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About Some Controllability Properties of Linear Discrete-Time Systems in Probabilistic Metric Spaces

Manuel DE LA SEN
Institute of Research and Development of Processes IIDP
Faculty of Science and Technology, University of the Basque Country
PO Box 644 Bilbao, Barrio Sarriena, 48940 Leioa (Bizkaia), Spain
e-mail: manuel.delasen@ehu.es

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Abstract. This paper investigates in a formal context some fundamental controllability properties "from" and "to" the origin of probabilistic discrete-time dynamic systems as well as their uniform versions and complete controllability in a class of probabilistic metric spaces or probabilistic normed spaces, in particular, in probabilistic Menger spaces. Some related approximate probabilistic controllability properties are also investigated for the case when a nominal controllable system is subject to either parametrical perturbations or unmodelled dynamics. In this context, the approximate controllability of a perturbed system is a robustness-type approximate controllability provided that the nominal system is controllable. Some illustrative examples are also given.

Key words: probabilistic metric spaces, Menger spaces, controllabillity, dynamic systems.

1. Introduction and Preliminaries

A clear discussion and characterization of the basic controllability properties of deterministic linear dynamic systems is given in Kailath (1980). A lack of their parallel characterization in the literature for the case of probabilistic dynamic systems is observed. Probabilistic metric spaces and probabilistic Banach spaces are of important interest nowadays in the context of Fixed Point Theory. See, for instance, Choudhury *et al*. (2011, 2012), Beg *et al.* (2001), Beg and Abbas (2005) and Mihet (2004, 2009). In probabilistic metric spaces, the deterministic notion of distance translates as follows. Given any two points x and y of a metric space, a measure of the distance between them is a probabilistic metric $F_{x,y}(t)$, rather than the deterministic distance $d(x, y)$, which is interpreted as the probability of the distance between x and y being less than t $(t > 0)$ (Pap *et al.*, 1996; Sehgal and Bharucha-Reid, 1972; Schweizer and Sklar, 1960). On the other hand, Menger probabilistic metric spaces are a special case of probabilistic metric spaces which are endowed with a triangular norm. See Pap *et al.* (1996), Sehgal and Bharucha-Reid (1972), Choudhury *et al*. (2011, 2012), Beg and Abbas (2005), Mihet (2009), Khan *et al.* (1984), Choudhury and Das (2008), Gopal *et al.* (2014). Several kind of contraction as the socalled B and C – type contractions have been proved to be useful for single and multivalued mappings. On the other hand, 2-cyclic ϕ-contractions on intersecting subsets of complete Menger spaces were discussed in Choudhury *et al.* (2011) for contractions based on control ϕ-functions. See also Beg and Abbas (2005). It was found that fixed points are unique. Also, ϕ-contractions in complete probabilistic Menger spaces have been also studied in Mihet (2009) through the use of altering distances. On the other hand, probabilistic Banach spaces versus Fixed Point Theory were discussed in Beg *et al.* (2001). The concept of probabilistic complete metric space was adapted to the formalism of Banach spaces defined with norms being defined by triangular functions and under a suitable ordering in the considered space. A parallel background literature related to best proximity points and fixed points in cyclic mappings in metric and Banach spaces is exhaustive. See, for instance, Eldred and Veeramani (2006), De la Sen (2010), De la Sen *et al.* (2013), De la Sen and Agarwal (2011), De la Sen (2013a, 2013b), Karpagam and Agrawal (2009), Suzuki (2006), Di Bari *et al.* (2008), Rezapour *et al.* (2011), Derafshpour *et al.* (2010), Al-Thagafi and Shahzad (2009), Sanhan *et al*. (2012), De la Sen and Karapinar (2013), Chandok and Postolache (2013) and references therein.

It is well-known that Fixed Point Theory has also been widely applied to stability and equilibrium problems, in particular, to the analysis of equilibrium points and related stability studies of discrete-time, continuous-time and hybrid dynamic systems since, even based on intuitionist ideas, the convergence of trajectory-solutions of differential or difference equations or dynamic systems to an equilibrium point can be typically associated to the convergence of sequences to fixed points, see, for instance, De la Sen and Karapinar (2013), Takahashi and Takahashi (2007), Kim *et al.* (2014) and references therein, and to ergodic processes (Kim *et al.*, 2001). Several probabilistic approaches are also of relevance for the appropriate treatment, appropriate selection and compacting of information in large databases like, for instance, data-mining and network control. See, for instance, Pragarauskaite and Dzemyda (2013), Rahim *et al.* (2010), Anrig and Baziukaite (2005) and references therein.

It is well-known how important in the analysis and design of deterministic control systems the controllability property is. See, for instance, De la Sen (2007), Marchenko (2012, 2013), Louati and Ouzhara (2014), Karampetakis and Gregoriadou (2014), Shi *et al.* (2012), Balasubramaniam *et al.* (2014) and Xu *et al.* (1995). In particular, and under its most advantageous property versions of uniform complete controllability, it is known that: (a) it is possible to transfer any state at any given time to any targeted one along a finite time interval; (b) it is possible to design model-matching stabilizing output-feedback controllers with a fully free-design characteristic closed-loop equation, and then additionally, with a prescribed closed-loop stability degree, if the given linear controlled object is controllable and observable. It can be pointed out that optimal controllers for piecewise affine systems with sampled-data switching strategies have been synthesized in Azuma and Imura (2006) and some references therein, while model predictive control for constrained discrete-type Markovian switching systems has been proposed and investigated in Patrinos *et al.* (2014) and some references therein. Also, an inverse adaptive control scheme using learning techniques based on neural networks has been recently discussed and synthesized in Calvo-Rolle *et al.* (2014). The more precise purpose here is to characterize several kinds of controllability properties in a probabilistic concepts in linear discrete-time probabilistic metric spaces and with some extensions to the framework of Menger probabilistic metric spaces, those last ones being endowed with a triangular norm. We can point out that triangular norms operate in the probabilistic context in a close way as the triangle inequality operates in normed spaces, the main difference being that the first one manipulates inequalities of the type greater than or equal to since one is dealing with increasing with time probabilities of achieving a zero distance in-between the current state trajectory sequence and the prefixed targeted value tending to converge to one (i.e. to reach the certainty). Note that the idea of increasing the probability towards the certainty in computation processes is a useful computing tool, for instance, related to genetic algorithms with random insertion which can preserve the stochastic characteristics and main properties of the genetic algorithm while it preserves feasibility of generated individuals and it increases the probability to find the global optimum. See, for instance, Vaira and Kurasova (2014). In the above context, this paper investigates some fundamental controllability properties of non-necessarily linear probabilistic discrete dynamic systems from and to the origin as well as their uniform versions and complete controllability in a class of probabilistic metric spaces or probabilistic normed spaces. Their extensions to the approximate controllability properties for a class of linearized probabilistic perturbed discrete dynamic system in the state and controls provided that the nominal system has the corresponding property are also addressed in this research. The problem is intuitively based on the following idea. If the system is controllable to or from the origin, then the probability of the distance (or norm) from the targeted state (in particular, the zero-state or any other predefined state) to the current state should be one for some injected admissible control after a finite number of samples exceeding a certain lower-bound threshold. Otherwise, we can say that the system is not controllable in a probabilistic context. The paper body is organized in two main sections which follow this introductory one and a final conclusion section. Section 2 is devoted to give the basic notions, revisited from the deterministic context, together with their mathematical characterizations, of controllability and reachability and their uniform counterparts of linear discrete-time systems in a probabilistic context and a set of related fundamental results are established together with some illustrative examples. Section 3 gives some approximate robustness-type extensions to the case when the system is perturbed (roughly speaking, by parametrical perturbations, unmodelled dynamics or both) while the nominal system keeps the corresponding property. In this context, some approximate notions of controllability for the perturbed probabilistic system are given and some associate results are established which are based on the assumption of exact controllability of the nominal system together plus certain contractive probabilistic conditions for the error dynamics caused by the perturbation.

We will denote by $\boldsymbol{R}_{0+} = \{z \in \boldsymbol{R} : z \geqslant 0\} = \boldsymbol{R}_+ \cup \{0\}$ and $\boldsymbol{Z}_+ = \{z \in \boldsymbol{Z} : z > 0\} = \boldsymbol{Z}_{0+} \cup \{0\}$, $\bar{n} = \{1, 2, \ldots, n\}$ and denote by $\boldsymbol{L}$, the set of distribution functions $F : \boldsymbol{R} \to [0, 1]$ which are non-decreasing and left continuous such that $F(0) = 0$ and $\sup_{t \in \boldsymbol{R}} F(t) = 1$. Let X be a nonempty set and let $\boldsymbol{F} : X \times X \to \boldsymbol{L}$ be a mapping from $X \times X$, where X is an abstract set of elements, into the set of distribution functions $\boldsymbol{L}$

which are symmetric functions of elements $F_{x,y}$ for every $(x, y) \in X \times X$, referred to as the probabilistic metric (or probability density). Then, the ordered pair (X, F) is a probabilistic metric space (PM), Pap *et al.* (1996), Sehgal and Bharucha-Reid (1972), Schweizer and Sklar (1960), if

$$
\begin{aligned}
&(1)\ \forall x, y \in X\big(\big(F_{x,y}(t) = 1;\ \forall t \in \boldsymbol{R}_+\big) \Leftrightarrow (x = y)\big),\\
&(2)\ F_{x,y}(t) = F_{y,x}(t), \quad \forall x, y \in X,\ \forall t \in \boldsymbol{R},\\
&(3)\ \forall x, y, z \in X,\ \forall t_1, t_2 \in \boldsymbol{R}_+, \quad \big(\big(F_{x,y}(t_1) = F_{y,z}(t_2) = 1\big) \Rightarrow (F_{x,z}(t_1 + t_2) = 1)\big).
\end{aligned} \tag{1.1}
$$

Note that an interpretation is that $F : X \times X \to L$ is a set of distribution functions. A particular distribution function $F_{x,y} \in F$ is a probabilistic metric (or distance) which takes values $F_{x,y}(t) = H(t)$ identified with a mapping $H : \boldsymbol{R} \to [0, 1]$ in the set of all the distribution functions $\boldsymbol{L}$ is denoted a probabilistic metric which is a mapping from $X \times X$ to a probability density function $F : \boldsymbol{R} \to [0, 1]$.

A Menger PM-space is a triplet (X, F, Δ), where (X, F) is a PM-space which satisfies:

$$
F_{x,y}(t_1 + t_2) \geqslant \Delta\big(F_{x,z}(t_1), F_{z,y}(t_2)\big), \quad \forall x, y \in X,\ \forall t_1, t_2 \in \boldsymbol{R}_{0+}, \tag{1.2}
$$

under $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is a t-norm (or triangular norm) belonging to the set T of t-norms which satisfy the properties:

$$
\begin{aligned}
&(1)\ \Delta(a, 1) = a,\\
&(2)\ \Delta(a, b) = \Delta(b, a),\\
&(3)\ \Delta(c, d) \geqslant \Delta(a, b) \quad \text{if } c \geqslant a,\ d \geqslant b,\\
&(4)\ \Delta\big(\Delta(a, b), c\big) = \Delta\big(a, \Delta(b, c)\big).
\end{aligned} \tag{1.3}
$$

A consequence of the above results is $\Delta(a, 0) = 0$.

2. Controllability of a Probabilistic Discrete-Time Dynamic System

Consider the following probabilistic discrete dynamic system (*PDDS*) $\Omega := (X, U, Y, F)$ by:

$$
x_{n+1} = T(x_n, u_n), \quad \forall n \in \mathbf{Z}_{0+},\ x_0 \in X, \tag{2.1}
$$

$$
y_n = f(x_n, u_n), \quad \forall n \in \mathbf{Z}_{0+} \tag{2.2}
$$

where U, X and Y are Banach spaces, $\{u_n\} \subset U$, $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ are, respectively, the control, state and output sequences in the and $T : X \times U \to X$ and $f : X \times U \to Y$ are mappings which, together with the *PM* space (X, F), define the *PDDS* Ω. Each member of the control, state and output sequences is, respectively, referred to as a control, state and output sample. A constant sequence $\{g_n\}$ of values g in any of the spaces U, X, Y is denoted by $\hat{g}$ for exposition simplicity. Assume that $0 \in X$ and denote

$$\bar{u}(n, n+1) = (u_n, u_{n+1}) n \in Z_{0+}, \text{ so that}$$
$$x_{n+2} = T^2\big(x_n, \bar{u}(n, n+1)\big) = T^2(x_n, u_n, u_{n+1}) := T\big(T(x_n, u_n), u_{n+1}\big), \quad n \in \mathbf{Z}_{0+}.$$

In general, the control string of $(m+1)$ controls $\bar{u}(n, n+m) = (u_n, u_{n+1}, \ldots, u_{n+m})$, for any $n \in Z_{0+}$and $m \in \mathbf{Z}_+$, generates a state trajectory sequence which satisfies:

$$\begin{aligned} x_{n+m+1} &:= T^m\big(x_n, \bar{u}(n, n+m)\big) = T\big(T(x_{n+m-2}, u_{n+m-1}), u_{n+m}\big) \\ &= T\big(T\big(T(x_{n+m-3}, u_{n-m-3}), u_{n+m-2}\big), u_{n+m-1}\big) \\ &= \cdots = T\big(T\big(\ldots\big(T(x_n, u_n), u_{n+1}\overbrace{\big)}^{m}, u_{n+2}, \ldots, u_{n+m-1}\big), u_{n+m}\big) \end{aligned}$$
$$\text{for all } n \in \mathbf{Z}_{0+} \text{ and } m \in \mathbf{Z}_+ \tag{2.3}$$

$$\begin{aligned} y_{n+m} &:= f\big(x_{n+m}, \bar{u}(n+m, n+m)\big) \\ &= f\big(T\big(\ldots\big(T(x_n, u_n), u_{n+1}\overbrace{\big)}^{m}, u_{n+2}, \ldots, u_{n+m-1}\big), u_{n+m}\big) \end{aligned}$$
$$\text{for all } n \in \mathbf{Z}_{0+} \text{ and } m \in \mathbf{Z}_+ \tag{2.4}$$

with $y_0 = f(x_0, u_0)$. The semigroup property (or state-transition property) of the mapping $T : X \times U \to X$ defining the evolution state-trajectory sequence of the *PDDS* Ω becomes:

$$\begin{aligned} x_{n+m+j} &:= T^{n+m}(x_j, \bar{u}(j, j+n+m-1)) \\ &= T^n\big(T^m(x_j, \bar{u}(j, j+n-1)), \bar{u}(j+n, j+n+m-1)\big), \ \forall n, m, j \in \mathbf{Z}_{0+} \end{aligned} \tag{2.5}$$

under the conventions $u_{-1} = 0$, $\bar{u}(j, j-1) = 0$, $\forall j \in \mathbf{Z}_{0+}$ and T^0 being identity. Note that $\bar{u}(n, n) = u_n$, $\forall n \in \mathbf{Z}_{0+}$.

Definition 1. (**1**) Ω is point-state controllable from the origin (p. s. c. f. o.) to $a \in X$ if there are some integer $p_a = p_a(a) \in \mathbf{Z}_+$ and some control sequence $\{u_n\}$ such that

$$F_{a, T^{p_a}(0, \bar{u}(0, p_a - 1))}(t) = 1, \quad \forall t \in \boldsymbol{R}_+ \tag{2.6}$$

(**2**) Ω is uniformly point-state controllable from the origin (u. p. s. c. f. o.) to $a \in X$ if there are some integer $\hat{p}_a = \hat{p}_a(a) \in Z_+$ and some control sequence $\{u_n\}$ such that

$$F_{a, T^{\hat{p}_a}(j, \bar{u}(j, j+\hat{p}_a - 1))}(t) = 1, \quad \forall t \in \boldsymbol{R}_+, \ \forall j \in \mathbf{Z}_{0+}. \tag{2.7}$$

Note that the uniformity of Definition 1(2) refers to the controllability property being just dependent on the sampling interval but not on the initial time instant and it guarantees the point-state controllability from the origin of Definition 1(1). On the other hand, Definition 1(1) may be understood as a controllability property from the origin at zero initial time instant. The idea can be directly extended to point-state controllability from the origin at any initial sampling time instant j by replacing $0 \to j$ and $p_a - 1 \to j + p_a - 1$. Similar considerations apply for the concepts of uniform controllability in the next definitions.

DEFINITION 2. (**1**) Ω is point-state controllable to the origin (p. s. c. t. o.) from $a \in X$ if there are some integer $q_a = q_a(a) \in \boldsymbol{Z}_+$ and some control sequence $\{u_n\}$ such that

$$F_{0,T^{q_a}(a,\bar{u}(0,q_a-1))}(t) = 1, \quad \forall t \in \boldsymbol{R}_+ \tag{2.8}$$

(**2**) Ω is uniformly point-state controllable to the origin (u. p. s. c. t. o.) from $a \in X$ if there are some integer $\hat{q}_a = \hat{q}_a(a) \in \boldsymbol{Z}_+$ and some control sequence $\{u_n\}$ such that

$$F_{0,T^{\hat{q}_a}(a,\bar{u}(j,j+\hat{q}_a-1))}(t) = 1, \quad \forall t \in \boldsymbol{R}_+, \ \forall j \in \boldsymbol{Z}_{0+}. \tag{2.9}$$

Note that uniformly point-state controllability to the origin guarantees point-state controllability to the origin. Note that the controllability of Definitions 2 is guaranteed for a finite control strip $\bar{u}(0, q_a - 1)$, irrespective of the remaining elements of the control sequence $\{u_n\}$. Therefore, it is said that Ω is p. s. c. t. o. in q_a samples from $a \in X$ through the control strip $\bar{u}(0, q_a - 1)$. Note also that, in deterministic linear dynamic systems, point-state controllability from the origin is equivalent to reachability while being only equivalent to point-state controllability to the origin if the matrix of dynamics is nonsingular.

Theorem 1. *Assume that $T(0, u^*) = 0$ for some u^* such that the constant control sequence $\hat{u}^* = \{u^*\} \subset U$ (i.e. $u_n = u^*$, $\forall n \in \boldsymbol{Z}_{0+}$) and that Ω is p. s. c. t. o. in q_a samples from $x = a (\in X)$ through a control strip $\bar{u}(0, q_a - 1)$. Then, Ω is also p. s. c. t. o. in $\bar{q}_a$ samples from $x = a$ for any $\bar{q}_a (\in \boldsymbol{Z}_+) \geqslant q_a$.*

Proof. Define $\bar{q}_a(j) = q_a + j$ for any given $j \in \boldsymbol{Z}_{0+}$. It turns out that Ω is p. s. c. t. o. in $\bar{q}_a(0) = q_a$ samples from $a \in X$. Proceed by complete induction to prove that Ω is p. s. c. t. o. in $\bar{q}_a(j)$ samples from $a \in X$ for any $j \in \boldsymbol{Z}_+$. It holds for $\bar{q}_a(0)$ samples and assume it also holds for $\bar{q}_a(j)$ samples from $a \in X$ and some given $j \in \boldsymbol{Z}_+$. Thus, there is a control strip $\bar{u}(0, \bar{q}_a(j))$ such that $F_{0,T^{\bar{q}_a(j)}(a,\bar{u}(0,\bar{q}_a(j)-1))}(t/2) = 1$, $\forall t \in \boldsymbol{R}_+$. Also, $F_{0,T(0,u^*)}(t/2) = 1$, $\forall t \in \boldsymbol{R}_+$, since $T(0, u^*) = 0$, from the first property in (1.1) of the *PM*-space (X, F). From the semigroup property, one has for the control strip $\bar{u}(0, \bar{q}_a(j)) = (\bar{u}(0, \bar{q}_a(j) - 1), u^*)$ that

$$T^{\bar{q}_a(j)+1}\big(a, \bar{u}\big(0, \bar{q}_a(j)\big)\big) = T\big(T^{\bar{q}_a(j)}\big(a, \bar{u}(0, \bar{q}_a(j) - 1)\big), u^*\big), \quad \forall t \in \boldsymbol{R}_+$$

and, from the third property in (1.1) of the *PM*-space (X, F), it follows that

$$F_{0,T^{\bar{q}_a(j)}(a,\bar{u}(0,\bar{q}_a(j)-1))}(t/2) = F_{0,T(0,u^*)}(t/2) = 1, \quad \forall t \in \boldsymbol{R}_+ \text{ implies that}$$

$$F_{0,T^{\bar{q}_a(j)+1}(a,\bar{u}(0,\bar{q}_a(j)))}(t) = F_{0,T(T^{\bar{q}_a(j)}(a,\bar{u}(0,\bar{q}_a(j)-1)),u^*)}(t) = 1, \quad \forall t \in \boldsymbol{R}_+$$

so that if Ω is p. s. c. t. o. in $\bar{q}_a(j)$ samples from $a \in X$ through the control strip $\bar{u}(0, \bar{q}_a(j))$ then it is also p. s. c. t. o. in $\bar{q}_a(j) + 1$ samples from $a \in X$ trough the control strip $\bar{u}(0, \bar{q}_a(j) + 1) = (\bar{u}(0, \bar{q}_a(j)), u^*)$ and the proof follows by complete induction. □

A close proof to that of Theorem 1 is valid for its next stronger parallel uniformity result:

Theorem 2. *Assume that $T(0, u^*) = 0$ for some u^* such that the constant control sequence $\hat{u}^* = \{u^*\} \subset U$ and that Ω is u. p. s. c. t. o. in q_a samples along any discrete time interval $[j, j + \hat{q}_a]$, $\forall j \in \mathbf{Z}_{0+}$ from $x = a(\in X)$ through a control strip $\bar{u}(j, j + \hat{q}_a - 1)$, $\forall j \in \mathbf{Z}_{0+}$. Then, Ω is u. p. s. c. t. o. in $\bar{q}_a$ samples from $x = a$ for any $\bar{\hat{q}}_a(\in \mathbf{Z}_+) \geqslant \hat{q}_a$.*

Theorem 3. *Assume that $T(a, 0) \neq 0$ if $a(\in X) \neq 0$ and $T(0, u^*) = 0$ for some constant sequence $\hat{u}^* \subset U$ of value u^*. If Ω is p. s. c. t. o. from $x = a(\in X)$ through a control sequence $\{u_n\}$, then $q_{am} \in \mathbf{Z}_+$ exists* (*the minimum necessary length of a control strip for controllability to the origin*) *such that*:

(1) *Ω is p. s. c. t. o. in q_a samples from $x = a$ through any admissible finite control strip $\bar{u}(0, q_a - 1)$ if $q_a > q_{am}$*;
(2) *The control sequence $\{u_n\}$ has the constraint $u_n = u^*$ for all $n \geqslant q_{am}$.*

Proof. If Ω is p. s. c. t. o. from $a \neq 0$, then it is not p. s. c. t. o. in 0 samples, then through the control strip $\bar{u}(0, -1) = 0$ since $T(a, \bar{u}(0, -1)) = T(a, 0) \neq 0$. So, a minimum $q_{am} \in \mathbf{Z}_+$ exists such that Ω is p. s. c. t. o. in $q_a \geqslant q_{am}$ samples for some admissible control strip $\bar{u}(0, q_a - 1)$. From Theorem 1 Ω is p. s. c. t. o. in any number of samples $\bar{q}_a(\geqslant q_{am}) \in \mathbf{Z}_+$ with a control strip $\bar{u}(0, \bar{q}_a) = (\bar{u}(0, \bar{q}_a - 1), u^*)$ since $T(0, u^*) = 0$. □

A case of important practical interest is when $\hat{0} \subset U$ and $T(0, 0) = 0$. In this case, the origin is reached after a minimum finite number of samples with distribution function of probability one for any $t \in \boldsymbol{R}_+$ and the state is kept with probability one for any $t \in \boldsymbol{R}_+$ for any greater number of samples under zero control.

Example 1. Define a set of constant controls $U^* = \{\hat{u} \subset U\}$. If $\hat{0} \subset U^*$, then the point-state controllability to the origin in a deterministic controllability context holds for any control strip of zero elements of the form $\bar{u}(0, k) = u(u_0, u_1, \ldots, u_{q_{am}-1}, 0, \ldots, 0)$ for any integer $k \geqslant q_{am}$. A parallel counterpart of a deterministic controllability to the origin problem can be visualized for the linear and time-invariant discrete-time n-th dimensional system $x_{k+1} = Ax_k + Bu_k$, $k \in \mathbf{Z}_{0+}$, $A \in \boldsymbol{R}^{n\times n}$ and $B \in \boldsymbol{R}^{n\times m}$ for any given nonzero $x_0 = a \in R^n \equiv X$ as follows. Assume that the pair (A, B) is controllable and that it has rank n, i.e. A is non-singular so that of rank n. Then the controllability matrix $C=(B, AB, \dots, A^{\mu-1}B)$ has rank n, where $1 \leqslant \mu \leqslant n$ is the degree of the minimal polynomial of A (Kailath, 1980; De la Sen, 2007). There is a control strip $\bar{u}(0, \mu-1)=u(u_0, u_1, \dots, u_{\mu-1})$ such that $x_\mu=0$ and any other control strip of larger size of the form $\bar{u}(0, k)=u(u_0, u_1, \dots, u_{\mu-1}, 0, \dots, 0)$ keeps the state $x_k=0, \forall k(\geqslant \mu-1) \in \boldsymbol{Z}_{0+}$. In this simple case, it turns out that $U^*=\{\hat{0}\}$ and that the point-state controllability to the origin is uniform. If $\mu=n$, the pair (A, b) is controllable and $b \in \boldsymbol{R}^n$, then the minimum size control strip $\bar{u}(0, k-1)$ for controllability to the origin is unique for any nonzero initial state since the controllability matrix is square and non-singular. The minimum size control strip is defined by the column vector $\bar{u}(0, \mu-1)^T=(u_0, u_1, \dots, u_{\mu-1})^T=-\boldsymbol{C}_p^{-1}A^\mu a$, where $\boldsymbol{C}_p$ is a permutation of $\boldsymbol{C}$ defined with its columns written in reverse order, so that the initial state can be uniquely reconstructed as:

$$a=-A^{-\mu}\boldsymbol{C}_p\bar{u}(0, \mu-1)^T=-\big(A^{-1}b, A^{-2}b, \dots, A^{-\mu}b\big)\bar{u}(0, \mu-1)^T$$

in the case when $A \in \boldsymbol{R}^{n\times n}$ is non-singular. The necessary and sufficient condition for point-state and uniform controllability from an initial state $a \in X$ to a state $b \in X$ by in general non unique $\bar{u}(0, \mu-1)$, is that $\operatorname{rank}(\boldsymbol{C})=\operatorname{rank}(\boldsymbol{C}, b-A^\mu a)$ from the Rouché–Froebenius theorem from Linear Algebra. A sufficient condition for the property to hold for any given $a, b \in X$ is that $\operatorname{rank}(\boldsymbol{C})=n$ under which a unique control strip $\bar{u}(0, \mu-1)^T=\boldsymbol{C}_p^{-1}(b-A^\mu a)$ achieves the control objective. Note that if $b=0$ and A is nilpotent, then $b-A^\mu a=0$ for any $a \in X$ so that the system is p. s. c. t. o. under zero control. Detailed controllability results and associate discussions in a deterministic context are given for various kinds of dynamic systems (linear continuous-time, linear discrete-time, linear hybrid positive or not and some special non linear systems), for instance, in De la Sen (2007), Marchenko (2012, 2013), Louati and Ouzhara (2014), Karampetakis and Gregoriadou (2014), Shi *et al.* (2012, 2013) and some references therein.

Example 2. Consider the problem of Example 1 in a probabilistic context with probability density function $F_{x,y}(t)=\frac{\alpha t}{\alpha t+\|x-y\|}, \forall x, y \in X, \forall t \in \boldsymbol{R}_+$ and some given $\alpha \in \boldsymbol{R}_+$ where the state control and output spaces are normed linear spaces. A state sequence $\{x_n\}$ is driven from $x=a$ to the origin $x=0$ by some control strip $\bar{u}(0, \mu-1)$ which reaches it at the μ-th sample under the probability density function $F_{x_j,0}(t)=\frac{\alpha t}{\alpha t+\|x_j\|}$, $k=0, 1, \dots, , k$, $t \in \boldsymbol{R}_+$ with $k(\geqslant \mu-1) \in \boldsymbol{Z}_{0+}$ and $x_k=A^k a+\sum_{j=0}^{k-1} A^{k-1-j}u_j$, $n \in \boldsymbol{Z}_{0+}$ with $\bar{u}(0, \mu-1)^T=(u_0, u_1, \dots, u_{\mu-1})^T=-\boldsymbol{C}_p^{-1}A^\mu a$, $x_k=0$ and $u_k=0$ for $k(\geqslant \mu-1) \in \boldsymbol{Z}_+$. Then, $F_{x_k,0}(t)=1$, $k(\geqslant \mu-1) \in \boldsymbol{Z}_{0+}$, $\forall t \in \boldsymbol{R}_+$. By the property (1) of the PM-space $x_k=0$, $k(\geqslant \mu-1) \in \boldsymbol{Z}_{0+}$ so that the system is p. s. c. t. o. and u. p. s. c. t. o. Note that for the given probability density function, the *PDDS* Ω is p. s. c. t. o. if its deterministic version has the same property.

Theorems 1 to 3 can be directly extended to controllability from the origin under similar proofs and the changes $a \to 0$, $0 \to a$, $\bar{q}_a \to \bar{p}_a$. In particular the counterpart of Theorem 3 is the following:

Theorem 4. *Assume that $T(0,a) \neq 0$ if $a(\in X) \neq 0$ and $T(a,u^*) = a$ for some constant sequence $\hat{u}^* \subset U$. If Ω is p. s. c. f. o. through a control sequence $\{u_n\}$, then $p_{am} \in \mathbf{Z}_+$ exists (the minimum necessary length of a control strip for controllability from the origin to $x = a$) such that*:

(1) *Ω is p. s. c. f. o. in p_a samples from $x = 0$ to $x = a$ through any admissible finite control strip $\bar{u}(0, p_a - 1)$ if $p_a > p_{am}$*;
(2) *The control sequence $\{u_n\}$ has the constraint $u_n = u^*$ for all $n \geqslant p_{am}$.*

Outline of proof. It is quite close to the proof of Theorems 1–3 based in the semigroup property in terms of:

$$T^{\bar{p}_a(j)+1}\big(0, \bar{u}\big(\big(0, \bar{p}_a(j)\big)\big)\big) = T\big(T^{\bar{q}_a(j)}\big(0, \bar{u}(0, \bar{p}_a(j)-1)\big), u^*\big), \quad \forall t \in \boldsymbol{R}_+,$$

and, from the third property in (1.1) of the *PM*-space (X, F) in terms of

$$F_{a,T^{\bar{p}_a(j)}(0,\bar{u}(0,\bar{q}_a(j)-1))}(t/2) = F_{a,T(a,u*)}(t/2) = 1, \quad \forall t \in \boldsymbol{R}_+. \qquad \square$$

Example 3. If the controllability matrix is non-singular, then Examples 1 and 2 can be reformulated in terms of controllability from the origin under a control sequence driving the origin to $x = a$ obtained from the control strip $\bar{u}(0, \mu-1)^T = -\boldsymbol{C}_p^{-1}a$. The probability density function becomes

$$F_{x_j,a}(t) = \frac{\alpha t}{\alpha t + \|x_j - a\|}, \quad j = 0, 1, \ldots, k, \ t \in \boldsymbol{R}_+$$

with $k(\geqslant \mu - 1) \in \mathbf{Z}_+$ for $x_0 = a$.

Definition 3. (**1**) Ω is point-state completely controllable from the origin (p. s. c. c. f. o.) if it is p. s. c. f. o. for any $a \in X$.

(**2**) Ω is uniformly point-state completely controllable from the origin (u. p. s. c. c. f. o.) if it is u. p. s. c. f. o. for any $a \in X$.

Definition 4. (**1**) Ω is point-state completely controllable to the origin (p. s. c. c. t. o.) if it is p. s. c. t. o. for any $a \in X$.

(**2**) Ω is uniformly point-state completely controllable to the origin (u. p. s. c. c. t. o.) if it is u. p. s. c. t. o. for any $a \in X$.

Definition 5. (**1**) Ω is completely controllable (c. c.) in $P_a \times P_b \subset X \times X$ if, for any given ordered pair $(a, b) \in P_a \times P_b$, there are some integer $z_{ab} = z_{ab}(a, b) \in \mathbf{Z}_+$ and some control sequence $\{u_n\}$ such that

$$F_{b,T^{z_{ab}}(a,\bar{u}(0,z_{ab}-1))}(t) = 1, \quad \forall t \in \boldsymbol{R}_+. \tag{2.10}$$

(**2**) Ω is uniformly completely controllable (u. c. c.) if, for any given $(a, b) \in P_a \times P_b$, there are some integer $\hat{z}_{ab} = \hat{z}_{ab}(a, b) \in \boldsymbol{Z}_+$ and some control sequence $\{u_n\}$ such that

$$F_{b,T^{\hat{z}_{ab}}(a,\bar{u}(j,j+\hat{z}_{ab}-1))}(t) = 1, \quad \forall t \in \boldsymbol{R}_+,\ \forall j \in \mathbf{Z}_{0+}. \tag{2.11}$$

Note that controllable deterministic time-invariant dynamic systems are uniformly controllable. A basic result in the probabilistic context follows:

Theorem 5. *Ω is u. c. c. in $P_a \times P_b$ if and only if it is both u. p. s. c. c. f. o. in $P_a \times P_b$ and u. p. s. c. c. t. o. in P.*

Proof. If Ω is u. c. c. in $P_a \times P_b$, then $F_{b,T^{\hat{z}_{ab}}(a,\bar{u}(j,j+\hat{z}_{ab}-1))}(t) = 1$, $\forall t \in \boldsymbol{R}_+$ for some $\bar{u}(j, j+\hat{z}_{ab}-1)$, $\forall j \in \mathbf{Z}_{0+}$, $\forall a, b \in P_a \times P_b$ for some $\hat{z}_{ab} = \hat{z}_{ab}(a, b) \in \boldsymbol{R}_+$, $\bar{u}(j, j+\hat{z}_{ab}-1)$. If $b = 0$ and $\hat{z}_{ab} = \hat{q}_a$, then (2.9) holds and Ω is u. p. s. c. t. o. in $P_a \times P_b$. Also, if $a = 0$ and $\hat{z}_{ab} = \hat{p}_b$, then $F_{b,T^{\hat{p}_b}(j,\bar{u}(j,j+\hat{p}_b-1))}(t) = 1$, $\forall t \in \boldsymbol{R}_+$, $\forall j \in \mathbf{Z}_{0+}$, then (2.7) holds and Ω is u. p. s. c. f. o in P. Thus, if Ω is u. c. c. in $P_a \times P_b$, then it is both p. s. c. c. f. o. in $P_a \times P_b$ and p. s. c. c. t. o. in $P_a \times P_b$. Conversely, if Ω is both u. p. s. c. f. o. in $P_a \times P_b$ and u. p. s. c. t. o. in $P_a \times P_b$, then

$$\begin{aligned} &F_{b,T^{\hat{p}_b}(0,\bar{u}(j,j+\hat{p}_b-1))}(t/2) \\ &\quad = F_{0,T^{\hat{q}_a}(a,\bar{u}(j+\hat{q}_a,j+\hat{q}_a+\hat{p}_b-1))}(t/2) = 1, \quad \forall t \in \boldsymbol{R}_+,\ \forall j \in \mathbf{Z}_{0+}. \end{aligned} \tag{2.12}$$

Then, one gets from (2.12), the properties 2–3 of the PM space (X, F) and the semigroup property (2.5) that

$$F_{b,T^{\hat{p}_b+\hat{q}_a}(a,\bar{u}(j,j+\hat{q}_a+\hat{p}_b-1))}(t) = 1, \quad \forall t \in \boldsymbol{R}_+,\ \forall j \in \mathbf{Z}_{0+} \tag{2.13}$$

so that Ω is u. c. c. in $P_a \times P_b$ with $\hat{z}_{ab} = \hat{q}_a + \hat{p}_b$. Now, the first property of the *PM* space and (2.13) imply that

$$\begin{aligned} b &= T^{\hat{p}_b+\hat{q}_a}\big(a, \bar{u}(j, j+\hat{p}_b+\hat{q}_a-1)\big) \\ &= T^{\hat{p}_b}\big(T^{\hat{q}_a}(a, \bar{u}(j, j+\hat{p}_b-1)), \bar{u}(j+\hat{p}_b, j+\hat{p}_b+\hat{q}_a-1)\big), \quad \forall j \in \mathbf{Z}_{0+}. \end{aligned} \tag{2.14}$$

Thus, the proof is complete. □

Definitions 1–5 can be directly extended to the parallel concepts of output controllability and uniform output controllability in the probabilistic senses. In particular, Definition 5 is extended as Definition 6 below. The remaining Definitions 1–4 are directly extendable in close senses of output controllability from or to the origin related to the amended following acronyms:
p. s. o. c. f. o., u. p. s. o. c. f. o., p. s. o. c. t. o., u. p. s. o. c. t. o., p. s. o. c. c. f. o., u. p. s. o. c. c. f. o., p. s. o. c. c. t. o., u. p. s. o. c. c. t. o.
which are omitted in terms of explicit formal definitions for the sake of simplicity.

EXAMPLE 4. Let $A \in \boldsymbol{R}^{n\times n}$ and $B \in \boldsymbol{R}^{n\times m}$ be the matrices of the deterministic Example 1 and consider the subset of $X \times X$ of ordered pairs defined as follows $\{(a,b) \in X \times X : rank(B) = rank(b - Aa, B)\}$ where $P_a, P_b \subset X$ are defined by the elements of such pairs. If the pair (A, B), then the controllability matrix $\boldsymbol{C} = (B, AB, \ldots, A^{\mu-1}B)$ and its column permuted matrix $\boldsymbol{C}_p$ have rank n and $\bar{u}(0, \mu - 1)^T = (u_0, u_1, \ldots, u_{\mu-1})^T = \boldsymbol{C}_p^{-1}(b - A^{\mu}a)$ drives $x = a$ to $x = b$ in μ samples with μ being the degree of the minimal polynomial of A. Since a, b are such that $rank(B) = rank(b - Aa, B)$, one has from the Rouché–Froebenius theorem from Linear Algebra that a constant control $u_k = u_\mu = u^* = u^*(a, b)$ exists for $k \geqslant \mu$ which guarantees $T(a, u^*) = b$ and any, in general non-unique (being unique if the controllability matrix is of full rank), control strip $\bar{u}(0, k)^T = (u_0, u_1, \ldots, u_{\mu-1}, u^*, \ldots, u^*)^T = (\boldsymbol{C}_p^{-1}(b - A^{\mu}a), u^*, \ldots, u^*)$ for any given $k(\geqslant \mu - 1) \in \boldsymbol{Z}_+$ guarantees that $x_k = b$ for any $k(\geqslant \mu - 1) \in \boldsymbol{Z}_{0+}$ if $x_0 = a$. Note that such a constant control $u^*(a, b)$ is not unique, in general, for the given a, b. In particular, we can choose $P_a = X$ and $P_b = \{b \in X : rank(B) = rank(b - Aa, B),\ \forall a \in X\}$ and $P_b = X$ and $P_a = \{a \in X : rank(B) = rank(b - Aa, B),\ \forall b \in X\}$. Note that, if $m \geqslant n$ (i.e. there are non less number of inputs than the dimension of the system) and $rank\, B = m$, then $P_a = P_b = X$. If $m = n$ and B is non-singular, then $u * (a, b)$ is unique for each pair (a, b).

The probabilistic counterpart behaves as follows based on Example 2. The probability density function of Example 2 is

$$F_{x_j,b}(t) = \frac{\alpha t}{\alpha t + \|x_j - b\|}, \quad k = 0, 1, \ldots, k,\ t \in \boldsymbol{R}_+$$

with $k(\geqslant \mu - 1) \in \boldsymbol{Z}_+$ and $x_0 = a$. Then, $F_{x_k,b}(t) = 1$, $k(\geqslant \mu - 1) \in \boldsymbol{Z}_{0+}$, $\forall t \in \boldsymbol{R}_+$. By the property (1) of the PM-space $x_k = b$, $k(\geqslant \mu - 1) \in \boldsymbol{Z}_+$ so that the system is p. s. c. f. o. and u. p. s. c. f. t. o. for the set of pairs $\{(a, b) \in X \times X : rank(B) = rank(b - Aa, B)\}$ and the system is u. c. c. in $P_a \times P_b$.

DEFINITION 6. (**1**) Ω is completely output-controllable (c. o. c.) if, for any given $a, b \in X$, there are some integer $z_{oab} = z_{oab}(a, b) \in \boldsymbol{Z}_+$ and some control sequence $\{u_n\}$ such that

$$F_{f(b,u_{z_{oab}}),fT^{z_{0ab}}(a,\bar{u}(0,z_{oab}-1))}(t) = 1, \quad \forall t \in \boldsymbol{R}_+. \tag{2.15}$$

(**2**) Ω is uniformly completely output-controllable (u. c. o. c.) if, for any given $a, b \in X$, there are some integer $\hat{z}_{oab} = \hat{z}_{oab}(a, b) \in \boldsymbol{Z}_+$ and some control sequence $\{u_n\}$ such that

$$F_{f(b,u_{j+zoab}),f(T^{\hat{z}_{oab}}(a,\bar{u}(j,j+\hat{z}_{oab}-1)),u_{j+\hat{z}_{oab}})}(t) = 1, \quad \forall t \in \boldsymbol{R}_+,\ \forall j \in \boldsymbol{Z}_{0+}. \tag{2.16}$$

Note that it turns out that if $\{u_n\}$ exists such that its associate control string $\bar{u}(j, j + z_{oab} - 1)$ leads to

$$F_{b,T^{\hat{z}_{oab}}(a,\bar{u}(j,j+\hat{z}_{oab}-1))}(t) = 1, \quad \forall t \in \boldsymbol{R}_+,\ \forall j \in \boldsymbol{Z}_{0+}. \tag{2.17}$$

Then, one has from the output constraint (2.2) of Ω for any given $u_{z_{oab}} \in U$ that

$$F_{b,T^{\hat{z}_{oab}}(a,\bar{u}(j,j+\hat{z}_{oab}-1))}(t) = 1 \Rightarrow F_{f(b,u_{j+\hat{z}_{oab}}),f(T^{\hat{z}_{oab}}(a,\bar{u}(j,j+\hat{z}_{oab}-1)),u_{j+\hat{z}_{oab}})}(t) = 1, \quad \forall t \in \boldsymbol{R}_+ \tag{2.18}$$

and the following result which is a consequence of Theorem 5:

Corollary 1. *The following properties hold*:

(1) *If Ω is u. c. c., then Ω is u. o. c. c.*;
(2) *If Ω is u. p. s. c. c. f. o. and u. p. s. c. c. t. o., then Ω is u. o. c. c.*;
(3) *If Ω is u. p. s. o. c. c. f. o. and u. p. s. c. c. t. o., then Ω is u. o. c. c.*;
(4) *If Ω is u. p. s. c. c. f. o. and u. p. s. o. c. c. t. o., then Ω is u. o. c. c.*

Proof. From (2.2), $f(b, u_{j+\hat{z}_{oab}}) = f(T^{\hat{z}_{oab}}(a, \bar{u}(j, j + \hat{z}_{oab} - 1)), u_{j+\hat{z}_{oab}})$, $\forall j \in \mathbf{Z}_{0+}$ for any given $a, b \in X$, thus, if Ω is u. c. c., then Ω is u. o. c. c. from (2.18) and the first property of the corollary is proved. If Ω is u. p. s. c. c. f. o. and u. p. s. c. c. t. o., then Ω is u. c. c. from Theorem 1, then the first property of the implication of (2.18) holds and the second property of this corollary follows from the first assertion.

The third property of Corollary 1 can be proved by requiring to Ω to be u. p. s. o. c. c. f. o., instead of the stronger condition of being u. p. s. c. c. f. o., and u. p. s. c. c. t. o. This leads to

$$F_{f(b,u_{\hat{z}_{oab}}),fT^{\hat{z}_{o0b}}(0,\bar{u}(j,j+\hat{z}_{a0b}-1),u_{j+\hat{z}_{a0b}})}(t/2) = F_{0,T^{\hat{q}_a}(a,\bar{u}(0,\hat{q}_a-1))}(t/2)$$
$$= F_{f(b,u_{\hat{z}_{oab}}),fT^{\hat{z}_{oab}}(a,\bar{u}(0,\hat{q}_a-1),u_{j+\hat{z}_{a0b}})}(t) = 1, \quad \forall t \in \boldsymbol{R}_+,\ \forall j \in \mathbf{Z}_{0+} \tag{2.19}$$

for some control string $\bar{u}(j, j+\hat{z}_{a0b}) = (\bar{u}(j, j+\hat{z}_{a0b}-1), u_{j+\hat{z}_{a0b}})$ since $f(b, u_{j+\hat{z}_{oab}}) = f(T^{\hat{z}_{oab}}(a, \bar{u}(j, j+\hat{z}_{oab}-1)), u_{j+\hat{z}_{oab}})$ from (2.2). The fourth property is proved closely to the third one according to:

$$F_{f(b,u_{\hat{z}_{oab}}),fT^{\hat{z}_{oab}}(a,\bar{u}(j,j+\hat{z}_{oab}-1))}(t/2) = F_{a,T^{\hat{q}_a}(0,\bar{u}(0,\hat{p}_a-1))}(t/2)$$
$$= F_{f(b,u_{\hat{z}_{oab}}),fT^{\hat{z}_{oab}}(a,\bar{u}(0,\hat{q}_a-1))}(t) = 1, \quad \forall t \in \boldsymbol{R}_+,\ \forall j \in \mathbf{Z}_{0+}. \quad \square \tag{2.20}$$

The converses of the properties of Corollary 1 are not true, in general.

3. Approximate Controllability of a Probabilistic Perturbed Linearized Discrete Dynamic System

Consider the following perturbed probabilistic discrete dynamic system (*PPDDS*) $\Omega_p := (\bar{X}, \bar{U}, \bar{Y}, F)$ defined by:

$$\bar{x}_{n+1} = \bar{T}_n(\bar{x}_n, \bar{u}_n) = \bar{S}_n\bar{x}_n + \bar{R}_n\bar{u}_n, \quad \forall n \in \mathbf{Z}_{0+},\ \bar{x}_0 \in X, \tag{3.1}$$

$$\bar{y}_n = \bar{f}_n(\bar{x}_n, \bar{u}_n) = \bar{Q}_n\bar{x}_n + \bar{W}_n\bar{u}_n, \quad \forall n \in \mathbf{Z}_{0+} \tag{3.2}$$

of nominal version $\Omega := (X, U, Y, F)$ being a particular case of (2.1)–(2.2):

$$x_{n+1} = T_n(x_n, u_n) = S_n x_n + R_n u_n, \quad \forall n \in \mathbf{Z}_{0+},\ x_0 \in X, \tag{3.3}$$

$$y_n = f_n(x_n, u_n) = Q_n x_n + W_n u_n, \quad \forall n \in \mathbf{Z}_{0+}. \tag{3.4}$$

Define errors $\tilde{x}_n = \bar{x}_n - x_n$, $\tilde{T}_n = \bar{T}_n - T_n$ and $\tilde{S}_n = \bar{S}_n - S_n$, $\forall n \in Z_{0+}$. Direct calculations with (3.1)–(3.2) yield the following error description:

$$\tilde{x}_{n+1} = S_n \tilde{x}_n + g_n = \bar{S}_n \tilde{x}_n + h_n, \quad \forall n \in \mathbf{Z}_{0+} \tag{3.5}$$

where

$$g_n = \tilde{S}_n \bar{x}_n + \tilde{R}_n \bar{u}_n, \qquad h_n = \tilde{S}_n x_n + \tilde{R}_n u_n. \tag{3.6}$$

A case of particular interest for comparison of the perturbed system with its nominal (unperturbed) one is that when $\tilde{T}_n = \bar{T}_n - T$ and $\tilde{S}_n = \bar{S}_n - T$ converge point-wise to zero as $n \to \infty$, simply denoted as $\tilde{T}_n \to 0$ and $\tilde{S}_n \to 0$ as $n \to \infty$. Definitions 2 are generalized for approximate controllability to the origin of Ω_p as follows:

Definition 7. (1) A *PDDS* Ω is approximately point-state controllable to the origin (a. p. s. c. t. o.) from $a \in X$ if, for any given real constants $\varepsilon \in \boldsymbol{R}_+$ and $\lambda \in (0, 1)$, there is a positive integer $n_a = n_a(a, \varepsilon, \lambda)$ and some control sequence $\{u_n\}$ such that

$$F_{0,T^n(a,\bar{u}(0,n-1))}(\varepsilon) > 1 - \lambda, \quad \forall n (\geqslant n_a) \in \mathbf{Z}_+. \tag{3.7}$$

(2) A *PDDS* Ω is approximately point-state controllable to the origin (a. p. s. c. t. o.) from $a \in X$ if, for any given real constants $\varepsilon \in \boldsymbol{R}_+$ and $\lambda \in (0, 1)$, there is a positive integer $\hat{n}_a = \hat{n}_a(a, \varepsilon, \lambda)$ and some control sequence $\{u_n\}$ such that

$$F_{j,T^n(a,\bar{u}(j,j+n-1))}(\varepsilon) > 1 - \lambda, \quad \forall n (\geqslant \hat{n}_a) \in \mathbf{Z}_+, \quad \forall j \in \mathbf{Z}_{0+}. \tag{3.8}$$

The interpretation of the approximate point-state controllability to the origin from $a \in X$ is that in a finite number of samples exceeding a certain minimum threshold $n \geqslant n_a$ the probability of $\|T^n(a, \bar{u}(0, n-1))\| = d(0, T^n(a, \bar{u}(0, n-1))) < \varepsilon$ is close to one for some admissible control sequence since we can chose both ε and λ to be arbitrarily small positive real constants. The above definition and associated idea are now used in a probabilistic robustness controllability context in the sense that it gives a sufficient condition for Ω_p (3.1)–(3.2) to be a. p. s. c. t. o. if Ω (3.3)–(3.4) is p. s. c. t. o.

Theorem 6. *Consider the PDDS Ω, Eqs.* (3.3)–(3.4)*, and the PPDDS Ω_p, Eqs.* (3.1)–(3.2)*, and assume that*:

(1) (X, F, Δ) *is a Menger PM-space, where (X, F) is a PM-space and Δ is the minimum triangular norm and the probability density function $F : X \times X \to L$ satisfies*

(a) $F_{x+z,y+z}(t) = F_{x,y}(t)$ *for any* $x, y, z \in X$, $\forall t \in \boldsymbol{R}_+$,

(b) *The contractive condition* $F_{S_{n+1}\tilde{x}_{n+1},S_n\tilde{x}_n}(Kt) \geqslant F_{\tilde{x}_{n+1},\tilde{x}_n}(t)$ *holds for some real constant* $K \in [0, 1)$, $\forall n \in \mathbf{Z}_{0+}$, $\forall t \in \boldsymbol{R}_+$ *where* $\{x_n\}$ *is a solution sequence of* (3.1) *for any* $x_0 \in X$,

(2) $\hat{0} \subset U$ *and* $S_n 0 = 0$ *and* $R_n 0 = 0$, $\forall n(\geqslant q_a - 1) \in \mathbf{Z}_{0+}$,

(3) Ω *is p. s. c. t. o. in* q_a *samples from* $x = a(\in X)$ *through some control strip* $\bar{u}(0, q_a - 1)$ *so that* $T(0, u^*) = 0$ *for some* u^* *such that the constant control sequence* $\hat{u}^* = \{u^*\} \subset U$,

(4) *The sequence* $\{g_n\}$ *converges.*

Then, Ω_p *is a. p. s. c. t. o. in any finite number of samples being non less than a minimum lower threshold* $n_a \geqslant q_a$ *under control strips of the form* $\bar{u}(0, k + q_a - 1) = (\bar{u}(0, q_a - 1), \overbrace{u^*, u^*, \ldots \ldots, u^*}^{k})$, $\forall k(\geqslant n_{qa} - q_a) \in \mathbf{Z}_{0+}$ *and with a prescribed guaranteed minimum probability error* $F_{\tilde{x}_{n+1},\tilde{x}_n}(\varepsilon) > 1 - \lambda$ *for any arbitrary given real constants* $\varepsilon \in \boldsymbol{R}_+$ *and* $\lambda \in (0, 1)$ *and all integer* $n \geqslant n_{qa}$.

Proof. Note that $T_n(0, 0) = S_n 0 + R_n 0 = 0$, $\forall n(\geqslant \mu - 1) \in \mathbf{Z}_{0+}$. From Theorem 1, Ω is p. s. c. t. o. in $\bar{q}_a$ samples from $x = a$ for any $\bar{q}_a(\in \mathbf{Z}_+) \geqslant q_a$ and $F_{x_n,a}(t) = 1$ for any integer $n \geqslant \bar{q}_a \geqslant q_a$ through a control strip $\bar{u}(0, \bar{q}_a - 1) = (\bar{u}(0, q_a - 1), 0, \ldots, 0)$. Note that

$$\tilde{x}_{n+m+1} = \bar{S}(n, n+m)\tilde{x}_n + \bar{g}(n, n+m); \quad \forall n, m \in \mathbf{Z}_{0+} \tag{3.9}$$

where

$$\bar{S}(n, n+m) = \prod_{j=0}^{m}[S_{n+j}], \qquad \bar{g}(n, n+m) = \sum_{j=0}^{m}\left(\prod_{i=j+1}^{m}\left[S_{n+i}\right]\right)g_{n+j}. \tag{3.10}$$

Then,

$$F_{\bar{S}(n,n+m+1)\tilde{x}_n,\bar{S}(n,n+m)\tilde{x}_n}(t) \geqslant F_{\tilde{x}_{n+1},\tilde{x}_n}(K^{-m-1}t), \quad \forall n, m \in \mathbf{Z}_{0+},\ \forall t \in \boldsymbol{R}_+ \tag{3.11}$$

and

$$F_{\tilde{x}_{n+m+2},\tilde{x}_{n+m+1}}(t) = F_{\bar{S}(n,n+m+1)\tilde{x}_n,\bar{S}(n,n+m)\tilde{x}_n+\bar{g}(n,n+m)-\bar{g}(n,n+m+1)}(t) \tag{3.12}$$

so that

$$\begin{aligned} F_{\tilde{x}_{n+m+2},\tilde{x}_{n+m+1}}(2t) &\geqslant \Delta\big(F_{\bar{S}(n,n+m+1)\tilde{x}_n,\bar{S}(n,n+m)\tilde{x}_n}(t), F_{\bar{g}_n-\bar{g}_{n+1},0}(t)\big) \\ &\geqslant \Delta\big(F_{\tilde{x}_{n+1},\tilde{x}_n}(K^{-m-1}t), F_{\bar{g}(n,n+m)-\bar{g}(n,n+m),0}(t)\big) \\ &\geqslant \Delta\big(F_{\tilde{x}_{n+1},\tilde{x}_n}(K^{-m-1}t), F_{\bar{g}(n,n+m),\bar{g}(n,n+m)}(t)\big), \end{aligned} \tag{3.13}$$

$\forall n, m \in \mathbf{Z}_{0+}$, $\forall t \in \boldsymbol{R}_+$ since $\{g_n\}$ converges then $\{\bar{g}(n, n+m)\} \to 0$ and $F_{\bar{g}(n,n+m),\bar{g}(n,n+m)}(t) \to 1$ as $n \to \infty$, $\forall m \in \mathbf{Z}_{0+}$, $\forall t \in \boldsymbol{R}_+$ and $F_{\tilde{x}_{n+1},\tilde{x}_n}(K^{-m-1}t) \to 1$

as $n, m \to \infty$, $\forall t \in \boldsymbol{R}_+$, that is $\{\tilde{x}_{n+1} - \tilde{x}_n\} \to 0$ as $n \to \infty$ with probability one from the first property of (1.1). Furthermore, for any given $\varepsilon \in \boldsymbol{R}_+$ and $\lambda \in (0, 1)$, there exists $n_{01} = n_{01}(\varepsilon, \lambda) \in \mathbf{Z}_{0+}$ such that

$$\begin{aligned} F_{\tilde{x}_{n+m+2},\tilde{x}_{n+m+1}}(2\varepsilon) &\geqslant \Delta(F_{\tilde{x}_{n+1},\tilde{x}_n}(K^{-m-1}t), F_{\bar{g}(n,n+m+1),\bar{g}(n,n+m)}(t)) \\ &> \Delta(1-\lambda, 1-\lambda) = 1-\lambda, \end{aligned} \tag{3.14}$$

$\forall n(\geqslant n_0) \in \mathbf{Z}_{0+}$, $\forall m \in \mathbf{Z}_{0+}$ so that $\{\tilde{x}_n\}$ is a Cauchy sequence which is, furthermore, convergent. Since $\{x_n\} = a$, $\forall n(\geqslant q_a)$, then for $\forall n(\geqslant \max(n_{01}, q_a)) \in \mathbf{Z}_+$

$$\begin{aligned} F_{\tilde{x}_{n+m+2},\tilde{x}_{n+m+1}}(2\varepsilon) &= F_{x_{n+m+2},x_{n+m+1}+\tilde{x}_{n+m+1}-\tilde{x}_{n+m+2}}(2\varepsilon) \\ &\geqslant \Delta\big(F_{x_{n+m+2},x_{n+m+1}}(\varepsilon), F_{\tilde{x}_{n+m+2},\tilde{x}_{n+m+1}}(\varepsilon)\big) \\ &= \Delta\big(F_{0,0}(\varepsilon), F_{\tilde{x}_{n+m+2},\tilde{x}_{n+m+1}}(\varepsilon)\big) \\ &= \Delta\big(1, F_{\tilde{x}_{n+1},\tilde{x}_n}(\varepsilon)\big) \\ &= \Delta\Big(1, \Delta\big(F_{\tilde{x}_{n+1},\tilde{x}_n}\big(K^{-m-1}\varepsilon\big), F_{\bar{g}(n,n+m+1),\bar{g}(n,n+m)}(\varepsilon/2)\big)\Big) \\ &= \Delta\big(F_{\tilde{x}_{n+1},\tilde{x}_n}\big(K^{-m-1}\varepsilon\big), F_{\bar{g}(n,n+m+1),\bar{g}(n,n+m)}(\varepsilon/2)\big) \\ &\geqslant \Delta\big(F_{\tilde{x}_{n+1},\tilde{x}_n}(K^{-m-1}\varepsilon), F_{\bar{g}(n,n+m+1),\bar{g}(n,n+m)}\big(K^{-m-1}\varepsilon\big)\big) \end{aligned} \tag{3.15}$$

$\forall n(\geqslant \max(n_{01}, q_a)) \in \mathbf{Z}_+$ provided that $m \geqslant \frac{|\ln 2\varepsilon|}{|\ln K|} - 1$ for any given real $\varepsilon \in (0, 1/2]$ since then $m \geqslant \frac{|\ln 2\varepsilon|}{|\ln K|} - 1$ is equivalent to $2k^{-m-1}\varepsilon \leqslant 1$ and then $F_{\bar{g}(n,n+m+1),\bar{g}(n,n+m)}(\varepsilon/2) \geqslant F_{\bar{g}(n,n+m+1),\bar{g}(n,n+m)}(K^{-m-1}\varepsilon)$. Thus, since the minimum triangular norm is continuous,

$$\begin{aligned} &\liminf_{n\to\infty} F_{\tilde{x}_{n+m+2},\tilde{x}_{n+m+1}}(2\varepsilon) \\ &\quad\geqslant \Delta\Big(\liminf_{n\to\infty} F_{\tilde{x}_{n+1},\tilde{x}_n}\big(K^{-m-1}\varepsilon\big), \liminf_{n\to\infty} F_{\bar{g}(n,n+m+1),\bar{g}(n,n+m)}\big(K^{-m-1}\varepsilon\big)\Big) \\ &\quad= \Delta\Big(\liminf_{n\to\infty} F_{\tilde{x}_{n+1},\tilde{x}_n}\big(K^{-m-1}\varepsilon\big), \liminf_{n\to\infty} F_{0,0}\big(K^{-m-1}\varepsilon\big)\Big) \\ &\quad= \Delta\Big(\liminf_{n\to\infty} F_{\tilde{x}_{n+1},\tilde{x}_n}\big(K^{-m-1}\varepsilon\big), 1\Big) \\ &\quad\geqslant \liminf_{n\to\infty} F_{\tilde{x}_{n+1},\tilde{x}_n}\big(K^{-m-1}\varepsilon\big) > 1-\lambda \end{aligned} \tag{3.16}$$

for any given real constants $\varepsilon \in (0, 1/2]$, $\lambda \in (0, 1)$ and any nonnegative integer $m \geqslant \max(n_{02}, \frac{|\ln 2\varepsilon|}{|\ln K|} - 1)$, for some existing nonnegative integer $n_{02} = n_{02}(\varepsilon, \lambda)$. The above constraints lead to the existence of the limit below:

$$\lim_{n\to\infty} F_{\tilde{x}_{n+2},\tilde{x}_{n+1}}\big(0^+\big) = \lim_{n,m\to\infty} F_{\tilde{x}_{n+m+2},\tilde{x}_{n+m+1}}(\varepsilon) = \lim_{n\to\infty} F_{\tilde{x}_{n+1},\tilde{x}_n}(\infty) = 1 \tag{3.17}$$

and

$$F_{\tilde{x}_{n+m+2},\tilde{x}_{n+m+1}}(2\varepsilon) = F_{\tilde{x}_{n+m+2},\tilde{x}_{n+m+1}}(2\varepsilon) \geqslant F_{\tilde{x}_{n+1},\tilde{x}_n}\big(K^{-m-1}\varepsilon\big) > 1-\lambda \tag{3.18}$$

for any given real constants $\varepsilon \in (0, 1/2]$, $\lambda \in (0, 1)$ and any nonnegative integer $n \geqslant n_a = \max(n_{01}, n_{02}, q_a, \frac{|\ln 2\varepsilon|}{|\ln K|} - 1)$ and $\{\bar{x}_n - x_n\} = \{\tilde{x}_n\} \to 0$ as $n \to \infty$ with probability one. Thus, $F_{0,0}(\varepsilon) = F_{0,T^{(0,\bar{n})}(a,\bar{u}(0,\bar{n}-1))}(\varepsilon) = 1 > 1-\lambda$, $\forall n(\geqslant n_{qa} \geqslant q_a) \in Z_+$ if $T^{(0,n)}$ denotes the composite operator of the sequence $\{T_n\}$ driving the initial state $x = a$ of (3.3) to $x = 0$ at $n \geqslant n_{qa}$-th sample. Thus, one gets for $n \geqslant n_{qa}$

$$\begin{aligned} F_{\tilde{x}_{n+2},\tilde{x}_{n+1}}(\varepsilon) &= F_{x_{n+2}+\tilde{x}_{n+2},x_{n+1}+\tilde{x}_{n+1}}(\varepsilon) \geqslant \Delta\big(F_{x_{n+2},x_{n+1}}(\varepsilon/2), F_{\tilde{x}_{n+2},\tilde{x}_{n+1}}(\varepsilon/2)\big) \\ &= \Delta\big(F_{0,0}(\varepsilon/2), F_{\tilde{x}_{n+2},\tilde{x}_{n+1}}(\varepsilon/2)\big) = \Delta\big(1, F_{\tilde{x}_{n+2},\tilde{x}_{n+1}}(\varepsilon/2)\big) \\ &\geqslant F_{\tilde{x}_{n+2},\tilde{x}_{n+1}}(\varepsilon/2) > 1-\lambda. \end{aligned} \tag{3.19}$$

Then, Ω_p is a. p. s. c. t. o. in some minimum finite number of samples $n_0 \geqslant n_{qa} \geqslant q_a$ with a prescribed guaranteed minimum probability error $F_{\tilde{x}_{n+1},\tilde{x}_n}(2\varepsilon) > 1-\lambda$ for any arbitrary given real constants $\varepsilon \in \boldsymbol{R}_+$ and $\lambda \in (0,1)$ and all integer $n \geqslant n_{qa} = \max(n_{01}, n_{02}, q_a, \frac{|\ln 2\varepsilon|}{|\ln K|} - 1)$ under control strips of the form $\bar{u}(0, k+q_a-1) = (\bar{u}(0, q_a-1), u^*, u^*, \ldots \overbrace{}^{k} \ldots, u^*)$, $\forall k(\geqslant n_{qa} - q_a) \in \mathbf{Z}_{0+}$. □

Remark 1. Theorem 6 invokes the hypothesis that $F_{x+z,y+z}(t) = F_{x,y}(t)$, $\forall x, y, z \in X$, $\forall t \in \boldsymbol{R}_{0+}$. This assumption is not restrictive in some standard cases. For instance, if X is a linear space so that $(X, \|\ \|)$ is a normed space, then $\|x-y\| = \|x+z-(y+z)\|$ so that $F_{x+z,y+z}(t) = F_{x,y}(t) = F_{x,y}(0^+) = 1$, $\forall t \in \boldsymbol{R}_+$ if and only if $x = y$. This is the probabilistic counterpart of the deterministic result $\|x-y\| = \|x+z-(y+z)\| = 0$, $\forall z \in X$ if and only if $x = y$ for any $x, y \in X$. If (X, d) is a metric space for a homogeneous and translation-invariant metric $d : X \times X \to \boldsymbol{R}_{0+}$, then the same property holds since $d(x, y) = d(x+z, y+z)$. Obviously, the same idea appears for a norm-induced metric in a normed space since $(X, \|\ \|) \equiv (X, d)$ and also, in the case of a metric space (X, d) under a homogeneous and translation-invariant metric since X can be endowed with a metric-induced norm $\|\ \|$ so that $(X, d) \equiv (X, \|\ \|)$ (De la Sen, 2013a).

In the same way, we can obtain the following parallel results, whose proofs are omitted, for the approximate uniform controllability to the origin of the *PPDDS* Ω_p:

Theorem 7. *Consider the PDDS* Ω*, Eqs.* (3.3)–(3.4)*, and the PPDDS* Ω_p*, Eqs.* (3.1)–(3.2)*, and assume that the conditions* (1) , (2) *and* (4) *of Theorem* 6 *hold and, furthermore*:
3′) Ω *is u. p. s. c. t. o. in* $\hat{q}_a$ *samples from* $x = a(\in X)$ *through a control strip* $\bar{u}(j, j+\hat{q}_a-1)$ *along the discrete time-interval* $[j, j+\hat{q}_a-1]$ *for any* $j \in \mathbf{Z}_{0+}$ *with* $T(0, u^*) = 0$ *for some* u^* *such that the constant control sequence* $\hat{u}^* = \{u^*\} \subset U$.

Then, Ω_p *is a. u. p. s. c. t. o. in any finite number of samples being non less than a minimum lower threshold* $\hat{n}_{qa} \geqslant \hat{q}_a$ *under control strips of the form* $\bar{u}(j, j+k+\hat{q}_a-1) =$

$(\bar{u}(j, j+\hat{q}_a-1), u^*, u^*, \ldots \overbrace{\quad}^{k} \ldots, u^*)$, $\forall k(\geqslant \hat{n}_{qa}-\hat{q}_a) \in \mathbf{Z}_{0+}$, $\forall j \in \mathbf{Z}_{0+}$ *and with a prescribed guaranteed minimum probability error* $F_{\tilde{x}_{n+1},\tilde{x}_n}(\varepsilon) > 1-\lambda$ *for any arbitrary given real constants* $\varepsilon \in \boldsymbol{R}_+$ *and* $\lambda \in (0,1)$ *and all integer* $n \geqslant \hat{n}_{qa}$.

The concept of approximate controllability from the origin is a direct "ad hoc" modification of Definitions 7 to the light of Definitions 1 for the nominal Ω. The approximate controllability from the origin (a. p. s. c. f. o) of the perturbed system if the nominal one is controllable to the origin is characterized in the next result with no specific proof since it becomes quite close to that of Theorem 6:

Theorem 8. *Assume that* $T(0,a) \neq 0$ *if* $a(\in X) \neq 0$ *and* $T(a,u^*) = a$ *for some constant sequence* $\hat{u}^* \subset U$ *of value* u^*. *If* Ω *is a. p. s. c. f. o. through a control sequence* $\{u_n\}$, *then* $p_{am} \in Z_+$ *exists (the minimum necessary length of a control strip for controllability from the origin to* $x=a$*). Consider the PDDS* Ω, *Eqs.* (3.3)–(3.4), *and the PPDDS* Ω_p, *Eqs.* (3.1)–(3.2), *and assume that the conditions* (1), (2) *and* (4) *of Theorem* 3.1 *hold and, furthermore*:

$3'$) *Assume that* $T(0,a) \neq 0$ *if* $a(\in X) \neq 0$ *and* $T(a,u^*) = a$ *for some constant sequence* $\hat{u}^* \subset U$ *of value* u^* *and that* Ω *is p. s. c. f. o. in a number of* p_a *samples through some control sequence* $\{u_n\}$.

Then, Ω_p *is a. p. s. c. f. o. in any finite number of samples being non less than a minimum lower threshold* $n_{pa} \geqslant p_a$ *under control strips of the form* $\bar{u}(0, k+p_a-1) = (\bar{u}(0, p_a-1), u^*, u^*, \ldots \overbrace{\quad}^{k} \ldots, u^*)$, $\forall k(\geqslant n_{pa}-p_a) \in \mathbf{Z}_{0+}$ *and with a prescribed guaranteed minimum probability error* $F_{\tilde{x}_{n+1},\tilde{x}_n}(\varepsilon) > 1-\lambda$ *for any arbitrary given real constants* $\varepsilon \in \boldsymbol{R}_+$ *and* $\lambda \in (0,1)$ *and all integer* $n \geqslant n_{pa}$.

Also, if Ω *is u. p. s. c. f. o. in any finite number of samples being non less than a minimum lower threshold* $\hat{p}_a$, *then* Ω_p *is a. u. p. s. c. f. o. in any finite number of samples being non less than a minimum lower threshold* $\hat{n}_{pa} \geqslant \hat{p}_a$ *under control strips of the form* $\bar{u}(j, j+k+p_a-1) = (\bar{u}(j, j+p_a-1), u^*, u^*, \ldots \overbrace{\quad}^{k} \ldots, u^*)$, $\forall k(\geqslant \hat{n}_{pa}-\hat{p}_a)$, $j \in \mathbf{Z}_{0+}$ *and with a prescribed guaranteed minimum probability error* $F_{\tilde{x}_{n+1},\tilde{x}_n}(\varepsilon) > 1-\lambda$ *for any arbitrary given real constants* $\varepsilon \in \boldsymbol{R}_+$ *and* $\lambda \in (0,1)$ *and all integer* $n \geqslant \hat{n}_{pa}$.

The definitions for approximate complete controllability and approximate uniform complete controllability are also direct extensions for the perturbed system Ω_p from those given for the nominal system Ω in Section 2. Those definitions lead to very close results to those in Theorems 6–8.

EXAMPLE 5. Consider the nominal deterministic linear discrete-time system given by the difference equation:

$$A(q^{-1})y_k = B(q^{-1})u_k, \tag{3.20}$$

$\forall k \in \mathbf{Z}_{0+}$ under any given initial conditions $y_{-n_0+1}, y_{-n_0+2}, \ldots, y_0$, where $\{y_k\}$ and $\{u_k\}$ are, respectively, the output and input sequences and $B(q^{-1})$ and $A(q^{-1})$ are polynomials of respective degrees m and $n \geqslant m$ and q^{-1} is the one-step delay operator so that

$z_k = q^{-1} z_{k+1}$ for any discrete sequence $\{z_k\}$. Since the nominal discrete transfer function $G(z^{-1}) = B(z^{-1})/A(z^{-1})$ is realizable, then there are non-unique state-space description associated to (3.20). It is well-known that if the polynomials $B(q^{-1})$ and $A(q^{-1})$ have no common polynomial factor, then any n_0-th dimensional state-space realization is completely controllable ("from" and "to" the origin for some finite control strip) and observable and even if there is some polynomial transfer-function cancellation, there are non-unique n_0-th dimensional state-space realizations which can be chosen to be completely controllable, that is, their controllability matrix is full rank. Consider the following time-varying perturbed version of (3.20):

$$\bar{A}_k(q^{-1})\bar{y}_k = \bar{B}_k(q^{-1})\bar{u}_k, \quad \forall k \in \mathbf{Z}_{0+} \tag{3.21}$$

with

$$\begin{aligned}\bar{A}_k(q^{-1}) &= A(q^{-1}) + \delta_k \tilde{A}_k(q^{-1}),\\ \bar{B}_k(q^{-1}) &= B(q^{-1}) + \delta_k \tilde{B}_k(q^{-1}), \quad \forall k \in \mathbf{Z}_{0+},\end{aligned} \tag{3.22}$$

equivalently, the zero-state response of the perturbed system is:

$$\begin{aligned}\bar{y}_k &= \left(1 + \frac{\delta_k \tilde{A}_k(q^{-1})}{A(q^{-1})}\right)^{-1} \left(\frac{B(q^{-1})}{A(q^{-1})} + \delta_k \frac{\tilde{B}_k(q^{-1})}{A(q^{-1})}\right)\bar{u}_k\\ &= \frac{B(q^{-1}) + \delta_k \tilde{B}_k(q^{-1})}{A(q^{-1}) + \delta_k \tilde{A}_k(q^{-1})}\bar{u}_k, \quad \forall k \in \mathbf{Z}_{0+}\end{aligned} \tag{3.23}$$

where the coefficients of the perturbation polynomial sequences $\{\tilde{A}_k(q^{-1})\}$ and $\{\tilde{B}_k(q^{-1})\}$, $\forall k \in \mathbf{Z}_{0+}$ are uniformly bounded, the bounded real sequence $\{\delta_k\} \to 0$ and the polynomial degree constraint $\deg(\tilde{B}_k(q^{-1})) \leqslant \max(\deg A(q^{-1}), \deg, (\tilde{A}_k(q^{-1})))$ holds, $\forall k \in \mathbf{Z}_{0+}$. Note that such a constraint ensures that the constraint below also holds:

$$\begin{aligned}&\max\left(\deg B(q^{-1}), \deg\left(\tilde{B}_k(q^{-1})\right)\right) \leqslant \max\left(\deg A(q^{-1}), \deg\left(\tilde{A}_k(q^{-1})\right)\right),\\ &\forall k \in \mathbf{Z}_{0+}\end{aligned} \tag{3.24}$$

since $m = \deg B(q^{-1}) \leqslant n = \deg A(q^{-1})$ so that the perturbed system (3.23) is state-space realizable. The output error response of the perturbed system with respect to the nominal one, i.e. $\tilde{y}_k = \bar{y}_k - y_k, \forall k \in \mathbf{Z}_{0+}$is:

$$\begin{aligned}\tilde{y}_k &= \tilde{y}_k^0 + \left[\left(1 + \delta_k \frac{\tilde{A}_k(q^{-1})}{A(q^{-1})}\right)^{-1} - 1\right]\left(\frac{B(q^{-1})}{A(q^{-1})}u_k + \delta_k \frac{\tilde{B}_k(q^{-1})}{A(q^{-1})}u_k\right)\\ &= \tilde{y}_k^0 + \left[\left(1 + \delta_k \frac{\tilde{A}_k(q^{-1})}{A(q^{-1})}\right)^{-1} - 1\right]\left(y^* + \delta_k \frac{\tilde{B}_k(q^{-1})}{A(q^{-1})}u_k\right), \quad \forall k (\geqslant n_\delta) \in \mathbf{Z}_{0+}\end{aligned} \tag{3.25}$$

provided that the input sequence for the perturbed system is identical to that injected to the system so as to fulfil the given controllability objective for the nominal one where $\{\tilde{y}_k^0\}$ is the sequence error output response to initial conditions corresponding to the error state free response $\{\tilde{x}_k^0\}$ (i.e. the zero-input sequence response), the second right-hand-side term is the zero-state, or forced, sequence response and y^* is the nominal output corresponding to the controllability targeted nominal state x^* reached in finite time by the nominal system. In particular, $x^* = 0$ for controllability to the origin from zero initial conditions (in this case, $\{\tilde{y}_k^0\} \equiv 0$ and $n_\delta = n_{qa}$ if the initial condition of the state-space realization is $x_0 = \bar{x}_0 = a$) and some desired value distinct from zero in other cases. Note that a way of choosing the nominal, perturbed and error state vectors from the respective output sequences is simply $x_k = (y_k, y_{k-1}, \dots, y_{k-n+1})^T$, $\bar{x}_k = (\bar{y}_k, \bar{y}_{k-1}, \dots, \bar{y}_{k-n+1})^T$ and $\tilde{x}_k = (\tilde{y}_k, \tilde{y}_{k-1}, \dots, \tilde{y}_{k-n+1})^T$. Since $\{\tilde{A}_k(q^{-1})\}$ and $\{\tilde{B}_k(q^{-1})\}$ are uniformly bounded, $\forall k \in \mathbf{Z}_{0+}$ and the bounded real sequence $\{\delta_k\} \to 0$, then $\lim_{k\to\infty}(\tilde{y}_k - \tilde{y}_k^0) = 0$. If $\{\tilde{y}_k^0\} \equiv 0$, if $x^* = 0$ (nominal controllability to the origin) or if it converges to zero, then $\{\tilde{y}_k\}$ is in a prescribed ball around zero after some sufficiently large time instant depending to the ball radius. If $A(q^{-1})$ is a Hurwitz polynomial, then from (3.25), the state and output error sequences, respectively, $\{\tilde{x}_k^0\} \to 0$ and $\{\tilde{y}_k^0\} \to 0$, since the sequence of polynomials $\{\bar{A}_k(q^{-1})\}$ has a Hurwitz subsequence for $k(\geqslant k_0) \in Z_{0+}$ and some $k_0 \in \mathbf{Z}_{0+}$ and the situation is similar.

Example 6. We now discuss a scalar probabilistic version of Example 5. Consider the following scalar perturbed *PPDDS* Ω_p and its corresponding nominal *PDDS* Ω given by:

$$\bar{x}_{k+1} = (a + \delta_k\tilde{a})\bar{x}_k + (b + \delta_k\tilde{b})u_k, \quad \forall k \in \mathbf{Z}_{0+}, \tag{3.26}$$

$$x_{k+1} = ax_k + bu_k, \quad \forall k \in \mathbf{Z}_{0+} \tag{3.27}$$

where $a, b(\neq 0)$, $\tilde{a}, \tilde{b} \in \boldsymbol{R}$ and $\{\delta_k\}(\subset R) \to 0$, $X = Y = U = \boldsymbol{R}$, the initial conditions are $\bar{x}_0 \in \boldsymbol{R}$ and $x_0 \in \boldsymbol{R}$ and $F_{x,y}(t) = \frac{t}{t+d(x,y)}$, $\forall x, y \in \boldsymbol{R}$, $\forall t \in \boldsymbol{R}_+$. Assume that the Euclidean metric is chosen. Define the state error between both systems as $\tilde{x}_k = \bar{x}_k - x_k$, $\forall k \in Z_{0+}$ so that one obtains from (3.26)–(3.27):

$$\tilde{x}_{k+1} = (a + \delta_k\tilde{a})\tilde{x}_k + \delta_k(\tilde{a}x_k + \tilde{b}u_k), \quad \forall k \in \mathbf{Z}_{0+} \tag{3.28}$$

with $\tilde{x}_0 = \bar{x}_0 - x_0$. The deterministic nominal version is uniformly controllable from and to the origin since $b \neq 0$. If the targeted state is x^*, then $u_0 = b^{-1}(x * -a\, x_0)$, $u_k = b^{-1}(1 - a)x^*$, $\forall k \in \mathbf{Z}_+$. If either $a \neq 1$ or $x^* \neq ax_0$, then the control sequence to keep the sate $x_k = x^*$, $\forall k \in \mathbf{Z}_+$ is non-zero. Define the error sequence $\{\varepsilon_k\}$ by $\varepsilon_k = \delta_k - \delta_{k+1}$, $k \in \mathbf{Z}_{0+}$

$$\begin{aligned}\tilde{x}_{k+2} - \tilde{x}_{k+1} = {} & (a + \delta_k\tilde{a})(\tilde{x}_{k+1} - \tilde{x}_k) + \delta_k\big[\tilde{a}(x_{k+1} - x_k) + \tilde{b}(u_{k+1} - u_k)\big] \\ & - \varepsilon_k(\tilde{a}x_{k+1} + \tilde{b}u_{k+1}), \quad \forall k \in \mathbf{Z}_{0+}.\end{aligned} \tag{3.29}$$

If $k = 0$, then

$$\begin{aligned}\tilde{x}_2 - \tilde{x}_1 = {} & (a + \delta_0\tilde{a})(\tilde{x}_1 - \tilde{x}_0) + \delta_0\big(\tilde{a} - \tilde{b}b^{-1}a\big)\big(x^* - x_0\big) \\ & - \varepsilon_0\big(\tilde{a} + \tilde{b}b^{-1}(1 - a)\big)x^*\end{aligned} \tag{3.30}$$

$$= (a+\delta_0\tilde{a})(\tilde{x}_1-\tilde{x}_0)-\varepsilon_0\big[\tilde{a}-\tilde{b}b^{-1}(a+1)\big]x^* + \delta_0\big(\tilde{b}b^{-1}a-\tilde{a}\big)\big(x_0-x^*\big), \tag{3.31}$$

$$\tilde{x}_{k+2}-\tilde{x}_{k+1} = (a+\delta_k\tilde{a})(\tilde{x}_{k+1}-\tilde{x}_k)-\varepsilon_k\big(\tilde{a}+\tilde{b}b^{-1}(1-a)\big)x^*, \quad \forall k\in \mathbf{Z}_+ \tag{3.32}$$

where, if $\{\lambda_k\}$ is defined by $\lambda_k = |(\delta_k-\delta_{k+1})(\tilde{a}+\tilde{b}b^{-1}(1-a))|$, $\forall k\in \mathbf{Z}_{0+}$

$$\varepsilon_k = \begin{cases} 0, & \text{if } (\tilde{a}+\tilde{b}b^{-1}(1-a))x^*=0, \\ \frac{\lambda_k|\tilde{x}_{k+1}-\tilde{x}_k|}{|\tilde{a}+\tilde{b}b^{-1}(1-a)|}, & \text{otherwise,} \end{cases} \quad \forall k\in \mathbf{Z}_+. \tag{3.33}$$

Then,

$$|\tilde{x}_{k+2}-\tilde{x}_{k+1}| \leqslant K|\tilde{x}_{k+1}-\tilde{x}_k|, \quad \forall k\in \mathbf{Z}_+, \tag{3.34}$$

$$F_{\tilde{x}_{k+2},\tilde{x}_{k+1}}(t) = \frac{t}{t+|\tilde{x}_{k+1}-\tilde{x}_k|} \geqslant \frac{1}{1+Kt^{-1}|\tilde{x}_{k+1}-\tilde{x}_k|} = F_{\tilde{x}_{k+1},\tilde{x}_k}\big(K^{-1}t\big), \quad \forall k\in \mathbf{Z}_+,\ \forall t\in \boldsymbol{R}_+, \tag{3.35}$$

so that $\lim_{k\to\infty} F_{\tilde{x}_{k+1},\tilde{x}_k}(0^+) = \lim_{k\to\infty} F_{\tilde{x}_{k+1},\tilde{x}_k}(t) = F_{\tilde{x}_1,\tilde{x}_0}(\infty) = 1$, $\forall t\in \boldsymbol{R}_+$, provided that $\limsup_{k\to\infty} |a+\delta_k\tilde{a}+\lambda_k| \leqslant K < 1$, $\forall k\in \mathbf{Z}_+$. A sufficient condition for that condition to hold is $|a| \leqslant K-\delta$ for any given arbitrarily small $\delta\in R_+$ since $\{\delta_k\}\to 0$ and $\{\lambda_k\}\to 0$. Also, for any given $\varepsilon\in \boldsymbol{R}_+$ and $\lambda\in(0,1)$, there is some finite $n_0 = n_0(\varepsilon,\lambda,a,b,\tilde{a},\tilde{b},x_0,x^*)\in \mathbf{Z}_{0+}$ such that $F_{\tilde{x}_{k+2},\tilde{x}_{k+1}}(\varepsilon) > 1-\lambda$, $\forall n(\geqslant n_0)\in \mathbf{Z}_{0+}$. Thus, one gets approximate complete controllability for any given initial and final states $x_0 = x_0(t_0)$ and $x^* = x^*(t_0+T)$, respectively, which is uniform in the sense that it does not depend on the initial time instant $t_0\in \boldsymbol{R}_{0+}$ and the minimum time interval to reach the targeted x^* is finite and independent from t_0.

4. Conclusions

This paper has dealt with the problem of controllability of linear discrete-time systems in a probabilistic context for metric spaces with some robustness-type extensions. Several controllability types are revisited in such a framework mainly based on the deterministic context of controllability to and from the origin. The idea is to achieve certainty of the probability density for a zero distance in-between the trajectory and the targeted point. The controllability robustness extensions are based on characterizing the approximate controllability in probabilistic terms to the nominally targeted point under perturbations of the nominal system basically relying on the achievement of certainty in some small region around the nominal targeted point. Worked examples are also discussed. The mathematical framework used for the problem statement and its solution relies on probabilistic metric spaces for a certain distribution function associated with a metric and, for deriving some of the obtained results, Menger probabilistic metric spaces, those ones endowed with triangular norms as it is well-known.

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M. De la Sen was born in Arrigorriaga, Bizkaia, Basque Country, Spain. He received his MS degree in applied physics (electronics and automation) and his PhD degree (in applied physics from the University of Basque Country, Bilbao, Spain, in 1975 and 1979, respectively, and the Doctorat d'Etat-ès-Sciences Physiques degree from the University of Grenoble, Grenoble, France, in 1987. He serves currently as a Professor of Systems Engineering and Automatic Control at the University of Basque Country, where he also serves as the Head of the Institute of Research and Development of Processes. He is currently serving as an Associate Editor of the journals: *Applied Mathematical Sciences*; *Nonlinear Analysis, Modeling and Control*; *Mathematics in Engineering, Science and Aerospace: The Transdisciplinary International Journal*; *Discrete Dynamics in Nature and Society*; *Nonlinear Studies: The International Journal*; *Fixed Point Theory and Applications and JNSA the Journal of Nonlinear Science and Applications*. His main interest research areas include discrete and sampled-data control systems, nonperiodic and adaptive sampling, adaptive control, fixed point theory, positive systems, stability, models for ecology, epidemic models, time-delay systems, artificial intelligence and heuristic tools for dynamic systems, and ordinary differential equations.

Apie kai kurias tiesinių diskrečiųjų sistemų valdomumo ypatybes tikimybinėse metrinėse erdvėse

Manuel DE LA SEN

Šis darbas formaliame kontekste tiria tam tikras pagrindines valdomumo ypatybes „nuo“ ir „iki“ tikimybinių diskrečiųjų sistemų ir jų vienarūšių versijų atsiradimo ir visiško valdomumo tikimybinėse metrinėse ar tikimybinėse normuotose erdvėse, ypač tikimybinėse Mengerio erdvėse. Taip pat kai kurios aproksimuotos tikimybinės valdomumo ypatybės yra ištirtos, kai nominali valdoma sistema yra priklausoma arba nuo parametrinės perturbacijos, arba nuo nesumodeliuotos dinamikos. Šiame kontekste aproksimuotas sutrikdytų sistemų valdomumas esti robustinio tipo, jei nominali sistema yra valdoma. Taip pat yra pateikti tam tikri iliustruojantys pavyzdžiai.