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Induced Uncertain Pure Linguistic Hybrid Averaging Aggregation Operator and Its Application to Group Decision Making

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Abstract. In this paper, we propose a new aggregation operator under uncertain pure linguistic environment called the induced uncertain pure linguistic hybrid averaging aggregation (IUPLHAA) operator. Some of the main advantages and properties of the new operator are studied. Moreover, in the situations where the given arguments about all the attribute weights, the attribute values and the expert weights are expressed in the form of linguistic labels variables, we develop an approach based on the IUPLHAA operator for multiple attribute group decision making with uncertain pure linguistic environment. Finally, an illustrative example is given to verify the developed approach and to demonstrate its feasibility and practicality.

Key words: pure linguistic, IUPLHAA operator, hybrid average, group decision making, aggregation.

1. Introduction

Yager's ordered weighted averaging (OWA) operator (Yager, 1988) is one of the most common aggregation operators found in the literature. It provides a method for aggregating operators that includes the maximum, the minimum, and the average, as special cases. Since its appearance, the OWA operator has been used in a wide range of applications, see, for example (Merigó and Casanovas, 2010b, 2011b; Merigó and Gil-Lafuente, 2011; Xu and Da, 2002; Xu, 2005a, 2005c, 2006b, 2006c; Yager, 1993, 2007, 2009, 2010; Yager and Filev, 1999; Zeng and Su, 2011; Zhou *et al.*, 2008).

Later, Yager and Filev (1999) introduced an interesting generalization of the OWA operator called the induced ordered weighted averaging (IOWA) operator which takes as their argument pairs, called OWA pairs, in which the first component is used to induce an ordering over the second component which is exact numerical values and then aggregate. Since its introduction, it has been studied by a lot of authors. For example, Merigó and Casanovas (2011a, 2011c) developed induced aggregation operators with distance mea-

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sures and considered some of their applications. Wei (2010) and Wei and Zhao (2012a) presented some induced geometric aggregation operators and some induced correlated aggregating operators with intuitionistic fuzzy information. Xia *et al.* (2011) studied the induced aggregation under confidence levels. Xu and Xia (2011) introduced the induced generalized intuitionistic fuzzy operators.

Another interesting research topic focus on the aggregation operator called the uncertain ordered weighted averaging (UOWA) operator, which proposed by Xu and Da (2002). It can be used in situations where the aggregated arguments are taken in the form of uncertain variables. In real life, most available information is vague or imprecise and the aggregation operators based on the exact numbers would lose its effect. Therefore, the UOWA operator is receiving more increasing attention by many authors.]inl6 developed an approach to group decision-making with uncertain preference ordinals. Xu and Cai (2010) introduced the uncertain power averaging aggregation operators with the interval fuzzy preference relations information. Zhou *et al.* (2012) proposed some uncertain generalized aggregation operators. The research on the UOWA operator and its applications has attracted great attentions (Chen *et al.*, 2011; Liu *et al.*, 2011a, 2011b; Peng *et al.*, 2012; Suo *et al.*, 2012; Wei, 2009, 2004, 2010; Xu and Chen, 2008; Zhou *et al.*, 2008).

Recently in the literature, a few authors have done some research on the hybrid averaging (HA) operators. The weighted averaging (WA) operator weights the variables themselves while the OWA operator weights the ordered positions of the variables, that is, weights represent different aspects in both the WA and the OWA operators. To overcome this drawback, Xu (2006a) introduced the linguistic hybrid arithmetic averaging (LHAA) operator, and developed an approach based on the LHAA operator to multiple attribute group decision making with linguistic information. Wei (2009) proposed a method for multiple attribute group decision making under uncertain linguistic environment based on the uncertain linguistic hybrid geometric mean (ULHG) operator. Motivated by the idea of the IUOWA operator, Merigó *et al.* (2012b) introduced the uncertain induced ordered weighted averaging-weighted averaging (UIOWAWA) operator, which unified the uncertain weighted average (UWA) and the uncertain induced ordered weighted averaging (UIOWA) operator in the same formulation. The research on this topic, refer to Wei (2011), Xu (2004, 2009).

As the increasing complexity of the socio-economic environment, in many situations, however, the decision information is expressed in the form of uncertain linguistic variables rather than numerical ones because of time pressure, people's limited expertise related to the problem domain and so on. Zadeh (1975a, 1975b, 1976) introduced the notion of a linguistic variable, whose values are words or sentences in a natural or artificial language, for example, 'good', 'poor', 'fair', 'very fast', 'extremely slow', etc. They are not only used to express fuzzy qualitative information but involved in the calculation. Many related investigations have been focused on group decision making with uncertain linguistic information. Taking into account the IOWA operator under uncertain linguistic environment, Xu (2006b) presented an induced uncertain linguistic OWA (IULOWA) operator version. Herrera *et al.* (2008) proposed an unbalanced linguistic computational model on the basis of the 2-tuple fuzzy linguistic computational model. Xu (2004) developed uncertain linguistic ordered weighted averaging (ULOWA) operator and uncertain linguistic hybrid aggregation (ULHA) operator. Xu (2006c) defined the concept of uncertain multiplicative linguistic preference relation, and developed an approach based on the uncertain LOWG and the induced uncertain LOWG operators to group decision making. For further research on group decision making with uncertain linguistic information and its applications, see, for example (Cordon *et al.*, 2002; Delgado *et al.*, 1993, 1998, 2002; Herrera, 1995; Herrera *et al.*, 1996, 2000; Herrera and Herrera-Viedma, 2000a, 2000b; Herrera and Martínez, 2001; Merigó and Casanovas, 2010a; Merigó et al., 2012a; Wei and Zhao, 2012b; Xu and Da, 2008; Xu, 1992, 2011; Yu *et al.*, 2010; Zhang and Guo, 2002).

However, in the real-life world, most of the decision information was given by the form of pure linguistic variables rather than part of the linguistic variables. Xu (2005b) developed a method based on some operators for multi-attribute group decision making with pure linguistic information under uncertainty. The prominent characteristic of the approach is straightforward and without losing any information. Therefore, research in this area has great significance and it is necessary to extend the linguistic operators to accommodate the induced uncertain pure linguistic situation, which is also the focus of this paper.

We also present a further generalization of the IULOWA operator by using the hybrid average (HA), which reflects the importance degrees of both the given uncertain linguistic variables and their ordered position. To do so, we are able to use the weighted average (WA) and the induced order weighed average (IOWA) in the same formulation with uncertain pure linguistic information.

For this purpose, we shall propose uncertain pure linguistic weighed averaging aggregation (UPLWAA) operator and induced uncertain pure linguistic ordered weighed averaging aggregation (IUPLOWAA) operator. Based on the operators, we shall develop an induced uncertain pure linguistic hybrid averaging aggregation (IUPLHAA) operator, in which the second components are uncertain linguistic variables, and then study some of its desirable properties. Moreover, in the situations where the information about all the attribute weights, the attribute values and the expert weights are taken in the form of linguistic variables, we shall develop an approach to multiple attribute group decision making based on the IUPLHAA operator under uncertainty.

This paper is structured as follows. In Section 2, we review the uncertain linguistic variables and introduce their operational laws. In Section 3, we briefly review the most common aggregation operators. In Section 4, we first develop uncertain pure linguistic weighed averaging aggregation (UPLWAA) operator and induced uncertain pure linguistic ordered weighed averaging aggregation (IUPLOWAA) operator and then, we develop the induced uncertain pure linguistic hybrid averaging aggregation (IUPLHAA) operator, and study some its desirable properties. In Section 5, we present an approach based on the developed operators for uncertain pure linguistic multiple attribute group decision making under uncertainty. Section 6 gives an illustrative example and presents a comparative analysis with other related decision making methods, and finally, the main conclusions of the paper are summarized in Section 7.

2. Uncertain Linguistic Variables and Their Operational Laws

A lot of information in our real-life world cannot be assessed in a quantitative form but may be in a qualitative one (Herrera and Herrera-Viedma, 2000a; Zadeh, 1975a, 1975b, 1976). A linguistic variable is a variable whose value is not crisp number but word or sentence in a natural language. And an approach based on the linguistic variable is an approximate technique, which represents qualitative aspects as linguistic values by means of linguistic variables. For more details, see Xu (1992).

Let $S = \{s_\alpha \mid \alpha = -t, \ldots, -1, 0, 1, \ldots, t\}$ be a finite and totally ordered discrete term set, where s_α represents a possible value for a linguistic variable. For example, S can be defined as:

$$S = \{s_{-4} = \textit{extremely poor}, s_{-3} = \textit{very poor}, s_{-2} = \textit{poor}, s_{-1} = \textit{slightly poor},$$
$$s_0 = \textit{fair}, s_1 = \textit{slightly good}, s_2 = \textit{good}, s_3 = \textit{very good}, s_4 = \textit{extremely good}\}.$$

And it is required that the linguistic label set should satisfy the following characteristics (Herrera *et al.*, 1996, 2000):

(1) the set is ordered: $s_\alpha > s_\beta$, if $\alpha > \beta$;
(2) there is the negative operator: $neg(s_\alpha) = s_{-\alpha}$. Especially, $neg(s_0) = s_0$;
(3) max operator: $\max(s_\alpha, s_\beta) = s_\alpha$ if $s_\alpha > s_\beta$;
(4) min operator: $\min(s_\alpha, s_\beta) = s_\alpha$ if $s_\alpha < s_\beta$.

Note that in the process of given information aggregating, some decision results may do not match any linguistic labels exactly. To preserve all the given information, Xu (2004) extended the discrete label set S to a continuous label set $\tilde{S} = \{s_\alpha \mid \alpha \in [-q, q]\}$, where $q\,(|q| > t)$ is a sufficiently large positive number. If $s_\alpha \in S$, then we call s_α original linguistic label, otherwise, we call s_α the virtual linguistic label. Generally, the decision maker (DM) uses the original linguistic labels to evaluate attributes and alternatives, and the virtual linguistic labels can only appear in calculation.

Xu and Da (2002) developed the uncertain linguistic variable and the degree of possibility of the uncertain linguistic variable, which can be defined as following:

Definition 1. Let $\tilde{s} = [s_\alpha, s_\beta]$, where $s_\alpha, s_\beta \in \tilde{S}$, s_α and s_β are the lower and the upper limits, respectively, then $\tilde{s}$ is called an uncertain linguistic variable. Especially, $\tilde{s}$ is a real linguistic variable, if $s_\alpha = s_\beta$.

Definition 2. Let $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$, $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}] \in \tilde{S}$ be two uncertain linguistic variables, and let $l_{\tilde{s}_1} = \beta_1 - \alpha_1$, $l_{\tilde{s}_2} = \beta_2 - \alpha_2$, then, the degree of possibility of $\tilde{s}_2 \geqslant \tilde{s}_1$ is defined as following:

$$p(\tilde{s}_2 \geqslant \tilde{s}_1) = \max\left\{1 - \max\left(\frac{\alpha_2 - \beta_1}{l_{\tilde{s}_1} + l_{\tilde{s}_2}}, 0\right), 0\right\}. \tag{1}$$

Consider any three uncertain linguistic variables $\tilde{s} = [s_\alpha, s_\beta]$, $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$ and $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}]$, and let $\lambda, \lambda_1, \lambda_2 > 0$, then, we define their operational laws as follows:

(1) $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}] \oplus [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1} \oplus s_{\alpha_2}, s_{\beta_1} \oplus s_{\beta_2}] = [s_{\alpha_1+\alpha_2}, s_{\beta_1+\beta_2}]$;
(2) $\tilde{s}_1 \otimes \tilde{s}_2 = [s_{\alpha_1}, s_{\beta_1}] \otimes [s_{\alpha_2}, s_{\beta_2}] = [s_{\alpha_1} \otimes s_{\alpha_2}, s_{\beta_1} \otimes s_{\beta_2}] = [s_{\alpha_1\alpha_2}, s_{\beta_1\beta_2}]$;
(3) $\lambda\tilde{s} = \lambda[s_\alpha, s_\beta] = [\lambda s_\alpha, \lambda s_\beta] = [s_{\lambda\alpha}, s_{\lambda\beta}]$;
(4) $\lambda(\tilde{s}_1 \oplus \tilde{s}_2) = \lambda\tilde{s}_1 \oplus \lambda\tilde{s}_2$;
(5) $(\lambda_1 + \lambda_2)\tilde{s} = \lambda_1\tilde{s} \oplus \lambda_2\tilde{s}$.

3. Some Basic Review

In this section, we briefly review some operators, the OWA operator, the IOWA operator, and the IULOWA operator.

3.1. *The OWA Operator*

The OWA operator (Yager, 1988) is the most common aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum. It is defined as follows:

Definition 3. An OWA operator of dimension n is a mapping OWA: $R^n \to R$ that has an associated n vector $w = (w_1, w_2, \dots, w_n)^T$ such that $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$ and

$$OWA_w(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{2}$$

where b_j is the jth largest of the a_i.

3.2. *The Induced OWA Operator*

An interesting generalization of the OWA operator called the induced ordered weighted averaging (IOWA) operator, which is proposed by Yager and Filev (1999). It is defined as follows:

Definition 4. An IOWA operator of dimension n is a mapping IOWA: $R^n \times R^n \to R$ that has an associated n vector $w = (w_1, w_2, \dots, w_n)^T$ such that $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$ and

$$IOWA_w\big(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \dots, \langle u_n, a_n\rangle\big) = \sum_{j=1}^{n} w_j b_j, \tag{3}$$

where b_j is the a_i value of the OWA pair $\langle u_i, a_i\rangle$ having the jth largest u_i, and u_i in $\langle u_i, a_i\rangle$ is referred to as the order inducing variable and a_i as the argument variable.

The IOWA operator takes as its argument pairs, called OWA pairs, in which the first component is used to induce an ordering over the second components which are then aggregated. Also, we can distinguish between the descending IOWA (DIOWA) operator and the ascending IOWA (AIOWA) operator.

Note that if there is a tie between the OWA pairs $\langle u_i, a_i \rangle$ and $\langle u_j, a_j \rangle$ with respect to order inducing variables, in this case, Yager and Filev (1999) present a policy, which is to replace each argument of the tied OWA pairs by their arithmetic average $(a_i + a_j)/2$. If k items are tied, then we can replace these by k replicas of their arithmetic average.

3.3. *The Induced Uncertain Linguistic OWA Operator*

The OWA and the IOWA operators can only be used in situations where the aggregated arguments are the exact numerical values. Xu (2006b) introduced the induced uncertain linguistic OWA (IULOWA) operator, which based on the IOWA operator and uncertain OWA (UOWA) operator (Xu and Da, 2002) under linguistic environment. As an extension of the OWA operator, the main difference is that the aggregated arguments given in the IULOWA operator is uncertain linguistic variable. The IULOWA operator is defined as follows:

Definition 5. Let IULOWA: $R^n \times \tilde{S}^n \to \tilde{S}$, if

$$IULOWA_w\big(\langle u_1, \tilde{s}_1 \rangle, \langle u_2, \tilde{s}_2 \rangle, \ldots, \langle u_n, \tilde{s}_n \rangle\big) = w_1\tilde{s}_{\beta_1} \oplus w_2\tilde{s}_{\beta_2} \oplus \cdots \oplus w_n\tilde{s}_{\beta_n}, \tag{4}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is a weighting vector, such that $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$, $\tilde{s}_{\beta_j}$ is the $\tilde{s}_i$ value of the uncertain linguistic OWA pair $\langle u_i, \tilde{s}_i \rangle$ having the jth largest u_i, and u_i in $\langle u_i, \tilde{s}_i \rangle$ is referred to as the order inducing variable and $\tilde{s}_i$ as the uncertain linguistic argument variable.

Similarly to IOWA operator, if there is a tie between the ULOWA pairs $\langle u_i, \tilde{s}_i \rangle$ and $\langle u_j, \tilde{s}_j \rangle$ with respect to order inducing variables, in this case, Xu (2006b) present a policy, which is to replace each argument of the tied ULOWA pairs by their average $(\tilde{s}_i \oplus \tilde{s}_j)/2$. If k items are tied, then we can replace these by k replicas of their average.

4. Induced Uncertain Pure Linguistic Hybrid Averaging Aggregation (IUPLHAA) Operator

In this section, we develop uncertain pure linguistic weighed averaging aggregation (UPLWAA) operator and induced uncertain pure linguistic ordered weighed averaging aggregation (IUPLOWAA) operator. And then, we propose an induced uncertain pure linguistic hybrid averaging aggregation (IUPLHAA) operator based on the developed operators, also we study some of its desirable properties.

4.1. *The UPLWAA and the IUPLOWAA Operators*

Many situations in our real-life, due to the increasing complexity of the socio-economic environment and the lack of knowledge or data about the practical problem domain, the input arguments may be uncertain pure linguistic variables, in which all the attribute weights, the attribute values and the expert weights are given by the form of linguistic labels variables. In the following, we shall develop pure linguistic weighed averaging aggregation operator and induced pure linguistic ordered weighed averaging aggregation operator under uncertain environment.

Definition 6. Let UPLWAA: $\tilde{S}^n \to \tilde{S}$, if

$$UPLWAA_{s_\omega}(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) = s_{\omega_1}\tilde{s}_1 \oplus s_{\omega_2}\tilde{s}_2 \oplus \cdots \oplus s_{\omega_n}\tilde{s}_n, \tag{5}$$

where $s_{\omega_i} = (s_{\omega_1}, s_{\omega_2}, \ldots, s_{\omega_n})^T \in \tilde{S}$ is the weighting vector of uncertain linguistic label variables $(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n)$, then UPLWAA is called the uncertain pure linguistic weighted averaging aggregation (UPLWAA) operator.

Definition 7. Let IUPLOWAA: $\tilde{S}^n \times \tilde{S}^n \to \tilde{S}$, if

$$IUPLOWAA_w\big(\langle s_{u_1}, \tilde{s}_1\rangle, \langle s_{u_2}, \tilde{s}_2\rangle, \ldots, \langle s_{u_n}, \tilde{s}_n\rangle\big) = w_1\tilde{s}_{\beta_1} \oplus w_2\tilde{s}_{\beta_2} \oplus \cdots \oplus w_n\tilde{s}_{\beta_n}, \tag{6}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is a weighting vector, such that $w_j \in [0, 1]$, $\sum_{j=1}^n w_j = 1$, $\tilde{s}_{\beta_j}$ is the $\tilde{s}_i$ value of the uncertain pure linguistic OWA pair $\langle s_{u_i}, \tilde{s}_i\rangle$ having the jth largest s_{u_i}, and s_{u_i} in $\langle s_{u_i}, \tilde{s}_i\rangle$ is referred to as the order inducing linguistic variable and $\tilde{s}_i$ as the uncertain linguistic argument variable.

Note that if there is a tie between the UPLOWA pairs $\langle s_{u_i}, \tilde{s}_i\rangle$ and $\langle s_{u_j}, \tilde{s}_j\rangle$ with respect to order inducing variables, in this case, we replace each argument of the tied UPLOWA pairs by their average $(\tilde{s}_i \oplus \tilde{s}_j)/2$. If k items are tied, then we can replace these by k replicas of their average.

Also, the IUPLOWAA operator can be considered as the generalization of the IULOWA operator. In fact, if we let the order inducing linguistic variable s_{u_i} of UPLOWA pairs $\langle s_{u_i}, \tilde{s}_i\rangle$ be the exact numerical values, the IUPLOWAA operator is reduced to the IULOWA operator. Moreover, similarly to IULOWA operator, the IUPLOWAA operator is commutative, idempotent, bounded and monotonic.

Example 1. Assume we have four UPLOWA pairs $\langle s_{u_i}, \tilde{s}_i\rangle$ given

$$\big\langle s_1, [s_{-2}, s_0]\big\rangle, \quad \big\langle s_{-2}, [s_0, s_1]\big\rangle, \quad \big\langle s_3, [s_{-1}, s_1]\big\rangle, \quad \big\langle s_{-1}, [s_0, s_2]\big\rangle.$$

Perform the ordering the UPLOWA pairs with respect to the first component, and we have

$$\big\langle s_3, [s_{-1}, s_1]\big\rangle, \quad \big\langle s_1, [s_{-2}, s_0]\big\rangle, \quad \big\langle s_{-1}, [s_0, s_2]\big\rangle, \quad \big\langle s_{-2}, [s_0, s_1]\big\rangle.$$

The ordering induces the ordered uncertain linguistic variables

$$\tilde{s}_{\beta_1} = [s_{-1}, s_1], \qquad \tilde{s}_{\beta_2} = [s_{-2}, s_0], \qquad \tilde{s}_{\beta_3} = [s_0, s_2], \qquad \tilde{s}_{\beta_4} = [s_0, s_1].$$

If the weighting vector $w = (0.4, 0.1, 0.3, 0.2)^T$, then we can get the aggregated result as following

$$\begin{aligned} & IUPLOWAA_w\big(\langle s_1, [s_{-2}, s_0]\rangle, \langle s_{-2}, [s_0, s_1]\rangle, \langle s_3, [s_{-1}, s_1]\rangle, \langle s_{-1}, [s_0, s_2]\rangle\big) \\ & \quad = 0.4 \times [s_{-1}, s_1] \oplus 0.1 \times [s_{-2}, s_0] \oplus 0.3 \times [s_0, s_2] \oplus 0.2 \times [s_0, s_1] \\ & \quad = [s_{-0.6}, s_{1.2}]. \end{aligned}$$

Usually, however, the order inducing linguistic variables s_{u_i} $(i = 1, 2, \ldots, n)$ take the form of uncertain linguistic variables $\tilde{s}_{u_i}$ $(i = 1, 2, \ldots, n)$, in this case, Xu and Da (2002) present a simple procedure for the ranking of the uncertain linguistic variables. At first, we can compare each variable $\tilde{s}_{u_i} \in \tilde{S}$ $(i = 1, 2, \ldots, n)$ with all arguments $\tilde{s}_{u_j} \in \tilde{S}$ $(j = 1, 2, \ldots, n)$ by using (1), and let $p_{ij} = p(\tilde{s}_{u_i} \geqslant \tilde{s}_{u_j})$. Then, we construct a complementary matrix $P = (p_{ij})_{n\times n}$, where $p_{ij} \geqslant 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = \frac{1}{2}$, $i, j = 1, 2, \ldots, n$. Sum all elements in each line of matrix $P = (p_{ij})_{n\times n}$, and we have $p_i = \sum_{j=1}^{n} p_{ij}$ $(i = 1, 2, \ldots, n)$. At last, we can rank the uncertain linguistic variables $\tilde{s}_{u_i}$ $(i = 1, 2, \ldots, n)$ in descending order in accordance with the values of p_i.

Example 2. Assume we have four UPLOWA pairs $\langle \tilde{s}_{u_i}, \tilde{s}_i\rangle$ given

$$\big\langle [s_{-1}, s_0], [s_1, s_2]\big\rangle, \quad \big\langle [s_0, s_2], [s_{-1}, s_1]\big\rangle, \quad \big\langle [s_{-1}, s_1], [s_1, s_3]\big\rangle, \quad \big\langle [s_0, s_1], [s_{-2}, s_0]\big\rangle.$$

First, we rank the order inducing linguistic variables $\tilde{s}_{u_i}$ $(i = 1, 2, 3, 4)$ of the UPLOWA pairs by using (1), and a complementary matrix is constructed as follows

$$P = \begin{bmatrix} 0.5 & 0 & 0.333 & 0 \\ 1 & 0.5 & 0.75 & 0.667 \\ 0.667 & 0.25 & 0.5 & 0.333 \\ 1 & 0.333 & 0.667 & 0.5 \end{bmatrix}.$$

Sum all elements in each line of the matrix, and we have

$$p_1 = 0.833, \qquad p_2 = 2.917, \qquad p_3 = 1.750, \qquad p_4 = 2.500.$$

Then we rank the order inducing linguistic variables $\tilde{s}_{u_i}$ $(i = 1, 2, 3, 4)$ in descending order in accordance with the values of p_i $(i = 1, 2, 3, 4)$:

$$\tilde{s}_{u_2} = [s_0, s_2], \qquad \tilde{s}_{u_4} = [s_0, s_1], \qquad \tilde{s}_{u_3} = [s_{-1}, s_1], \qquad \tilde{s}_{u_1} = [s_{-1}, s_0].$$

Perform the ordering the UPLOWA pairs with respect to the first component, and we have

$$\big\langle [s_0, s_2], [s_{-1}, s_1]\big\rangle, \quad \big\langle [s_0, s_1], [s_{-2}, s_0]\big\rangle, \quad \big\langle [s_{-1}, s_1], [s_1, s_3]\big\rangle, \quad \big\langle [s_{-1}, s_0], [s_1, s_2]\big\rangle.$$

The ordering induces the ordered uncertain linguistic variables

$$\tilde{s}_{\beta_1} = [s_{-1}, s_1], \qquad \tilde{s}_{\beta_2} = [s_{-2}, s_0], \qquad \tilde{s}_{\beta_3} = [s_1, s_3], \qquad \tilde{s}_{\beta_4} = [s_1, s_2].$$

If the weighting vector $w = (0.2, 0.3, 0.3, 0.2)^T$, then we can get the aggregated result as following

$$\begin{aligned} &IUPLOWAA_w\big(\langle \tilde{s}_{u_1}, \tilde{s}_1\rangle, \langle \tilde{s}_{u_2}, \tilde{s}_2\rangle, \langle \tilde{s}_{u_3}, \tilde{s}_3\rangle, \langle \tilde{s}_{u_4}, \tilde{s}_4\rangle\big) \\ &\quad = 0.2 \times [s_{-1}, s_1] \oplus 0.3 \times [s_{-2}, s_0] \oplus 0.3 \times [s_1, s_3] \oplus 0.2 \times [s_1, s_2] \\ &\quad = [s_{-0.3}, s_{1.5}]. \end{aligned}$$

4.2. *The IUPLHAA Operator*

We can know from Definitions 6 and 7 that the UPLWAA operator weights the uncertain linguistic variables while the IUPLOWAA operator weights the ordered positions of the uncertain linguistic variables instead of weighting the variables themselves. Therefore, weights represent different aspects in both the two operators. To overcome the drawback, we propose an induced uncertain pure linguistic hybrid averaging aggregation (IUPLHAA) operator, which is defined as follows:

Definition 8. An induced uncertain pure linguistic hybrid averaging aggregation (IUPLHAA) operator of dimension n is a mapping IUPLHAA: $\tilde{S}^n \times \tilde{S}^n \to \tilde{S}$, that has an associated n vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$ and

$$IUPLHAA_{s_\omega, w}\big(\langle s_{u_1}, \tilde{s}_1\rangle, \langle s_{u_2}, \tilde{s}_2\rangle, \ldots, \langle s_{u_n}, \tilde{s}_n\rangle\big) = w_1\tilde{s}_{\gamma_1} \oplus w_2\tilde{s}_{\gamma_2} \oplus \cdots \oplus w_n\tilde{s}_{\gamma_n}, \tag{7}$$

where $\tilde{s}_{\gamma_i}$ is the $\tilde{s}'_i$ value ($\tilde{s}'_i = n s_{\omega_i}\tilde{s}_i$, $i = 1, 2, \ldots, n$) of the uncertain pure linguistic OWA pair $\langle s_{u_i}, \tilde{s}_i\rangle$ having the jth largest s_{u_i}, and s_{u_i} in $\langle s_{u_i}, \tilde{s}_i\rangle$ is referred to as the order inducing linguistic variable. $s_{\omega_i} = (s_{\omega_1}, s_{\omega_2}, \ldots, s_{\omega_n})^T \in \tilde{S}$ is the weighting vector of uncertain linguistic variables $(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n)$, n is the balancing coefficient.

Remark 1. In Definition 8, if the weighting vectors s_{ω_i} $(i = 1, 2, \ldots, n)$ of uncertain linguistic variables $(\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n)$ and the order inducing linguistic variables s_{u_i} $(i = 1, 2, \ldots, n)$ take the form of uncertain linguistic variables $\tilde{s}_{\omega_i}$ $(i = 1, 2, \ldots, n)$ and $\tilde{s}_{u_i}$ $(i = 1, 2, \ldots, n)$, respectively, the IUPLHAA operator is extended to the pure uncertain environment, then we call an induced pure linguistic hybrid averaging aggregation operator under pure uncertain environment (IPLHAA-PU), which can be considered as the extension of the IUPLHAA operator.

Remark 2. If there is a tie between the UPLHAA pairs $\langle s_{u_i}, \tilde{s}_i\rangle$ and $\langle s_{u_j}, \tilde{s}_j\rangle$ with respect to order inducing variables, in this case, we replace each argument of the tied UPLHAA pairs by their average $(\tilde{s}_i \oplus \tilde{s}_j)/2$. If k items are tied, then we can replace these by k replicas of their average.

Remark 3. Especially, if $w = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$, then IUPLHAA is reduced to the UPLWAA operator; if $\omega = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$, we can have $\tilde{s}'_i = \tilde{s}_i$, then IUPLHAA is reduced to the IUPLOWAA operator.

Example 3. Assume we have four UPLOWA pairs $\langle s_{u_i}, \tilde{s}_i \rangle$ given

$$\langle s_1, [s_0, s_1] \rangle, \quad \langle s_2, [s_{-2}, s_0] \rangle, \quad \langle s_0, [s_1, s_2] \rangle, \quad \langle s_{-1}, [s_2, s_3] \rangle.$$

$s_{\omega_i} = ([s_0, s_1], [s_1, s_2], [s_{-1}, s_0], [s_0, s_1])^T$ is the weighting vector of uncertain linguistic label variables $(\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4)$.

Then we can have the $\tilde{s}'_i$ value by using $\tilde{s}'_i = n s_{\omega_i} \tilde{s}_i$ $(i = 1, 2, 3, 4)$:

$$\tilde{s}'_1 = [s_0, s_4], \qquad \tilde{s}'_2 = [s_{-8}, s_0], \qquad \tilde{s}'_3 = [s_{-4}, s_0], \qquad \tilde{s}'_4 = [s_0, s_{12}].$$

Perform the ordering the UPLOWA pairs with respect to the first component, and we have

$$\langle s_2, [s_{-8}, s_0] \rangle, \quad \langle s_1, [s_0, s_4] \rangle, \quad \langle s_0, [s_{-4}, s_0] \rangle, \quad \langle s_{-1}, [s_0, s_{12}] \rangle.$$

The ordering induces the ordered uncertain linguistic variables

$$\tilde{s}_{\gamma_1} = [s_{-8}, s_0], \qquad \tilde{s}_{\gamma_2} = [s_0, s_4], \qquad \tilde{s}_{\gamma_3} = [s_{-4}, s_0], \quad \tilde{s}_{\gamma_4} = [s_0, s_{12}].$$

The weight vector associated with the IUPLHAA operator $w = (0.15, 0.35, 0.35, 0.15)^T$, which is derived by the Gaussian distribution based method, for more details, refer to Xu (2005c). Then we can get the aggregated result as following

$$\begin{aligned}
&IUPLHAA_{s_\omega, w}\left(\langle s_1, [s_0, s_1] \rangle, \langle s_2, [s_{-2}, s_0] \rangle, \langle s_0, [s_1, s_2] \rangle, \langle s_{-1}, [s_2, s_3] \rangle\right) \\
&\quad = 0.15 \times [s_{-8}, s_0] \oplus 0.35 \times [s_0, s_4] \oplus 0.35 \times [s_{-4}, s_0] \oplus 0.15 \times [s_0, s_{12}] \\
&\quad = [s_{-2.6}, s_{3.2}].
\end{aligned}$$

The IUPLHAA operator has many desirable properties, which can be proved with the following theorems:

Theorem 1 (Boundedness).

$$\min_j \left(\tilde{s}'_j\right) \leqslant IUPLHAA_{s_\omega, w}\left(\langle s_{u_1}, \tilde{s}_1 \rangle, \langle s_{u_2}, \tilde{s}_2 \rangle, \dots, \langle s_{u_n}, \tilde{s}_n \rangle\right) \leqslant \max_j \left(\tilde{s}'_j\right).$$

Proof. Let $\min_j(\tilde{s}'_j) = \tilde{s}_\alpha$ and $\max_j(\tilde{s}'_j) = \tilde{s}_\beta$. Since $\tilde{s}'_j = n s_{\omega_j} \tilde{s}_j$, $j = 1, 2, \dots, n$, then

$$\begin{aligned}
&IUPLHAA_{s_\omega, w}\left(\langle s_{u_1}, \tilde{s}_1 \rangle, \langle s_{u_2}, \tilde{s}_2 \rangle, \dots, \langle s_{u_n}, \tilde{s}_n \rangle\right) \\
&\quad = w_1 \tilde{s}_{\gamma_1} \oplus w_2 \tilde{s}_{\gamma_2} \oplus \cdots \oplus w_n \tilde{s}_{\gamma_n} \geqslant w_1 \tilde{s}_\alpha \oplus w_2 \tilde{s}_\alpha \oplus \cdots \oplus w_n \tilde{s}_\alpha \\
&\quad = \tilde{s}_\alpha \sum_{i=1}^n w_i = \tilde{s}_\alpha,
\end{aligned}$$

and

$$\begin{aligned}
&IUPLHAA_{s_\omega,w}\big(\langle s_{u_1},\tilde{s}_1\rangle,\langle s_{u_2},\tilde{s}_2\rangle,\ldots,\langle s_{u_n},\tilde{s}_n\rangle\big)\\
&\quad= w_1\tilde{s}_{\gamma_1}\oplus w_2\tilde{s}_{\gamma_2}\oplus\cdots\oplus w_n\tilde{s}_{\gamma_n}\leqslant w_1\tilde{s}_\beta\oplus w_2\tilde{s}_\beta\oplus\cdots\oplus w_n\tilde{s}_\beta\\
&\quad=\tilde{s}_\beta\sum_{i=1}^n w_i=\tilde{s}_\beta.
\end{aligned}$$

This completes the proof of Theorem 1. □

Theorem 2 (Commutativity).

$$\begin{aligned}
&IUPLHAA_{s_\omega,w}\big(\langle s_{u_1},\tilde{s}_1\rangle,\langle s_{u_2},\tilde{s}_2\rangle,\ldots,\langle s_{u_n},\tilde{s}_n\rangle\big)\\
&\quad= IUPLHAA_{s_\omega,w}\big(\langle s'_{u_1},\tilde{s}'_1\rangle,\langle s'_{u_2},\tilde{s}'_2\rangle,\ldots,\langle s'_{u_n},\tilde{s}'_n\rangle\big)
\end{aligned}$$

where $(\langle s'_{u_1},\tilde{s}'_1\rangle,\langle s'_{u_2},\tilde{s}'_2\rangle,\ldots,\langle s'_{u_n},\tilde{s}'_n\rangle)$ *is any permutation of* $(\langle s_{u_1},\tilde{s}_1\rangle,\langle s_{u_2},\tilde{s}_2\rangle,\ldots,\langle s_{u_n},\tilde{s}_n\rangle)$.

Proof. Let

$$IUPLHAA_{s_\omega,w}\big(\langle s_{u_1},\tilde{s}_1\rangle,\langle s_{u_2},\tilde{s}_2\rangle,\ldots,\langle s_{u_n},\tilde{s}_n\rangle\big)= w_1\tilde{s}_{\gamma_1}\oplus w_2\tilde{s}_{\gamma_2}\oplus\cdots\oplus w_n\tilde{s}_{\gamma_n},$$

and

$$IUPLHAA_{s_\omega,w}\big(\langle s'_{u_1},\tilde{s}'_1\rangle,\langle s'_{u_2},\tilde{s}'_2\rangle,\ldots,\langle s'_{u_n},\tilde{s}'_n\rangle\big)= w_1\tilde{s}'_{\gamma_1}\oplus w_2\tilde{s}'_{\gamma_2}\oplus\cdots\oplus w_n\tilde{s}'_{\gamma_n}.$$

Since $(\langle s'_{u_1},\tilde{s}'_1\rangle,\langle s'_{u_2},\tilde{s}'_2\rangle,\ldots,\langle s'_{u_n},\tilde{s}'_n\rangle)$ is any permutation of $(\langle s_{u_1},\tilde{s}_1\rangle,\langle s_{u_2},\tilde{s}_2\rangle,\ldots,\langle s_{u_n},\tilde{s}_n\rangle)$, we have $\tilde{s}_{\gamma_j}=\tilde{s}'_{\gamma_j}$, $j=1,2,\ldots,n$.

This completes the proof of Theorem 2. □

Theorem 3 (Idempotency). *If* $\tilde{s}_j=\tilde{s}$ *for all* j*, then*

$$IUPLHAA_{s_\omega,w}\big(\langle s_{u_1},\tilde{s}_1\rangle,\langle s_{u_2},\tilde{s}_2\rangle,\ldots,\langle s_{u_n},\tilde{s}_n\rangle\big)=\tilde{s}.$$

Proof. Since $\tilde{s}_j=\tilde{s}$ for all j, we have

$$\begin{aligned}
&IUPLHAA_{s_\omega,w}\big(\langle s_{u_1},\tilde{s}_1\rangle,\langle s_{u_2},\tilde{s}_2\rangle,\ldots,\langle s_{u_n},\tilde{s}_n\rangle\big)\\
&\quad= w_1\tilde{s}_{\gamma_1}\oplus w_2\tilde{s}_{\gamma_2}\oplus\cdots\oplus w_n\tilde{s}_{\gamma_n}= w_1\tilde{s}\oplus w_2\tilde{s}\oplus\cdots\oplus w_n\tilde{s}\\
&\quad=\tilde{s}\sum_{i=1}^n w_i=\tilde{s}.
\end{aligned}$$

This completes the proof of Theorem 3. □

Theorem 4 (Monotonicity). *If $\tilde{s}_j \leqslant \tilde{s}'_j$ for all j, then*

$$IUPLHAA_{s_\omega,w}\big(\langle s_{u_1},\tilde{s}_1\rangle,\langle s_{u_2},\tilde{s}_2\rangle,\dots,\langle s_{u_n},\tilde{s}_n\rangle\big) \leqslant IUPLHAA_{s_\omega,w}\big(\langle s_{u_1},\tilde{s}'_1\rangle,\langle s_{u_2},\tilde{s}'_2\rangle,\dots,\langle s_{u_n},\tilde{s}'_n\rangle\big).$$

Proof. Let

$$IUPLHAA_{s_\omega,w}\big(\langle s_{u_1},\tilde{s}_1\rangle,\langle s_{u_2},\tilde{s}_2\rangle,\dots,\langle s_{u_n},\tilde{s}_n\rangle\big) = w_1\tilde{s}_{\gamma_1}\oplus w_2\tilde{s}_{\gamma_2}\oplus\cdots\oplus w_n\tilde{s}_{\gamma_n} \tag{8}$$

$$IUPLHAA_{s_\omega,w}\big(\langle s_{u_1},\tilde{s}'_1\rangle,\langle s_{u_2},\tilde{s}'_2\rangle,\dots,\langle s_{u_n},\tilde{s}'_n\rangle\big) = w_1\tilde{s}'_{\gamma_1}\oplus w_2\tilde{s}'_{\gamma_2}\oplus\cdots\oplus w_n\tilde{s}'_{\gamma_n} \tag{9}$$

Since $\tilde{s}_j \leqslant \tilde{s}'_j$ for all j, we can have $\tilde{s}_{\gamma_j} \leqslant \tilde{s}'_{\gamma_j}$, $j = 1,2,\dots,n$.

This completes the proof of Theorem 4. □

5. Multiple-Attribute Group Decision Making with the IUPLHAA Operator Under Uncertain Pure Linguistic Information

In this section, we shall develop an approach based on the IUPLHAA operator to multiple-attribute group decision making under uncertain pure linguistic information. Let $d_k \in D$ $(k = 1,2,\dots,m)$ be the set of decision makers, $s_\nu = (s_{\nu_1}, s_{\nu_2},\dots,s_{\nu_m})^T \in \tilde{S}$ be the weight vector of decision makers, $G = \{g_1, g_2,\dots,g_l\}$ be the set of attributes, and $s_\omega = (s_{\omega_1}, s_{\omega_2},\dots,s_{\omega_l})^T \in \tilde{S}$ be the weight vector of attributes. Then, we let $X = \{x_1, x_2,\dots,x_n\}$ be a discrete set of alternatives. Suppose that $\tilde{A}^{(k)} = (\tilde{a}^{(k)}_{ij})_{n\times l}$ is the decision matrix, where $\tilde{a}^{(k)}_{ij} \in \tilde{S}$ is a preference value, which takes the form of uncertain pure linguistic variable, given by the decision makers, for alternative $x_i \in X$ with respect to attribute $g_j \in G$. We shall utilize the IUPLHAA operator to propose an approach to multiple-attribute group decision making under uncertain pure linguistic information, which involves the following steps:

Step 1: Calculate the $\tilde{s}'_i$ value by using $\tilde{s}'_i = ns_\omega\tilde{s}_i$, $i = 1,2,\dots,m$ to aggregate all the decision matrices $\tilde{A}^{(k)} = (\tilde{a}^{(k)}_{ij})_{n\times l}$, where $\tilde{s}_i$ is the uncertain linguistic variables of the UPLOWA pair $\langle s_{u_i}, \tilde{s}_i\rangle$ and s_ω is the weighting vector of $\tilde{s}_i$.

Step 2: Utilize the IUPLHAA operator

$$\tilde{a}_j = IUPLHAA_{s_\omega,w}\big(\langle s_{u_1},\tilde{s}_1\rangle,\dots,\langle s_{u_m},\tilde{s}_m\rangle\big) = w_1\tilde{s}_{\gamma_{1j}}\oplus\cdots\oplus w_m\tilde{s}_{\gamma_{mj}}$$

to derive the collective overall preference value $\tilde{a}_j$ of alternative x_j $(j = 1,2,\dots,l)$ given by all the decision makers, where $w = (w_1, w_2,\dots,w_m)^T$ is the weight vector associated with the IUPLHAA operator, with $w_i \in [0,1]$, $\sum_{i=1}^m w_i = 1$.

Step 3: Compare each $\tilde{a}_j$ with all $\tilde{a}_i$, $i, j = 1, 2, \dots, l$ by using (1) and we let $p_{ij} = p\{\tilde{a}_i \geqslant \tilde{a}_j\}$, then we construct a complementary matrix $P = (p_{ij})_{l\times l}$, where $p_{ij} \geqslant 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = \frac{1}{2}$, $i, j = 1, 2, \dots, l$.
Step 4: Sum all elements in each line of matrix $P = (p_{ij})_{l\times l}$, and we have $p_i = \sum_{j=1}^{l} p_{ij}$ $(i = 1, 2, \dots, l)$. Then, we can rank the arguments $\tilde{a}_j$ $(j = 1, 2, \dots, l)$ in descending order in accordance with the values of p_i $(i = 1, 2, \dots, l)$.
Step 5: Rank all the alternatives x_j $(j = 1, 2, \dots, l)$ and select the best one(s) in accordance with $\tilde{a}_j$ $(j = 1, 2, \dots, l)$.

6. Illustrative Example

In the following, we shall develop a numerical example of the new approach. Let us suppose an investment company, which wants to invest a sum of money in the best option. There is a panel with six possible alternatives in which to invest the money:

(1) invest in a computer company called x_1;
(2) invest in a real estate company called x_2;
(3) invest in an insurance company called x_3;
(4) invest in a car company called x_4;
(5) invest in a food company called x_5;
(6) invest in an educational institution called x_6.

In order to assess these possible alternatives, the investment company must make a decision according to the following four attributes:

(1) g_1: the risk analysis;
(2) g_2: the growth analysis;
(3) g_3: the social-political impact analysis;
(4) g_4: other factors.

The six possible alternatives x_j $(j = 1, 2, 3, 4, 5, 6)$ are to be evaluated using the linguistic label term set

$$S = \{s_{-4} = \textit{extremely poor}, s_{-3} = \textit{very poor}, s_{-2} = \textit{poor}, s_{-1} = \textit{slightly poor},$$
$$s_0 = \textit{fair}, s_1 = \textit{slightly good}, s_2 = \textit{good}, s_3 = \textit{very good}, s_4 = \textit{extremely good}\}$$

by three decision makers d_k $(k = 1, 2, 3)$ (whose weight vector $s_v = (s_1, s_4, s_2)^T$) under four attributes above, as listed in Tables 1–3, respectively.

Calculate the $\tilde{s}_i'$ value by using $\tilde{s}_i' = s_{\omega_1}\tilde{s}_1 \oplus s_{\omega_2}\tilde{s}_2 \oplus s_{\omega_3}\tilde{s}_3 \oplus s_{\omega_4}\tilde{s}_4$, $i = 1, 2, 3$ to aggregate all the decision matrices $\tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{4\times 6}$, suppose that the weight vector of the four attributes is $s_\omega = (s_1, s_2, s_0, s_{-1})^T$, then we have the collective decision matrix $\tilde{A}$ as listed in Table 4.

Utilize the weight vector of decision makers in the form of linguistic labels $s_v = (s_1, s_4, s_2)^T \in \tilde{S}$, the weight vector associated with the UPLHAA operator $w =$

Table 1
Decision matrix $\tilde{A}^{(1)}$.

	x_1	x_2	x_3	x_4	x_5	x_6
g_1	$[s_1, s_2]$	$[s_{-3}, s_{-1}]$	$[s_{-2}, s_0]$	$[s_{-1}, s_0]$	$[s_0, s_1]$	$[s_2, s_3]$
g_2	$[s_0, s_1]$	$[s_1, s_2]$	$[s_1, s_2]$	$[s_2, s_3]$	$[s_1, s_2]$	$[s_{-1}, s_0]$
g_3	$[s_{-2}, s_{-1}]$	$[s_1, s_2]$	$[s_2, s_4]$	$[s_0, s_2]$	$[s_2, s_3]$	$[s_3, s_4]$
g_4	$[s_1, s_3]$	$[s_0, s_1]$	$[s_{-1}, s_0]$	$[s_3, s_4]$	$[s_2, s_3]$	$[s_1, s_2]$

Table 2
Decision matrix $\tilde{A}^{(2)}$.

	x_1	x_2	x_3	x_4	x_5	x_6
g_1	$[s_1, s_2]$	$[s_{-2}, s_0]$	$[s_1, s_2]$	$[s_0, s_1]$	$[s_{-1}, s_0]$	$[s_{-1}, s_1]$
g_2	$[s_0, s_2]$	$[s_2, s_3]$	$[s_0, s_1]$	$[s_1, s_2]$	$[s_2, s_3]$	$[s_1, s_2]$
g_3	$[s_0, s_1]$	$[s_0, s_1]$	$[s_{-1}, s_0]$	$[s_3, s_4]$	$[s_1, s_3]$	$[s_1, s_2]$
g_4	$[s_1, s_2]$	$[s_2, s_3]$	$[s_{-1}, s_1]$	$[s_1, s_2]$	$[s_0, s_2]$	$[s_2, s_3]$

Table 3
Decision matrix $\tilde{A}^{(3)}$.

	x_1	x_2	x_3	x_4	x_5	x_6
g_1	$[s_2, s_3]$	$[s_0, s_1]$	$[s_0, s_1]$	$[s_{-1}, s_0]$	$[s_{-1}, s_1]$	$[s_0, s_1]$
g_2	$[s_0, s_1]$	$[s_1, s_2]$	$[s_1, s_3]$	$[s_0, s_1]$	$[s_1, s_2]$	$[s_2, s_3]$
g_3	$[s_1, s_2]$	$[s_2, s_3]$	$[s_1, s_2]$	$[s_1, s_2]$	$[s_2, s_3]$	$[s_1, s_2]$
g_4	$[s_0, s_1]$	$[s_1, s_2]$	$[s_3, s_4]$	$[s_0, s_1]$	$[s_1, s_2]$	$[s_2, s_3]$

Table 4
The collective decision matrix $\tilde{A}$.

	x_1	x_2	x_3	x_4	x_5	x_6
$\tilde{s}'_1$	$[s_0, s_1]$	$[s_{-1}, s_2]$	$[s_1, s_4]$	$[s_0, s_2]$	$[s_0, s_2]$	$[s_{-1}, s_1]$
$\tilde{s}'_2$	$[s_0, s_4]$	$[s_0, s_3]$	$[s_2, s_3]$	$[s_1, s_3]$	$[s_3, s_4]$	$[s_{-1}, s_2]$
$\tilde{s}'_3$	$[s_2, s_4]$	$[s_1, s_3]$	$[s_{-1}, s_3]$	$[s_{-1}, s_1]$	$[s_0, s_3]$	$[s_2, s_4]$

$(0.24, 0.52, 0.24)^T$, which is derived by the Gaussian distribution based method (Xu, 2005c), and utilize the IUPLHAA operator to derive the collective overall preference value $\tilde{a}_j$ of alternative x_j given by the three decision makers. Comparing each $\tilde{a}_j$ with all $\tilde{a}_i$ by using (1), we let $p_{ij} = p\{\tilde{a}_i \geqslant \tilde{a}_j\}$, then we construct a complementary matrix $P = (p_{ij})_{6\times6}$. Sum all elements in each line of matrix $P = (p_{ij})_{6\times6}$, and we have the value of p_i $(i = 1, 2, 3, 4, 5, 6)$. Then we rank the arguments $\tilde{a}_j$ $(j = 1, 2, 3, 4, 5, 6)$ in descending order in accordance with p_i $(i = 1, 2, 3, 4, 5, 6)$.

Next, we present a comparative analysis with other related decision making methods. To do so, in the example, we consider the induced uncertain linguistic OWA (IULOWA) operator, the linguistic hybrid arithmetic averaging (LHAA) operator and the IUPLOWAA operator. The aggregated results are shown in Table 5. And then, we utilize Steps 3–4 proposed in Section 5, the ordering of the alternatives with the different aggregation operators can be obtained in Table 6.

Table 5
Aggregated results.

	x_1	x_2	x_3	x_4	x_5	x_6
IULOWA	$[s_{1.65}, s_{3.86}]$	$[s_{1.28}, s_{2.28}]$	$[s_{0.76}, s_{2.00}]$	$[s_{0.76}, s_{1.92}]$	$[s_{1.76}, s_{3.00}]$	$[s_{1.48}, s_{2.88}]$
LHAA	$[s_{0.48}, s_{3.28}]$	$[s_0, s_{2.76}]$	$[s_{0.76}, s_{3.52}]$	$[s_0, s_{2.00}]$	$[s_{0.72}, s_{3.00}]$	$[s_{-0.28}, s_{2.24}]$
IUPLOWAA	$[s_{2.04}, s_{6.52}]$	$[s_{0.51}, s_{5.42}]$	$[s_{0.38}, s_{6.22}]$	$[s_{-0.12}, s_{3.36}]$	$[s_{1.38}, s_{5.93}]$	$[s_{0.91}, s_{5.94}]$
IUPLHAA	$[s_{1.04}, s_{3.28}]$	$[s_{0.28}, s_{2.76}]$	$[s_{0.20}, s_{3.24}]$	$[s_{-0.28}, s_{1.72}]$	$[s_{0.72}, s_{3.00}]$	$[s_{0.56}, s_{2.80}]$

Table 6
Ordering of the alternatives.

	Ordering
IULOWA	$x_1 \succ x_5 \succ x_6 \succ x_2 \succ x_3 \succ x_4$
LHAA	$x_3 \succ x_1 \succ x_5 \succ x_2 \succ x_4 \succ x_6$
IUPLOWAA	$x_1 \succ x_5 \succ x_6 \succ x_3 \succ x_2 \succ x_4$
IUPLHAA	$x_1 \succ x_5 \succ x_3 \succ x_6 \succ x_2 \succ x_4$

From Table 6 we can see, for most of the cases the best alternative is x_1, however, with the different aggregation operators used in different methods, the aggregated results and the rankings of the alternatives may be different. In the situations where the information about the expert weights, the attribute weights and the attribute values are expressed in the form of linguistic labels variables, the IUPLHAA operator may be a better choice for the decision makers. Also, it considers not only the importance degrees of the uncertain linguistic variables but their ordered positions.

As we can see, depending on the aggregation operator used, the ordering of the alternatives may be different, and thus, the decision maker can select properly the aggregation operator according to his interests and the actual needs.

Note that s_0 in the term set S implies that a certain criterion is not taken into consideration in the aggregation process. However, as we can see, s_0 in $S = \{s_\alpha \mid \alpha = -t, \dots, -1, 0, 1, \dots, t\}$, which is an additive discrete linguistic term set. Based on the term set, we developed the arithmetic averaging operators above. Similarly, we can develop some geometric averaging operators and harmonic averaging operators based on integrability linguistic term set, in which, $s_1 =$ *fair* instead of s_0.

7. Conclusions

In this paper, we have presented a new aggregation operator called the IUPLHAA operator and we have focused on an application in a group decision making problem regarding the selection of investments. We also have studied several desirable properties of the new operator. Moreover, we have developed some uncertain pure linguistic aggregation operators such as the UPLWAA operator and the IUPLOWAA operator.

The IUPLHAA operator can be used in situations where the input arguments are uncertain pure linguistic variables and it reflects the importance degrees of both the given uncertain linguistic variables and their ordered positions. Also, the IUPLHAA operator can alleviate the influence of unduly large (or small) deviations on the aggregation results by assigning them low (or high) weights. Moreover, we have analyzed both the UPLWAA operator and the IUPLOWAA operator are the special case of the IUPLHAA operator. Note that the IUPLHAA operator can be extended to the pure uncertain environment, and then we call it the IPLHAA-PU operator, which can be considered as the extension of the IUPLHAA operator.

In future research, we expect to develop further extensions by adding new characteristics such as fuzzy numbers and probabilistic aggregations in the problem. We will also develop different types of applications in decision theory and other fields such as Economics, Statistics and so on.

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Indukuotasis neapibrėžtasis visiškai lingvistinis hibridinis vidurkio agregavimo operatorius ir jo taikymas grupiniame sprendimų priėmime

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Šiame straipsnyje siūlomas naujas agregavimo operatorius neapibrėžtoje visiškai lingvistinėje aplinkoje, vadinamas indukuotuoju neapibrėžtuoju visiškai lingvistiniu hibridiniu vidurkio (IUPLHAA) operatoriumi. Aptariami kai kurie pagrindiniai naujojo operatoriaus privalumai ir bruožai. Situacijoms, kuriose kriterijų svoriai, kriterijų reikšmės ir ekspertų svoriai išreiškiami lingvistiniais kintamaisiais, pasiūlyta grupinio daugiakriterinio vertinimo metodika, paremta IUPLHAA operatoriumi. Galiausiai pateikiamas pavyzdys, parodantis metodikos taikymo galimybes ir tinkamumą.