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Some Single Valued Neutrosophic Number Heronian Mean Operators and Their Application in Multiple Attribute Group Decision Making

Yanhua LI[1], Peide LIU[1,2*], Yubao CHEN[1]
[1]*School of Economics and Management, Civil Aviation University of China Tianjin 300300, China*
[2]*School of Management Science and Engineering, Shandong University of Finance and Economics Jinan Shandong 250014, China*
e-mail: Peide.liu@gmail.com

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Abstract. Heronian mean (HM) has the characteristic of capturing the correlations of the aggregated arguments and the neutrosophic set can express the incomplete, indeterminate and inconsistent information, in this paper, we applied the Heronian mean to the neutrosophic set, and proposed some Heronian mean operators. Firstly, we presented some operational laws and their properties of single valued neutrosophic numbers (SVNNs), and analyzed the shortcomings of the existing weighted HM operators which have not idempotency, then we propose the improved generalized weighted Heronian mean (IGWHM) operator and improved generalized weighted geometric Heronian mean (IGWGHM) operator based on crisp numbers, and prove that they can satisfy some desirable properties, such as reducibility, idempotency, monotonicity and boundedness Further, we proposed the single valued neutrosophic number improved generalized weighted Heronian mean (NNIGWHM) operator and single valued the neutrosophic number improved generalized weighted geometric Heronian mean (NNIGWGHM) operator, and some desirable properties and special cases of them are discussed. Moreover, with respect to multiple attribute group decision making (MAGDM) problems in which attribute values take the form of SVNNs, the decision making approaches based on the proposed operators are developed. Finally, an application example has been given to show the decision making steps and to discuss the influence of different parameter values on the decision-making results.

Key words: multiple attribute group decision making (MAGDM), neutrosophic set, Heronian mean, geometric Heronian mean, the generalized Heronian mean (GHM) operator.

1. Introduction

Multiple attribute decision group making (MAGDM) problems widely exist in the fields of management, economy, military and engineering techniques. Because of the complexity of object things and fuzziness of human thinking, the attribute values involved in the decision problems are often incomplete, indeterminate and inconsistent. With respect

*Corresponding author.

to the fuzzy information, Zadeh (1965) firstly proposed the fuzzy set theory to process this kind of information. On the basis of fuzzy set theory, Atanassov (1986, 1989) proposed the intuitionistic fuzzy set (IFS) by adding a non-membership function to overcome the shortcoming in which fuzzy set only has a membership function. The intuitionistic fuzzy set is composed of the membership (or called truth-membership) $T_A(x)$ and non-membership (or called falsity-membership) $F_A(x)$, and satisfies the conditions $T_A(x),\ F_A(x) \in [0, 1]$ and $0 \leqslant T_A(x) + F_A(x) \leqslant 1$. However, IFSs can only handle incomplete information not the indeterminate information and inconsistent information. In IFSs, the indeterminacy (or called Hesitation degree) is $1 - T_A(x) - F_A(x)$ by default. Further, Smarandache (1999) proposed the neutrosophic set (NS) by adding an independent indeterminacy-membership, i.e., NS is composed of the truth-membership $T_A(x)$, falsity-membership $F_A(x)$ and indeterminacy-membership $I_A(x)$. Obviously, NS is a generalization of FS and IFSs. In NS, the indeterminacy is quantified explicitly, and truth-membership, indeterminacy membership, and false-membership are completely independent. Recently, NSs have attracted the wide concerns. Wang *et al.* (2005b) further proposed a single valued neutrosophic set (SVNS) by changing to the conditions $T_A(x)$,$I_A(x)$, $F_A(x) \in [0, 1]$, and $0 \leqslant T_A(x) + I_A(x) + F_A(x) \leqslant 3$. Obviously, the SVNS is an instance of the neutrosophic set, and SVNSs can easier apply in scientific and engineering problems than NSs since neutrosophic components $T,\ I,\ F$ in NSs are nonstandard interval $]0, 1[$ and these components $T,\ I,\ F$ in SVNSs are standard interval $[0, 1]$. Similar to extension from IFS to interval-valued intuitionistic fuzzy set (IVIFS) (Atanassov, 1994; Atanassov and Gargov, 1989), Wang *et al.* (2005a) gave the definition of the interval neutrosophic sets (INSs) in which the truth-membership, indeterminacy-membership, and false-membership were extended to interval numbers, and various properties of INSs were discussed. Ye (2014a) defined the similarity measures between INSs on the basis of the Hamming and Euclidean distances, and based on the similarity measures, a multi-criteria decision-making method was proposed.

The information aggregation operators are an interesting research topic, and have been widely applied in MAGDM problems (Liu, 2013, 2014; Liu and Jin, 2012; Liu and Wang, 2014; Liu and Yu, 2014; Liu *et al.*, 2014). In general, they are divided into two types, i.e., arithmetic aggregation operators and geometric aggregation operators. Xu (2007), Xu and Yager (2006) proposed some arithmetic aggregation operators and geometric aggregation operators for intuitionistic fuzzy information; however, these operators cannot consider the correlations of the aggregated arguments. Heronian mean (HM) operator is an important aggregation operator which has the characteristic of capturing the correlations of the aggregated arguments. Beliakov *et al.* (2007) had firstly proved that Heronian mean was an aggregation operator, but he did not do further researches. Sykora (2009a, 2009b) further extended to the generalized Heronian means, and discussed two special cases of them. Yu and Wu (2012) extended Heronian mean, which can only deal with crisp numbers, to process intuitionistic fuzzy numbers, and proposed a generalized interval-valued intuitionistic fuzzy Heronian mean (GIIFHM) and a generalized interval-valued intuitionistic fuzzy weighted Heronian mean (GIIFWHM). However, the GIIFWHM has not idempotency and reducibility which seem to be counterintuitive. Liu and Pei (2012) extended

HM to the generalized Heronian OWA operator, which were similar to Bonferroni mean operator and BON-OWA operator which are originally proposed by Bonferroni (1950) and Yager (2009). Yu (2013) proposed some intuitionistic fuzzy aggregation operators based on HM, including the intuitionistic fuzzy geometric Heronian mean (IFGHM) operator and the intuitionistic fuzzy geometric weighed Heronian mean (IFGWHM) operator. Similarly, IFGWHM operator has also not reducibility and idempotency.

As mentioned above, in the real decision making problems, the interactions phenomena among the attribute values commonly exists. Because Heronian mean operator can deal with the interactions among the attribute values and the SVNNs can easier express the incomplete, indeterminate and inconsistent information. Therefore, in this paper, we will extend the Heronian mean to SVNNs, and propose some Heronian mean operators for SVNNs, including the improved generalized weighted Heronian mean (IGWHM) operator and generalized weighted geometric Heronian mean (IGWGHM) operator which can satisfy some desirable properties, such as reducibility, idempotency, monotonicity and boundedness, then applies them to multi-attribute group decision-making problems.

To do this, the structure of this paper is shown as follows. In Section 2, we briefly review some basic concepts and operational rules of SVNNs, and on the basis of analyzing the shortcoming of the generalized weighted Heronian mean (GWHM) operator and the generalized weighted geometric Heronian mean (GWGHM) operator, we propose the improved generalized weighted Heronian mean (IGWHM) operator and the improved generalized weighted geometric Heronian mean (IGWGHM) operator. Section 3 will extend IGWHM and IGWGHM operators to SVNNS, and proposes the neutrosophic number improved generalized weighted Heronian mean (NNIGWHM) operator and the neutrosophic number improved generalized weighted geometric Heronian mean (NNIGWGHM) operator. In Section 4, we develop the decision making methods for multi-criteria group decision making based on the proposed operators. Section 5 gives an example to illustrate the decision steps and discusses the influence of different parameters in these operators on the decision-making results. In Section 6, we give the conclusions and future research directions.

2. Preliminaries

2.1. *The Single Valued Neutrosophic Set*

Definition 1. (See Wang *et al.*, 2005b.) Let X be a universe of discourse, with a generic element in X denoted by x. A single valued neutrosophic set$A$ in$X$ is

$$A = \{x(T_A(x), I_A(x), F_A(x)) | x \in X\} \tag{1}$$

where, T_A is the truth-membership function, I_A is the indeterminacy-membership function, and F_A is the falsity-membership function. For each point x in X, we have $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$, and $0 \leqslant T_A(x) + I_A(x) + F_A(x) \leqslant 3$.

For convenience, we can simply use $x = (T_x, I_x, F_x)$ to represent an element x in SVNS, and the element x can be called a single valued neutrosophic number (SVNN).

In order to compare two SVNNs, Smarandache and Vladareanu (2011) gave the definition of the partial order relationship on the neutrosophic numbers shown as follows.

Definition 2. (See Smarandache and Vladareanu, 2011.) Suppose $x = (T_1, I_1, F_1)$ and $y = (T_2, I_2, F_2)$ are two SVNNs, iff (if and only if) $T_1 \leqslant T_2$, $I_1 \geqslant I_2$, $F_1 \geqslant F_2$ then $x \leqslant y$.

Obviously, in real applications, it is very difficult to meet the above conditions for many cases. With respect to these, Ye (2014b) proposed a comparison method based on the cosine similarity measure for a SVNN $x = (T, I, F)$ to ideal solution $(1, 0, 0)$, and gave the definition of the cosine similarity $S(x) = \frac{T}{\sqrt{T^2+I^2+F^2}}$.

Definition 3. (See Ye, 2014b.) Suppose $x = (T_1, I_1, F_1)$ and $y = (T_2, I_2, F_2)$ are two SVNNs, if $S(x) \leqslant S(y)$, then $x \leqslant y$.

Definition 4. Let $x = (T_1, I_1, F_1)$ and $y = (T_2, I_2, F_2)$ be two SVNNs, then the operational laws are defined as follows.

(1) The complement of x is $\overline{x} = (F_1, 1 - I_1, T_1)$, (2)

(2) $x \oplus y = (T_1 + T_2 - T_1T_2, I_1I_2, F_1F_2)$, (3)

(3) $x \otimes y = (T_1T_2, I_1 + I_2 - I_1I_2, F_1 + F_2 - F_1F_2)$, (4)

(4) $nx = \big(1 - (1 - T_1)^n, (I_1)^n, (F_1)^n\big), \quad n > 0$, (5)

(5) $x^n = \big((T_1)^n, 1 - (1 - I_1)^n, 1 - (1 - F_1)^n\big), \quad n > 0$. (6)

Theorem 1. *Let $x = (T_1, I_1, F_1)$ and $y = (T_2, I_2, F_2)$ be two SVNNs, and $\eta, \eta_1, \eta_2 > 0$, then we have*

(1) $x \oplus y = y \oplus x$, (7)

(2) $x \otimes y = y \otimes x$, (8)

(3) $\eta(x \oplus y) = \eta x \oplus \eta y$, (9)

(4) $\eta_1 x \oplus \eta_2 x = (\eta_1 + \eta_2)x$, (10)

(5) $x^\eta \otimes y^\eta = (x \otimes y)^\eta$, (11)

(6) $x^{\eta_1} \otimes x^{\eta_2} = x^{\eta_1+\eta_2}$. (12)

2.2. *Heronian Mean (HM) Operator*

Heronian mean (HM) operator, which can capture the interrelationship of the individual arguments, was defined as follows (Liu and Pei, 2012; Sykora, 2009a).

Definition 5. (See Liu and Pei, 2012; Sykora, 2009a.) A HM operator of dimension n is a mapping $HM : I^n \to I$. Such that,

$$HM(x_1, x_2, \dots, x_n) = \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} \sqrt{x_i x_j}, \tag{13}$$

where $I = [0, 1]$. Then the function HM is called Heronian mean (HM) operator.

Definition 6. (See Liu and Pei, 2012; Sykora, 2009a.) A GHM operator of dimension n is a mapping $GHM : I^n \to I$. Such that,

$$GHM(x_1, x_2, \dots, x_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} x_i^p x_j^q \right)^{\frac{1}{p+q}}, \tag{14}$$

where $p, q \geqslant 0$ and $I = [0, 1]$. Then the function $GHM^{p,q}$ is called generalized Heronian mean (GHM) operator.

It is easy to prove that the GHM operator has the following properties (Liu and Pei, 2012).

Theorem 2 (Idempotency). *Let $x_j = x$ for all $j = 1, 2, \dots, n$, then $GHM^{p,q}(x_1, x_2, \dots, x_n) = x$.*

Theorem 3 (Monotonicity). *Let $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ be two collections of the nonnegative numbers, if $x_j \leqslant y_j$ for all $j = 1, 2, \dots, n$, then $GHM^{p,q}(x_1, x_2, \dots, x_n) \leqslant GHM^{p,q}(y_1, y_2, \dots, y_n)$.*

Theorem 4 (Bounded). *GHM operator lies between the max and min operators, i.e.*

$$MIN(x_1, x_2, \dots, x_n) \leqslant GHM^{p,q}(x_1, x_2, \dots, x_n) \leqslant MAX(x_1, x_2, \dots, x_n).$$

Since the HM and GHM operators only consider the interrelationship of the input arguments and don't take their own weights into account. In the following, we will introduce another Heronian mean operator which is called the weighted generalized Heronian mean (WGHM) operator to overcome this shortcoming.

Yu and Wu (2012) proposed the generalized weighted Heronian mean (GWHM) operator shown as follows.

Definition 7. (See Yu and Wu, 2012.) Let $p, q \geqslant 0$, and x_i $(i = 1, 2, \dots, n)$ be a collection of nonnegative numbers. $W = (w_1, w_2, \dots, w_n)^T$ is the weight vector of x_i $(i = 1, 2, \dots, n)$, and satisfies $w_i \geqslant 0$, $\sum_{i=1}^{n} w_i = 1$. If

$$GWHM^{p,q}(x_1, x_2, \dots, x_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} (w_i x_i)^p \big(w_j x_j^q \big) \right)^{\frac{1}{p+q}} \tag{15}$$

then $GWHM^{p,q}$ is called a generalized weighted Heronian mean (GWHM) operator.

Obviously, $GWHM^{p,q}$ operator has not the idempotency. It seems to be counterintuitive. Liu (2012) propose an improved generalized weighted Heronian mean (IGWHM) operator to overcome this drawback.

Definition 8. (See Liu, 2012.) Let $p, q \geqslant 0$, and x_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. $W = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of x_i $(i = 1, 2, \ldots, n)$, and satisfies $w_i > 0$, $\sum_{i=1}^{n} w_i = 1$. If

$$IGWHM^{p,q}(x_1, x_2, \ldots, x_n) = \frac{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j x_i^p x_j^q\right)^{\frac{1}{p+q}}}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j\right)^{\frac{1}{p+q}}} \tag{16}$$

then $IGWHM^{p,q}$ is called the improved generalized weighted Heronian mean (IGWHM) operator.

The *IGWHM* operator has the properties, such as idempotency, monotonicity and boundedness (Liu, 2012).

Theorem 5 (Idempotency). *Let $x_j = x$, $j = 1, 2, \ldots, n$ then*

$$IGWHM^{p,q}(x_1, x_2, \ldots, x_n) = x. \tag{17}$$

Theorem 6 (Monotonicity). *Let x_i $(i = 1, 2, \ldots, n)$ and y_i $(i = 1, 2, \ldots, n)$ be two collections of nonnegative numbers. If $x_i \geqslant y_i$ for all i, then*

$$IGWHM^{p,q}(x_1, x_2, \ldots, x_n) \geqslant IGWHM^{p,q}(y_1, y_2, \ldots, y_n). \tag{18}$$

Theorem 7 (Boundedness). *The $IGWHM^{p,q}$ operator lies between the max and min operators, i.e.,*

$$\min(x_1, x_2, \ldots, x_n) \leqslant IGWHM^{p,q}(x_1, x_2, \ldots, x_n) \leqslant \max(x_1, x_2, \ldots, x_n). \tag{19}$$

In the following, we can analyze some special cases of the IGWHM operator.

(1) When $q = 0$, then

$$IGWHM^{p,0}(x_1, x_2, \ldots, x_n) = \frac{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j x_i^p\right)^{\frac{1}{p}}}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j\right)^{\frac{1}{p}}}. \tag{20}$$

Further, when $p = 1$, there is

$$IGWHM^{1,0}(x_1, x_2, \ldots, x_n) = \frac{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j x_i}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}. \tag{21}$$

(2) When $p = 0$, then

$$IGWHM^{0,q}(x_1, x_2, \dots, x_n) = \frac{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j x_j^q\right)^{\frac{1}{q}}}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j\right)^{\frac{1}{q}}}. \tag{22}$$

From here, we see that the parameters p and q don't have the interchangeability.

(3) When $p = q = 1$, then

$$IGWHM^{1,1}(x_1, x_2, \dots, x_n) = \frac{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j x_i x_j\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j\right)^{\frac{1}{2}}}. \tag{23}$$

2.3. *The Geometric Heronian Mean (GHM) Operator*

Based on HM and GHM operators, Yu (2013) propose the generalized geometric Heronian mean (GGHM) operator shown as follows.

Definition 9. (See Yu, 2013.) Let $p, q \geqslant 0$, and x_i $(i = 1, 2, \dots, n)$ be a collection of nonnegative numbers. If

$$GGHM^{p,q}(x_1, x_2, \dots, x_n) = \frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}(px_i + qx_j)^{\frac{2}{n(n+1)}} \tag{24}$$

then $GGHM^{p,q}$ is called the generalized geometric Heronian mean (GGHM) operator.

Similar to GHM operator, the GGHM operator also only takes the correlations of the aggregated arguments into account and ignores their own weights. Yu (2013) further proposed the generalized geometric weighted Heronian mean (GGWHM) operator.

Definition 10. (See Yu, 2013.) Let $p, q \geqslant 0$, and x_i $(i = 1, 2, \dots, n)$ be a collection of nonnegative numbers. $W = (w_1, w_2, \dots, w_n)^T$ is the weight vector of x_i $(i = 1, 2, \dots, n)$ and satisfies $w_i \geqslant 0$, $\sum_{i=1}^{n} w_i = 1$. If

$$GGWHM^{p,q}(x_1, x_2, \dots, x_n) = \frac{1}{p+q}\prod_{i=1}^{n}\prod_{j=i}^{n}\left((px_i)^{w_i} + (qx_j)^{w_j}\right)^{\frac{2}{n(n+1)}} \tag{25}$$

then $GGWHM^{p,q}$ is called the generalized geometric weighted Heronian mean (GGWHM) operator.

Similarly, the GGWHM operator has not the reducibility and idempotency, and it seem to be counterintuitive. Further, Liu (2012) proposed the improved generalized geometric weighted Heronian mean (IGGWHM) operator.

Definition 11. (See Liu, 2012.) Let $p, q \geqslant 0$, and x_i $(i = 1, 2, \dots, n)$ be a collection of nonnegative numbers. $W = (w_1, w_2, \dots, w_n)^T$ is the weight vector of and satisfies

$w_i \geqslant 0$, $\sum_{i=1}^{n} w_i = 1$. If

$$IGGWHM^{p,q}(x_1, x_2, \ldots, x_n) = \frac{1}{p+q} \prod_{i=1}^{n} \prod_{j=i}^{n} (px_i + qx_j)^{\frac{2(n+1-i)}{n(n+1)} \frac{w_j}{\sum_{k=i}^{n} w_k}} \tag{26}$$

then $IGGWHM^{p,q}$ is called the improved generalized geometric weighted Heronian mean (IGGWHM) operator.

The IGGWHM has the properties, such as reducibility, idempotency, monotonicity and boundedness (Liu, 2012).

Theorem 8 (Reducibility). *Let* $W = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ *then*

$$IGGWHM^{p,q}(x_1, x_2, \ldots, x_n) = GGHM^{p,q}(x_1, x_2, \ldots, x_n). \tag{27}$$

Theorem 9 (Idempotency). *Let* $x_j = x$, $j = 1, 2, \ldots, n$, *then*

$$IGGWHM^{p,q}(x_1, x_2, \ldots, x_n) = x. \tag{28}$$

Theorem 10 (Monotonicity). *Let* x_i $(i = 1, 2, \ldots, n)$ *and* y_i $(i = 1, 2, \ldots, n)$ *be two collections of nonnegative numbers. If* $x_i \geqslant y_i$ *for all* i, *then*

$$IGGWHM^{p,q}(x_1, x_2, \ldots, x_n) \geqslant IGGWHM^{p,q}(y_1, y_2, \ldots, y_n). \tag{29}$$

Theorem 11 (Boundedness). *The* $IGGWHM^{p,q}$ *operator lies between the max and min operators, i.e.,*

$$\min(x_1, x_2, \ldots, x_n) \leqslant IGGWHM^{p,q}(x_1, x_2, \ldots, x_n) \leqslant \max(x_1, x_2, \ldots, x_n). \tag{30}$$

In the following, we can analyze some special cases of the IGGWHM operator.

(1) When $q = 0$, then

$$IGGWHM^{p,0}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} (x_i)^{\frac{2(n+1-i)}{n(n+1)}}. \tag{31}$$

From here, we see that $WGGWHM^{p,0}$ does not have any relationship with p.

(2) When $p = 0$, then

$$IGGWHM^{0,q}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} \prod_{j=i}^{n} (x_j)^{\frac{2(n+1-i)}{n(n+1)} \frac{w_j}{\sum_{k=i}^{n} w_k}}. \tag{32}$$

Similarly, $IGGWHM^{0,q}$ does not have any relationship with q.

(3) When $p = q = 1$, then

$$IGGWHM^{1,1}(x_1, x_2, \dots, x_n) = \frac{1}{2}\prod_{i=1}^{n}\prod_{j=i}^{n}(x_i + x_j)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n} w_k}}. \tag{33}$$

3. Some Heronian Mean Operators Based on the Single Valued Neutrosophic Number

As mentioned above, the IGWHM and IGGWHM operators have better properties than GWHM and GGWHM ones. However, they can only aggregate the input arguments which take the form of crisp numbers, and cannot aggregate the single valued neutrosophic numbers. In this section, we will extend the IGWHM and IGGWHM operators to aggregate the single valued neutrosophic numbers, and propose a single valued neutrosophic number improved generalized weighted Heronian mean (NNIGWHM) operator and a single valued neutrosophic number improved generalized geometric weighted Heronian mean (NNIGGWHM) operator which can be described as follows.

3.1. *The NNIGWHM Operator*

Definition 12. Let $p, q \geqslant 0$, and $\widetilde{a}_j = (T_j, I_j, F_j)$ $(j = 1, 2, \dots, n)$ be a collection of SVNNs with the weight vector $W = (w_1, w_2, \dots, w_n)^T$ such that $w_j \geqslant 0$ and $\sum_{j=1}^{n} w_j = 1$, then a single valued neutrosophic number imprvoed generalized weighted Heronian mean (NNIGWHM) operator of dimension n is a mapping NNIGWHM: $\Omega^n \to \Omega$, and has

$$NNIGWHM^{p,q}(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n) = \left(\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(w_i w_j \widetilde{a}_i^{p} \otimes \widetilde{a}_j^{q}\right)\right)^{\frac{1}{p+q}}, \tag{34}$$

where Ω is the set of all SVNNs.

Based on the operational rules of the SVNNs, we can derive the result shown as Theorem 12.

Theorem 12. *Let $p, q \geqslant 0$, and $\widetilde{a}_j = (T_j, I_j, F_j)$ $(j = 1, 2, \dots, n)$ be a collection of SVNNs with the weight vector $W = (w_1, w_2, \dots, w_n)^T$ such that $w_j \geqslant 0$ and $\sum_{j=1}^{n} w_j = 1$, then, the result aggregated from Definition 12 is still a SVNN, and even*

$$NNIGWHM^{p,q}(\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n) = \left(\left(1 - \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1 - T_i^{p} T_j^{q}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p+q}},$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p(1-I_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}},$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_i)^p(1-F_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}\Bigg). \quad (35)$$

Proof. Since

$$\begin{aligned}
\widetilde{a}_i^p &= \left(T_i^p, 1-(1-I_i)^p, 1-(1-F_i)^p\right),\\
\widetilde{a}_j^q &= \left(T_j^q, 1-(1-I_j)^q, 1-(1-F_j)^q\right),\\
\widetilde{a}_i^p\widetilde{a}_j^q &= \left(T_i^pT_j^q, 1-(1-I_i)^p(1-I_j)^q, 1-(1-F_i)^p(1-F_j)^q\right)
\end{aligned}$$

and

$$\begin{aligned}
w_iw_j\widetilde{a}_i^p\otimes\widetilde{a}_j^q = &\left(1-\left(1-T_i^pT_j^q\right)^{w_iw_j}, \left(1-(1-I_i)^p(1-I_j)^q\right)^{w_iw_j},\right.\\
&\left.\left(1-(1-F_i)^p(1-F_j)^q\right)^{w_iw_j}\right)
\end{aligned}$$

then

$$\begin{aligned}
&\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(w_iw_j\widetilde{a}_i^p\otimes\widetilde{a}_j^q\right)\\
&=\Bigg(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^pT_j^q\right)^{w_iw_j}, \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p(1-I_j)^q\right)^{w_iw_j},\\
&\quad\prod_{i=1}^{n}\prod_{j=i}^{n}1-(1-F_i)^p(1-F_j)^q\Bigg)^{w_iw_j}.
\end{aligned}$$

Further

$$\begin{aligned}
&\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(w_iw_j\widetilde{a}_i^p\otimes\widetilde{a}_j^q\right)\\
&=\Bigg(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^pT_j^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}},\\
&\quad\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p(1-I_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}},\\
&\quad\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_i)^p(1-F_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\Bigg)
\end{aligned}$$

and

$$\left(\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(w_i w_j \widetilde{a}_i^p \otimes \widetilde{a}_j^q\right)\right)^{\frac{1}{p+q}}$$

$$= \left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^p T_j^q\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p+q}},\right.$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p(1-I_j)^q\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p+q}},$$

$$\left.1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_i)^p(1-F_j)^q\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p+q}}\right)$$

so,

$$NNIGWHM^{p,q}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n)$$

$$= \left(\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{n}\left(w_i w_j \widetilde{a}_i^p \otimes \widetilde{a}_j^q\right)\right)^{\frac{1}{p+q}}$$

$$= \left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^p T_j^q\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p+q}},\right.$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p(1-I_j)^q\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p+q}},$$

$$\left.1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_i)^p(1-F_j)^q\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p+q}}\right)$$

which completes the proof of Theorem 12. □

Moreover, the NNIGWHM operator also has the following properties.

Theorem 13 (Idempotency). *Let* $\widetilde{a}_j = (T, I, F)$ $(j = 1, 2, \ldots, n)$*, then*

$$NNIGWHM^{p,q}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) = (T, I, F). \tag{36}$$

Proof. Since $\widetilde{a}_j = (T, I, F)$ $(j = 1, 2, \ldots, n)$, then according to (35), we have

$$NNIGWHM^{p,q}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n)$$

$$= \left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T^p T^q\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p+q}},\right.$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I)^{p}(1-I)^{q}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p+q}},$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F)^{p}(1-F)^{q}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p+q}}\Bigg)$$

$$=\left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T^{p+q}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p+q}},\right.$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I)^{p+q}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p+q}},$$

$$\left.1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F)^{p+q}\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p+q}}\right)$$

$$=\Big(\big(1-\big(\big(1-T^{p+q}\big)^{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\big)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\big)^{\frac{1}{p+q}},$$

$$1-\big(1-\big(\big(1-(1-I)^{p+q}\big)^{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\big)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\big)^{\frac{1}{p+q}},$$

$$1-\big(1-\big(\big(1-(1-F)^{p+q}\big)^{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}\big)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\big)^{\frac{1}{p+q}}\Big)$$

$$=\Big(\big(1-\big(1-T^{p+q}\big)\big)^{\frac{1}{p+q}}, 1-\big(1-\big(1-(1-I)^{p+q}\big)\big)^{\frac{1}{p+q}},$$

$$1-\big(1-(1-(1-F)^{p+q})\big)^{\frac{1}{p+q}}\Big)$$

$$=\Big(\big(T^{p+q}\big)^{\frac{1}{p+q}}, 1-\big((1-I)^{p+q}\big)^{\frac{1}{p+q}}, 1-\big((1-F)^{p+q}\big)^{\frac{1}{p+q}}\Big)$$

$$=(T, I, F)$$

which completes the proof of Theorem 13. □

Theorem 14 (Monotonicity). *Let $\widetilde{a}_j=(T_j, I_j, F_j)$ and $\widetilde{a}'_j=(T'_j, I'_j, F'_j)$ $(j=1,2,\ldots,n)$ be two collections of SVNNs. If $\widetilde{a}_j \geqslant \widetilde{a}'_j$ for all j (suppose $T_j \geqslant T'_j$, $I_j \leqslant I'_j$ and $F_j \leqslant F'_j$), then*

$$NNIGWHM^{p,q}(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n) \geqslant NNIGWHM^{p,q}\big(\widetilde{a}'_1,\widetilde{a}'_2,\ldots,\widetilde{a}'_n\big). \tag{37}$$

Proof. (1) Since $T_j \geqslant T'_j$ for all j, and $p, q > 0$, then we have

$$T_i^p T_j^q \geqslant T_i'^{\,p} T_j'^{\,q}, \qquad 1-T_i^p T_j^q \leqslant 1-T_i'^{\,p} T_j'^{\,q},$$

$$\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^pT_j^q\right)^{w_iw_j} \leqslant \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i'^pT_j'^q\right)^{w_iw_j}$$

and

$$\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^pT_j^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}} \leqslant \left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i'^pT_j'^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}},$$

$$1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^pT_j^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}} \geqslant 1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i'^pT_j'^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}$$

so

$$\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^pT_j^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}} \geqslant \left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i'^pT_j'^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}.$$

(2) Since $I_j \leqslant I_j'$ for all j, and $p, q > 0$, then we have

$$(1-I_i)^p \geqslant \left(1-I_i'\right)^p$$

and

$$(1-I_j)^q \geqslant \left(1-I_j'\right)^q$$

then

$$(1-I_i)^p(1-I_j)^q \geqslant \left(1-I_i'\right)^p\left(1-I_j'\right)^q,$$

$$1-(1-I_i)^p(1-I_j)^q \leqslant 1-\left(1-I_i'\right)^p\left(1-I_j'\right)^q,$$

$$\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p(1-I_j)^q\right)^{w_iw_j} \leqslant \prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-I_i'\right)^p\left(1-I_j'\right)^q\right)^{w_iw_j},$$

$$\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p(1-I_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}$$

$$\leqslant\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-I_i'\right)^p\left(1-I_j'\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}},$$

$$1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p(1-I_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}$$

$$\geqslant 1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-I_i'\right)^p\left(1-I_j'\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}},$$

$$\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p(1-I_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}$$

$$\geqslant\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-I_i'\right)^p\left(1-I_j'\right)^qV\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}},$$

so

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p(1-I_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}$$

$$\leqslant 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-I_i'\right)^p\left(1-I_j'\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}.$$

(3) Similar to (2), we can prove

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_i)^p(1-F_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}$$

$$\leqslant 1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-F_i'\right)^p\left(1-F_j'\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}.$$

According to (1)–(3), we can get

$$\left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^pT_j^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}},\right.$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p(1-I_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}},$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_i)^p(1-F_j)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}\right)$$

$$\geqslant\left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i'^{\,p}T_j'^{\,q}\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}},\right.$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-I_i'\right)^p\left(1-I_j'\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}},$$

$$\left.1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-\left(1-F_i'\right)^p\left(1-F_j'\right)^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{p+q}}\right)$$

i.e., $NNIGWHM^{p,q}(\widetilde{a}_1,\widetilde{a}_2,\dots,\widetilde{a}_n)\geqslant NNIGWHM^{p,q}(\widetilde{a}_1',\widetilde{a}_2',\dots,\widetilde{a}_n')$ which completes the proof of Theorem 14. □

Theorem 15 (Boundedness). *Let* $\widetilde{a}_j=(T_j,I_j,F_j)$ $(j=1,2,\dots,n)$ *be a collection of SVNNs, and*

$$\widetilde{a}^-=\left(\min(T_j),\max(I_j),\max(F_j)\right),\qquad \widetilde{a}^+=\left(\max(T_j),\min(I_j),\min(F_j)\right)$$

then

$$\widetilde{a}^-\leqslant NNIGWHM^{p,q}(\widetilde{a}_1,\widetilde{a}_2,\dots,\widetilde{a}_n)\leqslant\widetilde{a}^+. \tag{38}$$

Proof. Since $\widetilde{a}_j\geqslant\widetilde{a}^-$, then based on Theorems 13 and 14, we have

$$NNIGWHM^{p,q}(\widetilde{a}_1,\widetilde{a}_2,\dots,\widetilde{a}_n)\geqslant NNIGWHM^{p,q}(\widetilde{a}^-,\widetilde{a}^-,\dots,\widetilde{a}^-)=\widetilde{a}^-.$$

Likewise, we can get

$$NNIGWHM^{p,q}(\widetilde{a}_1,\widetilde{a}_2,\dots,\widetilde{a}_n)\leqslant NNIGWHM^{p,q}\left(\widetilde{a}^+,\widetilde{a}^+,\dots,\widetilde{a}^+\right)=\widetilde{a}^+.$$

Then

$$\widetilde{a}^-\leqslant NNIGWHM^{p,q}(\widetilde{a}_1,\widetilde{a}_2,\dots,\widetilde{a}_n)\leqslant\widetilde{a}^+$$

which completes the proof of Theorem 15. □

In the following, we will discuss some specials of the NNIGWHM with respect to the parameters p and q.

(1) When $p=0$, then

$$NNIGWHM^{0,q}(\widetilde{a}_1,\widetilde{a}_2,\dots,\widetilde{a}_n)$$
$$=\left(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_j^q\right)^{w_iw_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n}w_iw_j}}\right)^{\frac{1}{q}},\right.$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_j)^q\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{q}},$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_j)^q\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{q}}\Bigg). \tag{39}$$

(2) When $q=0$, then

$$NNIGWHM^{p,0}(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n)$$
$$=\Bigg(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-T_i^p\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p}},$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)^p\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p}},$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_i)^p\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{p}}\Bigg). \tag{40}$$

(3) When $p=q=1$, then

$$NNIGWHM^{1,1}(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n)$$
$$=\Bigg(\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}(1-T_iT_j)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{2}},$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-I_i)(1-I_j)\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{2}},$$

$$1-\left(1-\left(\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-F_i)(1-F_j)\right)^{w_i w_j}\right)^{\frac{1}{\sum_{i=1}^{n}\sum_{j=i}^{n} w_i w_j}}\right)^{\frac{1}{2}}\Bigg). \tag{41}$$

3.2. *NNIGWGHM Operator*

Definition 13. Let $p,q\geqslant 0$, and $\widetilde{a}_j=(T_j,I_j,F_j)$ $(j=1,2,\ldots,n)$ be a collection of SVNNs with the weight vector $W=(w_1,w_2,\ldots,w_n)^T$ such that $w_j\geqslant 0$ and $\sum_{j=1}^{n} w_j=1$, then a single valued neutrosophic number imprvoed generalized weighted geometric Heronian mean (NNIGWGHM) operator of dimension n is a mapping NNIGWGHM: $\Omega^n\rightarrow\Omega$, and has

$$NNIGWGHM^{p,q}(\widetilde{a}_1,\widetilde{a}_2,\ldots,\widetilde{a}_n)=\frac{1}{p+q}\bigotimes_{i=1}^{n}\bigotimes_{j=i}^{n}(p\widetilde{a}_i\oplus\widetilde{a}_j)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}} \tag{42}$$

where Ω is the set of all SVNNs.

Based on the operational rules of the SVNNs, we have the following Theorem 16.

Theorem 16. *Let $p, q \geqslant 0$, and $\widetilde{a}_j = (T_j, I_j, F_j)$ $(j = 1, 2, \ldots, n)$ be a collection of SVNNs with the weight vector $W = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \geqslant 0$ and $\sum_{j=1}^{n} w_j = 1$, then the aggregated value by* (42) *can be expressed as*

$$\begin{aligned}
&NNIGWGHM^{p,q}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n)\\
&= \Bigg(1 - \Bigg(1 - \prod_{i=1}^{n}\prod_{j=i}^{n}\big(1 - (1 - T_i^p)(1 - T_j^q)\big)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n} w_k}}\Bigg)^{\frac{1}{p+q}},\\
&\Bigg(1 - \prod_{i=1}^{n}\prod_{j=i}^{n}\big(1 - I_i^p I_j^q\big)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n} w_k}}\Bigg)^{\frac{1}{p+q}},\\
&\Bigg(1 - \prod_{i=1}^{n}\prod_{j=i}^{n}\big(1 - F_i^p F_j^q\big)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n} w_k}}\Bigg)^{\frac{1}{p+q}}\Bigg).
\end{aligned} \tag{43}$$

Similar to the proof of Theorem 12, the proof of Theorem 12 is omitted.

Moreover, similar to the proofs of Theorems 13–15, it is easy to prove that the NNIGWGHM operator also has the following properties.

Theorem 17 (Reducibility). *Let $W = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then*

$$NNIGWGHM^{p,q}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) = NNGGHM^{p,q}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n). \tag{44}$$

Theorem 18 (Idempotency). *Let $\widetilde{a}_j = (T, I, F)$ $(j = 1, 2, \ldots, n)$, then*

$$NNIGWGHM^{p,q}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) = (T, I, F). \tag{45}$$

Theorem 19 (Monotonicity). *Let $\widetilde{a}_j = (T_j, I_j, F_j)$ and $\widetilde{a}'_j = (T'_j, I'_j, F'_j)$ $(j = 1, 2, \ldots, n)$ be two collections of SVNNs. If $\widetilde{a}_j \geqslant \widetilde{a}'_j$ for all j (suppose $T_j \geqslant T'_j$, $I_j \leqslant I'_j$ and $F_j \leqslant F'_j$), then*

$$NNIGWGHM^{p,q}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) \geqslant NNIGWGHM^{p,q}(\widetilde{a}'_1, \widetilde{a}'_2, \ldots, \widetilde{a}'_n). \tag{46}$$

Theorem 20 (Boundedness). *Let $\widetilde{a}_j = (T, I, F)$ $(j = 1, 2, \ldots, n)$ be a collection of SVNNs, and*

$$\widetilde{a}^- = \big(\min(T_j), \max(I_j), \max(F_j)\big), \qquad \widetilde{a}^+ = \big(\max(T_j), \min(I_j), \min(F_j)\big)$$

then

$$\widetilde{a}^- \leqslant NNIGWHM^{p,q}(\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_n) \leqslant \widetilde{a}^+. \tag{47}$$

In the following, we will discuss some specials of the NNIGWGHM with respect to the parameters p and q.

(1) When $p=0$, then

$$\begin{aligned} &NNIGWGHM^{0,q}(\widetilde{a}_1,\widetilde{a}_2,\dots,\widetilde{a}_n)\\ &=\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-T_j)^q\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{\frac{1}{q}},\right.\\ &\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-I_j^q\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{\frac{1}{q}},\\ &\left.\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-F_j^q\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{\frac{1}{q}}\right). \end{aligned} \tag{48}$$

(2) When $q=0$, then

$$\begin{aligned} &NNIGWGHM^{p,0}(\widetilde{a}_1,\widetilde{a}_2,\dots,\widetilde{a}_n)\\ &=\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-T_i)^p\right)^{\frac{2(n+1-i)}{n(n+1)}}\right)^{\frac{1}{p}},\right.\\ &\left.\left(1-\prod_{i=1}^{n}\left(1-I_i^q\right)^{\frac{2(n+1-i)}{n(n+1)}}\right)^{\frac{1}{p}},\left(1-\prod_{i=1}^{n}\left(1-F_i^q\right)^{\frac{2(n+1-i)}{n(n+1)}}\right)^{\frac{1}{p}}\right). \end{aligned} \tag{49}$$

Obviously, when $q=0$, $NNIGWGHM^{p,0}$ does not have any relationship with w. In addition, the parameters p and q don't have the interchangeability.

(3) When $p=q=1$, then

$$\begin{aligned} &NNIGWGHM^{1,1}(\widetilde{a}_1,\widetilde{a}_2,\dots,\widetilde{a}_n)\\ &=\left(1-\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-(1-T_i)(1-T_j)\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{\frac{1}{2}},\right.\\ &\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}(1-I_iI_j)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{\frac{1}{2}},\\ &\left.\left(1-\prod_{i=1}^{n}\prod_{j=i}^{n}\left(1-F_iF_j\right)^{\frac{2(n+1-i)}{n(n+1)}\frac{w_j}{\sum_{k=i}^{n}w_k}}\right)^{\frac{1}{2}}\right). \end{aligned} \tag{50}$$

4. The Approach to Multiple Attribute Group Decision Making with SVNNs

In this section, we shall propose the approach to multiple attribute group decision making with SVNNs by NNIGWHM operator or NNIGWGHM operator.

Consider a multiple attribute group decision making problem with SVNNs. Let $A = \{A_1, A_2, \dots, A_m\}$ be the collection of alternatives, $C = \{C_1, C_2, \dots, C_n\}$ be the collection of attributes, and $E = \{e_1, e_2, \dots, e_\lambda\}$ be the collection of decision makers. Suppose that $r_{ij}^k = (T_{ij}^k, I_{ij}^k, F_{ij}^k)$ is an attribute value given by the decision maker e_k for the alternative A_i with respect to the attribute C_j which is expressed by a SVNN, $w = (w_1, w_2, \dots, w_n)$ is the weight vector of attribute set $C = \{C_1, C_2, \dots, C_n\}$, and $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$. Let $\omega = (\omega_1, \omega_2, \dots, \omega_\lambda)$ be the weight vector of decision makers $\{e_1, e_2, \dots, e_\lambda\}$, and $\omega_k \in [0, 1]$, $\sum_{k=1}^{\lambda} \omega_k = 1$. Then we use the attribute weights, the decision makers' weights, and the attribute values to rank the order of the alternatives.

The method involves the following steps:

Step 1: Utilize the *NNIGWHM* operator

$$r_i^k = \left(T_i^k, I_i^k, F_i^k\right) = NNIGWHM\left(r_{i1}^k, r_{i2}^k, \dots, r_{in}^k\right) \tag{51}$$

or *NNIGWGHM* operator

$$r_i^k = \left(T_i^k, I_i^k, F_i^k\right) = NNIGWGHM\left(r_{i1}^k, r_{i2}^k, \dots, r_{in}^k\right) \tag{52}$$

to derive the comprehensive values r_i^k $(i = 1, 2, \dots, m; k = 1, 2, \dots, \lambda)$ of each decision maker.

Step 2: Utilize the *NNIGWHM* operator

$$r_i = (T_i, I_i, F_i) = NNIGWHM\left(r_i^1, r_i^2, \dots, r_i^\lambda\right) \tag{53}$$

or *NNIGWGHM* operator

$$r_i = (T_i, I_i, F_i) = NNIGWGHM\left(r_i^1, r_i^2, \dots, r_i^\lambda\right) \tag{54}$$

to derive the collective overall values r_i $(i = 1, 2, \dots, m)$.

Step 3: Calculate the cosine similarity $S(r_i)$ $(i = 1, 2, \dots, m)$ by Definition 3.

Step 4: Rank all the alternatives $A = \{A_1, A_2, \dots, A_m\}$ by the cosine similarity $S(r_i)$ $(i = 1, 2, \dots, m)$.

Step 5: End.

5. An Application Example

In order to demonstrate the application of the proposed method to multi-attribute group decision making problems, in this part, we will cite an example about the air

Table 1
Air quality data from station e_1.

	C_1	C_2	C_3
A_1	$(0.265, 0.350, 0.385)$	$(0.330, 0.390, 0.280)$	$(0.245, 0.275, 0.480)$
A_2	$(0.345, 0.245, 0.410)$	$(0.430, 0.290, 0.280)$	$(0.245, 0.375, 0.380)$
A_3	$(0.365, 0.300, 0.335)$	$(0.480, 0.315, 0.205)$	$(0.340, 0.370, 0.290)$
A_4	$(0.430, 0.300, 0.270)$	$(0.460, 0.245, 0.295)$	$(0.310, 0.520, 0.170)$

Table 2
Air quality data from station e_2.

	C_1	C_2	C_3
A_1	$(0.125, 0.470, 0.405)$	$(0.220, 0.420, 0.360)$	$(0.345, 0.490, 0.165)$
A_2	$(0.355, 0.315, 0.330)$	$(0.300, 0.370, 0.330)$	$(0.205, 0.630, 0.165)$
A_3	$(0.315, 0.380, 0.305)$	$(0.330, 0.565, 0.105)$	$(0.280, 0.520, 0.200)$
A_4	$(0.365, 0.365, 0.270)$	$(0.355, 0.320, 0.325)$	$(0.425, 0.485, 0.090)$

Table 3
Air quality data from station e_3.

	C_1	C_2	C_3
A_1	$(0.260, 0.425, 0.315)$	$(0.220, 0.450, 0.330)$	$(0.255, 0.500, 0.245)$
A_2	$(0.270, 0.370, 0.360)$	$(0.320, 0.215, 0.465)$	$(0.135, 0.575, 0.290)$
A_3	$(0.245, 0.465, 0.290)$	$(0.250, 0.570, 0.180)$	$(0.175, 0.660, 0.165)$
A_4	$(0.390, 0.340, 0.270)$	$(0.305, 0.475, 0.220)$	$(0.465, 0.485, 0.050)$

quality evaluation (adapted from Yue, 2011). To find out the trends of the air quality in Guangzhou for the 16th Asian Olympic Games, there are 3 air-quality monitoring stations expressed by (e_1, e_2, e_3) to collect the air quality data in Guangzhou for the Novembers of 2006, 2007, 2008 and 2009. The 3 air-quality monitoring stations can be seen as decision makers which have weights $\omega = (0.314, 0.355, 0.331)^T$, and there are 3 measured indexes, namely, SO2 (C_1), NO2 (C_2) and PM10 (C_3), and their weight $W = (0.40, 0.20, 0.40)^T$. The measured values under these indexes from air-quality monitoring stations are shown in Tables 1, 2 and 3, and they can be expressed by SVNNs (note: the original data take the form of intuitionistic fuzzy numbers, we can get SVNNs by $I = 1 - T - F$). Let $(A_1, A_2, A_3, A_4) = \{$November of 2006, November of 2007, November of 2008, November of 2009$\}$ be the set of alternatives, please give the rank of air quality from 2006 to 2009.

5.1. *The Evaluation Steps by NNIGWHM Operator*

The steps are shown as follows:

(1) Calculate the comprehensive evaluation values r_i^k $(i = 1, 2, 3, 4;\ k = 1, 2, 3)$ of each decision maker by formula (51) of the *NNIGWHM* operator (suppose $p = q = 1$),

we can get

$$r_1^1 = (0.269, 0.324, 0.415), \qquad r_2^1 = (0.322, 0.303, 0.390),$$
$$r_3^1 = (0.376, 0.330, 0.308), \qquad r_4^1 = (0.388, 0.369, 0.226),$$
$$r_1^2 = (0.240, 0.469, 0.288), \qquad r_2^2 = (0.287, 0.438, 0.253),$$
$$r_3^2 = (0.303, 0.466, 0.247), \qquad r_4^2 = (0.389, 0.403, 0.183),$$
$$r_1^3 = (0.251, 0.459, 0.286), \qquad r_2^3 = (0.227, 0.412, 0.335),$$
$$r_3^3 = (0.218, 0.558, 0.228), \qquad r_4^3 = (0.408, 0.420, 0.149).$$

(2) Calculate the collective overall values r_i $(i = 1, 2, 3, 4)$ by formula (53) of the *NNIGWHM* operator (suppose $p = q = 1$), we can get

$$r_1 = (0.253, 0.418, 0.360), \qquad r_2 = (0.279, 0.385, 0.360),$$
$$r_3 = (0.300, 0.448, 0.275), \qquad r_4 = (0.395, 0.398, 0.195).$$

(3) Calculate the cosine similarity $S(r_i)$ $(i = 1, 2, 3, 4)$ of the collective overall values r_i $(i = 1, 2, 3, 4)$, we can get

$$S(r_1) = 0.417, \qquad S(r_2) = 0.468, \qquad S(r_3) = 0.496, \qquad S(r_4) = 0.665.$$

(4) Rank the alternatives.

According to the cosine similarity r_i $(i = 1, 2, 3, 4)$, we can rank the alternatives $\{A_1, A_2, A_3, A_4\}$ shown as follows

$$A_4 \succ A_3 \succ A_2 \succ A_1.$$

So, the best alternative is A_4, i.e., the best air quality in Guangzhou is November of 2009 among the Novembers of 2006, 2007, 2008, and 2009.

5.2. *The Evaluation Steps by NNIGWGHM Operator*

The steps are shown as follows:

(1) Calculate the comprehensive evaluation values r_i^k $(i = 1, 2, 3, 4;\ k = 1, 2, 3)$ of each decision maker by formula (52) of the *NNIGWGHM* operator (suppose $p = q = 1$), we can get

$$r_1^1 = (0.276, 0.341, 0.390), \qquad r_2^1 = (0.333, 0.303, 0.367),$$
$$r_3^1 = (0.387, 0.327, 0.289), \qquad r_4^1 = (0.398, 0.368, 0.250),$$
$$r_1^2 = (0.207, 0.462, 0.329), \qquad r_2^2 = (0.287, 0.452, 0.285),$$
$$r_3^2 = (0.308, 0.487, 0.231), \qquad r_4^2 = (0.380, 0.394, 0.239),$$

$$r_1^3 = (0.246, 0.457, 0.299), \qquad r_2^3 = (0.234, 0.411, 0.374),$$
$$r_3^3 = (0.223, 0.564, 0.228), \qquad r_4^3 = (0.385, 0.431, 0.200).$$

(2) Calculate the collective overall values r_i $(i = 1, 2, 3, 4)$ by formula (54) of the *NNIGWHM* operator (suppose $p = q = 1$), we can get

$$r_1 = (0.243, 0.419, 0.347), \qquad r_2 = (0.288, 0.390, 0.344),$$
$$r_3 = (0.309, 0.461, 0.254), \qquad r_4 = (0.388, 0.395, 0.233).$$

(3) Calculate the cosine similarity $S(r_i)$ $(i = 1, 2, 3, 4)$ of the collective overall values r_i $(i = 1, 2, 3, 4)$, we can get

$$S(r_1) = 0.408, \qquad S(r_2) = 0.485, \qquad S(r_3) = 0.506, \qquad S(r_4) = 0.646.$$

(4) Rank the alternatives.

According to the cosine similarity r_i $(i = 1, 2, 3, 4)$, we can rank the alternatives $\{A_1, A_2, A_3, A_4\}$ shown as follows

$$A_4 \succ A_3 \succ A_2 \succ A_1.$$

So, the best alternative is A_4, i.e., the best air quality in Guangzhou is November of 2009 among the Novembers of 2006, 2007, 2008, and 2009.

Obviously, this ranking result is the same as that in Yue (2011). However, the proposed method can consider the correlations of the aggregated arguments.

5.3. *The Influence of the Parameters p and q on Decision Making of this Example*

In order to illustrate the influence of the parameters p and q on decision making of this example, we use the different values p and q in steps 1 and 2 of above methods to rank the alternatives. The ranking results are shown in Tables 4 and 5.

From Tables 4 and 5, we know parameters p and q can influence the aggregation result in the *NNIGWHM* and *NNIGWGHM* operators. In general, we can take the values of the two parameters as $p = q = 1$, which is not only intuitive and simple but also the correlations of the aggregated arguments can be fully taken into account.

6. Conclusions

The neutrosophic set can be easier and better to express the incomplete, indeterminate and inconsistent information, and Heronian mean can capture the correlations of the aggregated arguments, in this paper, we applied the Heronian mean to the neutrosophic set, and proposed some Heronian mean operators based on SVNNs. Further, we applied the proposed operators to multi-attribute group decision making problems, and proposed some decision making methods. Firstly, with respect to the defects of the existing generalized

Table 4
Ordering of the alternatives by the different parameters p and q in NNIGWHM operator.

p, q	Cosine similarity $S(r_i)$ $(i = 1, 2, 3, 4)$	Ranking
$p = 0$ $q = 0.01$	$S(r_1) = 0.428$, $S(r_2) = 0.387$ $S(r_3) = 0.417$, $S(r_4) = 0.665$	$A_4 \succ A_1 \succ A_3 \succ A_2$
$p = 0$ $q = 1$	$S(r_1) = 0.436$, $S(r_2) = 0.406$ $S(r_3) = 0.429$, $S(r_4) = 0.670$	$A_4 \succ A_1 \succ A_3 \succ A_2$
$p = 0$ $q = 2.1$	$S(r_1) = 0.446$, $S(r_2) = 0.429$ $S(r_3) = 0.445$, $S(r_4) = 0.676$	$A_4 \succ A_1 \succ A_3 \succ A_2$
$p = 0$ $q = 2.2$	$S(r_1) = 0.44830$, $S(r_2) = 0.43418$ $S(r_3) = 0.44832$, $S(r_4) = 0.67743$	$A_4 \succ A_3 \succ A_1 \succ A_2$
$p = 0$ $q = 10$	$S(r_1) = 0.5272$, $S(r_2) = 0.5817$ $S(r_3) = 0.5821$, $S(r_4) = 0.7297$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 0.01$ $q = 0$	$S(r_1) = 0.3911$, $S(r_2) = 0.5168$ $S(r_3) = 0.5482$, $S(r_4) = 0.6764$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 1$ $q = 0$	$S(r_1) = 0.4021$, $S(r_2) = 0.5251$ $S(r_3) = 0.5558$, $S(r_4) = 0.6808$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 10$ $q = 0$	$S(r_1) = 0.4864$, $S(r_2) = 0.5902$ $S(r_3) = 0.6391$, $S(r_4) = 0.7287$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 1$ $q = 1$	$S(r_1) = 0.4167$, $S(r_2) = 0.4677$ $S(r_3) = 0.4956$, $S(r_4) = 0.6654$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 2$ $q = 1$	$S(r_1) = 0.4226$, $S(r_2) = 0.4972$ $S(r_3) = 0.5257$, $S(r_4) = 0.6728$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 10$ $q = 1$	$S(r_1) = 0.4878$, $S(r_2) = 0.5796$ $S(r_3) = 0.6278$, $S(r_4) = 0.7227$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 1$ $q = 2$	$S(r_1) = 0.4306$, $S(r_2) = 0.4667$ $S(r_3) = 0.4881$, $S(r_4) = 0.6706$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 1$ $q = 10$	$S(r_1) = 0.5147$, $S(r_2) = 0.5816$ $S(r_3) = 0.5893$, $S(r_4) = 0.7274$	$A_4 \succ A_3 \succ A_2 \succ A_1$

weighted Heronian mean operator and generalized weighted geometric Heronian mean operator, for example, there have not reducibility and idempotency which seem to be counterintuitive, we have proposed the improved generalized weighted Heronian mean (IGWHM) operator and generalized weighted geometric Heronian mean (IGWGHM) operator. Further, we extended IGWHM and IGWGHM operators to SVNNs, and proposed the neutrosophic number improved generalized weighted Heronian mean (NNIGWHM) operator and the neutrosophic number improved generalized weighted geometric Heronian mean (NNIGWGHM) operator, and some desirable properties of these operators have been investigated in detail, including idempotency, monotonicity and boundedness. At the same time, some special cases of them with respect to the parameter values p and q are discussed. Moreover, with respect to multiple attribute decision making group problems in which the attribute values take the form of SVNNs, some approaches based on the developed operators are proposed. The important characteristic of the proposed approaches is that they could consider the correlations of the aggregated arguments. Finally, an appli-

Table 5
Ordering of the alternatives by the different parameters p and q in NNIGWGHM operator.

p, q	Cosine similarity $S(r_i)$ $(i = 1, 2, 3, 4)$	Ranking
$p = 0$ $q = 0.01$	$S(r_1) = 0.4308$, $S(r_2) = 0.3740$ $S(r_3) = 0.4088$, $S(r_4) = 0.6534$	$A_4 \succ A_1 \succ A_3 \succ A_2$
$p = 0$ $q = 1$	$S(r_1) = 0.4242$, $S(r_2) = 0.3616$ $S(r_3) = 0.4015$, $S(r_4) = 0.6430$	$A_4 \succ A_1 \succ A_3 \succ A_2$
$p = 0$ $q = 2$	$S(r_1) = 0.4165$, $S(r_2) = 0.3486$ $S(r_3) = 0.3935$, $S(r_4) = 0.6309$	$A_4 \succ A_1 \succ A_3 \succ A_2$
$p = 0$ $q = 10$	$S(r_1) = 0.3628$, $S(r_2) = 0.2805$ $S(r_3) = 0.3395$, $S(r_4) = 0.5706$	$A_4 \succ A_1 \succ A_3 \succ A_2$
$p = 0.01$ $q = 0$	$S(r_1) = 0.4076$, $S(r_2) = 0.5574$ $S(r_3) = 0.5785$, $S(r_4) = 0.6757$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 1$ $q = 0$	$S(r_1) = 0.4019$, $S(r_2) = 0.5476$ $S(r_3) = 0.5669$, $S(r_4) = 0.6668$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 2$ $q = 0$	$S(r_1) = 0.3956$, $S(r_2) = 0.5354$ $S(r_3) = 0.5535$, $S(r_4) = 0.6571$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 10$ $q = 0$	$S(r_1) = 0.3447$, $S(r_2) = 0.4257$ $S(r_3) = 0.4636$, $S(r_4) = 0.5896$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 1$ $q = 1$	$S(r_1) = 0.4171$, $S(r_2) = 0.4688$ $S(r_3) = 0.4918$, $S(r_4) = 0.6491$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 2$ $q = 1$	$S(r_1) = 0.4080$, $S(r_2) = 0.4845$ $S(r_3) = 0.5061$, $S(r_4) = 0.6456$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 10$ $q = 1$	$S(r_1) = 0.3515$, $S(r_2) = 0.4135$ $S(r_3) = 0.4529$, $S(r_4) = 0.5896$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 1$ $q = 2$	$S(r_1) = 0.4130$, $S(r_2) = 0.4258$ $S(r_3) = 0.4554$, $S(r_4) = 0.6341$	$A_4 \succ A_3 \succ A_2 \succ A_1$
$p = 1$ $q = 10$	$S(r_1) = 0.3631$, $S(r_2) = 0.3041$ $S(r_3) = 0.3567$, $S(r_4) = 0.5737$	$A_4 \succ A_3 \succ A_2 \succ A_1$

cation example has been given to show the steps of the proposed methods and to discuss the influence of different parameter values on the decision-making results.

In the future, we shall further consider the relationship among attributes and generalize some operators by using the well-known quasi-arithmetic, Choquet integral and Dempster–Shafer belief structure, or extend the potential applications of the developed operators and methods to other domains, such as pattern recognition, supply chain management, etc.

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Y. Li obtained her Doctoral degree in Technological Economics in Tianjin University. Now she is Vice-Dean of School of Economics and Management in Civil Aviation University of China, and Director of the Research Institute of aviation industry. She has authored or coauthored more than 20 publications. Her main research fields are Aviation Transport Economics and Management, decision support, production and operation management.

P. Liu obtained his Doctoral degree in Management Science and Engineering in the Beijing Jiaotong University, obtained his Master's degree in Signal and Information Processing in the Southeast University, and obtained his Bachelor's degree in Signal and Information Processing in the Southeast University. He is an Associate Editor of Journal of Intelligent and Fuzzy Systems, a member of Editorial Board of Technological and Economic Development of Economy, The Scientific World Journal, etc. He has authored or coauthored more than 100 publications. His main research fields are decision analysis and decision support, applied mathematics, expert systems, technology and information management, intelligent information processing.

Y. Chen obtained his MBA degree in School of Management in Northwestern Polytechnical University, and obtained his Bachelor's degree in Department of Economics in Nanjing University. Now, he is Dean of School of Economics and Management in Civil Aviation University of China, and he has authored or coauthored more than 20 publications. Her main research fields are decision support, technology and information management, production and operation management.

Kai kurie vienareikšmiai neutrosofistinių skaičių Herono vidurkio operatoriai ir jų taikymas daugiarodikliams grupiniams sprendimams priimti

Yanhua Lia, Peide Liu, Yubao Chena

Herono vidurkis (HM) turi savybę atspindėti agreguotų argumentų ir neutrosofistinių aibių koreliaciją. Jis gali atspindėti neišsamią, neapibrėžtą ir nenuoseklią informaciją. Šiame straipsnyje mes taikėme Herono vidurkį neutrosofistinėms aibėms, ir pasiūlėme, kai kuriuos Herono vidurkio operatorius. Mes pateikėme keletą operacijų dėsnių ir jų vienareikšmių neutrosofistinių skaičių (SVNNs) savybių, analizavome egzistuojančių svertinių (pasvertų) HM operatorių, kurie nėra pakankamai tinkami, trūkumus, pasiūlėme geresnį apibendrintą svertinį Herono operatoriaus vidurkį (IGWHM) ir patobulintą apibendrintą svertinį geometrinį Herono vidurkio (IGWGHM) operatorių, pagrįstus tiksliaisiais skaičiais, įrodome, kad jie gali tenkinti, kai kurias pageidautinas savybes, tokias kaip redukuojamumas, pakankamas tinkamumas, monotoniškumas ir apribojimas. Be to, mes pasiūlėme vienintele reikšme išreikštu ir neutrosofistiniu skaičiumi patobulintą apibendrintą svertinį Herono vidurkio (NNIGWHM) operatorių, ir viena reikšme įvertintą, neurosofistiniu skaičiumi patobulintą, apibendrintą svertinį geometrinio vidurkio Herono (NNIGWGHM) operatorių. Aptarėme kai kurias pageidaujamas jų savybes ir išskirtinius atvejus.

Atsižvelgiant į daugiarodiklių grupinių sprendimų priėmimo (MAGDM) uždavinius, kuriuose rodiklių reikšmės įgyja SVNNs formą, yra sukurti sprendimų priėmimo metodai, pagrįsti siūlomais operatoriais. Buvo pateiktas taikymo pavyzdys sprendimų priėmimo žingsniams parodyti, aptarta įvairių parametrų reikšmių įtaka sprendimų priėmimo rezultatams.