objective hesitant fuzzy mathematical programming model was constructed to objectively determine the weights of attributes.
(3) For solving the constructed bi-objective programming model, an effective method was developed and the weights of attributes were obtained. Using the overall closeness of alternatives computed by the relative projection, alternatives were ranked and the best one was selected.

The presented method in this paper can not only solve the supplier selection problem but can be applied to other related fields, such as investment projects selection, personal selection and material selection. Meanwhile, the proposed method may provide the DM with more choices in decision making process and also contributes to the theoretical investigation of hesitant fuzzy programming.

However, the PIS and NIS in this paper are given a priori. If the PIS and NIS are unknown in advance, how to effectively construct fuzzy programming models to determine them is a valuable and interesting topic. In addition, extending the proposed method to Pythagorean fuzzy set (Zeng *et al.*, 2016a) is also worth researching. These two topics will be investigated in the near future.

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Atspariojo neraiškiojo programavimo metodas hibridiniam daugiakriteriniam vertinimui su nevisa kriterijų svorių informacija

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Šiame straipsnyje nagrinėjama daugiakriterinio sprendimų priėmimo problema su nevisa informacija apie kriterijų svorius. Pasiūlytas atsparaus neraiškiojo programavimo modelis, paremtas daugiametės pirmenybių analizės (LINMAP) metodu. Spendimų priėmėjai pateikia informaciją apie pirmenybes alternatyvų atžvilgiu porinio palyginimo su atspariaisiais neraiškiaisiais skaičiais būdu, o alternatyvos aprašomos realiaisiais skaičiais, intervaliniais skaičiais, intuitionistiniais skaičiais, lingvistiniais kintamaisiais ir atspariaisiais neraiškiaisiais skaičiais. Remiantis alternatyvų projekcijomis į idealiuosius sprendinius, alternatyvoms nustatomi santykiniai atstumai. Tuomet apibrėžiami atsparieji neraiškieji suderintumo ir nesuderintumo matai. Minimizuojant nesuderintumo ir maksimizuojant suderintumo matus, pasiūlytas naujas dvitikslis atspariojo neraiškiojo programavimo modelis. Taigi kriterijų svoriai nustatomi objektyviai. Alternatyvos ranguojamos pagal jų santykinius atstumus. Tiekėjo pasirinkimo uždavinys pateikiamas kaip pasiūlyto metodo taikymo pavyzdys.