INFORMATICA, 2016, Vol. 27, No. 1, 141–159

DOI: http://dx.doi.org/10.15388/Informatica.2016.73

Some Properties and Applications of Fuzzy Quasi-Pseudo-Metric Spaces

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Received: November 2014; accepted: January 2015

Abstract. In this paper we establish some properties of fuzzy quasi-pseudo-metric spaces. An important result is that any partial ordering can be defined by a fuzzy quasi-metric, which can be applied both in theoretical computer science and in information theory, where it is usual to work with sequences of objects of increasing information. We also obtain decomposition theorems of a fuzzy quasi-pseudo metric into a right continuous and ascending family of quasi-pseudo metrics. We develop a topological foundation for complexity analysis of algorithms and programs, and based on our results a fuzzy complexity space can be considered. Also, we built a fertile ground to study some types of fuzzy quasi-pseudo-metrics on the domain of words, which play an important role on denotational semantics, and on the poset BX of all closed formal balls on a metric space.

Key words: fuzzy quasi-pseudo-metric space, fuzzy quasi-metric space, fuzzy metric space, partial ordering, complexity space.

1. Introduction

The concept of metric space introduced by Maurice René Fréchet in 1906 was generalized in the following years. Thus, Wilson (1931) introduced the notion of quasi-metric space, Kim (1968) introduced pseudo-quasi-metric spaces, and Matthews (1994) introduced the concept of partial metric spaces. Recently Amini-Harandi (2012) introduced a new generalization of partial metric space which is called metric-like spaces.

Those generalizations occurred due to both the needs of internal development of Mathematics and, especially, due to the fact that Mathematics was summoned to answer some issues raised by other domains of sciences and the progress in Mathematical Sciences has become essential for a better understanding of the natural world and our relationship with it. For example the study of nonsymmetric structures was boosted by numerous applications in Computer Science (see for instance, Smyth, 1987; Schellekens, 1995; Romaguera and Schellekens, 1999, etc.).

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On the other hand, both in theoretical computer science and in information theory (see Garcia *et al.*, 2011) it is usual to work with sequences (x_n) of objects of increasing information. In this situation the relation $x_n \leqslant x_{n+1}$ is understood as x_{n+1} must have at least as much information as (x_n). The relation $\leqslant$ is a partial ordering. What we aim is to find those suitable metrics d to model the partial order relation $\leqslant$, meaning that, starting from a metric d, to define a partial order relation $\leqslant_d$ which to be the same with the order relation $\leqslant$. We note that a quasi-metric d on a set X induces a partial order $\leqslant_d$ on X (called the specialization order) defined by $y \leqslant_d x \Leftrightarrow d(y, x) = 0$.

It is well known fact that, in practice, the distance $d(x, y)$ between two points cannot be precisely measured. This fact had lead to at least two approaches enabling to describe and to handle somehow this situation. The first, probabilistic and statistical approach was developed by Menger (1942). The other, fuzzy approach, was introduced by Kramosil and Michálek (1975). We note that George and Veeramani (1994) modified the concept of fuzzy metric introduced by I. Kramosil and J. Michálek and defined a Hausdorff topology on this space. Another approach for fuzzy metric was introduced by Kaleva and Seikkala (1984), by setting the distance between two points to be a non-negative, upper semi-continuous normal and convex fuzzy number.

In recent years, different types of generalized fuzzy metrics were considered by different authors in different approaches. Thus Gregori and Romaguera (2004) and Cho *et al.* (2006) introduced the notions of fuzzy quasi-metric space and fuzzy quasi-pseudo-metric space, generalizing in this way the notion of fuzzy metric introduced by I. Kramosil and J. Michálek and by A. George and P. Veeramani. New concepts of generalized fuzzy metrics and some fixed points theorems are established in the papers of Ray and Saha (2010), Sun and Yang (2010), Bag (2013), Pleabaniak (2014), Tripathy *et al.* (2014), etc.

The structure of the paper is as follows: after the preliminary section, in Section 3, we give some important examples of fuzzy quasi-pseudo-metric spaces. In Section 4 we prove that any partial order relation can be modeled by a fuzzy quasi-metric. Also, any equivalence relation can be modeled by a fuzzy pseudo-metric. In Section 5, we apply the techniques introduced in paper of Nădăban and Dzitac (2014), in order to obtain that there exists a bijective correspondence between a fuzzy quasi-pseudo-metric and a right continuous and ascending family of quasi-pseudo-metrics. As the domain of words play an important role in denotational semantics, in Section 6, we show that a fuzzy quasi-pseudo-metric can be considered on the domain of words. Finally, we show that a fuzzy quasi-metric can be considered on the set of all closed formal balls $BX = X \times [0, \infty)$, which leads to new connection between the theory of metric spaces and domain theory, the two basic mathematical structures in computer science.

2. Preliminaries

Definition 1. (See Schweizer and Sklar, 1960.) A binary operation

$$* : [0, 1] \times [0, 1] \to [0, 1]$$

is called triangular norm (t-norm) if it satisfies the following condition:

1. $a * b = b * a$, $(\forall) a, b \in [0, 1]$;
2. $a * 1 = a$, $(\forall) a \in [0, 1]$;
3. $(a * b) * c = a * (b * c)$, $(\forall) a, b, c \in [0, 1]$;
4. If $a \leqslant c$ and $b \leqslant d$, with $a, b, c, d \in [0, 1]$, then $a * b \leqslant c * d$.

Example 1. Three basic examples of continuous t-norms are $\wedge, \cdot, *_L$, which are defined by $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$ (usual multiplication in $[0, 1]$) and $a *_L b = \max\{a + b - 1, 0\}$ (the Lukasiewicz t-norm).

Our basic reference for t-norms is Klement *et al.* (2000).

Definition 2. (See Nădăban, 2015.) Let $*, *'$ be two t-norms. We say that $*'$ dominates $*$ and we denote $*' \gg *$ if

$$\left(x_1 *' x_2\right) * \left(y_1 *' y_2\right) \leqslant (x_1 * y_1) *' (x_2 * y_2), \quad (\forall) x_1, x_2, y_1, y_2 \in [0, 1].$$

Remark 1. (See Nădăban, 2015.) For any t-norm $*$ we have $\wedge \gg *$.

Definition 3. Let X be a nonempty set. A mapping $p : X \times X \to [0, \infty)$ is called quasi-pseudo-metric if it satisfies the following conditions:

(p1) $p(x, x) = 0$, $(\forall) x \in X$;
(p2) $p(x, z) \leqslant p(x, y) + p(y, z)$, $(\forall) x, y, z \in X$.

The pair (X, p) will be called quasi-pseudo-metric space.

A quasi-metric is a quasi-pseudo-metric p which satisfies the additional condition:

(p3) $p(x, y) = p(y, x) = 0 \Rightarrow x = y$.

The pair (X, p) will be called quasi-metric space.

A pseudo-metric is a quasi-pseudo-metric p which satisfies the additional condition:

(p4) $p(x, y) = p(y, x)$, $(\forall) x, y \in X$.

The pair (X, p) will be called pseudo-metric space.

If the mapping p satisfies (p1)–(p4), then p is called metric and the pair (X, p) will be called metric space.

Our basic references for quasi-pseudo-metric spaces are the works of Fletcher and Lindgren (1982), Künzi (1992, 2001).

Definition 4. (See Gregori and Romaguera, 2004.) Let X be a nonempty set and $*$ be a continuous t-norm. A fuzzy set M in $X \times X \times [0, \infty)$ is called fuzzy quasi-pseudo-metric if it satisfies, for all $x, y, z \in X$, the following conditions:

(M1) $M(x, y, 0) = 0$;
(M2) $M(x, x, t) = 1$, $(\forall) t > 0$;
(M3) $M(x, z, t + s) \geqslant M(x, y, t) * M(y, z, s)$, $(\forall) t, s \geqslant 0$;

(M4) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous and $\lim_{t\to\infty} M(x, y, t) = 1$.

The triple $(X, M, *)$ will be called fuzzy quasi-pseudo-metric space.

A fuzzy quasi-metric is a fuzzy quasi-pseudo-metric such that

(M5) $[M(x, y, t) = M(y, x, t) = 1, (\forall)t > 0] \Rightarrow x = y$.

In this case the triple $(X, M, *)$ will be called fuzzy quasi-metric space.

A T_1 fuzzy quasi-metric is a fuzzy quasi-pseudo-metric such that

(M5)$'$ $[M(x, y, t) = 1, (\forall)t > 0] \Rightarrow x = y$.

In this case the triple $(X, M, *)$ will be called T_1 fuzzy quasi-metric space.

A fuzzy pseudo-metric is a fuzzy quasi-pseudo-metric such that

(M6) $M(x, y, t) = M(y, x, t), (\forall)t \geqslant 0$.

In this case the triple $(X, M, *)$ will be called fuzzy pseudo-metric space.

If M satisfies (M1)–(M6), then M will be called fuzzy metric and the triple $(X, M, *)$ will be called fuzzy metric space.

Remark 2. We note that the fuzzy metric spaces are exactly the fuzzy metric spaces in the sense of Kramosil and Michálek.

Remark 3. (See Gregori and Romaguera, 2004.) If M is a fuzzy quasi-(pseudo-)metric then M^{-1} defined by $M^{-1}(x, y, t) = M(y, x, t)$ is also a fuzzy quasi-(pseudo-)metric, called the conjugate of M.

If M is a fuzzy quasi-pseudo-metric, then for each $x, y \in X$ the mapping $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is nondecreasing.

Let $(X, M, *)$ be a fuzzy quasi-pseudo-metric space. For $x \in X$, $r \in (0, 1)$, $t > 0$ we define the open ball $B(x, r, t) := \{y \in X : M(x, y, t) > 1 - r\}$. Let

$$\mathcal{T}_M := \{T \subset X : x \in T \text{ iff } (\exists)t > 0,\ r \in (0, 1) : B(x, r, t) \subseteq T\}.$$

Then $\mathcal{T}_M$ is a topology on X.

If $(X, M, *)$ is a fuzzy quasi-metric space, then $\mathcal{T}_M$ is a T_0 topology. If $(X, M, *)$ is a T_1 fuzzy quasi-metric space, then $\mathcal{T}_M$ is a T_1 topology. If $(X, M, *)$ is a fuzzy metric space, then $\mathcal{T}_M$ is a Hausdorff topology.

3. Some Examples of Fuzzy Quasi-Pseudo-Metric Spaces

In this section we give some examples in order to show that any quasi-(pseudo-)metric induces in a natural way a fuzzy quasi-(pseudo-)metric. Also we give an example of fuzzy quasi-metric space which is not a fuzzy metric space. Another example shows that there exists a fuzzy quasi-pseudo-metric space which is not a fuzzy quasi-metric space and it is not a fuzzy pseudo-metric space. Finally, we give an example of a fuzzy pseudo-metric space which is not a fuzzy metric space.

Example 2. Let (X, p) be a quasi-(pseudo-)metric space and $k, m, n \in \mathbb{N}$. Let M be a fuzzy set in $X \times X \times [0, \infty)$ defined by

$$M(x, y, t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{kt^n}{kt^n + mp(x,y)} & \text{if } t > 0. \end{cases}$$

Then $(X, M, \cdot)$ and $(X, M, \wedge)$ are fuzzy quasi-(pseudo-)metric spaces. If (X, p) is a pseudo-metric space, then $(X, M, \cdot)$ and $(X, M, \wedge)$ are fuzzy pseudo-metric spaces.

Remark 4. In particular, for $k = m = n = 1$, we get

$$M_p(x, y, t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{t}{t + p(x,y)} & \text{if } t > 0. \end{cases}$$

This fuzzy quasi-(pseudo-)metric induces by a quasi-(pseudo-)metric p will be called standard fuzzy quasi-(pseudo-)metric.

Example 3. Let $X = \mathbb{R}$ and M be a fuzzy set in $X \times X \times [0, \infty)$ defined by

$$M(x, y, t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{t}{t + x - y} & \text{if } t > 0. \end{cases}$$

Then $(X, M, \wedge)$ is fuzzy quasi-metric space which is not a fuzzy metric space.

Example 4. Let $\mathcal{C}$ be the set of all convergent real sequences. For $(x_n), (y_n) \in \mathcal{C}$ we define

$$M((x_n), (y_n), t) = \begin{cases} 0 & \text{if } t = 0, \\ \frac{t}{t + \lim_{n\to\infty}(x_n - y_n)} & \text{if } t > 0. \end{cases}$$

Then $(\mathcal{C}, M, \wedge)$ is a fuzzy quasi-pseudo-metric space but not a fuzzy quasi-metric space and not a fuzzy pseudo-metric space.

Proof. It is obvious that M satisfies (M1), (M2) and (M4). We check (M3), namely

$$M\big((x_n), (z_n), t + s\big) \geqslant M\big((x_n), (y_n), t\big) \wedge M\big((y_n), (z_n), s\big).$$

We suppose that $M((x_n), (y_n), t) \leqslant M((y_n), (z_n), s)$ (the case $M((x_n), (y_n), t) \geqslant M((y_n), (z_n), s)$ is similar). Therefore

$$\frac{t}{t + \lim\limits_{n\to\infty}(x_n - y_n)} \leqslant \frac{s}{s + \lim\limits_{n\to\infty}(y_n - z_n)}$$
$$\Rightarrow t \lim_{n\to\infty}(y_n - z_n) \leqslant s \lim_{n\to\infty}(x_n - y_n) \Rightarrow (t + s) \lim_{n\to\infty} y_n \leqslant s \lim_{n\to\infty} x_n + t \lim_{n\to\infty} z_n.$$

We show that

$$
\begin{aligned}
&M\big((x_n),(z_n),t+s\big) \geqslant M\big((x_n),(y_n),t\big)\\
&\quad\Leftrightarrow \frac{t+s}{t+s+\lim_{n\to\infty}(x_n-z_n)} \geqslant \frac{t}{t+\lim_{n\to\infty}(x_n-y_n)}\\
&\quad\Leftrightarrow t\lim_{n\to\infty}(x_n-y_n)+s\lim_{n\to\infty}(x_n-y_n) \geqslant t\lim_{n\to\infty}(x_n-z_n)\\
&\quad\Leftrightarrow t\lim_{n\to\infty}x_n - t\lim_{n\to\infty}y_n + s\lim_{n\to\infty}x_n - s\lim_{n\to\infty}y_n \geqslant t\lim_{n\to\infty}x_n - t\lim_{n\to\infty}z_n\\
&\quad\Leftrightarrow t\lim_{n\to\infty}z_n + s\lim_{n\to\infty}x_n \geqslant (t+s)\lim_{n\to\infty}y_n.
\end{aligned}
$$

We note that M does not satisfy (M5). Indeed,

$$M\big((x_n),(y_n),t\big)=M\big((y_n),(x_n),t\big)=1,\quad (\forall)t>0 \Rightarrow \lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n,$$

which does not mean that $(x_n)=(y_n)$.

M does not satisfy (M6). Indeed, $M((x_n),(y_n),t)=M((y_n),(x_n),t)$ is equivalent to $\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n$, which is not true for arbitrary convergent sequences. □

EXAMPLE 5. Let $\mathcal{C}$ be the set of all convergent real sequences. For $(x_n),(y_n)\in\mathcal{C}$ we define

$$M\big((x_n),(y_n),t\big)=\begin{cases}0 & \text{if } t=0,\\ \frac{t}{t+|\lim_{n\to\infty}(x_n-y_n)|} & \text{if } t>0.\end{cases}$$

Then $(\mathcal{C},M,\wedge)$ is a fuzzy pseudo-metric space but not a fuzzy metric space.

Proof. It is obvious that M satisfies (M1), (M2) and (M4). We check (M3), namely

$$M\big((x_n),(z_n),t+s\big)\geqslant M\big((x_n),(y_n),t\big)\wedge M\big((y_n),(z_n),s\big).$$

We suppose that $M((x_n),(y_n),t)\leqslant M((y_n),(z_n),s)$. Thus,

$$
\begin{aligned}
&\frac{t}{t+|\lim_{n\to\infty}(x_n-y_n)|}\leqslant\frac{s}{s+|\lim_{n\to\infty}(y_n-z_n)|}\\
&\quad\Rightarrow t\Big|\lim_{n\to\infty}(y_n-z_n)\Big|\leqslant s\Big|\lim_{n\to\infty}(x_n-y_n)\Big|.
\end{aligned}
$$

We show that

$$
\begin{aligned}
&M\big((x_n),(z_n),t+s\big)\geqslant M\big((x_n),(y_n),t\big)\\
&\quad\Leftrightarrow\frac{t+s}{t+s+|\lim_{n\to\infty}(x_n-z_n)|}\geqslant\frac{t}{t+|\lim_{n\to\infty}(x_n-y_n)|}\\
&\quad\Leftrightarrow (t+s)\Big|\lim_{n\to\infty}(x_n-y_n)\Big|\geqslant t\Big|\lim_{n\to\infty}(x_n-z_n)\Big|.
\end{aligned}
$$

We have that

$$\begin{aligned} t\Big|\lim_{n\to\infty}(x_n - z_n)\Big| &= t\Big|\lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n + \lim_{n\to\infty} y_n - \lim_{n\to\infty} z_n\Big| \\ &\leqslant t\Big|\lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n\Big| + t\Big|\lim_{n\to\infty} y_n - \lim_{n\to\infty} z_n\Big| \\ &\leqslant t\Big|\lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n\Big| + s\Big|\lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n\Big| \\ &= (t+s)\Big|\lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n\Big|. \end{aligned}$$

We check now (M6).

$$\begin{aligned} &M\big((x_n),(y_n),t\big) = M\big((y_n),(x_n),t\big), \\ &\quad (\forall)t \geqslant 0 \Leftrightarrow \Big|\lim_{n\to\infty}(x_n - y_n)\Big| = \Big|\lim_{n\to\infty}(y_n - x_n)\Big|, \end{aligned}$$

which is true. Thus, $(\mathcal{C}, M, \wedge)$ is a fuzzy pseudo-metric space.

We note that M does not satisfy (M5). Indeed,

$$M\big((x_n),(y_n),t\big) = M\big((y_n),(x_n),t\big) = 1, \quad (\forall)t > 0 \Rightarrow \Big|\lim_{n\to\infty}(x_n - y_n)\Big| = 0,$$

which is not true for arbitrary convergent sequences. Thus, $(\mathcal{C}, M, \wedge)$ is not a fuzzy metric space. □

Theorem 1. *Let $(X, Q, *)$ be a fuzzy quasi-(pseudo-)metric space and $*'$ be a continuous t-norm such that $*' \gg *$. Let M be a fuzzy set in $X \times X \times [0,\infty)$ defined by $M(x,y,t) = Q(x,y,t) *' Q^{-1}(x,y,t)$. Then $(X, M, *)$ is a fuzzy (pseudo-)metric space.*

Proof. It is easy to check (M1), (M2) and (M4). We prove (M3).

$$\begin{aligned} M(x,z,t+s) &= Q(x,z,t+s) *' Q^{-1}(x,z,t+s) \\ &\geqslant \big[Q(x,y,t) * Q(y,z,s)\big] *' \big[Q^{-1}(x,y,t) * Q^{-1}(y,z,s)\big] \\ &\geqslant \big[Q(x,y,t) *' Q^{-1}(x,y,t)\big] * \big[Q(y,z,s) *' Q^{-1}(y,z,s)\big] \\ &= M(x,y,t) * M(y,z,s). \end{aligned}$$

Now we check (M6).

$$\begin{aligned} M(y,x,t) &= Q(y,x,t) *' Q^{-1}(y,x,t) = Q(y,x,t) *' Q(x,y,t) \\ &= Q(x,y,t) *' Q(y,x,t) = Q(x,y,t) *' Q^{-1}(x,y,t) = M(x,y,t). \end{aligned}$$

Thus M is a fuzzy pseudo-metric. If, in addition, Q is a fuzzy quasi-metric we will show that M is a fuzzy metric, namely M also satisfies (M5). If Q is a fuzzy quasi-metric, from (M5) we obtain that for $x \neq y$, there exists $t_{xy} > 0$ such that $Q(x, y, t_{xy}) \neq 1$ or $Q(y, x, t_{xy}) \neq 1$. Thus $M(x, y, t_{xy}) \neq 1$ and $M(y, x, t_{xy}) \neq 1$. Therefore M satisfies (M5). □

Corollary 1. *Let* $(X, Q, *)$ *be a fuzzy quasi-(pseudo-)metric space and* $M(x, y, t) = \min\{Q(x, y, t), Q(y, x, t)\}$. *Then* $(X, M, *)$ *is a fuzzy (pseudo-)metric space.*

Proof. We apply previous proposition for $*' = \wedge \gg *$. □

4. Partial Ordering Relations and Equivalence Relations Generated by Fuzzy Quasi-Pseudo-Metrics

In theoretical computer science and in information theory it is usual to work with sequences (x_n) of objects of increasing information. In this case the relation $x \leqslant y$ is understood in the sense that y contains at least as much information as x. In this section, we prove that any partial order relation can be modeled by a fuzzy quasi-metric. Also, any equivalence relation can be modeled by a fuzzy pseudo-metric.

Theorem 2. *If* $(X, M, *)$ *is a fuzzy quasi-pseudo-metric space, then the relation* $\leqslant_M$ *on* X *defined by*

$$x \leqslant_M y \text{ if and only if } M(x, y, t) = 1, \quad (\forall) t > 0$$

is reflexive and transitive.

If $(X, M, *)$ *is a fuzzy quasi-metric space, then the relation* $\leqslant_M$ *is a partial ordering.*

Proof. $x \leqslant_M x \Leftrightarrow M(x, x, t) = 1$, $(\forall) t > 0 \Leftrightarrow$ (M2).

If $x \leqslant_M y$ and $y \leqslant_M z$, then $M(x, y, t) = 1$, $(\forall) t > 0$ and $M(y, z, s) = 1$, $(\forall) s > 0$. From (M3) we obtain that $M(x, z, t + s) = 1$, $(\forall) t, s > 0$. Thus $x \leqslant_M z$.

We assume now that $(X, M, *)$ is a fuzzy quasi-metric space. Let $x, y \in X$ such that $x \leqslant_M y$ and $y \leqslant_M x$. Then $M(x, y, t) = M(y, x, t) = 1$, $(\forall) t > 0$. Using (M5) we obtain that $x = y$. □

Theorem 3. *Let* $\leqslant$ *be a reflexive and transitive relation on a set* X. *Then there exists a fuzzy quasi-pseudo-metric* $(M, \wedge)$ *on* X *such that the relation* $\leqslant_M$ *generated by* M *is the same with the relation* $\leqslant$.

Proof. Let $M : X \times X \times [0, \infty) \to [0, 1]$ defined by

$$M(x, y, t) = \begin{cases} 1 & \text{if } x \leqslant y,\ t > 0, \\ 1 - e^{-t} & \text{otherwise.} \end{cases}$$

We show that $(X, M, \wedge)$ is a fuzzy quasi-pseudo-metric space. (M1), (M2) and (M4) are obvious. We check (M3).

Let $x, y, z \in X$ and $t, s \geqslant 0$. It $t = 0$ or $s = 0$, then $M(x, y, t) = 0$ or $M(y, z, s) = 0$ and the inequality $M(x, z, t + s) \geqslant (x, y, t) \wedge M(y, z, s)$ holds. We suppose that $t > 0$, $s > 0$. If $x \leqslant z$, then $M(x, z, t + s) = 1$ and the inequality $M(x, z, t + s) \geqslant M(x, y, t) \wedge M(y, z, s)$ is obvious.

If $x > z$, then $x > y$ or $y > z$. We assume, without restricting the general case, that $x > y$.

Case 1. $x > z$, $x > y$, $y > z$.

In this case $M(x, z, t + s) = 1 - e^{-(t+s)}$, $M(x, y, t) = 1 - e^{-t}$, $M(y, z, s) = 1 - e^{-s}$. We suppose that $t \leqslant s$ (the case $s \leqslant t$ is similar). Then $1 - e^{-t} \leqslant 1 - e^{-s}$. Thus $M(x, y, t) \wedge M(y, z, s) = 1 - e^{-t}$ and

$$M(x, z, t + s) \geqslant M(x, y, t) \wedge M(y, z, s) \Leftrightarrow 1 - e^{-(t+s)} \geqslant 1 - e^{-t},$$

which is true.

Case 2. $x > z$, $x > y$, $y \leqslant z$.

In this case $M(x, z, t + s) = 1 - e^{-(t+s)}$, $M(x, y, t) = 1 - e^{-t}$, $M(y, z, s) = 1$. Thus $M(x, y, t) \wedge M(y, z, s) = 1 - e^{-t}$ and

$$M(x, z, t + s) \geqslant M(x, y, t) \wedge M(y, z, s) \Leftrightarrow 1 - e^{-(t+s)} \geqslant 1 - e^{-t},$$

which is true.

Therefore $(X, M, \wedge)$ is fuzzy quasi-pseudo-metric space.

Finally, $x \leqslant_M y \Leftrightarrow M(x, y, t) = 1$, $(\forall)t > 0 \Leftrightarrow x \leqslant y$. □

Theorem 4. *Let $\leqslant$ be a partial ordering on a set X. Then, there exists a fuzzy quasi-metric space $(X, M, \wedge)$ such that the relation $\leqslant_M$ generated by M is the same with the relation $\leqslant$.*

Proof. Let $(X, M, \wedge)$ be the fuzzy quasi-pseudo-metric space from previous theorem. We check $(M5)$.

$$M(x, y, t) = M(y, x, t) = 1, \quad (\forall)t > 0 \Rightarrow x \leqslant y \text{ and } y \leqslant x \Rightarrow x = y.$$

Therefore $(X, M, \wedge)$ is a fuzzy quasi-metric space. □

Theorem 5. *If $(X, M, *)$ is a fuzzy pseudo-metric space then the relation $\sim_M$ on X defined by: $x \sim_M y$ if and only if $M(x, y, t) = 1$, $(\forall)t > 0$ is an equivalence relation.*

Proof. We obtain that $\sim_M$ is reflexive and transitive as in the proof of Theorem 2. We show that $\sim_M$ is symmetric. Let $x, y \in X$ such that $x \sim_M y$. Thus $M(x, y, t) = 1$, $(\forall)t > 0$. As M satisfies (M6) we obtain that $M(y, x, t) = 1$, $(\forall)t > 0$. Thus $y \sim_M x$. □

Theorem 6. *Let $\sim$ be an equivalence relation on a set X. Then there exists a fuzzy pseudo-metric space $(X, M, \wedge)$ such that the relation $\sim_M$ generated by M is the same with the relation $\sim$.*

Proof. Applying Theorem 3, we obtain that $M : X \times X \times [0, \infty) \to [0, 1]$ defined by

$$M(x, y, t) = \begin{cases} 1 & \text{if } x \sim y,\ t > 0, \\ 1 - e^{-t} & \text{otherwise} \end{cases}$$

is a fuzzy quasi-pseudo-metric with the property that the relation $\sim_M$ generated by M is the same with the relation $\sim$. We check (M6) and we obtain that M is a fuzzy pseudo-metric.

$$\begin{aligned} M(y, x, t) &= \begin{cases} 1 & \text{if } y \sim x,\ t > 0 \\ 1 - e^{-t} & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } x \sim y,\ t > 0 \\ 1 - e^{-t} & \text{otherwise} \end{cases} = M(x, y, t). \end{aligned}$$ □

5. Decomposition Theorems of Fuzzy Quasi-Pseudo-Metrics

In Section 5, we apply the techniques introduced in Nădăban and Dzitac (2014), where we obtain decomposition theorems of fuzzy norms on a linear space into a family of crisp semi-norms. Thus, we obtain that there exists a bijective correspondence between a fuzzy quasi-pseudo-metric and a right continuous and ascending family of quasi-pseudo-metrics.

Theorem 7. *Let $(X, M, \wedge)$ be a fuzzy quasi-pseudo-metric space and*

$$p_\alpha(x, y) := \inf\{t > 0 : M(x, y, t) > \alpha\}, \quad \alpha \in (0, 1).$$

Then $\mathcal{P} = \{p_\alpha\}_{\alpha \in (0,1)}$ is an ascending family of quasi-pseudo-metrics on X.

Proof. (p1) $p_\alpha(x, x) = \inf\{t > 0 : M(x, x, t) > \alpha\} = 0$.

(p2) $p_\alpha(x, y) + p_\alpha(y, z)$

$$\begin{aligned} &= \inf\{t > 0 : M(x, y, t) > \alpha\} + \inf\{s > 0 : M(y, z, s) > \alpha\} \\ &= \inf\{t + s > 0 : M(x, y, t) > \alpha,\ M(y, z, s) > \alpha\} \\ &= \inf\{t + s > 0 : M(x, y, t) \wedge M(y, z, s) > \alpha\} \\ &\geqslant \inf\{t + s > 0 : M(x, z, t + s) > \alpha\} = p_\alpha(x, z). \end{aligned}$$

It remains to prove that $\mathcal{P} = \{p_\alpha\}_{\alpha \in (0,1)}$ is an ascending family. Let $\alpha_1 \leqslant \alpha_2$. Then $\{t > 0 : M(x, y, t) > \alpha_2\} \subseteq \{t > 0 : M(x, y, t) > \alpha_1\}$. Thus

$$\inf\{t > 0 : M(x, y, t) > \alpha_2\} \geqslant \inf\{t > 0 : M(x, y, t) > \alpha_1\},$$

namely $p_{\alpha_2}(x, y) \geqslant p_{\alpha_1}(x, y)$, $(\forall)(x, y) \in X \times X$. □

Theorem 8. *Let* $(X, M, \wedge)$ *be a fuzzy pseudo-metric space and*

$$p_\alpha(x, y) := \inf\{t > 0 : M(x, y, t) > \alpha\}, \quad \alpha \in (0, 1).$$

Then $\mathcal{P} = \{p_\alpha\}_{\alpha\in(0,1)}$ *is an ascending family of pseudo-metrics on* X.

Proof. We must check (p4).

$$\begin{aligned} p_\alpha(x, y) &:= \inf\{t > 0 : M(x, y, t) > \alpha\} \\ &= \inf\{t > 0 : M(y, x, t) > \alpha\} = p_\alpha(y, x). \end{aligned}$$

□

Proposition 1. *Let* $(X, M, *)$ *be a fuzzy quasi-pseudo-metric space and*

$$p_\alpha(x, y) := \inf\{t > 0 : M(x, y, t) > \alpha\}, \quad \alpha \in (0, 1).$$

Then, for $x, y \in X$, $s > 0$, $\alpha \in (0, 1)$, *we have*

$$p_\alpha(x, y) < s \text{ if and only if } M(x, y, s) > \alpha.$$

Proof. "⇒" We prove that $s \in \{t > 0 : M(x, y, t) > \alpha\}$. We suppose that $s \notin \{t > 0 : M(x, y, t) > \alpha\}$. Then there exists $t_0 \in \{t > 0 : M(x, y, t) > \alpha\}$ such that $t_0 < s$. (Contrary, $s \leqslant t$, $(\forall)t \in \{t > 0 : M(x, y, t) > \alpha\}$. Hence $s \leqslant \inf\{t > 0 : M(x, y, t) > \alpha\}$, i.e. $s \leqslant p_\alpha(x, y)$, which is a contradiction.) As $t_0 \in \{t > 0 : M(x, y, t) > \alpha\}$, $t_0 < s$ and $M(x, y, \cdot)$ is nondecreasing, we have that $M(x, y, s) > \alpha$, which leads to a contradiction.

"⇐" $M(x, y, s) > \alpha \Rightarrow s \in \{t > 0 : M(x, y, t) > \alpha\} \Rightarrow p_\alpha(x, y) \leqslant s$.

We suppose that $p_\alpha(x, y) = s$. As $M(x, y, \cdot)$ is left continuous in s, we have $\lim_{t\to s, t<s} M(x, y, t) = N(x, s)$. Thus there exists $t_0 < s$ such that $M(x, y, t_0) > \alpha$. But $t_0 < s$ and $M(x, y, t_0) > \alpha$ are in contradiction with the fact that $s = \inf\{t > 0 : M(x, y, t) > \alpha\}$. Hence $p_\alpha(x, y) \neq s$. Thus $p_\alpha(x, y) < s$. □

Definition 5. An ascending family $\{p_\alpha\}_{\alpha\in(0,1)}$ of quasi-pseudo-metric on a set X is called right continuous if for any decreasing sequence (α_n) in $(0, 1)$, $\alpha_n \to \alpha \in (0, 1)$, we have $p_{\alpha_n}(x, y) \to p_\alpha(x, y)$, $(\forall)x, y \in X$.

Theorem 9. *Let* $(X, M, \wedge)$ *be a be a fuzzy quasi-pseudo-metric space and*

$$p_\alpha(x, y) := \inf\{t > 0 : M(x, y, t) > \alpha\}, \quad \alpha \in (0, 1).$$

Then $\mathcal{P} = \{p_\alpha\}_{\alpha\in(0,1)}$ *is right continuous.*

Proof. Let $x, y \in X$ and (α_n) a decreasing sequence in $(0, 1)$, $\alpha_n \to \alpha \in (0, 1)$. Let $s > p_\alpha(x, y)$. Then $M(x, y, s) > \alpha$. As (α_n) a decreasing sequence and $\alpha_n \to \alpha$, there exists $n_0 \in \mathbb{N}$ such that $\alpha_n < M(x, y, s)$, $(\forall)n \geqslant n_0$. Therefore $p_{\alpha_n}(x, y) < s$, $(\forall)n \geqslant n_0$. Thus $p_{\alpha_n}(x, y) \to p_\alpha(x, y)$. □

Theorem 10. *Let $\{q_\alpha\}_{\alpha\in(0,1)}$ be an ascending family of quasi-pseudo-metrics on a set X. We agree that if $\{\alpha \in (0,1) : q_\alpha(x,y) < t\} = \emptyset$, then we put* $\sup\{\alpha \in (0,1) : q_\alpha(x,y) < t\} = 0$. *Let $M' : X \times X \times [0,\infty) \to [0,1]$, defined by*

$$M'(x,y,t) = \begin{cases} \sup\{\alpha \in (0,1) : q_\alpha(x,y) < t\} & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Then $(X, M', \wedge)$ is a fuzzy quasi-pseudo-metric space.

Proof. First, we note that $M'(x,y,\cdot)$ is nondecreasing. Indeed, for $t_1 < t_2$, we have $\{\alpha \in (0,1) : q_\alpha(x,y) < t_1\} \subseteq \{\alpha \in (0,1) : q_\alpha(x,y) < t_2\}$. Thus

$$\sup\left\{\alpha \in (0,1) : q_\alpha(x,y) < t_1\right\} \leqslant \sup\left\{\alpha \in (0,1) : q_\alpha(x,y) < t_2\right\}.$$

Hence $M'(x,y,t_1) \leqslant N'(x,y,t_2)$.

(M1) $M'(x,y,0) = 0$, $(\forall)x,y \in X$ is obvious.

(M2) $M'(x,x,t) = \sup\{\alpha \in (0,1) : q_\alpha(x,x) < t\} = \sup\{\alpha \in (0,1) : 0 < t\} = 1$.

(M3) The inequality $M'(x,z,t+s) \geqslant M'(x,y,t) \wedge M'(y,z,s)$ is obvious for $t = 0$ or $s = 0$. Let $t > 0$, $s > 0$. We suppose that $M'(x,z,t+s) < M'(x,y,t) \wedge M'(y,z,s)$. Then there exists $\alpha_0 \in (0,1)$ such that

$$M'(x,z,t+s) < \alpha_0 < M'(x,y,t) \wedge M'(y,z,s).$$

As $M'(x,y,t) > \alpha_0$, there exists $\beta_1 \in \{\alpha \in (0,1) : q_\alpha(x,y) < t\}$ such that $\beta_1 > \alpha_0$. As $M'(y,z,s) > \alpha_0$, there exists $\beta_2 \in \{\alpha \in (0,1) : q_\alpha(y,z) < s\}$ such that $\beta_2 > \alpha_0$. Let $\beta_0 = \min\{\beta_1, \beta_2\}$. Then $\beta_0 > \alpha_0$ and $q_{\beta_0}(x,y) < t$, $q_{\beta_0}(y,z) < s$. Thus $q_{\beta_0}(x,z) \leqslant q_{\beta_0}(x,y) + q_{\beta_0}(y,z) < t+s$. Hence

$$\beta_0 \in \left\{\alpha \in (0,1) : q_\alpha(x,z) < t+s\right\} \Rightarrow M'(x,z,t+s) \geqslant \beta_0 > \alpha_0,$$

which is in contradiction. Hence $M'(x,z,t+s) \geqslant M'(x,y,t) \wedge M'(y,z,s)$.

(M4) We prove that $\lim_{t\to\infty} M'(x,y,t) = 1$. Let $\alpha_0 \in (0,1)$ arbitrary. We show that there exists $t_0 > 0$ such that $M'(x,y,t_0) > \alpha_0$. Let $t_0 > q_{\alpha_1}(x,y)$, where $\alpha_1 = \frac{1+\alpha_0}{2} \in (\alpha_0, 1)$. Then

$$M'(x,y,t_0) = \sup\left\{\alpha \in (0,1) : q_\alpha(x,y) < t_0\right\} \geqslant \alpha_1 > \alpha_0.$$

We prove now that $M'(x,y,\cdot)$ is left continuous in $t > 0$.

Case 1. $M'(x,y,t) = 0$. Thus, for all $s \leqslant t$, we have $M'(x,y,s) = 0$. Therefore $\lim_{s\to t, s<t} M'(x,y,s) = 0 = M'(x,y,t)$.

Case 2. $M'(x,y,t) > 0$. Let $\alpha_0 : 0 < \alpha_0 < M'(x,y,t)$. Let (t_n) be a sequence such that $t_n \to t$, $t_n < t$. We show that there exists $n_0 \in \mathbb{N}$ such that

$$M'(x,y,t_n) > \alpha_0, \quad (\forall)n \geqslant n_0.$$

As $0 < \alpha_0 < M'(x, y, t)$, there exists $\beta_0 \in \{\alpha \in (0, 1) : q_\alpha(x, y) < t\}$ such that $\beta_0 > \alpha_0$. As $q_{\beta_0}(x, y) < t$ and $t_n \to t$, $t_n < t$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, we have $t_n > q_{\beta_0}(x, y)$. Thus $M'(x, y, t_n) \geqslant \beta_0 > \alpha_0$, $(\forall) n \geqslant n_0$. □

Theorem 11. *Let* $(X, M, \wedge)$ *be a fuzzy quasi-pseudo-metric space and*

$$p_\alpha(x, y) := \inf\big\{t > 0 : M(x, y, t) > \alpha\big\}, \quad \alpha \in (0, 1).$$

Let $M' : X \times X \times [0, \infty) \to [0, 1]$*, defined by*

$$M'(x, y, t) = \begin{cases} \sup\{\alpha \in (0, 1) : p_\alpha(x, y) < t\} & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Then

1. $\mathcal{P} = \{p_\alpha\}_{\alpha \in (0,1)}$ *is a right continuous and ascending family of quasi-pseudo-metrics on* X;
2. $(X, M', \wedge)$ *is a fuzzy quasi-pseudo-metric space*;
3. $M' = M$.

Proof.
1. It results from Theorems 7 and 9.
2. It results from Theorem 10.
3. For $t = 0$, we have $M'(x, y, t) = 0 = M(x, y, t)$. For $t > 0$, we have

$$\begin{aligned} M'(x, y, t) &= \sup\big\{\alpha \in (0, 1) : p_\alpha(x, y) < t\big\} \\ &= \sup\big\{\alpha \in (0, 1) : M(x, y, t) > \alpha\big\} = M(x, y, t). \end{aligned}$$ □

Theorem 12. *Let* $\{q_\alpha\}_{\alpha \in (0,1)}$ *be an ascending family of quasi-pseudo-metrics on a set* X*. Let* $M' : X \times X \times [0, \infty) \to [0, 1]$*, defined by*

$$M'(x, y, t) = \begin{cases} \sup\big\{\alpha \in (0, 1) : q_\alpha(x, y) < t\big\} & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Let $p_\alpha : X \to [0, \infty)$ *defined by*

$$p_\alpha(x, y) := \inf\big\{t > 0 : M'(x, y, t) > \alpha\big\}, \quad \alpha \in (0, 1).$$

Then

1. $(X, M', \wedge)$ *is a fuzzy quasi-pseudo-metric space*;
2. $\mathcal{P} = \{p_\alpha\}_{\alpha \in (0,1)}$ *is a right continuous and ascending family of quasi-pseudo-metrics on* X;
3. $p_\alpha = q_\alpha$, $(\forall)\alpha \in (0, 1)$ *if and only if* $\{q_\alpha\}_{\alpha \in (0,1)}$ *is right continuous.*

Proof.

1. It results from Theorem 10.
2. It results from Theorems 7 and 9.
3. "⇒" It is obvious.

"⇐" We suppose that there exists $\alpha_0 \in (0,1)$ such that $p_{\alpha_0} \neq q_{\alpha_0}$. Then there exists $x, y \in X$ such that $p_{\alpha_0}(x,y) \neq q_{\alpha_0}(x,y)$.

Case A. $p_{\alpha_0}(x,y) < q_{\alpha_0}(x,y)$. Let $s > 0$ such that $p_{\alpha_0}(x,y) < s < q_{\alpha_0}(x,y)$. As $p_{\alpha_0}(x,y) < s$, we have $M'(x,y,s) > \alpha_0$. We suppose that $\alpha_0 < \sup\{\alpha \in (0,1) : q_\alpha(x,y) < s\}$. Then there exists $\beta \in \{\alpha \in (0,1) : q_\alpha(x,y) < s\} : \alpha_0 < \beta$. Thus $q_{\alpha_0}(x,y) \leqslant q_\beta(x,y) < s$, which contradicts the fact that $q_{\alpha_0}(x,y) > s$. Hence $\alpha_0 \geqslant \sup\{\alpha \in (0,1) : q_\alpha(x,y) < s\}$, i.e. $\alpha_0 \geqslant M'(x,y,t)$, which is a contradiction.

Case B. $q_{\alpha_0}(x,y) < p_{\alpha_0}(x,y)$. Let $\beta \in (\alpha_0, 1)$. We will show that $p_{\alpha_0}(x,y) \leqslant q_\beta(x,y)$. We suppose that $p_{\alpha_0}(x,y) > q_\beta(x,y)$. Let $s > 0 : q_\beta(x,y) < s < p_{\alpha_0}(x,y)$. As $q_\beta(x,y) < s$, we have $M'(x,y,s) \geqslant \beta > \alpha_0$. Thus $p_{\alpha_0}(x,y) < s$, which is a contradiction. Hence $p_{\alpha_0}(x,y) \leqslant q_\beta(x,y)$, $(\forall)\beta \in (\alpha_0, 1)$. Thus $p_{\alpha_0}(x,y) \leqslant_{\beta \to \alpha_0, \beta > \alpha_0} q_\beta(x,y)$. Therefore $p_{\alpha_0}(x,y) \leqslant q_{\alpha_0}(x,y)$, which is a contradiction. □

6. Applications and Further Works

The domain of words play an important role in denotational semantics. In this section, we show that a fuzzy quasi-pseudo-metric can be consider on the domain of words. Schellekens (1995) introduced the so-called complexity space in order to develop a topological foundation for complexity analysis of algorithms and programs. Based on our results, a fuzzy complexity space can be considered in future papers. Edalat and Heckmann (1998) established new connection between the theory of metric spaces and domain theory, the two basic mathematical structures in computer science. In this paper we show that a fuzzy quasi-metric can be considered on the set of all closed formal balls $BX = X \times [0,\infty)$. Similarly, we can introduce and analyze, in a future paper, a partial ordering relation on the set $BX = X \times [0,\infty) \times [0,1]$.

6.1. *Applications to the Domain of Words*

Let Σ^∞ be the domain of words, namely the set of all finite and infinite sequences ("words") over a nonempty alphabet Σ. We denote by $\emptyset$ the empty sequence. On Σ^∞ we denote by $\sqsubseteq$ the prefix order, i.e. $x \sqsubseteq y \Leftrightarrow x$ is a prefix of y. Let $x \cap y$ be the common prefix of x and y and for $x \in \Sigma^\infty$ we denote by $l(x)$ the length of x. We have that $l(x) \in [1,\infty]$, for $x \neq \emptyset$ and $l(\emptyset) = 0$.

Smyth (1987) introduced the following quasi-pseudo-metric:

$$d_\sqsubseteq : \Sigma^\infty \times \Sigma^\infty \to [0,\infty) \text{ defined by } d_\sqsubseteq(x,y) = \begin{cases} 0 & \text{if } x \sqsubseteq y, \\ 2^{-l(x\cap y)} & \text{otherwise.} \end{cases}$$

We must note that this quasi-pseudo-metric is a slight modification of Baire metric on Σ^∞, which is defined by

$$d_B(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2^{-l(x \cap y)} & \text{if } x \neq y, \end{cases}$$

where we adopt the convention $2^{-\infty} = 0$.

As every quasi-pseudo-metric induces, in a natural way, a fuzzy quasi-pseudo-metric, the problem of extending Smyth's work to fuzzy context has to be analyzed in future papers.

If we are in this context of Σ^∞ space, we should also consider an issue mentioned in Bukatin *et al.* (2009) and which leads us to the study of those metrics d for which $d(x, x)$ may not be zero, but a strictly positive number or, more general it may lead to the study of some types of fuzzy metric-like spaces.

Let us assume that we are interested to write a computer program to print out the values $x_0, x_1, x_2, \dots$ of a sequence x. As x is an infinite sequence we will not be able to print out all its values and so a computer scientist is interested in how the sequence x is formed by its part, the finite sequence $\langle x_0 \rangle$, $\langle x_0, x_1 \rangle$, $\langle x_0, x_1, x_2 \rangle$ etc. If the metric d_B is extended to Σ^* of all finite sequence over Σ we will have that $d_B(x, x) = 2^{-k}$ for some number $k < \infty$, which is not zero.

6.2. *Complexity Space*

Let f be a partial recursive function and $[f]$ be the set of all programs computing a partial recursive function which approximates f. Let C_P be the complexity function of a program P. Schellekens (1995) introduced the so-called "complexity space" in order to develop a topological foundation for complexity analysis of algorithms and programs. The "complexity distance", defined below, between programs P and Q measures the relative progress made in lowering the complexity by replacing any program P with the complexity function C_P by any program Q with the complexity function C_Q:

$$d(P, Q) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \max\left(\frac{1}{C_Q(n)} - \frac{1}{C_P(n)}, 0\right).$$

We must note that we adopt the convention $\frac{1}{\infty} = 0$ and the distance is normalized by the factor $\frac{1}{2^n}$ to guarantee the convergence of the series. This distance is defined on the complexity space

$$\mathcal{C} := \left\{ f : \mathbb{N}^* \to (0, \infty] : \sum_{n=1}^{\infty} \frac{1}{f(n)} 2^{-n} < \infty \right\}.$$

We mention that the complexity distance is a quasi-pseudo-metric and d is not necessarily a quasi-metric, that is there may be programs P and Q such that $d(P, Q) = 0$ and $P \neq Q$.

As every quasi-pseudo-metric induces in a natural way a fuzzy quasi-pseudo-metric, the study of fuzzy complexity space has to be considered in future papers.

We also note that, recently, Romaguera and Tirado (2015) obtained a general fixed point theorem in the settings of the complexity space, from which they deduced, in a unified and fast way, the existence of solution for a large class of algorithms defined by recurrence equations that includes Hanoi, Largetwo (average case) and Quicksort (worst case). The extension of these results is a real challenge.

6.3. *The Poset BX of Formal Balls*

Edalat and Heckmann (1998) constructed a computational model for metric spaces based on the notion of formal ball. In this way, they established new connections between the theory of metric spaces and domain theory, the two basic mathematical structures in computer science.

Given a metric space (X, d), the set of closed formal balls is given by $BX := X \times [0, \infty)$. Then $(BX, \subseteq)$ is a partial order set (shortly poset), where $\subseteq$ is the partial order given by

$$(x, r) \subseteq (y, s) \Leftrightarrow d(x, y) \leqslant r - s, \quad (\forall)(x, r), (y, s) \in BX.$$

Later on, Heckmann (1999) proved that the map $p : BX \times BX \to [0, \infty)$ given by $p((x, r), (y, s)) = \max\{d(x, y), |r - s|\} + r + s$ is a partial metric in the sense of Matthews (1994), such that the topology generated by p coincides with the Scott topology on $(BX, \subseteq)$.

Since every partial metric p induces a quasi-metric q_p, we deduce that the map

$$q_p((x, r), (y, s)) = \max\{d(x, y), |r - s|\} + s - r$$

is a quasi-metric on BX such that the topology $\mathcal{T}_{q_p}$ coincides with the Scott topology on BX.

There are several ways of generalizations. On one hand we can start from q_p and to obtain a fuzzy quasi-metric on BX. On the other hand, if we consider (BX, p) as a metric-like space, we can extend partially metric p and obtain a fuzzy metric-like space. Eventually, motivated by Edalat and Heckmann's ideas we can considered a fuzzy metric space $(X, M, *)$ and we can also introduce and analyze a partial ordering relation on the set of closed formal balls $BX := X \times [0, \infty) \times [0, 1]$.

6.4. *The Interval Domain*

The interval domain $\mathcal{I}$ gives the set $\mathbb{R}$ of real numbers a computational structures. The interval domain is the collection of all compact intervals, endowed with a least element:

$$\mathcal{I} = \big\{[a, b] \subset \mathbb{R} : a, b \in \mathbb{R}, a \leqslant b\big\} \cup \{\bot\}.$$

On $\mathcal{I}$ we have an order, which is the reversed inclusion, namely

$$[a, b] \subseteq [c, d] \Leftrightarrow a \leqslant c \text{ and } b \geqslant d.$$

We note that the least element $\perp$ is the set $\mathbb{R}$ and the maximal elements are the intervals $[a, a]$, that is the singleton sets.

We note that the mapping $d : \mathcal{I} \times \mathcal{I} \to [0, \infty)$ defined by

$$d([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\} + a - b$$

is a quasi-metric on $\mathcal{I}$, whose specialization order coincides with the reversed inclusion.

As every quasi-metric induces in a natural way a fuzzy quasi-metric, the problem of extending the paper of Edalat and Sünderhauf (1999) to fuzzy context has to be analyzed in future papers.

7. Conclusions

In this paper some properties of fuzzy quasi-pseudo-metric spaces were investigated and among other results established, there should be underline that any partial ordering can be defined by a fuzzy quasi-metric, which can be applied both in theoretical computer science and in information theory, where it is usual to work with sequences of objects of increasing information. The papers also deals with decomposition theorems of a fuzzy quasi-pseudo metric into a right continuous and ascending family of quasi-pseudo metrics. We intend to develop a topological foundation for complexity analysis of algorithms and programs, and based on our results the fuzzy complexity space has to be considered in future papers. Also the present paper provides a fertile ground for further study of some types of fuzzy quasi-pseudo metrics on the domain of words, which play an important role on denotational semantics, and on the poset BX of all closed formal balls on a metric space.

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Kai kurios neraiškiosios Kvazi-Pseudo-metrinės erdvės savybės ir taikymai

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Šiame darbe yra pateiktos kelios neraiškiosios Kvazi-Pseudo-metrinės erdvės savybės. Svarbus rezultatas yra tai, kad bet kuris dalinis rangavimas gali būti aprašytas neraiškioje Kvazi erdvėje, kuri gali būti taikoma tiek teorinėje informatikoje, tiek ir informacijos teorijoje, kur yra įprasta dirbti su augančios informacijos sekų objektais. Taip pat gautos neraiškiosios Kvazi-Pseudo-metrikos skaidymo į tęstinės ir augančios Kvazi-Pseudo-metrikos šeimą teoremos. Sukurtas topologinis algoritmų ir programų kompleksinės analizės pagrindas, ir, remiantis mūsų rezultatais, gali būti nagrinėjama neraiškiojo kompleksiškumo erdvė. Be to, sukurtas pagrindas studijuoti kai kurias neraiškiosios Kvazi-Pseudo-metrikos rūšis žodžių domene, kuris vaidina svarbų vaidmenį ženklinimo semantikoje ir dalinai sutvarkytoje aibėje BX iš metrikos erdvės visų uždarų formalių sferų.