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Entropy Measures for Interval-Valued Intuitionistic Fuzzy Information from a Comparative Perspective and Their Application to Decision Making

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Abstract. This paper reviews the existing definitions and formulas of entropy for interval-valued intuitionistic fuzzy sets (IVIFSs) and demonstrates that they cannot fully capture the uncertainty of IVIFSs. Then considering both fuzziness and intuitionism of IVIFSs, we introduce a novel axiomatic definition of entropy for IVIFSs and develop several entropy formulas. Example analyses show that the developed entropy formulas can fully reflect both fuzziness and intuitionism of IVIFSs. Furthermore, based on the entropy formulas of IVIFSs, a method is proposed to solve multi-attribute decision making problems with IVIFSs. Additionally, an investment alternative selection example is provided to validate the practicality and effectiveness of the method.

Key words: multi-attribute decision making, interval-valued intuitionistic fuzzy sets, fuzziness, intuitionism, entropy.

1. Introduction

In decision making, uncertainty is ubiquitous since objective things are uncertain and complex (Shahbazova, 2013), and the modelling of uncertain information is pivotal for the solving of the considered problem. To date, many techniques have been developed to portray the uncertainty involved in decision making, such as fuzzy set (Zadeh, 1965), interval-valued fuzzy set (IVFS) (Feng and Liu, 2012), interval-valued triangular fuzzy number (IVTFN) (Stanujkic, 2013), intuitionistic fuzzy set (IFS) (Atanassov, 1986), interval-valued intuitionistic fuzzy set (IVIFS) (Atanassov and Gargov, 1989) and interval-valued hesitant fuzzy set (Chen *et al.*, 2013c). IFS is more powerful and flexible than FS in dealing with uncertainty since it allocates to each element not only a membership degree but also a non-membership degree and a hesitancy degree. To date, IFS has been widely applied to many areas including decision making (Xu, 2011; Verma and Sharma, 2013; Zeng *et al.*, 2013), medical diagnosis (De *et al.*, 2001) and pattern recognition (Vlachos and Sergiadis, 2007), etc. However, in these applications, because of the increasing complexity of socioeconomic environment and a lack of knowledge on the problem domains,

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it is more and more difficult for decision makers to determine an exact membership degree and an exact non-membership degree for an element, but only specify an interval membership degree and an interval non-membership degree. Hence, Atanassov and Gargov (1989) extended IFS to introduce the notion of IVIFS, which provides a more reasonable and practical mathematical framework to manipulate imprecise facts or imperfect information.

In the theory of FS, entropy is a vital tool to measure the fuzziness of information. Zadeh (1968) proposed the concept of entropy for FSs for the first time. Subsequently, De Luca and Termini (1972) presented some axiomatic requirements that an entropy measure of FSs should comply with. Burillo and Bustince (1996) first introduced the notion of entropy for IFSs. Since then, the entropies of IFSs and IVIFSs have received great attention from scholars. Szmidt and Kacprzyk (2001) came up with a non-probabilistic-type entropy measure based on a geometric interpretation of IFSs. Afterwards, Hung and Yang (2006) gave an axiomatic definition of entropy for IFSs from the viewpoint of probability. On the basis of distance measures of IFSs, Hung (2003) and Zhang *et al.* (2009) constructed some entropy measures for IFSs. Recently, Pal *et al.* (2013) pointed out that the existing entropy measures of IFSs are unable to capture both fuzziness and lack of knowledge associated with IFSs, and thus developed new entropy measures in light of new constructive axioms. In a simple way, Mao *et al.* (2013) refined the axiomatic requirements of entropy for IFSs by considering both fuzziness and intuitionism of IFSs simultaneously and proposed a new entropy formula. However, it is worth noting that the entropy formula of IFSs developed by Mao *et al.* (2013) is meaningless when membership degrees are equal to the corresponding non-membership degrees of an IFS, which indicates that the entropy formula is invalid in calculating entropy values of some IFSs.

In order to measure the uncertainty of IVIFSs, many researchers have investigated the definition and formulation of entropy of IVIFSs from various aspects. For example, Liu *et al.* (2005) first introduced an axiomatic definition of entropy for IVIFSs. Based on this definition, Wei *et al.* (2011), Gao and Wei (2012) and Jin *et al.* (2014) constructed a variety of entropy formulas. Later on, Zhang and Jiang (2010) and Zhang *et al.* (2010) gave a new entropy concept of IVIFSs by De Luca and Termini (1972) extending definition of entropy for FSs and developed a family of entropy formulas. In light of Burillo and Bustince (1996) work for IFSs, Zhang *et al.* (2011) proposed a different definition of entropy for IVIFSs and presented a methodology whereby different forms of entropy formulas of IVIFSs can be constructed. Recently, Guo and Song (2014) and Zhang *et al.* (2014) redefined the concept of entropy for IVIFSs from the perspectives of amount of knowledge and distance between IVIFSs, respectively and developed new entropy formulas.

It is noteworthy that none of the above axiomatic definitions of entropy for IVIFSs (Liu *et al.*, 2005; Zhang *et al.*, 2010, 2011, 2014; Zhang and Jiang, 2010; Guo and Song, 2014) completely captures the uncertainty associated with an IVIFS. In our opinion, uncertain information of an IVIFS should be depicted by means of both fuzziness and intuitionism. The former is related to the differences between interval membership degrees and the corresponding interval non-membership degrees, and the latter is related to interval hesitancy degrees, which represent a lack of knowledge of whether an element belongs to a given set. Nevertheless, as demonstrated later in the third section, current definitions of entropy for IVIFSs only reflect one of the two types of uncertainties. For instance, the axiomatic definitions of entropy for IVIFSs presented in Liu *et al.* (2005), Zhang *et al.* (2010, 2014), Zhang and Jiang (2010), Guo and Song (2014) only take into account the fuzziness of IVIFSs, and the entropy defined by Zhang *et al.* (2011) only measures the intuitionism of IVIFSs. Consequently, most of the existing entropy formulas of IVIFSs will produce counterintuitive results when applied to some IVIFSs. Therefore, it necessitates us to design a new axiomatic definition of entropy for IVIFSs which takes both fuzziness and intuitionism of IVIFSs into full consideration and construct new entropy formulas which perform well in discriminating IVIFSs.

The paper is organized as follows: Section 2 recalls some basic notions of IFSs and IVIFSs. Section 3 presents a brief review of some existing studies on entropies of IFSs and IVIFSs and points out the disadvantages of some existing axiomatic definitions and formulas of entropies for IFSs and IVIFSs. In Section 4, we first introduce a general method to formulate the entropy of IFSs, by which the constructed entropy formulas of IFSs are able to well circumvent the drawbacks existing in the entropy formula developed by Mao *et al.* (2013). Then a new axiomatic definition of entropy for IVIFSs is given which fully considers both fuzziness and intuitionism of IVIFSs. Based on the axiomatic definition, we provide some methods of construction of entropy formulas of IVIFSs with the help of some simple functions. A comparative analysis is also made in this section to show the validity and superiority of the constructed entropy formulas. Section 5 proposes an entropy-based approach to solving the multi-attribute decision making problems with interval-valued intuitionistic fuzzy information, and presents an investment alternative selection example to validate the practicality. Finally, in Section 6, we conclude the paper.

2. Basic Notions of IFSs and IVIFSs

In order to describe the fuzzy nature of things more detailedly and comprehensively, Atanassov (1986) initiated the concept of IFSs by generalizing Zadeh's FSs (Zadeh, 1965). This section is devoted to reviewing some basic notions of IFSs and its generalization referred to as IVIFSs.

Definition 1. (See Atanassov, 1986.) An IFS I in a finite set X is an object with the following mathematical form:

$$I = \{\langle x, M_I(x), N_I(x)\rangle \mid x \in X\}$$

where $M_I, N_I : X \to [0, 1]$ satisfy the condition $M_I(x) + N_I(x) \leqslant 1$. Here $M_I(x)$ and $N_I(x)$ are called the membership and non-membership degrees of the element x to the set I, respectively, and $\pi_I(x) = 1 - M_I(x) - N_I(x)$ is called the hesitancy degree of x to I. Apparently, $\pi_I(x) \in [0, 1]$ for all $x \in X$. For convenience, Xu (2011) named $\alpha_I(x) = (M_I(x), N_I(x))$ an intuitionistic fuzzy value (IFV) and defined the complement of $\alpha_I(x)$ as $\alpha_I^c(x) = (N_I(x), M_I(x))$.

Considering that it is sometimes difficult to exactly ascertain the membership and non-membership degrees for an element to an IFS, Atanassov and Gargov (1989) generalized IFS to introduce the notion of IVIFS as follows:

Definition 2. (See Atanassov and Gargov, 1989.) Let $int(0,1)$ denote the set of all closed subintervals of the interval $[0,1]$. Then an IVIFS $\tilde{I}$ in a finite set X is an object having the following mathematical form:

$$\tilde{I} = \{\langle x, M_{\tilde{I}}(x), N_{\tilde{I}}(x)\rangle \mid x \in X\}$$

where $M_{\tilde{I}}, N_{\tilde{I}} : X \to int(0,1)$ satisfy the condition $\sup(M_{\tilde{I}}(x)) + \sup(N_{\tilde{I}}(x)) \leqslant 1$. Here $M_{\tilde{I}}(x)$ and $N_{\tilde{I}}(x)$ denote the interval membership and non-membership degrees of the element x to the set $\tilde{I}$, respectively. For simplicity, we here let $M_{\tilde{I}}(x) = [M_{\tilde{I}}^L(x), M_{\tilde{I}}^U(x)]$ and $N_{\tilde{I}}(x) = [N_{\tilde{I}}^L(x), N_{\tilde{I}}^U(x)]$ such that $M_{\tilde{I}}^U(x) + N_{\tilde{I}}^U(x) \leqslant 1$ for all $x \in X$, and call the interval $[1 - M_{\tilde{I}}^U(x) - N_{\tilde{I}}^U(x), 1 - M_{\tilde{I}}^L(x) - N_{\tilde{I}}^L(x)]$ the interval hesitancy degree of x to $\tilde{I}$, which is abbreviated by $[\pi_{\tilde{I}}^L(x), \pi_{\tilde{I}}^U(x)]$ and denoted by $\pi_{\tilde{I}}(x)$. Obviously, if $M_{\tilde{I}}^L(x) = M_{\tilde{I}}^U(x) = M_I(x)$ and $N_{\tilde{I}}^L(x) = N_{\tilde{I}}^U(x) = N_I(x)$ for all $x \in X$, then the IVIFS $\tilde{I}$ degenerates to an IFS. For convenience, Xu and Yager (2009) called $\tilde{\alpha}_{\tilde{I}}(x) = (M_{\tilde{I}}(x), N_{\tilde{I}}(x))$ an interval-valued intuitionistic fuzzy value (IVIFV) and defined the complement of $\tilde{\alpha}_{\tilde{I}}(x)$ as $\tilde{\alpha}_{\tilde{I}}^c(x) = (N_{\tilde{I}}(x), M_{\tilde{I}}(x))$.

3. Some Existing Studies on Entropies of IFSs and IVIFSs

So far, the entropies of IFSs and IVIFSs have been extensively investigated from different viewpoints. In this section, we briefly review some existing studies on entropies of IFSs and IVIFSs and make a detailed analysis of their disadvantages. First of all, for simplifying the notation, we from now on denote IFS(X) and IVIFS(X) as the sets of all IFSs and IVIFSs in X, respectively, where $X = \{x_1, x_2, \dots, x_n\}$ is the finite set.

3.1. *Entropy of IFSs*

Up to now, a lot of studies have been done on the entropy of IFSs involving its definition and formulation. Among these studies, there is a representative one which considers both fuzziness and intuitionism of IFSs simultaneously rather than separately. Let $\Delta_I(x_i) = |M_I(x_i) - N_I(x_i)|$ for $\forall x_i \in X$. Now we show it as follows:

Definition 3. (See Mao *et al.*, 2013.) An intuitionistic fuzzy entropy $E : \text{IFS}(X) \to [0,1]$ is a real-valued function related to $\Delta_I(x_i)$ and $\pi_I(x_i)$, where $x_i \in X$, and meets the following axiomatic requirements:

(E_M1) $E(I) = 0$ if and only if I is crisp, i.e., $I = \{\langle x_i, 1, 0\rangle \mid x_i \in X\}$ or $I = \{\langle x_i, 0, 1\rangle \mid x_i \in X\}$;

(E_M2) $E(I)=1$ if and only if $I=\{\langle x_i,0,0\rangle \mid x_i\in X\}$;
(E_M3) $E(I)=E(I^c)$;
(E_M4) $E(I)$ is monotonically decreasing with regard to $\Delta_I(x_i)$ and monotonically increasing with regard to $\pi_I(x_i)$ for all $x_i\in X$.

In Definition 3, the absolute deviations of membership degrees and the corresponding non-membership degrees are used to display the closeness of membership degrees and the corresponding non-membership degrees, which reflect the fuzziness of an IFS, and the intuitionism of an IFS is reflected by hesitancy degrees. The entropy formula of IFSs developed by Mao *et al.* (2013) is shown as:

$$E_M(I)=\frac{1}{2n\ln 2}\sum_{i=1}^{n}\left(\pi_I(x_i)\ln 2+\Delta_I(x_i)\ln\Delta_I(x_i)+\left(\Delta_I(x_i)+1\right)\ln\frac{2}{\Delta_I(x_i)+1}\right). \quad (1)$$

Clearly, Eq. (1) describes the uncertainty of an IFS by taking both fuzziness and intuitionism of the IFS into full consideration. However, they cannot be utilized to compute the entropy value of an arbitrary IFS. As to the IFS I, Eq. (1) is meaningless when $\Delta_I(x_i)=0$ for some $x_i\in X$, which implies that the entropy formula of IFSs proposed by Mao *et al.* (2013) is invalid when applied to the IFS I with $M_I(x_i)=N_I(x_i)$ for some $x_i\in X$, and does not satisfy the axiomatic requirement (E_M2) in Definition 3. It is possible to give some examples. For some $x_i\in X$, let $\alpha_{I_1}(x_i)=(0.3,0.3)$, $\alpha_{I_2}(x_i)=(0.5,0.5)$ and $\alpha_{I_3}(x_i)=(0.1,0.1)$ be three IFVs of three IFSs $I_i(i=1,2,3)$, respectively, then for each IFS I_i, we cannot derive its entropy value by Eq. (1) in that Eq. (1) is not meaningful for $\Delta_{I_i}(x_i)=0$.

3.2. *Entropy of IVIFSs*

We may classify the existing entropies of IVIFSs into two main classes according to their performance: fuzzy entropy and intuitionistic entropy. An entropy only being able to measure the fuzziness of an IVIFS is referred to as a fuzzy entropy, and the intuitionism of an IVIFS is measured through an intuitionistic entropy. In the following, we conduct a detailed analysis.

3.2.1. *Fuzzy Entropy of IVIFSs*

Liu *et al.* (2005) first introduced the concept of entropy for IVIFSs and gave some axiomatic requirements for the entropy of IVIFSs, which actually examine how far an IVIFS is from being a crisp set, listed as follows:

Definition 4. (See Liu *et al.*, 2005). A real-valued function $E:\mathrm{IVIFS}(X)\to[0,1]$ is called an entropy for IVIFSs, if it satisfies the following axiomatic requirements:

(E_L1) $E(\tilde I)=0$ if and only if $\tilde I$ is crisp;
(E_L2) $E(\tilde I)=1$ if and only if $[M_{\tilde I}^L(x_i),M_{\tilde I}^U(x_i)]=[N_{\tilde I}^L(x_i),N_{\tilde I}^U(x_i)]$ for all $x_i\in X$;
(E_L3) $E(\tilde I)=E(\tilde I^c)$;

(E_L4) $E(\tilde{I}_1) \leqslant E(\tilde{I}_2)$ if $\tilde{I}_1$ is less fuzzy than $\tilde{I}_2$, which is defined as:

$$M_{\tilde{I}_1}^L(x_i) \leqslant M_{\tilde{I}_2}^L(x_i), \quad M_{\tilde{I}_1}^U(x_i) \leqslant M_{\tilde{I}_2}^U(x_i),$$
$$N_{\tilde{I}_1}^L(x_i) \geqslant N_{\tilde{I}_2}^L(x_i), \quad N_{\tilde{I}_1}^U(x_i) \geqslant N_{\tilde{I}_2}^U(x_i), \quad \text{for}$$
$$M_{\tilde{I}_2}^L(x_i) \leqslant N_{\tilde{I}_2}^L(x_i) \quad \text{and} \quad M_{\tilde{I}_2}^U(x_i) \leqslant N_{\tilde{I}_2}^U(x_i) \quad \text{for all } x_i \in X,$$

or

$$M_{\tilde{I}_1}^L(x_i) \geqslant M_{\tilde{I}_2}^L(x_i), \quad M_{\tilde{I}_1}^U(x_i) \geqslant M_{\tilde{I}_2}^U(x_i),$$
$$N_{\tilde{I}_1}^L(x_i) \leqslant N_{\tilde{I}_2}^L(x_i), \quad N_{\tilde{I}_1}^U(x_i) \leqslant N_{\tilde{I}_2}^U(x_i), \quad \text{for}$$
$$M_{\tilde{I}_2}^L(x_i) \geqslant N_{\tilde{I}_2}^L(x_i) \quad \text{and} \quad M_{\tilde{I}_2}^U(x_i) \geqslant N_{\tilde{I}_2}^U(x_i) \quad \text{for all } x_i \in X.$$

Based on the above definition, different forms of entropy formulas of IVIFSs have been developed. For instance, Liu *et al.* (2005) proposed the following entropy formula:

$$E_L(\tilde{I}) = \frac{\sum_{i=1}^n [2 - M_{\tilde{I}}^L(x_i) \vee N_{\tilde{I}}^L(x_i) - M_{\tilde{I}}^U(x_i) \vee N_{\tilde{I}}^U(x_i)]}{\sum_{i=1}^n [2 - M_{\tilde{I}}^L(x_i) \wedge N_{\tilde{I}}^L(x_i) - M_{\tilde{I}}^U(x_i) \wedge N_{\tilde{I}}^U(x_i)]}. \tag{2}$$

Wei *et al.* (2011) generalized the entropy formula of IFSs (Huang and Liu, 2005) to the context of IVIFSs and presented an entropy formula:

$$E_W(\tilde{I}) = \frac{1}{n}\sum_{i=1}^n \frac{2 - |M_{\tilde{I}}^L(x_i) - N_{\tilde{I}}^L(x_i)| - |M_{\tilde{I}}^U(x_i) - N_{\tilde{I}}^U(x_i)| + \pi_{\tilde{I}}^L(x_i) + \pi_{\tilde{I}}^U(x_i)}{2 + |M_{\tilde{I}}^L(x_i) - N_{\tilde{I}}^L(x_i)| + |M_{\tilde{I}}^U(x_i) - N_{\tilde{I}}^U(x_i)| + \pi_{\tilde{I}}^L(x_i) + \pi_{\tilde{I}}^U(x_i)}. \tag{3}$$

Gao and Wei (2012) developed an entropy formula of IVIFSs on the foundation of the distance between an IVIFS and the crisp set:

$$E_{GW}(\tilde{I}) = \frac{\min\left\{\sum_{i=1}^n (2 - M_{\tilde{I}}^L(x_i) - M_{\tilde{I}}^U(x_i)), \sum_{i=1}^n (2 - N_{\tilde{I}}^L(x_i) - N_{\tilde{I}}^U(x_i))\right\}}{\max\left\{\sum_{i=1}^n (2 - M_{\tilde{I}}^L(x_i) - M_{\tilde{I}}^U(x_i)), \sum_{i=1}^n (2 - N_{\tilde{I}}^L(x_i) - N_{\tilde{I}}^U(x_i))\right\}}. \tag{4}$$

With the help of the continuous ordered weighted averaging operator (Yager, 2004), Jin *et al.* (2014) constructed two continuous entropy formulas of IVIFSs:

$$E_{J1}(\tilde{I}) = \frac{1}{n}\sum_{i=1}^n \frac{2 - |M_{\tilde{I}}^L(x_i) + M_{\tilde{I}}^U(x_i) - N_{\tilde{I}}^L(x_i) - N_{\tilde{I}}^U(x_i)| + \pi_{\tilde{I}}^L(x_i) + \pi_{\tilde{I}}^U(x_i)}{2 + |M_{\tilde{I}}^L(x_i) + M_{\tilde{I}}^U(x_i) - N_{\tilde{I}}^L(x_i) - N_{\tilde{I}}^U(x_i)| + \pi_{\tilde{I}}^L(x_i) + \pi_{\tilde{I}}^U(x_i)}, \tag{5}$$

$$E_{J2}(\tilde{I}) = \frac{1}{n}\sum_{i=1}^n \frac{\min\{M_{\tilde{I}}^L(x_i) + M_{\tilde{I}}^U(x_i), N_{\tilde{I}}^L(x_i) + N_{\tilde{I}}^U(x_i)\} + \pi_{\tilde{I}}^L(x_i) + \pi_{\tilde{I}}^U(x_i)}{\max\{M_{\tilde{I}}^L(x_i) + M_{\tilde{I}}^U(x_i), N_{\tilde{I}}^L(x_i) + N_{\tilde{I}}^U(x_i)\} + \pi_{\tilde{I}}^L(x_i) + \pi_{\tilde{I}}^U(x_i)}. \tag{6}$$

Then, Guo and Song (2014) modified the axiomatic requirement (E_L2) as:

(E_G2) $E(\tilde{I}) = 1$ if and only if $\left[M_{\tilde{I}}^L(x_i), M_{\tilde{I}}^U(x_i)\right] = \left[N_{\tilde{I}}^L(x_i), N_{\tilde{I}}^U(x_i)\right] = [0, 0]$, for all $x_i \in X$

and defined an entropy formula as:

$$E_G(\tilde{I}) = \frac{1}{n}\sum_{i=1}^{n}\left[1-\frac{1}{2}\left(\left|M_{\tilde{I}}^{L}(x_i)-N_{\tilde{I}}^{L}(x_i)\right|+\left|M_{\tilde{I}}^{U}(x_i)-N_{\tilde{I}}^{U}(x_i)\right|\right)\right] \times \frac{1+0.5(\pi_{\tilde{I}}^{L}(x_i)+\pi_{\tilde{I}}^{U}(x_i))}{2}. \tag{7}$$

Afterwards, Zhang *et al.* (2010) extended De Luca–Termini's axiom for fuzzy entropy (De Luca and Termini, 1972) and introduced an axiomatic definition of entropy for IVIFSs, in which the forth axiomatic requirement is different from that in Definition 4, exhibited as:

$(E_{ZJ}4)$ $E(\tilde{I}_1) \leqslant E(\tilde{I}_2)$ if $\tilde{I}_1$ is less fuzzy than $\tilde{I}_2$, which is defined by

$$\begin{aligned}
&M_{\tilde{I}_1}^{L}(x_i) \leqslant M_{\tilde{I}_2}^{L}(x_i), \quad M_{\tilde{I}_1}^{U}(x_i) \leqslant M_{\tilde{I}_2}^{U}(x_i),\\
&N_{\tilde{I}_1}^{L}(x_i) \geqslant N_{\tilde{I}_2}^{L}(x_i), \quad N_{\tilde{I}_1}^{U}(x_i) \geqslant N_{\tilde{I}_2}^{U}(x_i), \quad \text{for}\\
&M_{\tilde{I}_2}^{L}(x_i) \leqslant N_{\tilde{I}_2}^{L}(x_i) \quad \text{and} \quad M_{\tilde{I}_2}^{U}(x_i) \leqslant N_{\tilde{I}_2}^{U}(x_i) \quad \text{for all } x_i \in X,
\end{aligned}$$

or

$$\begin{aligned}
&M_{\tilde{I}_1}^{L}(x_i) \geqslant M_{\tilde{I}_2}^{L}(x_i), \quad M_{\tilde{I}_1}^{U}(x_i) \geqslant M_{\tilde{I}_2}^{U}(x_i),\\
&N_{\tilde{I}_1}^{L}(x_i) \leqslant N_{\tilde{I}_2}^{L}(x_i), \quad N_{\tilde{I}_1}^{U}(x_i) \leqslant N_{\tilde{I}_2}^{U}(x_i), \quad \text{for}\\
&M_{\tilde{I}_2}^{L}(x_i) \geqslant N_{\tilde{I}_2}^{L}(x_i) \quad \text{and} \quad M_{\tilde{I}_2}^{U}(x_i) \geqslant N_{\tilde{I}_2}^{U}(x_i) \quad \text{for all } x_i \in X,
\end{aligned}$$

or

$$\begin{aligned}
&M_{\tilde{I}_1}^{L}(x_i) \leqslant M_{\tilde{I}_2}^{L}(x_i), \quad M_{\tilde{I}_1}^{U}(x_i) \geqslant M_{\tilde{I}_2}^{U}(x_i),\\
&N_{\tilde{I}_1}^{L}(x_i) \geqslant N_{\tilde{I}_2}^{L}(x_i), \quad N_{\tilde{I}_1}^{U}(x_i) \leqslant N_{\tilde{I}_2}^{U}(x_i), \quad \text{for}\\
&M_{\tilde{I}_2}^{L}(x_i) \leqslant N_{\tilde{I}_2}^{L}(x_i) \quad \text{and} \quad M_{\tilde{I}_2}^{U}(x_i) \geqslant N_{\tilde{I}_2}^{U}(x_i) \quad \text{for all } x_i \in X,
\end{aligned}$$

or

$$\begin{aligned}
&M_{\tilde{I}_1}^{L}(x_i) \geqslant M_{\tilde{I}_2}^{L}(x_i), \quad M_{\tilde{I}_1}^{U}(x_i) \leqslant M_{\tilde{I}_2}^{U}(x_i),\\
&N_{\tilde{I}_1}^{L}(x_i) \leqslant N_{\tilde{I}_2}^{L}(x_i), \quad N_{\tilde{I}_1}^{U}(x_i) \geqslant N_{\tilde{I}_2}^{U}(x_i), \quad \text{for}\\
&M_{\tilde{I}_2}^{L}(x_i) \geqslant N_{\tilde{I}_2}^{L}(x_i) \quad \text{and} \quad M_{\tilde{I}_2}^{U}(x_i) \leqslant N_{\tilde{I}_2}^{U}(x_i) \quad \text{for all } x_i \in X.
\end{aligned}$$

Meanwhile, Zhang *et al.* (2010) and Zhang and Jiang (2010) established a family of entropy formulas of IVIFSs shown as:

$$E_{ZJ1}(\tilde{I}) = \frac{1}{n}\sum_{i=1}^{n}\frac{M_{\tilde{I}}^{L}(x_i)\wedge N_{\tilde{I}}^{L}(x_i)+M_{\tilde{I}}^{U}(x_i)\wedge N_{\tilde{I}}^{U}(x_i)}{M_{\tilde{I}}^{L}(x_i)\vee N_{\tilde{I}}^{L}(x_i)+M_{\tilde{I}}^{U}(x_i)\vee N_{\tilde{I}}^{U}(x_i)}, \tag{8}$$

$$E_{ZJ2}(\tilde{I}) = 1 - \frac{1}{2n}\sum_{i=1}^{n}\left(\left|M_{\tilde{I}}^{L}(x_i) - N_{\tilde{I}}^{L}(x_i)\right| + \left|M_{\tilde{I}}^{U}(x_i) - N_{\tilde{I}}^{U}(x_i)\right|\right), \tag{9}$$

$$E_{ZJ3}(\tilde{I}) = 1 - \frac{1}{n}\sum_{i=1}^{n}\left(\left|M_{\tilde{I}}^{L}(x_i) - N_{\tilde{I}}^{L}(x_i)\right| \vee \left|M_{\tilde{I}}^{U}(x_i) - N_{\tilde{I}}^{U}(x_i)\right|\right). \tag{10}$$

Moreover, Wei and Zhang (2015) proposed a different entropy formula, shown as below:

$$E_{WZ}(\tilde{I}) = \frac{1}{n}\sum_{i=1}^{n}\cos\frac{|M_{\tilde{I}}^{L}(x_i) - N_{\tilde{I}}^{L}(x_i)| + |M_{\tilde{I}}^{U}(x_i) - N_{\tilde{I}}^{U}(x_i)|}{2(2 + \pi_{\tilde{I}}^{L}(x_i) + \pi_{\tilde{I}}^{U}(x_i))}\pi. \tag{11}$$

Recently, Zhang *et al.* (2014) put forward a different axiomatic definition of entropy for IVIFSs from the perspective of distance between IVIFSs presented as below:

Definition 5. (See Zhang *et al.*, 2014.) A real-valued function $E: \mathrm{IVIFS}(X) \to [0, 1]$ is called an entropy for IVIFSs, if it accords with the following conditions:

($E_{ZX}1$) $E(\tilde{I}) = 0$ if $\tilde{I}$ is crisp;
($E_{ZX}2$) $E(\tilde{I}) = 1$ if and only if $[M_{\tilde{I}}^{L}(x_i), M_{\tilde{I}}^{U}(x_i)] = [N_{\tilde{I}}^{L}(x_i), N_{\tilde{I}}^{U}(x_i)] = [0.5, 0.5]$ for all $x_i \in X$;
($E_{ZX}3$) $E(\tilde{I}) = E(\tilde{I}^c)$;
($E_{ZX}4$) $E(\tilde{I}_1) \leqslant E(\tilde{I}_2)$ if $d(\tilde{I}_1, ([0.5, 0.5], [0.5, 0.5])_X) \geqslant d(\tilde{I}_2, ([0.5, 0.5], [0.5, 0.5])_X)$, where d is a distance measure of IVIFSs and $([0.5, 0.5], [0.5, 0.5])_X$ is the IVIFS with $M(x_i) = N(x_i) = [0.5, 0.5]$ for all $x_i \in X$.

By means of the normalized Hamming distance and the normalized Hamming distance induced by Hausdorff metric of IVIFSs, Zhang *et al.* (2014) constructed the following two entropy formulas, respectively:

$$\begin{aligned} E_{ZX1}(\tilde{I}) \;=\; & 1 - \frac{1}{2n}\sum_{i=1}^{n}\left(\left|M_{\tilde{I}}^{L}(x_i) - \frac{1}{2}\right| + \left|M_{\tilde{I}}^{U}(x_i) - \frac{1}{2}\right|\right. \\ & \left. + \left|N_{\tilde{I}}^{L}(x_i) - \frac{1}{2}\right| + \left|N_{\tilde{I}}^{U}(x_i) - \frac{1}{2}\right|\right), \end{aligned} \tag{12}$$

$$\begin{aligned} E_{ZX2}(\tilde{I}) \;=\; & 1 - \frac{1}{n}\sum_{i=1}^{n}\left(\left|M_{\tilde{I}}^{L}(x_i) - \frac{1}{2}\right| \vee \left|M_{\tilde{I}}^{U}(x_i) - \frac{1}{2}\right|\right. \\ & \left. + \left|N_{\tilde{I}}^{L}(x_i) - \frac{1}{2}\right| \vee \left|N_{\tilde{I}}^{U}(x_i) - \frac{1}{2}\right|\right). \end{aligned} \tag{13}$$

Analyzing the above-mentioned axiomatic definitions, we find that Liu *et al.* (2005) considered in the requirement (E_L2) that the entropy of an IVIFS takes the maximum value if and only if the interval membership degree of each element to the IVIFS equals to the corresponding interval non-membership degree. However, it is unreasonable since for different IVIFSs satisfying $[M_{\tilde{I}}^{L}(x_i), M_{\tilde{I}}^{U}(x_i)] = [N_{\tilde{I}}^{L}(x_i), N_{\tilde{I}}^{U}(x_i)]$ for all $x_i \in X$, the interval hesitancy degrees of x_i may be obviously different, that is to say, these IVIFSs have different amounts of uncertainties. Therefore, the requirement $(E_L 2)$ ignores the effects of interval hesitancy degrees on entropy values when the interval membership degree of each element equals to the corresponding interval non-membership degree. Furthermore, given an IVIFS with $[M_{\tilde{I}}^{L}(x_i), M_{\tilde{I}}^{U}(x_i)] = [N_{\tilde{I}}^{L}(x_i), N_{\tilde{I}}^{U}(x_i)] \neq [0, 0]$ for all $x_i \in X$, although it is difficult for us to make a decision according to the interval membership and non-membership degrees, the IVIFS contains more determinate information than the IVIFS $\tilde{I} = \{\langle x_i, [0, 0], [0, 0]\rangle \mid x_i \in X\}$. Because from the perspective of a voting, we know absolutely nothing about the vote for a candidate from the IVIFS $\tilde{I} = \{\langle x_i, [0, 0], [0, 0]\rangle \mid x_i \in X\}$, while according to the IVIFS with $[M_{\tilde{I}}^{L}(x_i), M_{\tilde{I}}^{U}(x_i)] = [N_{\tilde{I}}^{L}(x_i), N_{\tilde{I}}^{U}(x_i)] \neq [0, 0]$ for all $x_i \in X$, we know that the approval and rejection percentages are the same. Similarly, the requirement $(E_{ZX} 2)$ in Definition 5 is not in accordance with human's intuition. Because compared with the IVIFS $\tilde{I} = \{\langle x_i, [0, 0], [0, 0]\rangle \mid x_i \in X\}$, the IVIFS $\tilde{I} = \{\langle x_i, [0.5, 0.5], [0.5, 0.5]\rangle \mid x_i \in X\}$ possesses more determinate information, that is to say, the IVIFS $\tilde{I} = \{\langle x_i, [0.5, 0.5], [0.5, 0.5]\rangle \mid x_i \in X\}$ is not the most uncertain one. Consequently, it is improper to say that the closer an IVIFS is to the IVIFS $\tilde{I} = \{\langle x_i, [0.5, 0.5], [0.5, 0.5]\rangle \mid x_i \in X\}$, the more uncertain it is, that is to say, the requirement $(E_{ZX} 4)$ is unreasonable. Additionally, the requirements $(E_L 4)$ and $(E_{ZJ} 4)$ are also irrational since they only consider the effects of differences between interval membership degrees and the corresponding interval non-membership degrees on entropy values (for an IVIFS, the smaller the differences, the larger the entropy value), but ignore the influence of interval hesitancy degrees.

3.2.2. *Intuitionistic Entropy of IVIFSs*

Based on the axiomatic definition of entropy for IFSs (Burillo and Bustince, 1996), Zhang *et al.* (2011) proposed a set of axiomatic conditions of entropy for IVIFSs, which believe the entropy of an IVIFS as a measure of how far the IVIFS is from being a fuzzy set.

Definition 6. (See Zhang *et al.*, 2011.) A real-valued function $E : \mathrm{IVIFS}(X) \to [0, 1]$ is called an entropy for IVIFSs, if it meets the following conditions:

- $(E_{ZM} 1)$ $E(\tilde{I}) = 0$ if and only if $\tilde{I}$ is a fuzzy set;
- $(E_{ZM} 2)$ $E(\tilde{I}) = 1$ if and only if $[M_{\tilde{I}}^{L}(x_i), M_{\tilde{I}}^{U}(x_i)] = [N_{\tilde{I}}^{L}(x_i), N_{\tilde{I}}^{U}(x_i)] = [0, 0]$ for all $x_i \in X$;
- $(E_{ZM} 3)$ $E(\tilde{I}) = E(\tilde{I}^c)$;
- $(E_{ZM} 4)$ $E(\tilde{I}_1) \leqslant E(\tilde{I}_2)$ if $M_{\tilde{I}_1}^{L}(x_i) \geqslant M_{\tilde{I}_2}^{L}(x_i)$, $M_{\tilde{I}_1}^{U}(x_i) \geqslant M_{\tilde{I}_2}^{U}(x_i)$, $N_{\tilde{I}_1}^{L}(x_i) \geqslant N_{\tilde{I}_2}^{L}(x_i)$ and $N_{\tilde{I}_1}^{U}(x_i) \geqslant N_{\tilde{I}_2}^{U}(x_i)$ for all $x_i \in X$.

At the same time, the following equation is introduced from which a class of entropy formulas of IVIFSs can be obtained by installing different parameter values presented as:

$$E_{ZM}(\tilde{I}) = \frac{1}{n} \sum_{i=1}^{n} \big(1 - \overline{M_{\tilde{I}}}(x_i) - \overline{N_{\tilde{I}}}(x_i)\big) e^{1 - \overline{M_{\tilde{I}}}(x_i) - \overline{N_{\tilde{I}}}(x_i)}$$

where for all $x_i \in X$, $\overline{M}_{\tilde{I}}(x_i) = M_{\tilde{I}}^L(x_i) + \tau(M_{\tilde{I}}^U(x_i) - M_{\tilde{I}}^L(x_i))$, $\overline{N}_{\tilde{I}}(x_i) = N_{\tilde{I}}^L(x_i) + \tau(N_{\tilde{I}}^U(x_i) - N_{\tilde{I}}^L(x_i))$ and $\tau \in [0, 1]$. In fact, since $\pi_{\tilde{I}}^L(x_i) = 1 - M_{\tilde{I}}^U(x_i) - N_{\tilde{I}}^U(x_i)$ and $\pi_{\tilde{I}}^U(x_i) = 1 - M_{\tilde{I}}^L(x_i) - N_{\tilde{I}}^L(x_i)$ for all $x_i \in X$, then the above equation can be written as:

$$E_{ZM}(\tilde{I}) = \frac{1}{n}\sum_{i=1}^{n}\big(\tau\pi_{\tilde{I}}^L(x_i) + (1-\tau)\pi_{\tilde{I}}^U(x_i)\big)e^{\tau\pi_{\tilde{I}}^L(x_i)+(1-\tau)\pi_{\tilde{I}}^U(x_i)}.$$

Especially, let $\tau = 0.5$, then we have

$$E_{ZM}(\tilde{I}) = \frac{1}{2n}\sum_{i=1}^{n}\big(\pi_{\tilde{I}}^L(x_i) + \pi_{\tilde{I}}^U(x_i)\big)e^{\frac{\pi_{\tilde{I}}^L(x_i)+\pi_{\tilde{I}}^U(x_i)}{2}}. \tag{14}$$

From Definition 6, we see that Zhang *et al.* (2011) deemed in the requirement ($E_{ZM}1$) that the entropy of an IVIFS takes the minimum value if and only if the IVIFS is a fuzzy set. Nonetheless, it is unreasonable since a fuzzy set itself has fuzziness. Moreover, the requirement ($E_{ZM}4$) only focuses on the influence of interval hesitancy degrees in entropy values (for an IVIFS, the larger the interval hesitancy degrees, the larger the entropy value), but neglects the effects of differences between interval membership degrees and the corresponding interval non-membership degrees.

In a word, none of the existing axiomatic definitions of entropy for IVIFSs fully describes the uncertainty associated with an IVIFS, which involves both fuzziness and intuitionism. The definitions of entropy for IVIFSs shown in Liu *et al.* (2005), Zhang *et al.* (2010, 2014), Guo and Song (2014) only consider the fuzziness of IVIFSs, which is reflected by the differences between interval membership degrees and the corresponding interval non-membership degrees, and the entropy defined by Zhang *et al.* (2011) only measures the intuitionism of IVIFSs, which is related to the interval hesitancy degrees. Consequently, the existing entropy formulas of IVIFSs, all of which are built on the foundation of the above axiomatic definitions, will produce counterintuitive results when applied to some IVIFSs. Thus, it is very urgent and necessary to introduce a new axiomatic definition of entropy for IVIFSs which can simultaneously consider both fuzziness and intuitionism of IVIFSs, and construct new entropy formulas which can generate good results that are consistent with human's intuition.

4. Novel Entropy Model for IVIFSs

The uncertainty of an IVIFS originates from its inherent fuzziness and intuitionism, which are reflected by the closeness of interval membership degrees and the corresponding interval non-membership degrees and the interval hesitancy degrees, respectively. Nonetheless, as demonstrated previously, the existing axiomatic definitions of entropy for IVIFSs can only cope with one of the two types of uncertainty of IVIFSs. In this section, we attempt to introduce a new definition of entropy for IVIFSs in full consideration of both fuzziness and intuitionism of IVIFSs and develop some methods for formulating the entropy. Before that, for overcoming the drawbacks existing in Eq. (1), we first come up with a method to construct the entropy formulas of IFSs fulfilling the axiomatic requirements in Definition 3.

4.1. *Improved Entropy Formulas of IFSs*

In order to circumvent the drawbacks in Eq. (1), we propose the following method by which a family of entropy formulas of IFSs satisfying the requirements in Definition 3 can be constructed.

Theorem 1. *Let* $O = \{(x, y) \in [0,1] \times [0,1] | x + y \leqslant 1\}$ *and* $f : O \to [0,1]$ *be a continuous function. Then the function* $E : \mathrm{IFS}(X) \to [0,1]$ *defined as*

$$E(I) = \frac{1}{n} \sum_{i=1}^{n} f\big(\Delta_I(x_i), \pi_I(x_i)\big) \tag{15}$$

satisfies the requirements $(E_M 1)$–$(E_M 4)$ *if and only if* f *possesses the following properties*:

- (i) $f(x, y) = 0$ *if and only if* $x = 1$ *and* $y = 0$;
- (ii) $f(x, y) = 1$ *if and only if* $x = 0$ *and* $y = 1$;
- (iii) f *is monotonically decreasing regarding the first variable and monotonically increasing regarding the second variable.*

Proof. Suppose that $E(I)$ defined by Eq. (15) meets the axiomatic requirements $(E_M 1)$–$(E_M 4)$. We below illustrate that f has the properties (i)–(iii).

(1) Suppose that $f(x, y) = 0$ with $x + y \leqslant 1$, $x, y \in [0,1]$, then we take $\Delta_I(x_i) = x$ and $\pi_I(x_i) = y$ for every $x_i \in X$. Thus, for the constructed IFSs I, we obtain

$$E(I) = \frac{1}{n} \sum_{i=1}^{n} f\big(\Delta_I(x_i), \pi_I(x_i)\big) = 0. \tag{16}$$

By the requirement $(E_M 1)$, we get that Eq. (16) holds if and only if I is crisp, i.e., $\Delta_I(x_i) = 1$ and $\pi_I(x_i) = 0$ for every $x_i \in X$, i.e., $x = 1$ and $y = 0$. Then at this time, we show that f satisfies the property (i).

(2) Assume that $f(x, y) = 1$ with $x + y \leqslant 1$, $x, y \in [0,1]$, then for the IFSs I defined by $\Delta_I(x_i) = x$ and $\pi_I(x_i) = y$ for every $x_i \in X$, we have

$$E(I) = \frac{1}{n} \sum_{i=1}^{n} f\big(\Delta_I(x_i), \pi_I(x_i)\big) = 1. \tag{17}$$

From the requirement $(E_M 2)$, we know that Eq. (17) holds if and only if $\Delta_I(x_i) = 0$ and $\pi_I(x_i) = 1$ for every $x_i \in X$, i.e., $x = 0$ and $y = 1$. Then we complete the proof of (ii).

(3) Concerning (iii), suppose that there exist $z_1, z_2, y \in [0,1]$ with $z_1 \leqslant z_2$, $z_1 + y \leqslant 1$ and $z_2 + y \leqslant 1$ such that $f(z_1, y) \leqslant f(z_2, y)$. Considering the IFSs I_1 given by $\Delta_{I_1}(x_i) = z_1$ and $\pi_{I_1}(x_i) = y$ for every $x_i \in X$, and the IFSs I_2 given by $\Delta_{I_2}(x_i) = z_2$ and $\pi_{I_2}(x_i) = y$ for every $x_i \in X$, by Eq. (15), we derive

$$\begin{aligned} E(I_1) &= \frac{1}{n}\sum_{i=1}^{n} f\big(\Delta_{I_1}(x_i), \pi_{I_1}(x_i)\big) = f(z_1, y) < f(z_2, y) \\ &= \frac{1}{n}\sum_{i=1}^{n} f\big(\Delta_{I_2}(x_i), \pi_{I_2}(x_i)\big) = E(I_2) \end{aligned}$$

which contradicts the axiomatic requirement (E_M4). Then we have validated (iii). The converse is easily proven. □

Two concrete examples are shown as:

Case 1. Let $f: O \to [0,1]$ be $f(x,y) = 1 - \frac{x^2+(1-y)^2}{2}$, then f satisfies the properties in Theorem 1. Thus, according to Theorem 1, the corresponding entropy formula of IFSs is

$$E_1(I) = 1 - \frac{1}{n}\sum_{i=1}^{n} \frac{|M_I(x_i) - N_I(x_i)|^2 + (1 - \pi_I(x_i))^2}{2}.$$

Case 2. Given $f: O \to [0,1]$ as $f(x,y) = \frac{(1-x)(1+y)}{2}$. Then f fulfills the properties in Theorem 1, and the derived entropy formula of IFSs is

$$E_2(I) = \frac{1}{n}\sum_{i=1}^{n} \frac{(1 - |M_I(x_i) - N_I(x_i)|)(1+\pi_I(x_i))}{2}.$$

Evidently, it is a bit difficult to look for the bivariate function f described in Theorem 1. In what follows, we try to simplify it to the combinations of univariate functions.

Theorem 2. *Let $S: [0,1]^2 \to [0,1]$ be a symmetric aggregation function such that $S(x,\cdot): [0,1] \to [0,1]$ is strictly increasing for every $x \in [0,1]$, and $h: [0,1] \to [0,1]$ be a continuous function. Then the function $f(x,y) = S(1-h(x), h(y))$ satisfies the properties in Theorem 1 if and only if h fulfills the following properties*:

(i) $h(x) = 0$ *if and only if* $x = 0$;
(ii) $h(x) = 1$ *if and only if* $x = 1$;
(iii) $E_i(\tilde{I}_6) > E_i(\tilde{I}_2)$ *is monotonically increasing in* $[0,1]$.

Proof. Firstly, since S is symmetric and $S(x,\cdot)$ is strictly increasing, then it can be deduced that $S(x,y) = 0$ if and only if $x = y = 0$ and $S(x,y) = 1$ if and only if $x = y = 1$. Suppose that $f(x,y) = S(1-h(x), h(y))$ meets the properties in Theorem 1. Then we have $f(1,0) = S(1-h(1), h(0)) = 0$. From the properties of S, we further obtain $h(1) = 1$ and $h(0) = 0$. Assume that there exists $x_1 \neq 1$ such that $h(x_1) = 1$, then we have $f(x_1, 0) = S(1-h(x_1), h(0)) = S(0,0) = 0$, which is contradictory. Again assume that there exists $x_2 \neq 0$ such that $h(x_2) = 0$, then we obtain $f(x_2, 1) = S(1 - h(x_2), h(1)) = S(1,1) = 1$, which is also a contradiction. Then at this time, (i) and (ii) hold. At last, with respect to (iii), we suppose that there exists $0 \leqslant x \leqslant y \leqslant 1$ such that $h(x) > h(y)$. Then in this case, we get $f(x, 1-y) = S(1-h(x), h(1-y))$ and $f(y, 1-y) = S(1-h(y), h(1-y))$. By the strict monotonicity of the function $S(x,\cdot)$, it follows that $f(x, 1-y) \leqslant f(y, 1-y)$, which contradicts the property (iii) in Theorem 1. It is easy to prove the converse. □

Some illustrative examples are presented as follows:

Case 3. Define $h : [0,1] \to [0,1]$ as $h(x) = x^2$, then h fulfills the properties in Theorem 2. Again define $S : [0,1]^2 \to [0,1]$ as $S(x,y) = \frac{x+y}{2}$, then S is symmetric and $S(x,\cdot)$ is strictly increasing for every $x \in [0,1]$. In this case, by Theorems 1 and 2, the corresponding entropy formula of IFSs is

$$E_3(I) = \frac{1}{n}\sum_{i=1}^{n} \frac{1 - |M_I(x_i) - N_I(x_i)|^2 + \pi_I^2(x_i)}{2}.$$

Case 4. Let $h : [0,1] \to [0,1]$ be $h(x) = \sin\frac{\pi x}{2}$ and S be defined as that in Case 3, then in light of Theorems 1 and 2, the obtained entropy formula of IFSs is

$$E_4(I) = \frac{1}{n}\sum_{i=1}^{n} \frac{1 - \sin(\frac{\pi}{2}|M_I(x_i) - N_I(x_i)|) + \sin(\frac{\pi}{2}\pi_I(x_i))}{2}.$$

It is clear that the developed entropy formulas for IFSs in this subsection meet the axiomatic requirements shown in Definition 3 and manage to overcome the drawbacks existing in Eq. (1).

Remark. It is worth pointing out that although Pal *et al.* (2013) represented the fuzziness and intuitionism of IFSs by using a two-tuple entropy model, they did not propose any total measure of uncertainty for IFSs, i.e., we cannot measure the total amount of uncertainty associated with an IFS with reference to Pal *et al.* (2013).

4.2. *New Entropy of IVIFSs*

Now we give a new definition of entropy for IVIFSs by taking into consideration both fuzziness and intuitionism of IVIFSs simultaneously, which generalizes the axiomatic framework of entropy for IFSs (Mao *et al.*, 2013). Firstly, for any IVIFS $\tilde{I}$, let $\Delta_{\tilde{I}}^L(x_i) = |M_{\tilde{I}}^L(x_i) - N_{\tilde{I}}^L(x_i)|$ and $\Delta_{\tilde{I}}^U(x_i) = |M_{\tilde{I}}^U(x_i) - N_{\tilde{I}}^U(x_i)|$ for $\forall x_i \in X$.

Definition 7. An interval-valued intuitionistic fuzzy entropy $E : \text{IVIFS}(X) \to [0, 1]$ is a real-valued function associated with $\Delta_{\tilde{I}}^{L}(x_i)$, $\Delta_{\tilde{I}}^{U}(x_i)$, $\pi_{\tilde{I}}^{L}(x_i)$ and $\pi_{\tilde{I}}^{U}(x_i)$, and satisfies the following axiomatic requirements:

- $(E1)$ $E(\tilde{I}) = 0$ if and only if $\tilde{I}$ is a crisp set, i.e., $\tilde{I} = \{\langle x_i, [1, 1], [0, 0]\rangle \mid x_i \in X\}$ or $\tilde{I} = \{\langle x_i, [0, 0], [1, 1]\rangle \mid x_i \in X\}$;
- $(E2)$ $E(\tilde{I}) = 1$ if and only if $\tilde{I} = \{\langle x_i, [0, 0], [0, 0]\rangle \mid x_i \in X\}$;
- $(E3)$ $E(\tilde{I}) = E(\tilde{I}^c)$;
- $(E4)$ $E(\tilde{I})$ is monotonically decreasing with respect to $\Delta_{\tilde{I}}^{L}(x_i)$ and $\Delta_{\tilde{I}}^{U}(x_i)$, respectively, and monotonically increasing with respect to $\pi_{\tilde{I}}^{L}(x_i)$ and $\pi_{\tilde{I}}^{U}(x_i)$, respectively, for all $x_i \in X$.

From Definition 7, it is not hard to find that the new axiomatic definition of entropy for IVIFSs has the following rationality and advantages: (i) the requirement $(E1)$ overcomes the drawbacks of $(E_{ZM}1)$. Because for an element, it either fully belongs to the crisp set or fully does not belong to the crisp set and there are no uncertainties. Consequently, it is reasonable to say that the amount of uncertainty of the crisp set is 0; (ii) the requirement $(E2)$ circumvents the weaknesses of $(E_L 2)$ and $(E_{ZX}2)$. Since from the IVIFS $\tilde{I} = \{\langle x_i, [0, 0], [0, 0]\rangle \mid x_i \in X\}$, we cannot get any information about the relationship between the element $x_i \in X$ and the IVIFS $\tilde{I}$, then it accords with human's intuition to deem that the amount of uncertainty of the IVIFS $\tilde{I} = \{\langle x_i, [0, 0], [0, 0]\rangle \mid x_i \in X\}$ is maximum; (iii) since for an arbitrary IVIFS, it contains the same amount of information with its complement, then the requirement $(E3)$ is rational; (iv) the requirement $(E4)$ overcomes the shortcomings of $(E_L 4)$, $(E_{ZJ}4)$, $(E_{ZX}4)$, $(E_{ZM}4)$. In the requirement $(E4)$, we believe that the uncertainty of an IVIFS is related to both the differences between interval membership degrees and the corresponding interval non-membership degrees, and the interval hesitancy degrees. The entropy value will increase with the weakened differences under the same interval hesitancy degrees, and equivalently the entropy value will increase with the enhanced interval hesitancy degrees under the same differences. This requirement is logical since for two IVIFSs, when the interval hesitancy degrees of each element are the same, the closer the interval membership degree of the element to one IVIFS is to the corresponding interval non-membership degree, the more uncertain the relationship between the element and the IVIFS, and when the differences between interval membership degrees and the corresponding interval non-membership degrees of each element are the same, the larger the interval hesitancy degree of the element to one IVIFS, the less determinate the relationship between the element and the IVIFS. In short, the new axiomatic definition of entropy for IVIFSs overcomes the drawbacks of the existing ones and fully describes the two facets of uncertainty associated with IVIFSs, i.e., fuzziness and intuitionism, which is more intuitive and reasonable.

Now the question we are faced with is to construct the entropy formulas of IVIFSs which fulfill the axiomatic requirements in Definition 7. In the first place, a general result is shown as follows:

Theorem 3. *Let* $D = \{(y_1, y_2, z_1, z_2) \in [0,1] \times [0,1] \times [0,1] \times [0,1] \mid y_1 + z_1 \leqslant 1, y_2 + z_2 \leqslant 1\}$ *and* $F : D \to [0,1]$ *be a continuous function. Then the function* $E : \mathrm{IVIFS}(X) \to [0,1]$ *given by*

$$E(\tilde{I}) = \frac{1}{n}\sum_{i=1}^{n} F\big(\Delta_{\tilde{I}}^{L}(x_i), \Delta_{\tilde{I}}^{U}(x_i), \pi_{\tilde{I}}^{L}(x_i), \pi_{\tilde{I}}^{U}(x_i)\big)$$

satisfies the requirements $(E1)$–$(E4)$ *if and only if* F *owns the following properties*:

(i) $F(y_1, y_2, z_1, z_2) = 0$ *if and only if* $y_1 = y_2 = 1$ *and* $z_1 = z_2 = 0$;
(ii) $F(y_1, y_2, z_1, z_2) = 1$ *if and only if* $y_1 = y_2 = 0$ *and* $z_1 = z_2 = 1$;
(iii) F *is monotonically decreasing regarding the first two variables and monotone increasing regarding the latter two variables.*

Proof. The proof is analogous to that of Theorem 1, hence we here omit it. □

We below consider two illustrative examples:

CASE 5. Let $F : D \to [0,1]$ be $F(y_1, y_2, z_1, z_2) = 1 - \frac{y_1^2 + y_2^2 + (1-z_1)^2 + (1-z_2)^2}{4}$, then F has the properties listed in Theorem 3. Consequently, the following entropy formula of IVIFSs can be constructed, shown as:

$$E_5(\tilde{I}) = 1 - \frac{1}{n}\sum_{i=1}^{n} \frac{|M_{\tilde{I}}^{L}(x_i) - N_{\tilde{I}}^{L}(x_i)|^2 + |M_{\tilde{I}}^{U}(x_i) - N_{\tilde{I}}^{U}(x_i)|^2 + (1 - \pi_{\tilde{I}}^{L}(x_i))^2 + (1 - \pi_{\tilde{I}}^{U}(x_i))^2}{4}. \tag{18}$$

CASE 6. Given $F : D \to [0,1]$ as $F(y_1, y_2, z_1, z_2) = \frac{(2 - y_1 - y_2)(2 + z_1 + z_2)}{8}$, then F possesses the properties in Theorem 3, and the corresponding entropy formula of IVIFSs is

$$E_6(\tilde{I}) = \frac{1}{n}\sum_{i=1}^{n} \frac{(2 - |M_{\tilde{I}}^{L}(x_i) - N_{\tilde{I}}^{L}(x_i)| - |M_{\tilde{I}}^{U}(x_i) - N_{\tilde{I}}^{U}(x_i)|)(2 + \pi_{\tilde{I}}^{L}(x_i) + \pi_{\tilde{I}}^{U}(x_i))}{8}. \tag{19}$$

It is worthwhile to notice that Eq. (19) is identical to Eq. (7), which is to say, the entropy formula of IVIFSs developed by Guo and Song (2014) can be constructed by our proposed method.

Observing that it is not easy to find such a function as described in Theorem 3, we below attempt to simplify it.

Theorem 4. *Let* $S : [0,1]^2 \to [0,1]$ *be a symmetric aggregation function such that* $S(x, \cdot) : [0,1] \to [0,1]$ *is strictly increasing for every* $x \in [0,1]$ *and* $H : [0,1] \to [0,1]$ *be a continuous function. Then the function* $F(y_1, y_2, z_1, z_2) = S(1 - H(y_1, y_2), H(z_1, z_2))$ *meets the properties listed in Theorem* 3 *if and only if* H *has the following properties*:

(i) $H(x, y) = 0$ *if and only if* $x = y = 0$;
(ii) $H(x, y) = 1$ *if and only if* $x = y = 1$;
(iii) H *is monotone increasing in both variables.*

Proof. By the help of the proof of Theorem 2, the theorem is easily proven. □

Furthermore, we may replace the bivariate function H with the combinations of univariate functions.

Theorem 5. *Let* $S : [0, 1]^2 \to [0, 1]$ *be a symmetric aggregation function such that* $S(x, \cdot) : [0, 1] \to [0, 1]$ *is strictly increasing for every* $x \in [0, 1]$*, and* $h : [0, 1] \to [0, 1]$ *be a continuous function. Then the function* $H(x, y) = S(h(x), h(y))$ *has the properties in Theorem 4 if and only if h fulfills the following properties*:

(i) $h(x) = 0$ *if and only if* $x = 0$;
(ii) $h(x) = 1$ *if and only if* $x = 1$;
(iii) h *is monotone increasing in* $[0, 1]$.

Proof. It follows from an easy deduction. □

Case 7. Again let $S : [0, 1]^2 \to [0, 1]$ be $S(x, y) = \frac{x+y}{2}$, then in accordance with Theorems 3–5, different forms of entropy formulas of IVIFSs can be generated using different univariate functions h, shown as:

1) Select $h(x) = x^2$, then the constructed entropy formula of IVIFSs is

$$E_7(\tilde{I}) = \frac{1}{n}\sum_{i=1}^{n}\frac{2 - |M_{\tilde{I}}^L(x_i) - N_{\tilde{I}}^L(x_i)|^2 - |M_{\tilde{I}}^U(x_i) - N_{\tilde{I}}^U(x_i)|^2 + (\pi_{\tilde{I}}^L(x_i))^2 + (\pi_{\tilde{I}}^U(x_i))^2}{4}; \tag{20}$$

2) Define $h(x) = \sin\frac{\pi x}{2}$, then the corresponding entropy formula of IVIFSs is

$$\begin{aligned} &E_8(\tilde{I}) \\ &\quad = \frac{1}{n}\sum_{i=1}^{n}\frac{2 - \sin(\frac{\pi}{2}|M_{\tilde{I}}^L(x_i) - N_{\tilde{I}}^L(x_i)|) - \sin(\frac{\pi}{2}|M_{\tilde{I}}^U(x_i) - N_{\tilde{I}}^U(x_i)|) + \sin(\frac{\pi}{2}\pi_{\tilde{I}}^L(x_i)) + \sin(\frac{\pi}{2}\pi_{\tilde{I}}^U(x_i))}{4}; \end{aligned} \tag{21}$$

3) Let $h(x) = \frac{2x}{1+x}$, then the established entropy formula of IVIFSs is shown as:

$$\begin{aligned} E_9(\tilde{I}) = \frac{1}{2n}\sum_{i=1}^{n} 1 &- \frac{|M_{\tilde{I}}^L(x_i) - N_{\tilde{I}}^L(x_i)|}{1 + |M_{\tilde{I}}^L(x_i) - N_{\tilde{I}}^L(x_i)|} - \frac{|M_{\tilde{I}}^U(x_i) - N_{\tilde{I}}^U(x_i)|}{1 + |M_{\tilde{I}}^U(x_i) - N_{\tilde{I}}^U(x_i)|} \\ &+ \frac{\pi_{\tilde{I}}^L(x_i)}{1 + \pi_{\tilde{I}}^L(x_i)} + \frac{\pi_{\tilde{I}}^U(x_i)}{1 + \pi_{\tilde{I}}^U(x_i)}; \end{aligned} \tag{22}$$

Table 1
Results derived from the existing entropy formulas of IVIFSs.

	E_L	E_W	E_{GW}	E_{J1}	E_{J2}	E_G	E_{ZJ1}	E_{ZJ2}	E_{ZJ3}	E_{WZ}	E_{ZX1}	E_{ZX2}	E_{ZM}
$\tilde{I}_1$	0.6875	0.6875	0.6875	0.6875	0.6875	0.5063	0.4444	0.75	0.7	0.9579	0.55	0.4	$0.35e^{0.35}$
$\tilde{I}_2$	0.7059	0.7059	0.7059	0.7059	0.7059	0.5438	0.375	0.75	0.7	0.9635	0.55	0.4	$0.45e^{0.45}$
$\tilde{I}_3$	1	1	1	1	1	1	×	1	1	1	0	0	e
$\tilde{I}_4$	1	1	1	1	1	0.75	1	1	1	1	0.5	0.5	$0.5e^{0.5}$
$\tilde{I}_5$	1	1	1	1	1	0.5	1	1	1	1	1	1	0
$\tilde{I}_6$	0.9333	0.9333	0.9333	0.9333	0.9333	0.6888	0.8333	0.95	0.9	0.9985	0.55	0.4	$0.45e^{0.45}$

Note: "×" means "meaningless", and the numbers with red or blue color denote counterintuitive results.

4) Define $h(x) = \frac{\lg(1+x)}{\lg 2}$, then the corresponding entropy formula of IVIFSs is

$$E_{10}(\tilde{I}) = \frac{1}{2n}\sum_{i=1}^{n} 1 - \frac{\lg(1+|M_{\tilde{I}}^{L}(x_i) - N_{\tilde{I}}^{L}(x_i)|)}{2\lg 2} - \frac{\lg(1+|M_{\tilde{I}}^{U}(x_i) - N_{\tilde{I}}^{U}(x_i)|)}{2\lg 2} + \frac{\lg(1+\pi_{\tilde{I}}^{L}(x_i))}{2\lg 2} + \frac{\lg(1+\pi_{\tilde{I}}^{U}(x_i))}{2\lg 2}. \tag{23}$$

Notice that the entropy formulas of IVIFSs shown by Eqs. (18)–(23) fully depict both fuzziness and intuitionism of IVIFSs by use of all three components linked to IVIFSs, i.e., the interval membership degree, the interval non-membership degree and the interval hesitancy degree. It can be also observed that when the IVIFS $\tilde{I}$ is simplified to an IFS I, each entropy formula of IVIFSs introduced in this subsection reduces to the entropy formula of IFSs presented in Section 4.1, correspondingly. Moreover, as mentioned by Vlachos (2007), a good entropy measure E should be in possession of a desirable property that the average of entropy values of separate elements A_i in a set A should be equal to the entropy value $E(A)$ of the set A, i.e.,$E(A) = \frac{1}{n}\sum_{i=1}^{n} E(A_i)$. It is evident that all developed entropy formulas of IVIFSs in this subsection satisfy the aforementioned property, here for $i = 1, 2, \ldots, n$, $A_i = \{< x_i, [M_A^L(x_i), M_A^U(x_i)], [N_A^L(x_i), N_A^U(x_i)] >\}$, which states that our developed entropy formulas of IVIFSs are well defined.

4.3. *Comparative Analysis and Discussion*

In this part, we conduct a comparative analysis to show the validity and superiority of the entropy formulas of IVIFSs developed on the basis of the new axiomatic definition.

Example. Let $\tilde{I}_1 = \{\langle x_i, [0.3, 0.6], [0.1, 0.3]\rangle \mid x_i \in X\}$, $\tilde{I}_2 = \{\langle x_i, [0.1, 0.2], [0.3, 0.5]\rangle \mid x_i \in X\}$, $\tilde{I}_3 = \{\langle x_i, [0, 0], [0, 0]\rangle \mid x_i \in X\}$, $\tilde{I}_4 = \{\langle x_i, [0.25, 0.25], [0.25, 0.25]\rangle \mid x_i \in X\}$, $\tilde{I}_5 = \{\langle x_i, [0.5, 0.5], [0.5, 0.5]\rangle \mid x_i \in X\}$ and $\tilde{I}_6 = \{\langle x_i, [0.2, 0.3], [0.2, 0.4]\rangle \mid x_i \in X\}$ be six IVIFSs in the finite set X. Now we calculate their entropy values by the entropy formulas of IVIFSs shown in Eqs. (2)–(14), respectively, and list the results in Table 1.

From Table 1, it can be easily observed that the existing entropy formulas of IVIFSs except for the formula shown in Eq. (7) yield counterintuitive results. We below give an in-depth analysis.

According to Table 1, we first notice that applying entropy formulas E_L, E_W, E_{GW}, E_{J1}, E_{J2}, E_{ZJ1}, E_{ZJ2}, E_{ZJ3} and E_{WZ} to the IVIFSs $\tilde{I}_3$, $\tilde{I}_4$ and $\tilde{I}_5$ gives rise to the same result, which is unreasonable since the three IVIFSs own obviously different interval hesitancy degrees, that is to say, they have different intuitionism aspects of uncertainty. The reason is that the nine entropy formulas of IVIFSs are all constructed on the basis of the axiomatic requirement (E_L2), which neglects the changes of entropy values caused by interval hesitancy degrees when the interval membership degree of each element equals to the corresponding interval non-membership degree. Moreover, the entropy formula E_{ZJ1} is invalid when applied to the IVIFS $\tilde{I}_3$, and the same value is got when the entropy formulas E_{ZJ2} and E_{ZJ3} are respectively applied to the IVIFSs $\tilde{I}_1$ and $\tilde{I}_2$, which is also unreasonable since although the IVIFSs $\tilde{I}_1$ and $\tilde{I}_2$ possess the same absolute deviation of interval membership degrees and interval non-membership degrees, they possess different interval hesitancy degrees. The reason is that the formulas E_{ZJ2} and E_{ZJ3} are both developed under the condition of $(E_{ZJ}4)$, which only considers the effects of differences between interval membership degrees and corresponding interval non-membership degrees on entropy values, but neglects the influence of interval hesitancy degrees. In addition, from the eleventh and twelfth columns, we know that applying the entropy formulas E_{ZX1} and E_{ZX2} to the IVIFSs $\tilde{I}_1$, $\tilde{I}_2$ and $\tilde{I}_6$, which have the same distance from the IVIFS $\{\langle x_i, [0.5, 0.5], [0.5, 0.5]\rangle \mid x_i \in X\}$, generates the same result, and the calculated entropy values of $\tilde{I}_3$ and $\tilde{I}_5$ are 0 and 1, respectively. These are irrational since the three IVIFSs $\tilde{I}_1$, $\tilde{I}_2$ and $\tilde{I}_6$ have different interval hesitancy degrees or different absolute deviations of interval membership degrees and interval non-membership degrees. The IVIFS $\tilde{I}_5$ contains more determinate information than $\tilde{I}_3$ does because from the perspective of a voting, we know absolutely nothing about the vote for a candidate from the IVIFS $\tilde{I}_3$ while according to the IVIFS $\tilde{I}_5$, we have a knowledge that the approval and rejection percentages are both 0.5. These also illustrate that there exist some shortcomings in Definition 5. Finally, we obtain the same result when applying the entropy formula E_{ZM} to the IVIFSs $\tilde{I}_2$ and $\tilde{I}_6$ which have the same interval hesitancy degree but different absolute deviations of interval membership degrees and interval non-membership degrees. The main reason is that Definition 6 only captures the intuitionism aspect of uncertainty of IVIFSs but ignores the fuzziness aspect of uncertainty. From the seventh column, it can be seen that the entropy formula E_G is more appropriate although there are some drawbacks in the axiomatic definition proposed by Guo and Song (2014) (e.g., the axiomatic requirement (E_L4)).

Below we apply the developed entropy formulas of IVIFSs shown in Eqs. (18)–(23) to the above six IVIFSs, and list the obtained entropy values in Table 2.

From Table 2, it can be clearly seen that no matter which entropy formula of IVIFSs we adopt, the ranking orders of obtained entropy values of six IVIFSs are the same as $E_i(\tilde{I}_3) > E_i(\tilde{I}_4) > E_i(\tilde{I}_6) > E_i(\tilde{I}_2) > E_i(\tilde{I}_1) > E_i(\tilde{I}_5)$ for $i = 5, 6, \dots, 10$. It can be also noted that for two IVIFSs, when their interval hesitancy degrees are the same, the closer the interval membership degrees of one IVIFS are to the corresponding interval non-membership degrees, the larger its entropy value is, e.g., $E_i(\tilde{I}_6) > E_i(\tilde{I}_2)$ for $i = 5, 6, \dots, 10$; when the distances between interval membership degrees and the corresponding interval non-membership degrees are the same, the larger the interval hesitancy degrees of one IVIFS are, the larger its entropy value is, e.g., $E_i(\tilde{I}_3) > E_i(\tilde{I}_4) > E_i(\tilde{I}_5)$

Table 2
Results derived from new entropy formulas of IVIFSs.

	E_5	E_6	E_7	E_8	E_9	E_{10}
$\tilde{I}_1$	0.725	0.5063	0.56	0.5012	0.5342	0.5435
$\tilde{I}_2$	0.805	0.5438	0.58	0.5025	0.6042	0.6038
$\tilde{I}_3$	1	1	1	1	1	1
$\tilde{I}_4$	0.875	0.75	0.625	0.5062	0.8333	0.7925
$\tilde{I}_5$	0.5	0.5	0.5	0.5	0.5	0.5
$\tilde{I}_6$	0.835	0.6888	0.61	0.5049	0.7574	0.7298

and $E_i(\tilde{I}_2) > E_i(\tilde{I}_1)$ for $i = 5, 6, \ldots, 10$. These results are more rational and accord with human's intuition. All of these imply that the developed entropy formulas of IVIFSs perform well in discriminating IVIFSs and do not yield inconsistent orderings even though the counterintuitive cases of current entropy formulas of IVIFSs are taken into account.

5. Entropy-Based Interval-Valued Intuitionistic Fuzzy Multi-Attribute Decision Making Method

As the IVIFS is a very useful tool to model the uncertainty of objective things, it has been widely applied in dealing with the multi-attribute decision making (MADM) problems with imprecise, vague or uncertain information (Wan and Dong, 2015). In recent years, interval-valued intuitionistic fuzzy MADM has received great attention and many useful and valuable decision making approaches have been put forward, which can be mainly classified into the following six categories: (1) the ideal solution-based approach (Park *et al.*, 2011; Chen, 2014; Zhang and Xu, 2015; Tong and Yu, 2015), (2) the aggregation operator-based approach (Liu, 2014; Wu and Su, 2015), (3) the outranking-based approach (Chen, 2015; Hashemi *et al.*, 2016), (4) the interactive approach (Xu, 2012; Xu and Xia, 2012), (5) the psychological behavior-based approach (Wu and Chiclana, 2014; Meng *et al.*, 2015), and (6) others (Zavadskas *et al.*, 2015; Abdullah and Najib, 2016). In the development of these approaches, the entropy for IVIFSs plays an important role which has been successfully used to determine the weights of attributes (Xu and Zhao, 2016). For example, according to the principle that the smaller the entropy value of an attribute across alternatives, the bigger the weight should be assigned to the attribute, and otherwise the smaller the weight should be assigned to the attribute, Ye (2010), Zhang *et al.* (2013), Jin *et al.* (2014), and Xu and Shen (2014) established the entropy weight models to ascertain the weights of attributes when the weight information of attributes is completely unknown. In these models, different entropy measures for IVIFSs are utilized which can generate different ranking outcomes of attribute weights (Chen and Li, 2010). When the weight information of attributes is partially known, Chen et al. (2013a, 2013b), Jin *et al.* (2014) developed single-objective programming models to determine the optimal weights of attributes with the principle of minimizing the entropy values of the interval-valued intuitionistic fuzzy assessment information of all alternatives under all attributes. Besides, Meng and Chen (2016) developed several Shapley-weighted similarity measures for IVIFSs by using fuzzy measures, and established the models for the optimal fuzzy measure on the attribute set with the help of entropy measures for IVIFSs when the weight information of attributes is completely unknown or partially known.

It is worthwhile to mention that in the above studies, the entropy for IVIFSs is just applied to deal with the interval-valued intuitionistic fuzzy MADM problems with completely unknown or partially known weight information on attributes, and for those with completely known attribute weights, the entropy for IVIFSs is unhelpful. In this section, we shall investigate the applications of the entropy for IVIFSs in solving the interval-valued intuitionistic fuzzy MADM problems with completely known attribute weights.

5.1. *The Proposed Method*

The investment decision making problem involved in this section can be depicted as follows: suppose that $G = \{G_1, G_2, \ldots, G_m\}$ is a set of investment alternatives and $C = \{c_1, c_2, \ldots, c_n\}$ is a set of attributes with the weight vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$ for $j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$. Each investment alternative is assessed under each attribute and the assessment is expressed by an IVIFV, representing the assessment as to what interval degree an investment alternative is and is not an excellent investment as per an attribute. The objective of the decision making problem is to select the most desirable investment alternative from G or to get a ranking order of all investment alternatives.

In what follows, we intend to develop a method for solving the investment decision making problem based on the proposed entropy measures of IVIFSs. In the first place, it is worth pointing out that the information contained in a set includes two parts, one is the uncertainty information, and the other is the determinate information. Without loss of generality, suppose that the total amount of information contained in the IVIFS $\tilde{I}$ is 1, then the amount of determinate information in $\tilde{I}$ is $1 - E(\tilde{I})$. As mentioned earlier, all proposed entropy measures of IVIFSs fulfill the desirable property: $E(\tilde{I}) = \frac{1}{n}\sum_{i=1}^{n} E(I_i)$, where for $i = 1, 2, \ldots, n$, $I_i = \{\langle x_i, [M_{\tilde{I}}^L(x_i), M_{\tilde{I}}^U(x_i)], [N_{\tilde{I}}^L(x_i), N_{\tilde{I}}^U(x_i)]\rangle\}$, thus the amount of determinate information contained in the IVIFS $\tilde{I}$ can be calculated by the equation $D(\tilde{I}) = 1 - \frac{1}{n}\sum_{i=1}^{n} E(I_i)$. Usually, the weight of each element $x_i \in X$ should be taken into consideration. Hence, the weighted form of $D(\tilde{I})$ is $D^w(\tilde{I}) = 1 - \sum_{i=1}^{n} w_i E(I_i)$, where w_i is the weight of the x_i with $w_i \in [0, 1]$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$. Then by Eqs. (18)–(23), we can get the corresponding weighted determinate measures of the IVIFS $\tilde{I}$ used to measure the amount of determinate information in the IVIFS $\tilde{I}$ in terms of the weight information of the elements in the finite universe X. Here we take Eq. (19) for an example. By Eq. (19), the corresponding weighted determinate measure of the IVIFS $\tilde{I}$ is

$$D_6^w(\tilde{I}) = 1 - \sum_{i=1}^{n} w_i \frac{(2 - |M_{\tilde{I}}^L(x_i) - N_{\tilde{I}}^L(x_i)| - |M_{\tilde{I}}^U(x_i) - N_{\tilde{I}}^U(x_i)|)(2 + \pi_{\tilde{I}}^L(x_i) + \pi_{\tilde{I}}^U(x_i))}{8}.$$

In a similar manner, we can obtain other weighted determinate measures D_k^w $(k = 5, 7, \dots, 10)$.

Then, the procedures to deal with the aforementioned MADM problem with interval-valued intuitionistic fuzzy information are listed as follows:

Step 1: The decision maker provides the assessments on the set of investment alternatives G under the set of attributes C in the form of the interval-valued intuitionistic fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{m\times n} = (\langle [M_{ij}^L, M_{ij}^U], [N_{ij}^L, N_{ij}^U] \rangle)_{m\times n}$, where $[M_{ij}^L, M_{ij}^U], [N_{ij}^L, N_{ij}^U] \in int(0,1)$ denote to what interval degree the alternative G_i satisfies and does not satisfy the "excellence" requirement as per attribute c_j, respectively, under the condition $M_{ij}^U + N_{ij}^U \leqslant 1$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

Step 2: By Definition 2, we know that all the assessments of each investment alternative can be regarded as an IVIFS over the set of attributes. Thus, the amount of determinate information of the investment alternative G_i can be measured by

$$D^w(G_i) = 1 - \sum_{j=1}^{n} w_j E(\tilde{r}_{ij}). \tag{24}$$

Step 3: Rank the investment alternatives G_i $(i = 1, 2, \dots, m)$ in terms of the values of $D^w(G_i)$ $(i = 1, 2, \dots, m)$, and the larger the value of $D^w(G_i)$, the better the investment alternative G_i.

5.2. *Numerical Example*

Suppose that there is a panel with four possible alternatives for investment: (1) G_1 is a car company; (2) G_2 is a food company; (3) G_3 is a computer company; (4) G_4 is an arms company. The investment company must take a decision in light of the following three attributes: c_1 (risk), c_2 (growth), c_3 (environmental impacts). Assume that the weights of c_1, c_2 and c_3 are 0.35, 0.25 and 0.4, respectively, and the four investment alternatives are evaluated by the decision maker under the above three attributes using the interval-valued intuitionistic fuzzy information listed in the following matrix:

$$R = \begin{pmatrix} ([0.4, 0.5], [0.3, 0.4]) & ([0.4, 0.6], [0.2, 0.4]) & ([0.1, 0.3], [0.5, 0.6]) \\ ([0.6, 0.7], [0.2, 0.3]) & ([0.6, 0.7], [0.2, 0.3]) & ([0.4, 0.8], [0.1, 0.2]) \\ ([0.3, 0.6], [0.3, 0.4]) & ([0.5, 0.6], [0.3, 0.4]) & ([0.4, 0.5], [0.1, 0.3]) \\ ([0.7, 0.8], [0.1, 0.2]) & ([0.6, 0.7], [0.1, 0.3]) & ([0.3, 0.4], [0.1, 0.2]) \end{pmatrix}.$$

As stated above, each row of the decision making matrix corresponding to an investment alternative is deemed as an IVIFS over the universe consisting of three attributes $\{c_j \mid j = 1, 2, 3\}$, and the more the determinate information an investment alternative possesses as to the three attributes, the better the investment alternative is. Therefore, all investment alternatives can be ranked according to the determinate measure values of the corresponding IVIFSs computed by Eq. (24). Table 3 lists the obtained results by different weighted determinate measures of IVIFSs.

Table 3
Decision results derived from different weighted determinate measures of IVIFSs.

	D_5^w	D_6^w	D_7^w	D_8^w	D_9^w	D_{10}^w
G_1	0.3475	0.5485	0.496	0.5056	0.5094	0.5056
G_2	0.464	0.6833	0.562	0.6856	0.6759	0.6533
G_3	0.332	0.5135	0.476	0.4696	0.4795	0.4763
G_4	0.3853	0.6246	0.5355	0.5892	0.5851	0.5751
Rankings:			$G_2 \succ G_4 \succ G_1 \succ G_3$			

From Table 3, we can see that no matter which weighted determinate measure D_5^w–D_{10}^w of IVIFSs induced by the corresponding entropy measure E_5–E_{10} is utilized, the priority ranking of the four alternatives is the same as $G_2 \succ G_4 \succ G_1 \succ G_3$. Therefore, the best alternative for investment is G_2, which is in agreement with the result acquired by Nayagam *et al.* (2011).

The developed approach provides a simple way to address the MADM problems with interval-valued intuitionistic fuzzy information, which only needs to compute the determinate measure value of each alternative to make a decision. The larger the determinate measure value of an alternative is, the better the alternative is. However, Xu (2007) capitalized the interval-valued intuitionistic fuzzy weighted arithmetic operator or the interval-valued intuitionistic fuzzy weighted geometric operator to aggregate the assessments of each alternative and then ranked the alternatives based on the score and accuracy functions. Nayagam *et al.* (2011) ranked the alternatives by means of a novel accuracy function. It should be pointed out that the weighted aggregation process of all attribute values in the forms of IVIFVs is very complicated and the two methods do not always give a full ranking for the alternatives since the comparison mechanisms only employ the score and accuracy functions, which leads to the alternatives with equal scores and accuracy degrees incomparable. Although Zhang *et al.* (2014) proposed a decision making method on the basis of entropy measures of IVIFSs, yet it does not take account of the weight information of attributes and just believes the attributes have the same importance. To solve the interval-valued intuitionistic fuzzy MADM problems with incomplete attribute weight information, Wang *et al.* (2009) developed a linear programming model for determining the weights of attributes and ranked all alternatives by comparing their weighted aggregation values with the help of score and accuracy functions, the membership uncertainty index and the hesitation uncertainty index, which makes the decision making process complex and time-consuming. Li (2011) presented a closeness coefficient based nonlinear programming method. Nevertheless, in his methodology, nonlinear programming models are constructed on the concept of the closeness coefficient to calculate closeness IFSs of alternatives to the interval-valued intuitionistic fuzzy positive ideal solution, which are utilized to evaluate the optimal degrees of membership and hereby generate the ranking of alternatives. Obviously, the decision making process in Li (2011) method is also complicated. In short, our proposed method is much simpler and easier to use and understand, which corresponds to the decision making reality of the decision makers whose knowledge and ability are limited.

6. Conclusion

At present, a variety of axiomatic definitions and entropy formulas have been presented to reflect the uncertainty of interval-valued intuitionistic fuzzy sets (IVIFSs). This paper has first reviewed the existing axiomatic definitions and entropy formulas of IVIFSs and demonstrated that they are unable to simultaneously capture the fuzziness and intuitionism of IVIFSs. Therefore, this paper has then put forward a novel axiomatic definition of entropy for IVIFSs, which takes the fuzziness and intuitionism of IVIFSs into full consideration, and developed some methods whereby different forms of entropy formulas of IVIFSs can be constructed by using some very simple functions. The superiority of the constructed entropy formulas of IVIFSs over other formulas has been illustrated by a comparative analysis. Finally, based on the developed entropy formulas of IVIFSs, we have proposed an entropy-based decision making method to deal with the multi-attribute decision making problems with interval-valued intuitionistic fuzzy information. Our proposed method involves few steps and is easy to use and understand, which matches the decision making reality of the decision makers with limited knowledge and ability.

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Intervalais vertinamos intuityviosios neraiškiosios informacijos entropijos matavimų lyginamoji perspektyva ir jos taikymas sprendimams priimti

Na ZHAO, Zeshui XU

Šiame straipsnyje trumpai apžvelgiama kai kurios entropijos apibrėžimų aksiomos ir formulės, skirtos intervalais vertinamoms intuityviosioms neapibrėžtosioms aibėms (IVIFSs), ir parodoma, kad jos nesugeba pilnai išreikšti IVIFSs neapibrėžtumo. Atsižvelgiant tiek į neraiškumus, tiek ir į IVIFSs intuityvumą, pristatoma novatoriška aksioma IVIFSs entropijai apibrėžti. Pateikta keletas paprastų būdų suformuluoti IVIFSs entropiją, ir tai iliustruota keliomis konkrečiomis entropijos formulėmis. Pavyzdžio analizė parodė, kad naujai pateiktos IVIFSs entropijos formulės gali pilnai atspindėti tiek neaiškumus, tiek ir IVIFSs intuityvumą. IVIFSs entropijos formulių pagrindu yra pateiktas paprastas metodas daugiarodikliams sprendimų priėmimo uždaviniams spręsti, kuriuose informacija vertinama intervalinėmis intuityviosiomis neraiškiosiomis vertėmis. Investicijų alternatyvų atrankos pavyzdys yra pateiktas patvirtinti siūlomo metodo praktiškumą ir veiksmingumą.