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A Group Decision Making Approach Based on Interval-Valued Intuitionistic Uncertain Linguistic Aggregation Operators

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Abstract. This paper investigates group decision making problems in which the criterion values take the form of interval-valued intuitionistic uncertain linguistic numbers (IIULNs). First, some additive operational laws of IIULNs are defined. Subsequently, some new arithmetic aggregation operators, such as the interval-valued intuitionistic uncertain linguistic weighted averaging (IIULWA) operator, interval-valued intuitionistic uncertain linguistic ordered weighted averaging (IIULOWA) operator and interval-valued intuitionistic uncertain linguistic hybrid aggregation (IIULHA) operator, are proposed which are based on the operational laws. Furthermore, an approach to group decision making with interval-valued intuitionistic uncertain linguistic information is developed, which is based on the IIULWA and IIULHA operators. Finally, an illustrative example is provided to demonstrate the feasibility and effectiveness of the proposed method.

Key words: group decision making, interval-valued intuitionistic uncertain linguistic number (IIULN), IIULWA operator, IIULOWA operator, IIULHA operator.

1. Introduction

There are many different types of multi-criteria decision making (MCDM) problems, for example those that deal with issues such as: military system performance evaluation; personnel evaluation; venture capital; online auction; supply chain management; and medical diagnostics. Part of the decision making process involves forming an assessment and instead of using precise numerical values for this purpose, a more realistic approach may be to use linguistic assessments by means of linguistic variables, i.e., variables whose values are not exact numbers but linguistic terms, such as “very poor”, “poor”, “fair”, “good”, and “very good”. Therefore, the linguistic MCDM approach has been attracting increasing attention in recent years (Merig *et al.*, 2012; Wang *et al.*, 2014a, 2015a; Wei, 2011; Xu, 2012; Yang and Wang, 2013; Zeng *et al.*, 2012).

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An assumption exists when using linguistic variables, that the membership degree of an element to a linguistic term is 1, which does not adequately describe the decision-maker's confidence level when making a judgment. Wang and Li (2010) defined the concept of the intuitionistic linguistic number (ILN) based on the concept of the intuitionistic fuzzy set (Atanassov, 1986) and its applications in the MCDM field (Wang and Zhang, 2013; Xu and Cai, 2010; Zeng *et al.*, 2013; Zhao and Wei, 2013). These authors described the membership degree and non-membership degree of an element to a linguistic label, which can reflect the decision-maker's confidence level when they are making an evaluation. There has been a great deal of research in the area of intuitionistic linguistic MCDM problems, for example, Wang and Li (2010) proposed the intuitionistic linguistic weighted arithmetic averaging (ILWAA) operator and intuitionistic linguistic weighted geometric averaging (ILWGA) operator, and applied them to MCDM problems in which the criterion values take the form of ILNs; Liu (2013a) proposed the intuitionistic linguistic generalized dependent ordered weighted average (ILGDOWA) operator and intuitionistic linguistic generalized dependent hybrid weighted aggregation (ILGDHWA) operator, and applied them to group decision making with intuitionistic linguistic information; Wang *et al.* (2014c) developed the intuitionistic linguistic ordered weighted averaging (ILOWA) operator and intuitionistic linguistic hybrid aggregation (ILHA) operator, and developed a group decision making approach based on these operators.

It can be ascertained from the work of authors such as Liu (2013a), Wang and Li (2010), and Wang *et al.* (2014c), that an ILN is characterized by a linguistic term, a membership degree and a non-membership degree. It is noteworthy that, on one hand, in many situations the given linguistic arguments may not match any of the original linguistic terms and may be located between two of these terms. This could be due to a lack of knowledge, time pressure, and people's limited expertise when it comes to solving problems. In such cases, it is more suitable to deal with vagueness and uncertainty by letting them take the form of uncertain linguistic variables (Lan *et al.*, 2013; Wang *et al.*, 2015d, 2015c; Xu, 2006, 2004b). Alternatively, sometimes it is inappropriate to assume that the membership degree and non-membership degree have been defined precisely. The authors would also highlight another uncertainty, which is motivated by the interval-valued intuitionistic fuzzy set (see Atanassov and Gargov, 1989) whose fundamental characteristic is that the values of its membership function and non-membership function are interval numbers rather than exact numbers. This uncertainty arises because the membership degree or non-membership degree is no longer a number, but a whole interval. Therefore, based on uncertain linguistic variables (Lan *et al.*, 2013; Xu, 2006) and the interval-valued intuitionistic fuzzy set (Atanassov and Gargov, 1989; Huang *et al.*, 2013; Wang *et al.*, 2015b; 2014b; Yu, 2013), Liu (2013b) developed the following: the concept of interval-valued intuitionistic uncertain linguistic number (IIULN); some multiplicative operational laws of IIULNs; some interval-valued intuitionistic uncertain linguistic weighted geometric operators (such as the interval-valued intuitionistic uncertain linguistic weighted geometric average (IVIULWGA) operator, interval-valued intuitionistic uncertain linguistic ordered weighted geometric (IVIULOWG) operator and interval-valued intuitionistic uncertain linguistic hybrid geometric (IVIULHG) operator); and a group decision making approach with interval-valued intuitionistic uncertain linguistic information. In this paper, some additive operational laws of IIULNs will be defined and some interval-valued intuitionistic uncertain linguistic arithmetic aggregation operators will be proposed, before applying them to group decision making.

This paper is therefore organized as follows. In Section 2, some additive operational laws of IIULNs are defined, and a simple method for the comparison between two IIULNs is presented. In Section 3, some new arithmetic aggregation operators, such as the interval-valued intuitionistic uncertain linguistic weighted averaging (IIULWA) operator, interval-valued intuitionistic uncertain linguistic ordered weighted averaging (IIULOWA) operator and interval-valued intuitionistic uncertain linguistic hybrid aggregation (IIULHA) operator, are proposed, and various desirable properties of these operators are established. In Section 4, an approach to group decision making, in which the criterion values are expressed as IIULNs and the criterion weight information is known completely, which is based on the IIULWA operator and the IIULHA operator is developed. In Section 5, an example is provided in order to illustrate the application of the developed approach. Finally, conclusions are drawn in Section 6.

2. Preliminaries

Suppose that $S = \{s_\theta \mid \theta = 0, 1, \ldots, 2l\}$ is an additive linguistic term set (Herrera *et al.*, 1996; Herrera and Martnez, 2000), where l is a positive integer, s_θ represents a possible value for a linguistic variable, and s_θ has the following characteristics: (1) if $a > b$, then $s_a > s_b$; and (2) the negation operator is defined as: $\mathrm{neg}(s_a) = s_b$ such that $a + b = 2l$. For example, when $l = 4$, then S can be defined as:

$S = \{s_0 =$ extremely poor, $s_1 =$ very poor, $s_2 =$ poor, $s_3 =$ slightly poor, $s_4 =$ fair, $s_5 =$ slightly good, $s_6 =$ good, $s_7 =$ very good, $s_8 =$ extremely good$\}$.

To preserve all the information provided, Xu (2004a) proposed that the discrete linguistic term set S should be extended to a continuous linguistic term set $\bar{S} = \{s_\theta | \theta \in [0, q]\}$, in which $s_a > s_b$ if $a > b$, and q $(q > 2l)$ is a sufficiently large positive integer.

Definition 1. (See Xu, 2004b.) Let $\tilde{s} = [s_\theta, s_\phi]$, where $s_\theta, s_\phi \in \bar{S}$, s_θ and s_ϕ are the lower and the upper limits, respectively. Subsequently $\tilde{s}$ is called an additive uncertain linguistic variable.

Let $\tilde{S}$ be the set of all additive uncertain linguistic variables. For any two uncertain linguistic variables $\tilde{s}_1 = [s_{\theta_1}, s_{\phi_1}]$ and $\tilde{s}_2 = [s_{\theta_2}, s_{\phi_2}] \in \tilde{S}$, their operational laws are defined as follows (Xu, 2004b):

(1) $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{\theta_1+\theta_2}, s_{\phi_1+\phi_2}]$;

(2) $\lambda\tilde{s}_1 = [s_{\lambda\theta_1}, s_{\lambda\phi_1}]$, $\lambda \in [0, 1]$.

Definition 2. (See Wang and Li, 2010.) Let X be a universe of discourse, $s_{\theta(x)} \in \bar{S}$, then an ILN set A in X is an object having the following form:

$$A = \{(x, \langle s_{\theta(x)}, \mu_A(x), \nu_A(x)\rangle) | x \in X\}, \tag{1}$$

where $s_{\theta(x)}$ is a linguistic term, the functions $\mu_A(x)$ and $\nu_A(x)$ determine the degree of membership and the degree of non-membership of the element x to the linguistic evaluation $s_{\theta(x)}$, respectively, and

$$s_\theta : X \to \bar{S}, \quad x \mapsto s_{\theta(x)}; \tag{2}$$

$$\mu_A : X \to [0, 1], \quad x \mapsto \mu_A(x); \tag{3}$$

$$\nu_A : X \to [0, 1], \quad x \mapsto \nu_A(x), \tag{4}$$

with the condition $0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1$, $\forall x \in X$.

For convenience, Wang and Li (2010) called $\langle s_{\theta(x)}, \mu_A(x), \nu_A(x)\rangle$ the ILN. Let $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ for all $x \in X$, then $\pi_A(x)$ is called the degree of hesitancy of x to $s_{\theta(x)}$.

From Definition 2, it can be seen that the ILN $\langle s_{\theta(x)}, \mu_A(x), \nu_A(x)\rangle$ is characterized by the linguistic evaluation $s_{\theta(x)}$, the membership degree $\mu_A(x)$ and the non-membership degree $\nu_A(x)$ of the element x to the linguistic evaluation $s_{\theta(x)}$. However, in some situations, the linguistic evaluation may be expressed as uncertain linguistic variables rather than original linguistic terms (Xu, 2004b; 2004c). Furthermore, the membership degree and the non-membership degree may take the form of interval numbers rather than exact numbers (Atanassov and Gargov, 1989). Therefore, Liu (2013b) extended the ILN set, and defined the concept of an IIULN set as follows.

Definition 3. (See Liu, 2013b.) Let X be fixed, $\tilde{s}(x) \in \tilde{S}$, then an IIULN set $\tilde{A}$ in X is defined as follows:

$$\tilde{A} = \{(x, \langle \tilde{s}(x), \tilde{\mu}_{\tilde{A}}(x), \tilde{\nu}_{\tilde{A}}(x)\rangle) | x \in X\}, \tag{5}$$

where $\tilde{s}(x) \in \tilde{S}$, $\tilde{\mu}_{\tilde{A}}(x) \subset [0, 1]$ and $\tilde{\nu}_{\tilde{A}}(x) \subset [0, 1]$, with the condition $0 \leqslant \sup \tilde{\mu}_{\tilde{A}}(x) + \sup \tilde{\nu}_{\tilde{A}}(x) \leqslant 1$, for each $x \in X$. The intervals $\tilde{\mu}_{\tilde{A}}(x)$ and $\tilde{\nu}_{\tilde{A}}(x)$ represent, respectively, the membership degree and the non-membership degree of the element x to the uncertain linguistic evaluation $\tilde{s}(x)$.

Let $\tilde{\pi}_{\tilde{A}}(x) = 1 - \tilde{\mu}_{\tilde{A}}(x) - \tilde{\nu}_{\tilde{A}}(x)$ for all $x \in X$, then $\tilde{\pi}_{\tilde{A}}(x)$ is called the degree of hesitancy of x to $\tilde{s}(x)$.

For convenience, Liu (2013b) called $\langle \tilde{s}(x), \tilde{\mu}_{\tilde{A}}(x), \tilde{\nu}_{\tilde{A}}(x)\rangle$ the IIULN, and denoted it as $\beta = \langle [s_{\theta_\beta}, s_{\phi_\beta}], [a_\beta, b_\beta], [c_\beta, d_\beta]\rangle$, where $[s_{\theta_\beta}, s_{\phi_\beta}] \in \tilde{S}$, $[a_\beta, b_\beta] \subset [0, 1]$, $[c_\beta, d_\beta] \subset [0, 1]$ and $b_\beta + d_\beta \leqslant 1$. Let Ω be the set of all IIULNs.

DEFINITION 4. Let $\beta_1 = \langle[s_{\theta_{\beta_1}}, s_{\phi_{\beta_1}}],[a_{\beta_1}, b_{\beta_1}], [c_{\beta_1}, d_{\beta_1}]\rangle$ and $\beta_2 = \langle[s_{\theta_{\beta_2}}, s_{\phi_{\beta_2}}], [a_{\beta_2}, b_{\beta_2}], [c_{\beta_2}, d_{\beta_2}]\rangle$ be two IIULNs, then

(1) $\beta_1 \oplus \beta_2 = \langle[s_{\theta_{\beta_1}+\theta_{\beta_2}}, s_{\phi_{\beta_1}+\phi_{\beta_2}}],$
$[1-(1-a_{\beta_1})(1-a_{\beta_2}), 1-(1-b_{\beta_1})(1-b_{\beta_2})],$
$[(1-a_{\beta_1})(1-a_{\beta_2})-(1-a_{\beta_1}-c_{\beta_1})(1-a_{\beta_2}-c_{\beta_2}),$
$(1-b_{\beta_1})(1-b_{\beta_2})-(1-b_{\beta_1}-d_{\beta_1})(1-b_{\beta_2}-d_{\beta_2})]\rangle;$

(2) $\lambda\beta_1 = \langle[s_{\lambda\theta_{\beta_1}}, s_{\lambda\phi_{\beta_1}}], [1-(1-a_{\beta_1})^{\lambda}, 1-(1-b_{\beta_1})^{\lambda}],$
$[(1-a_{\beta_1})^{\lambda}-(1-a_{\beta_1}-c_{\beta_1})^{\lambda}, (1-b_{\beta_1})^{\lambda}-(1-b_{\beta_1}-d_{\beta_1})^{\lambda}]\rangle, \quad \lambda \in [0,1].$

It can be easily proven that all of the above results are also IIULNs and that the following relationships are true.

(1) $\beta_1 \oplus \beta_2 = \beta_2 \oplus \beta_1$;
(2) $(\beta_1 \oplus \beta_2) \oplus \beta_3 = \beta_1 \oplus (\beta_2 \oplus \beta_3)$;
(3) $\lambda(\beta_1 \oplus \beta_2) = \lambda\beta_1 \oplus \lambda\beta_2$, $\lambda \in [0,1]$;
(4) $\lambda_1\beta_1 \oplus \lambda_2\beta_1 = (\lambda_1+\lambda_2)\beta_1$, $\lambda_1, \lambda_2 \in [0,1]$.

Based on the expected value function and accuracy function of the IIULN proposed by Liu (2013b), a score function and a new accuracy function of IIULN, are now defined, and a method for comparing two IIULNs is also proposed.

DEFINITION 5. Let $\beta = \langle[s_{\theta_\beta}, s_{\phi_\beta}], [a_\beta, b_\beta], [c_\beta, d_\beta]\rangle$ be an IIULN, the score and the degree of accuracy of β can be evaluated by a score function $h(\beta)$ and a new accuracy function $u(\beta)$ shown respectively as follows:

$$h(\beta) = \frac{1}{8}(\theta_\beta + \phi_\beta)(2 + a_\beta + b_\beta - c_\beta - d_\beta), \tag{6}$$

$$u(\beta) = \frac{1}{4}(\theta_\beta + \phi_\beta)(a_\beta + b_\beta + c_\beta + d_\beta). \tag{7}$$

DEFINITION 6. Let $\beta_1 = \langle[s_{\theta_{\beta_1}}, s_{\phi_{\beta_1}}], [a_{\beta_1}, b_{\beta_1}], [c_{\beta_1}, d_{\beta_1}]\rangle$ and $\beta_2 = \langle[s_{\theta_{\beta_2}}, s_{\phi_{\beta_2}}], [a_{\beta_2}, b_{\beta_2}], [c_{\beta_2}, d_{\beta_2}]\rangle$ be two IIULNs, then

(1) if $h(\beta_1) < h(\beta_2)$, then β_1 is smaller than β_2, denoted by $\beta_1 < \beta_2$;
(2) if $h(\beta_1) = h(\beta_2)$, then
  (a) if $u(\beta_1) > u(\beta_2)$, then β_1 is bigger than β_2, denoted by $\beta_1 > \beta_2$;
  (b) if $u(\beta_1) < u(\beta_2)$, then β_1 is smaller than β_2, denoted by $\beta_1 < \beta_2$;
  (c) if $u(\beta_1) = u(\beta_2)$, then β_1 is equal to β_2, denoted by $\beta_1 = \beta_2$.

3. Interval-Valued Intuitionistic Uncertain Linguistic Aggregation Operators

To date, many aggregation operators have been proposed for aggregating information (Xu and Da, 2003). The weighted arithmetic averaging (WAA) operator (Harsanyi, 1955) and the ordered weighted averaging (OWA) operator (Yager, 1988) are two of most common operators used to aggregate arguments. In the 10 to 20 years, the WAA operator and the OWA operator have been extended to accommodate fuzzy and uncertain situations. For example: Bordogna *et al.* (1997) and Herrera and Herrera-Viedma (1997) extended these two operators to accommodate situations where the input arguments take the form of linguistic variables; Xu (2006, 2004b) extended them to accommodate situations where the input arguments are in the form of uncertain linguistic variables; and Xu (2007) extended them to accommodate situations where the input arguments are intuitionistic fuzzy numbers. In the following section, based on Definition 4, the WAA operator and the OWA operator are extended to accommodate situations where the input arguments are in the form of IIULNs. Moreover, some new arithmetic aggregation operators are developed, such as the IIULWA operator, the IIULOWA operator and the IIULHA operator.

3.1. *The IIULWA Operator and the IIULOWA Operator*

Definition 7. Let $\beta_j = \langle [s_{\theta_{\beta_j}}, s_{\phi_{\beta_j}}], [a_{\beta_j}, b_{\beta_j}], [c_{\beta_j}, d_{\beta_j}]\rangle$ $(j = 1, 2, \dots, n)$ be a collection of IIULNs, and $\mathbf{w} = (w_1, w_2, \dots, w_n)^T$ be the weight vector of β_j $(j = 1, 2, \dots, n)$, with $w_j \in [0, 1]$ and $\sum\limits_{j=1}^{n} w_j = 1$, and let IIULWA: $\Omega^n \to \Omega$, if

$$\mathrm{IIULWA}_{\mathbf{w}}(\beta_1, \beta_2, \dots, \beta_n) = w_1\beta_1 \oplus w_2\beta_2 \oplus \cdots \oplus w_n\beta_n, \tag{8}$$

then IIULWA is called an IIULWA operator of dimension n. Especially, if $\mathbf{w} = (1/n, 1/n, \dots, 1/n)^T$, then the IIULWA operator is reduced to the interval-valued intuitionistic uncertain linguistic arithmetic averaging (IIULAA) operator of dimension n, which is defined as follows:

$$\mathrm{IIULAA}_{\mathbf{w}}(\beta_1, \beta_2, \dots, \beta_n) = \frac{1}{n}(\beta_1 \oplus \beta_2 \oplus \cdots \oplus \beta_n). \tag{9}$$

Theorem 1. *Let $\beta_j = \langle [s_{\theta_{\beta_j}}, s_{\phi_{\beta_j}}], [a_{\beta_j}, b_{\beta_j}], [c_{\beta_j}, d_{\beta_j}]\rangle$ $(j = 1, 2, \dots, n)$ be a collection of IIULNs, and $\mathbf{w} = (w_1, w_2, \dots, w_n)^T$ be the weight vector of β_j $(j = 1, 2, \dots, n)$, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, then*

$$\begin{aligned}
&\mathrm{IIULWA}_{\mathbf{w}}(\beta_1, \beta_2, \dots, \beta_n) \\
&\quad = \Bigg\langle \big[s_{\sum_{j=1}^{n} w_j\theta_{\beta_j}}, s_{\sum_{j=1}^{n} w_j\phi_{\beta_j}}\big], \left[1 - \prod_{j=1}^{n}(1 - a_{\beta_j})^{w_j}, 1 - \prod_{j=1}^{n}(1 - b_{\beta_j})^{w_j}\right], \\
&\qquad \Bigg[\prod_{j=1}^{n}(1 - a_{\beta_j})^{w_j} - \prod_{j=1}^{n}(1 - a_{\beta_j} - c_{\beta_j})^{w_j}, \\
&\qquad \prod_{j=1}^{n}(1 - b_{\beta_j})^{w_j} - \prod_{j=1}^{n}(1 - b_{\beta_j} - d_{\beta_j})^{w_j}\Bigg]\Bigg\rangle.
\end{aligned} \tag{10}$$

Proof. In the following, (10) is proved using mathematical induction on n.

(1) When $n=2$, since

$$\begin{aligned} w_1\beta_1 &= \big\langle [s_{w_1\theta_{\beta_1}}, s_{w_1\phi_{\beta_1}}], \big[1-(1-a_{\beta_1})^{w_1}, 1-(1-b_{\beta_1})^{w_1}\big], \\ &\quad \big[(1-a_{\beta_1})^{w_1}-(1-a_{\beta_1}-c_{\beta_1})^{w_1}, (1-b_{\beta_1})^{w_1}-(1-b_{\beta_1}-d_{\beta_1})^{w_1}\big]\big\rangle, \\ w_2\beta_2 &= \big\langle [s_{w_2\theta_{\beta_2}}, s_{w_2\phi_{\beta_2}}], \big[1-(1-a_{\beta_2})^{w_2}, 1-(1-b_{\beta_2})^{w_2}\big], \\ &\quad \big[(1-a_{\beta_2})^{w_2}-(1-a_{\beta_2}-c_{\beta_2})^{w_2}, (1-b_{\beta_2})^{w_2}-(1-b_{\beta_2}-d_{\beta_2})^{w_2}\big]\big\rangle, \end{aligned}$$

then

$$\begin{aligned} &\mathrm{IIULWA}_{\mathbf{w}}(\beta_1, \beta_2) \\ &\quad = w_1\beta_1 \oplus w_2\beta_2 \\ &\quad = \big\langle [s_{w_1\theta_{\beta_1}+w_2\theta_{\beta_2}}, s_{w_1\phi_{\beta_1}+w_2\phi_{\beta_2}}], \\ &\qquad \big[1-(1-a_{\beta_1})^{w_1}(1-a_{\beta_2})^{w_2}, 1-(1-b_{\beta_1})^{w_1}(1-b_{\beta_2})^{w_2}\big], \\ &\qquad \big[(1-a_{\beta_1})^{w_1}(1-a_{\beta_2})^{w_2}-(1-a_{\beta_1}-c_{\beta_1})^{w_1}(1-a_{\beta_2}-c_{\beta_2})^{w_2}, \\ &\qquad (1-b_{\beta_1})^{w_1}(1-b_{\beta_2})^{w_2}-(1-b_{\beta_1}-d_{\beta_1})^{w_1}(1-b_{\beta_2}-d_{\beta_2})^{w_2}\big]\big\rangle \\ &\quad = \left\langle \left[s_{\sum_{j=1}^{2} w_j\theta_{\beta_j}}, s_{\sum_{j=1}^{2} w_j\phi_{\beta_j}}\right], \left[1-\prod_{j=1}^{2}(1-a_{\beta_j})^{w_j}, 1-\prod_{j=1}^{2}(1-b_{\beta_j})^{w_j}\right], \right. \\ &\qquad \left[\prod_{j=1}^{2}(1-a_{\beta_j})^{w_j}-\prod_{j=1}^{2}(1-a_{\beta_j}-c_{\beta_j})^{w_j}, \right. \\ &\qquad \left.\left. \prod_{j=1}^{2}(1-b_{\beta_j})^{w_j}-\prod_{j=1}^{2}(1-b_{\beta_j}-d_{\beta_j})^{w_j}\right]\right\rangle. \end{aligned}$$

Thus, (10) holds for $n=2$.

(2) If (10) holds for $n=k$, that is

$$\begin{aligned} &\mathrm{IIULWA}_{\mathbf{w}}(\beta_1, \beta_2, \dots, \beta_k) \\ &\quad = \left\langle \left[s_{\sum_{j=1}^{k} w_j\theta_{\beta_j}}, s_{\sum_{j=1}^{k} w_j\phi_{\beta_j}}\right], \left[1-\prod_{j=1}^{k}(1-a_{\beta_j})^{w_j}, 1-\prod_{j=1}^{k}(1-b_{\beta_j})^{w_j}\right], \right. \\ &\qquad \left[\prod_{j=1}^{k}(1-a_{\beta_j})^{w_j}-\prod_{j=1}^{k}(1-a_{\beta_j}-c_{\beta_j})^{w_j}, \right. \\ &\qquad \left.\left. \prod_{j=1}^{k}(1-b_{\beta_j})^{w_j}-\prod_{j=1}^{k}(1-b_{\beta_j}-d_{\beta_j})^{w_j}\right]\right\rangle. \end{aligned}$$

then, when $n = k + 1$, by the operational laws in Definition 4, we have

$$
\begin{aligned}
&\mathrm{IIULWA}_{\mathbf{w}}(\beta_1, \beta_2, \ldots, \beta_k, \beta_{k+1}) \\
&= \Bigg\langle \big[s_{\sum_{j=1}^{k} w_j \theta_{\beta_j} + w_{k+1}\theta_{\beta_{k+1}}}, s_{\sum_{j=1}^{k} w_j \phi_{\beta_j} + w_{k+1}\phi_{\beta_{k+1}}} \big], \\
&\quad \left[1 - \prod_{j=1}^{k} (1 - a_{\beta_j})^{w_j} (1 - a_{\beta_{k+1}})^{w_{k+1}}, 1 - \prod_{j=1}^{k} (1 - b_{\beta_j})^{w_j} (1 - b_{\beta_{k+1}})^{w_{k+1}} \right], \\
&\quad \Bigg[\prod_{j=1}^{k} (1 - a_{\beta_j})^{w_j} (1 - a_{\beta_{k+1}})^{w_{k+1}} \\
&\quad - \prod_{j=1}^{k} (1 - a_{\beta_j} - c_{\beta_j})^{w_j} (1 - a_{\beta_{k+1}} - c_{\beta_{k+1}})^{w_{k+1}}, \\
&\quad \prod_{j=1}^{k} (1 - b_{\beta_j})^{w_j} (1 - b_{\beta_{k+1}})^{w_{k+1}} \\
&\quad - \prod_{j=1}^{k} (1 - b_{\beta_j} - d_{\beta_j})^{w_j} (1 - b_{\beta_{k+1}} - d_{\beta_{k+1}})^{w_{k+1}} \Bigg] \Bigg\rangle \\
&= \Bigg\langle \big[s_{\sum_{j=1}^{k+1} w_j \theta_{\beta_j}}, s_{\sum_{j=1}^{k+1} w_j \phi_{\beta_j}} \big], \left[1 - \prod_{j=1}^{k+1} (1 - a_{\beta_j})^{w_j}, 1 - \prod_{j=1}^{k+1} (1 - b_{\beta_j})^{w_j} \right], \\
&\quad \Bigg[\prod_{j=1}^{k+1} (1 - a_{\beta_j})^{w_j} - \prod_{j=1}^{k+1} (1 - a_{\beta_j} - c_{\beta_j})^{w_j}, \\
&\quad \prod_{j=1}^{k+1} (1 - b_{\beta_j})^{w_j} - \prod_{j=1}^{k+1} (1 - b_{\beta_j} - d_{\beta_j})^{w_j} \Bigg] \Bigg\rangle,
\end{aligned}
$$

i.e. (10) holds for $n = k + 1$.

Therefore, (10) holds for all $n \in N$, which completes the proof of Theorem 1. □

Theorem 2 (Properties of IIULWA). *Let* $\beta_j = \langle [s_{\theta_{\beta_j}}, s_{\phi_{\beta_j}}], [a_{\beta_j}, b_{\beta_j}], [c_{\beta_j}, d_{\beta_j}] \rangle$ $(j = 1, 2, \ldots, n)$ *be a collection of IIULNs, and* $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T$ *be the weight vector of* β_j $(j = 1, 2, \ldots, n)$*, with* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$*, then the IIULWA operator has the following properties.*

(1) (*Idempotency*): *if* $\beta_j = \beta$ *for all* j*, then* $\mathrm{IIULWA}_{\mathbf{w}}(\beta_1, \beta_2, \ldots, \beta_n) = \beta$.

(2) (*Boundary*): $\min\{\beta_1, \beta_2, \ldots, \beta_n\} \leqslant \mathrm{IIULWA}_{\mathbf{w}}(\beta_1, \beta_2, \ldots, \beta_n) \leqslant \max\{\beta_1, \beta_2, \ldots, \beta_n\}$.

(3) (*Monotonicity*): *if* $\beta_j \leqslant \beta_j^*$ *for all* j*, and* β_j^* $(j = 1, 2, \ldots, n)$ *is a collection of IIULNs, then* $\mathrm{IIULWA}_{\mathbf{w}}(\beta_1, \beta_2, \ldots, \beta_n) \leqslant \mathrm{IIULWA}_{\mathbf{w}}(\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$.

DEFINITION 8. Let $\beta_j = \langle [s_{\theta_{\beta_j}}, s_{\phi_{\beta_j}}], [a_{\beta_j}, b_{\beta_j}], [c_{\beta_j}, d_{\beta_j}] \rangle$ $(j = 1, 2, \dots, n)$ be a collection of IIULNs. An IIULOWA operator of dimension n is a mapping IIULOWA: $\Omega^n \to \Omega$, that has an associated weight vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_n)^T$ such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$. Furthermore,

$$\mathrm{IIULOWA}_{\omega}(\beta_1, \beta_2, \dots, \beta_n) = \omega_1 \beta_{\tau(1)} \oplus \omega_2 \beta_{\tau(2)} \oplus \dots \oplus \omega_n \beta_{\tau(n)}, \tag{11}$$

where $(\tau(1), \tau(2), \dots, \tau(n))$ is a permutation of $(1, 2, \dots, n)$ such that $\beta_{\tau(j-1)} \leqslant \beta_{\tau(j)}$ for all j. Especially, if $\boldsymbol{\omega} = (1/n, 1/n, \dots, 1/n)^T$, then the IIULOWA operator is reduced to the IIULAA operator.

Similar to Theorem 1, the following can be stated.

Theorem 3. *Let* $\beta_j = \langle [s_{\theta_{\beta_j}}, s_{\phi_{\beta_j}}], [a_{\beta_j}, b_{\beta_j}], [c_{\beta_j}, d_{\beta_j}] \rangle$ $(j = 1, 2, \dots, n)$ *be a collection of IIULNs,then*

$$\begin{aligned} &\mathrm{IIULOWA}_{\boldsymbol{\omega}}(\beta_1, \beta_2, \dots, \beta_n) \\ &= \Bigg\langle \big[s_{\sum_{j=1}^n \omega_j \theta_{\beta_{\tau(j)}}}, s_{\sum_{j=1}^n \omega_j \phi_{\beta_{\tau(j)}}} \big], \\ &\quad \left[1 - \prod_{j=1}^n (1 - a_{\beta_{\tau(j)}})^{\omega_j}, 1 - \prod_{j=1}^n (1 - b_{\beta_{\tau(j)}})^{\omega_j} \right], \\ &\quad \left[\prod_{j=1}^n (1 - a_{\beta_{\tau(j)}})^{\omega_j} - \prod_{j=1}^n (1 - a_{\beta_{\tau(j)}} - c_{\beta_{\tau(j)}})^{\omega_j}, \right. \\ &\quad \left. \prod_{j=1}^n (1 - b_{\beta_{\tau(j)}})^{\omega_j} - \prod_{j=1}^n (1 - b_{\beta_{\tau(j)}} - d_{\beta_{\tau(j)}})^{\omega_j} \right] \Bigg\rangle, \end{aligned} \tag{12}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ *is the weight vector related to the IIULOWA operator, with* $\omega_j \in [0, 1]$ *and* $\sum_{j=1}^n \omega_j = 1$, *which can be determined similar to the OWA weights* (Xu, 2005).

Theorem 4 (Properties of IIULOWA). *Let* $\beta_j = \langle [s_{\theta_{\beta_j}}, s_{\phi_{\beta_j}}], [a_{\beta_j}, b_{\beta_j}], [c_{\beta_j}, d_{\beta_j}] \rangle$ $(j = 1, 2, \dots, n)$ *be a collection of IIULNs, and* $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_n)^T$ *be the weight vector related to the IIULOWA operator, with* $\omega_j \in [0, 1]$ *and* $\sum_{j=1}^n \omega_j = 1$, *then the IIULOWA operator has the following properties.*

(1) (*Idempotency*): *if* $\beta_j = \beta$ *for all* j, *then* $\mathrm{IIULOWA}_{\boldsymbol{\omega}}(\beta_1, \beta_2, \dots, \beta_n) = \beta$.
(2) (*Boundary*): $\min\{\beta_1, \beta_2, \dots, \beta_n\} \leqslant \mathrm{IIULOWA}_{\boldsymbol{\omega}}(\beta_1, \beta_2, \dots, \beta_n) \leqslant \max\{\beta_1, \beta_2, \dots, \beta_n\}$.
(3) (*Monotonicity*): *if* $\beta_j \leqslant \beta_j^*$ *for all* j, *and* β_j^* $(j = 1, 2, \dots, n)$ *is a collection of IIULNs, then* $\mathrm{IIULOWA}_{\boldsymbol{\omega}}(\beta_1, \beta_2, \dots, \beta_n) \leqslant \mathrm{IIULOWA}_{\boldsymbol{\omega}}(\beta_1^*, \beta_2^*, \dots, \beta_n^*)$.

(4) (*Commutativity*): *if* $(\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_n)$ *is any permutation of* $(\beta_1, \beta_2, \dots, \beta_n)$*, then* $\mathrm{IIULOWA}_{\boldsymbol{\omega}}(\beta_1, \beta_2, \dots, \beta_n) = \mathrm{IIULOWA}_{\boldsymbol{\omega}}(\tilde{\beta}_1, \tilde{\beta}_2, \dots, \tilde{\beta}_n)$.

From the above, it can be seen that the IIULOWA operator has commutativity, whereas the IIULWA operator does not have this property.

Apart from the properties outlined above, the IIULOWA operator has the following desirable results.

Theorem 5. *Let* $\beta_j = \langle [s_{\theta_{\beta_j}}, s_{\phi_{\beta_j}}], [a_{\beta_j}, b_{\beta_j}], [c_{\beta_j}, d_{\beta_j}] \rangle$ $(j = 1, 2, \dots, n)$ *be a collection of IIULNs, and* $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_n)^T$ *be the weight vector related to the IIULOWA operator, with* $\omega_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \omega_j = 1$*, then*

(1) *if* $\boldsymbol{\omega} = (1, 0, \dots, 0)^T$*, then* $\mathrm{IIULOWA}_{\boldsymbol{\omega}}(\beta_1, \beta_2, \dots, \beta_n) = \max\{\beta_1, \beta_2, \dots, \beta_n\}$.
(2) *if* $\boldsymbol{\omega} = (0, 0, \dots, 1)^T$*, then* $\mathrm{IIULOWA}_{\boldsymbol{\omega}}(\beta_1, \beta_2, \dots, \beta_n) = \min\{\beta_1, \beta_2, \dots, \beta_n\}$.
(3) *if* $\omega_j = 1$*,* $\omega_i = 0$*,* $i \neq j$*, and* $\beta_{\tau(j)}$ *is the* j*th largest of* β_i $(i = 1, 2, \dots, n)$*, then* $\mathit{IIULOWA}_{\boldsymbol{\omega}}(\beta_1, \beta_2, \dots, \beta_n) = \beta_{\tau(j)}$.

3.2. *The IIULHA Operator*

From Definitions 7 and 8, it is known that the IIULWA operator weights only the IIULNs, whereas the IIULOWA operator weights only the ordered positions of the IIULNs. To overcome this limitation, an IIULHA operator is now developed, which weights both the given IIULNs and their ordered positions.

Definition 9. Let $\beta_j = \langle [s_{\theta_{\beta_j}}, s_{\phi_{\beta_j}}], [a_{\beta_j}, b_{\beta_j}], [c_{\beta_j}, d_{\beta_j}] \rangle$ $(j = 1, 2, \dots, n)$ be a collection of IIULNs. An IIULHA operator of dimension n is a mapping IIULHA: $\Omega^n \to \Omega$, which has an associated vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_n)^T$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$, such that

$$\mathrm{IIULHA}_{\mathbf{w},\boldsymbol{\omega}}(\beta_1, \beta_2, \dots, \beta_n) = \omega_1 \beta'_{\tau(1)} \oplus \omega_2 \beta'_{\tau(2)} \oplus \dots \oplus \omega_n \beta'_{\tau(n)}, \tag{13}$$

where $\beta'_{\tau(j)}$ is the jth largest of weighted IIULNs $(nw_1\beta_1, nw_2\beta_2, \dots, nw_n\beta_n)$, $\mathbf{w} = (w_1, w_2, \dots, w_n)^T$ is the weight vector of β_j $(j = 1, 2, \dots, n)$, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient.

Furthermore, similar to Theorem 1, we have

$$\begin{aligned}
&\mathrm{IIULHA}_{\mathbf{w},\boldsymbol{\omega}}(\beta_1, \beta_2, \dots, \beta_n) \\
&\quad = \Big\langle \big[s_{\sum_{j=1}^{n} \omega_j \theta_{\beta'_{\tau(j)}}}, s_{\sum_{j=1}^{n} \omega_j \phi_{\beta'_{\tau(j)}}} \big], \\
&\quad \left[1 - \prod_{j=1}^{n} (1 - a_{\beta'_{\tau(j)}})^{\omega_j}, 1 - \prod_{j=1}^{n} (1 - b_{\beta'_{\tau(j)}})^{\omega_j} \right],
\end{aligned}$$

$$\left[\prod_{j=1}^{n}(1-a_{\beta'_{\tau(j)}})^{\omega_j}-\prod_{j=1}^{n}(1-a_{\beta'_{\tau(j)}}-c_{\beta'_{\tau(j)}})^{\omega_j},\right.$$

$$\left.\prod_{j=1}^{n}(1-b_{\beta'_{\tau(j)}})^{\omega_j}-\prod_{j=1}^{n}(1-b_{\beta'_{\tau(j)}}-d_{\beta'_{\tau(j)}})^{\omega_j}\right]\Big\rangle. \tag{14}$$

Theorem 6. *If $\boldsymbol{\omega}=(1/n,1/n,\ldots,1/n)^T$, then the IIULHA operator is reduced to the IIULWA operator.*

Proof. If $\boldsymbol{\omega}=(1/n,1/n,\ldots,1/n)^T$, then

$$\begin{aligned}
\mathrm{IIULHA}_{\mathbf{w},\boldsymbol{\omega}}(\beta_1,\beta_2,\ldots,\beta_n) &= \omega_1\beta'_{\tau(1)}\oplus\omega_2\beta'_{\tau(2)}\oplus\cdots\oplus\omega_n\beta'_{\tau(n)}\\
&= \frac{1}{n}\big(\beta'_1\oplus\beta'_2\oplus\cdots\oplus\beta'_n\big)\\
&= w_1\beta_1\oplus w_2\beta_2\oplus\cdots\oplus w_n\beta_n\\
&= \mathrm{IIULWA}_{\mathbf{w}}(\beta_1,\beta_2,\ldots,\beta_n),
\end{aligned}$$

which completes the proof of Theorem 6. □

Theorem 7. *If $\mathbf{w}=(1/n,1/n,\ldots,1/n)^T$, then the IIULHA operator is reduced to the IIULOWA operator.*

Proof. If $\mathbf{w}=(1/n,1/n,\ldots,1/n)^T$, then $\beta'_j=\beta_j$, $j=1,2,\ldots,n$, thus

$$\begin{aligned}
\mathrm{IIULHA}_{\mathbf{w},\boldsymbol{\omega}}(\beta_1,\beta_2,\ldots,\beta_n) &= \omega_1\beta'_{\tau(1)}\oplus\omega_2\beta'_{\tau(2)}\oplus\cdots\oplus\omega_n\beta'_{\tau(n)}\\
&= \omega_1\beta_{\tau(1)}\oplus\omega_2\beta_{\tau(2)}\oplus\cdots\oplus\omega_n\beta_{\tau(n)}\\
&= \mathrm{IIULOWA}_{\boldsymbol{\omega}}(\beta_1,\beta_2,\ldots,\beta_n).
\end{aligned}$$

This completes the proof of Theorem 7. □

From Theorems 6 and 7, it is known that the IIULHA operator generalizes both the IIULWA operator and the IIULOWA operator at the same time, and can reflect the importance degrees of both the given IIULNs and their ordered positions.

4. A Group Decision Making Method Based on the IIULWA and IIULHA Operators

Consider a group decision making problem with interval-valued intuitionistic uncertain linguistic information. Let $A=\{A_1,A_2,\ldots,A_m\}$ be the set of alternatives, $I=\{I_1,I_2,\ldots,I_n\}$ be the set of criteria, and $\mathbf{w}=(w_1,w_2,\ldots,w_n)^T$ be the weight vector of I_j $(j=1,2,\ldots,n)$ with $w_j\geqslant 0$ and $\sum_{j=1}^{n}w_j=1$. Let $D=\{d_1,d_2,\ldots,d_t\}$ be the set of decision-makers, and $\mathbf{e} = (e_1, e_2, \ldots, e_t)^T$ be the weight vector of d_k $(k = 1, 2, \ldots, t)$ with $e_k \geqslant 0$ and $\sum_{k=1}^t e_k = 1$. Suppose that $\beta_{ij}^{(k)} = \langle [s_{\theta_{\beta_{ij}^{(k)}}}, s_{\phi_{\beta_{ij}^{(k)}}}], [a_{\beta_{ij}^{(k)}}, b_{\beta_{ij}^{(k)}}], [c_{\beta_{ij}^{(k)}}, d_{\beta_{ij}^{(k)}}] \rangle$ are the criterion values of the alternatives A_i with respect to the criteria I_j given by the decision-makers d_k $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n;\ k = 1, 2, \ldots, t)$, and $\mathbf{R}^{(k)} = (\beta_{ij}^{(k)})_{m \times n} (k = 1, 2, \ldots, t)$ are the decision matrices, where $[s_{\theta_{\beta_{ij}^{(k)}}}, s_{\phi_{\beta_{ij}^{(k)}}}]$ are the uncertain linguistic evaluation values of A_i with respect to I_j given by d_k, $[a_{\beta_{ij}^{(k)}}, b_{\beta_{ij}^{(k)}}]$ and $[c_{\beta_{ij}^{(k)}}, d_{\beta_{ij}^{(k)}}]$ indicate the degree of membership and the degree of non-membership of A_i to $[s_{\theta_{\beta_{ij}^{(k)}}}, s_{\phi_{\beta_{ij}^{(k)}}}]$ with respect to I_j given by d_k, respectively $(i = 1, 2, \ldots, m;$ $j = 1, 2, \ldots, n;\ k = 1, 2, \ldots, t)$.

In the following steps, the IIULWA and IIULHA operators are applied in order to develop a group decision making approach, in which the criterion values take the form of IIULNs and the criterion weight information is known completely.

Step 1. Normalize the decision matrices $\mathbf{R}^{(k)} = (\beta_{ij}^{(k)})_{m \times n}$ to $\bar{\mathbf{R}}^{(k)} = (\bar{\beta}_{ij}^{(k)})_{m \times n}$ $(k = 1, 2, \ldots, t)$, where $\bar{\beta}_{ij}^{(k)} = \langle [s_{\theta_{\bar{\beta}_{ij}^{(k)}}}, s_{\phi_{\bar{\beta}_{ij}^{(k)}}}], [a_{\bar{\beta}_{ij}^{(k)}}, b_{\bar{\beta}_{ij}^{(k)}}], [c_{\bar{\beta}_{ij}^{(k)}}, d_{\bar{\beta}_{ij}^{(k)}}] \rangle$ $(k = 1, 2, \ldots, t)$. For the benefit-type criteria, there is nothing to do; for the cost-type criteria, the linguistic negation operator $s_{\theta(\bar{\beta}_{ij}^{(k)})} = \mathrm{neg}(s_{\theta(\beta_{ij}^{(k)})}) = s_{2l-\theta(\beta_{ij}^{(k)})}$ and $s_{\phi(\bar{\beta}_{ij}^{(k)})} = \mathrm{neg}(s_{\phi(\beta_{ij}^{(k)})}) = s_{2l-\phi(\beta_{ij}^{(k)})}$ is utilized to make the uncertain linguistic evaluation values normalized.

Step 2. Utilize the IIULWA operator

$$\bar{\beta}_i^{(k)} = \mathrm{IIULWA}_{\mathbf{w}}\big(\bar{\beta}_{i1}^{(k)}, \bar{\beta}_{i2}^{(k)}, \ldots, \bar{\beta}_{in}^{(k)}\big) \tag{15}$$

to aggregate the criterion values of the line of the decision matrices $\bar{\mathbf{R}}^{(k)} = (\bar{\beta}_{ij}^{(k)})_{m \times n}$ $(k = 1, 2, \ldots, t)$ and derive the individual overall values $\bar{\beta}_i^{(k)} = \langle [s_{\theta_{\bar{\beta}_i^{(k)}}}, s_{\phi_{\bar{\beta}_i^{(k)}}}], [a_{\bar{\beta}_i^{(k)}}, b_{\bar{\beta}_i^{(k)}}],$ $[c_{\bar{\beta}_i^{(k)}}, d_{\bar{\beta}_i^{(k)}}] \rangle$ of the alternatives A_i $(i = 1, 2, \ldots, m)$ given by the decision-makers d_k $(k = 1, 2, \ldots, t)$.

Step 3. Utilize the IIULHA operator

$$\begin{aligned} \bar{\beta}_i &= \mathrm{IIULHA}_{\mathbf{e},\mathbf{v}}\big(\bar{\beta}_i^{(1)}, \bar{\beta}_i^{(2)}, \ldots, \bar{\beta}_i^{(t)}\big) \\ &= v_1 \bar{\beta}_i^{\prime(\tau(1))} \oplus v_2 \bar{\beta}_i^{\prime(\tau(2))} \oplus \cdots \oplus v_t \bar{\beta}_i^{\prime(\tau(t))} \end{aligned} \tag{16}$$

to derive the collective overall values $\bar{\beta}_i = \langle [s_{\theta_{\bar{\beta}_i}}, s_{\phi_{\bar{\beta}_i}}], [a_{\bar{\beta}_i}, b_{\bar{\beta}_i}], [c_{\bar{\beta}_i}, d_{\bar{\beta}_i}] \rangle$ of the alternatives A_i $(i = 1, 2, \ldots, m)$, where $\mathbf{v} = (v_1, v_2, \ldots, v_t)^T$ is the weighting vector of the IIULHA operator with $v_k \geqslant 0$ and $\sum_{k=1}^t v_k = 1$, $\bar{\beta}_i^{\prime(\tau(k))} = \langle [s_{\theta_{\bar{\beta}_i^{\prime(\tau(k))}}}, s_{\phi_{\bar{\beta}_i^{\prime(\tau(k))}}}],$ $[a_{\bar{\beta}_i^{\prime(\tau(k))}}, b_{\bar{\beta}_i^{\prime(\tau(k))}}], [c_{\bar{\beta}_i^{\prime(\tau(k))}}, d_{\bar{\beta}_i^{\prime(\tau(k))}}] \rangle$ is the kth largest of weighted IIULNs $(te_1 \bar{\beta}_i^{(1)},$ $te_2 \bar{\beta}_i^{(2)}, \ldots, te_t \bar{\beta}_i^{(t)})$, $(\tau(1), \tau(2), \ldots, \tau(t))$ is a permutation of $(1, 2, \ldots, t)$, and t is the balancing coefficient.

Step 4. Utilize (6) to calculate the scores $h(\bar{\beta}_i)$ of the collective overall values $\bar{\beta}_i$ of the alternatives A_i $(i = 1, 2, \ldots, m)$.

Step 5. Rank all the alternatives A_i $(i = 1, 2, \ldots, m)$, and then select the best one in accordance with $h(\bar{\beta}_i)$ $(i = 1, 2, \ldots, m)$ (if there is no difference between two scores $h(\bar{\beta}_i)$ and $h(\bar{\beta}_p)$, then it is necessary to calculate the accuracy degrees $u(\bar{\beta}_i)$ and $u(\bar{\beta}_p)$ of the collective overall values $\bar{\beta}_i$ and $\bar{\beta}_p$ by using (7), and then rank the alternatives A_i and A_p, in accordance with the accuracy degrees $u(\bar{\beta}_i)$ and $u(\bar{\beta}_p)$).

5. An Illustrative Example

Suppose that there is a risk investment company, which wants to invest a sum of money in the best option. There is a panel with four alternative enterprises A_i $(i = 1, 2, 3, 4)$ in which to invest the money: A_1 is a computer company; A_2 is a car company; A_3 is an arms company; and A_4 is a bicycle company. Three decision-makers d_k $(k = 1, 2, 3)$, whose weighting vector is $\mathbf{e} = (0.3, 0.4, 0.3)^T$, will be completely responsible for this investment. In assessing the potential contribution of each enterprise, three factors are considered: profitability (I_1), competitiveness (I_2), and risk affordability (I_3), whose weighting vector is $\mathbf{w} = (0.3727, 0.3500, 0.2773)^T$, and the characteristic information of the alternatives A_i $(i = 1, 2, 3, 4)$ with respect to the criteria I_j $(j = 1, 2, 3)$ given by decision-makers d_k $(k = 1, 2, 3)$ are represented by the IIULNs $\beta_{ij}^{(k)}$ $(i = 1, 2, 3, 4,\ j = 1, 2, 3,\ k = 1, 2, 3)$ listed in Tables 1, 2, and 3.

Table 1
Decision matrix $\mathbf{R}^{(1)}$.

	I_1	I_2	I_3
A_1	$\langle[s_3, s_5]$, [0.5,0.7], [0.2,0.3]$\rangle$	$\langle[s_5, s_6]$, [0.7,0.8], [0.1,0.2]$\rangle$	$\langle[s_6, s_7]$, [0.6,0.7], [0.1,0.2]$\rangle$
A_2	$\langle[s_6, s_7]$, [0.7,0.8], [0.2,0.2]$\rangle$	$\langle[s_5, s_6]$, [0.6,0.8], [0.1,0.1]$\rangle$	$\langle[s_4, s_5]$, [0.7,0.8], [0.2,0.2]$\rangle$
A_3	$\langle[s_5, s_6]$, [0.5,0.8], [0.1,0.2]$\rangle$	$\langle[s_5, s_7]$, [0.6,0.7], [0.2,0.3]$\rangle$	$\langle[s_5, s_8]$, [0.8,0.9], [0.1,0.1]$\rangle$
A_4	$\langle[s_4, s_5]$, [0.6,0.7], [0.1,0.2]$\rangle$	$\langle[s_5, s_6]$, [0.7,0.9], [0.1,0.1]$\rangle$	$\langle[s_5, s_7]$, [0.6,0.7], [0.2,0.3]$\rangle$

Table 2
Decision matrix $\mathbf{R}^{(2)}$.

	I_1	I_2	I_3
A_1	$\langle[s_4, s_6]$, [0.8,0.9], [0.1,0.1]$\rangle$	$\langle[s_5, s_6]$, [0.6,0.7], [0.1,0.2]$\rangle$	$\langle[s_4, s_5]$, [0.7,0.8], [0.1,0.2]$\rangle$
A_2	$\langle[s_3, s_4]$, [0.5,0.7], [0.1,0.2]$\rangle$	$\langle[s_5, s_7]$, [0.5,0.7], [0.1,0.3]$\rangle$	$\langle[s_5, s_6]$, [0.8,0.9], [0.1,0.1]$\rangle$
A_3	$\langle[s_4, s_5]$, [0.6,0.7], [0.2,0.3]$\rangle$	$\langle[s_7, s_8]$, [0.7,0.8], [0.1,0.2]$\rangle$	$\langle[s_5, s_7]$, [0.6,0.7], [0.1,0.3]$\rangle$
A_4	$\langle[s_5, s_6]$, [0.7,0.8], [0.1,0.2]$\rangle$	$\langle[s_4, s_5]$, [0.8,0.9], [0.1,0.1]$\rangle$	$\langle[s_5, s_7]$, [0.7,0.8], [0.1,0.2]$\rangle$

Table 3
Decision matrix $\mathbf{R}^{(3)}$.

	I_1	I_2	I_3
A_1	$\langle[s_4, s_5]$, [0.6,0.7], [0.2,0.3]$\rangle$	$\langle[s_6, s_7]$, [0.6,0.7], [0.2,0.3]$\rangle$	$\langle[s_5, s_6]$, [0.8,0.9], [0.1,0.1]$\rangle$
A_2	$\langle[s_3, s_4]$, [0.6,0.7], [0.1,0.2]$\rangle$	$\langle[s_5, s_7]$, [0.7,0.8], [0.1,0.2]$\rangle$	$\langle[s_6, s_7]$, [0.7,0.8], [0.1,0.2]$\rangle$
A_3	$\langle[s_5, s_6]$, [0.7,0.8], [0.2,0.2]$\rangle$	$\langle[s_7, s_8]$, [0.8,0.9], [0.1,0.1]$\rangle$	$\langle[s_5, s_7]$, [0.6,0.7], [0.1,0.3]$\rangle$
A_4	$\langle[s_3, s_5]$, [0.8,0.9], [0.1,0.1]$\rangle$	$\langle[s_5, s_6]$, [0.6,0.7], [0.1,0.2]$\rangle$	$\langle[s_6, s_7]$, [0.7,0.8], [0.2,0.2]$\rangle$

To abtain the best investment enterprise using the developed approach, the following steps are taken.

Step 1. Because all criteria I_j $(j = 1, 2, 3)$ are of a benefit-type, their values need not to be normalized.

Step 2. Utilize (15) to aggregate the criterion values of the line of the decision matrices $\mathbf{R}^{(k)}$ and derive the individual overall evaluation values $\beta_i^{(k)}$ of the alternatives A_i $(i = 1, 2, 3, 4)$ given by the decision-makers d_k $(k = 1, 2, 3)$:

$$\beta_1^{(1)} = \langle [s_{4.5319}, s_{5.9046}], [0.6069, 0.7397], [0.1327, 0.2603] \rangle,$$
$$\beta_2^{(1)} = \langle [s_{5.0954}, s_{6.0954}], [0.6682, 0.8000], [0.1849, 0.2000] \rangle,$$
$$\beta_3^{(1)} = \langle [s_{5.0000}, s_{6.9046}], [0.6413, 0.8098], [0.1450, 0.1902] \rangle,$$
$$\beta_4^{(1)} = \langle [s_{4.6273}, s_{5.9046}], [0.6383, 0.7958], [0.1291, 0.2042] \rangle,$$
$$\beta_1^{(2)} = \langle [s_{4.3500}, s_{5.7227}], [0.7148, 0.8220], [0.1072, 0.1780] \rangle,$$
$$\beta_2^{(2)} = \langle [s_{4.2546}, s_{5.6046}], [0.6122, 0.7788], [0.1155, 0.2212] \rangle,$$
$$\beta_3^{(2)} = \langle [s_{5.3273}, s_{6.6046}], [0.6383, 0.7397], [0.1379, 0.2603] \rangle,$$
$$\beta_4^{(2)} = \langle [s_{4.6500}, s_{5.9273}], [0.7397, 0.8431], [0.1034, 0.1569] \rangle,$$
$$\beta_1^{(3)} = \langle [s_{4.9773}, s_{5.9773}], [0.6699, 0.7788], [0.1650, 0.2212] \rangle,$$
$$\beta_2^{(3)} = \langle [s_{4.5319}, s_{5.8819}], [0.6660, 0.7674], [0.1013, 0.2326] \rangle,$$
$$\beta_3^{(3)} = \langle [s_{5.7000}, s_{6.9773}], [0.7181, 0.8244], [0.1463, 0.1756] \rangle,$$
$$\beta_4^{(3)} = \langle [s_{4.5319}, s_{5.9046}], [0.7148, 0.8220], [0.1384, 0.1780] \rangle.$$

Step 3. Utilize (16) to aggregate the individual overall evaluation values $\beta_i^{(1)}, \beta_i^{(2)}, \beta_i^{(3)}$ and derive the collective overall evaluation values β_i of the alternatives A_i $(i = 1, 2, 3, 4)$, where the weight vector of the IIULHA operator is $\mathbf{v} = (0.2429, 0.5142, 0.2429)^T$, which is determined by the weighting method based on normal distribution (Xu, 2005):

$$\beta_1 = \langle [s_{4.5621}, s_{5.7250}], [0.6613, 0.7758], [0.1429, 0.2242] \rangle,$$
$$\beta_2 = \langle [s_{4.5889}, s_{5.7403}], [0.6417, 0.7776], [0.1535, 0.2224] \rangle,$$
$$\beta_3 = \langle [s_{5.2837}, s_{6.6635}], [0.6693, 0.7899], [0.1477, 0.2101] \rangle,$$
$$\beta_4 = \langle [s_{4.4642}, s_{5.7510}], [0.6973, 0.8147], [0.1282, 0.1853] \rangle.$$

Step 4. Utilize (6) to calculate the scores $h(\beta_i)$ of the collective overall values β_i of the alternatives A_i $(i = 1, 2, 3, 4)$:

$$h(\beta_1) = 3.9478, \qquad h(\beta_2) = 3.9296, \qquad h(\beta_3) = 4.6316, \qquad h(\beta_4) = 4.0842.$$

Thus

$$h(\beta_3) > h(\beta_4) > h(\beta_1) > h(\beta_2).$$

Step 5. Rank all the alternatives A_i $(i = 1, 2, 3, 4)$ in accordance with $h(\beta_i)$ $(i = 1, 2, 3, 4)$: $A_3 \succ A_4 \succ A_1 \succ A_2$. Therefore, the best investment enterprise is A_3.

For comparison purposes and for convenience, in the following steps, the IVIULWGA and IVIULHG operators proposed by Liu (2013b) are used to solve the above group decision making problem.

Step 1′. See step 1 above.

Step 2′. Utilize the IVIULWGA operator

$$\beta_i^{(k)} = \text{IVIULWGA}_{\mathbf{w}}(\beta_{i1}^{(k)}, \beta_{i2}^{(k)}, \beta_{i3}^{(k)})$$

to aggregate the criterion values of the line of the decision matrices $\mathbf{R}^{(k)}$ and derive the individual overall evaluation values $\beta_i^{(k)}$ of the alternatives A_i $(i = 1, 2, 3, 4)$ given by the decision-makers d_k $(k = 1, 2, 3)$:

$$\begin{aligned}
\beta_1^{(1)} &= \langle [s_{4.3475}, s_{5.8507}], [0.5917, 0.7335], [0.1387, 0.2388] \rangle, \\
\beta_2^{(1)} &= \langle [s_{5.0305}, s_{6.0415}], [0.6632, 0.8000], [0.1663, 0.1663] \rangle, \\
\beta_3^{(1)} &= \langle [s_{5.0000}, s_{6.8585}], [0.6071, 0.7888], [0.1363, 0.2112] \rangle, \\
\beta_4^{(1)} &= \langle [s_{4.6010}, s_{5.8507}], [0.6333, 0.7644], [0.1289, 0.1966] \rangle, \\
\beta_1^{(2)} &= \langle [s_{4.3249}, s_{5.7042}], [0.6971, 0.7977], [0.1000, 0.1641] \rangle, \\
\beta_2^{(2)} &= \langle [s_{4.1332}, s_{5.4444}], [0.5696, 0.7505], [0.1000, 0.2112] \rangle, \\
\beta_3^{(2)} &= \langle [s_{5.1760}, s_{6.4704}], [0.6333, 0.7335], [0.1387, 0.2665] \rangle, \\
\beta_4^{(2)} &= \langle [s_{4.6244}, s_{5.8749}], [0.7335, 0.8337], [0.1000, 0.1663] \rangle, \\
\beta_1^{(3)} &= \langle [s_{4.9042}, s_{5.9166}], [0.6498, 0.7505], [0.1734, 0.2495] \rangle, \\
\beta_2^{(3)} &= \langle [s_{4.3475}, s_{5.6822}], [0.6609, 0.7612], [0.1000, 0.2000] \rangle, \\
\beta_3^{(3)} &= \langle [s_{5.6249}, s_{6.9254}], [0.7028, 0.8034], [0.1387, 0.1966] \rangle, \\
\beta_4^{(3)} &= \langle [s_{4.3475}, s_{5.8507}], [0.6971, 0.7977], [0.1289, 0.1641] \rangle.
\end{aligned}$$

Step 3′. Utilize the IVIULHG operator

$$\beta_i = \text{IVIULHG}_{\mathbf{e},\mathbf{v}}\big(\beta_i^{(1)}, \beta_i^{(2)}, \beta_i^{(3)}\big) = \big(\beta_i^{\prime(\tau(1))}\big)^{v_1} \otimes \big(\beta_i^{\prime(\tau(2))}\big)^{v_2} \otimes \big(\beta_i^{\prime(\tau(3))}\big)^{v_3}$$

to aggregate the individual overall evaluation values $\beta_i^{(1)}$, $\beta_i^{(2)}$, $\beta_i^{(3)}$ and derive the collective overall evaluation values β_i of the alternatives A_i $(i = 1, 2, 3, 4)$, where $\beta_i^{\prime(\tau(k))}$ is the kth largest of the weighted IIULNs $((\beta_i^{(1)})^{te_1}, (\beta_i^{(2)})^{te_2}, (\beta_i^{(3)})^{3e_3})$, $(\tau(1), \tau(2), \tau(3))$ is a permutation of $(1, 2, 3)$, and $\mathbf{v} = (0.2429, 0.5142, 0.2429)^T$ is the weight vector of IVIULHG operator:

$$\beta_1 = \langle [s_{4.4104}, s_{5.5646}], [0.6574, 0.7661], [0.1406, 0.2171] \rangle,$$

$$\beta_2 = \langle [s_{4.4040}, s_{5.5076}], [0.6411, 0.7815], [0.1288, 0.1830] \rangle,$$

$$\beta_3 = \langle [s_{5.1057}, s_{6.4286}], [0.6667, 0.7839], [0.1347, 0.2161] \rangle,$$

$$\beta_4 = \langle [s_{4.3065}, s_{5.5836}], [0.6996, 0.8055], [0.1173, 0.1679] \rangle.$$

Step 4′. Utilize (6) to calculate the scores $h(\beta_i)$ of the collective overall values β_i of the alternatives A_i $(i = 1, 2, 3, 4)$:

$$h(\beta_1) = 3.8228, \qquad h(\beta_2) = 3.8540, \qquad h(\beta_3) = 4.4692, \qquad h(\beta_4) = 3.9806.$$

Thus

$$h(\beta_3) > h(\beta_4) > h(\beta_2) > h(\beta_1).$$

Step 5′. Rank all the alternatives A_i $(i = 1, 2, 3, 4)$ in accordance with $h(\beta_i)$ $(i = 1, 2, 3, 4)$: $A_3 \succ A_4 \succ A_2 \succ A_1$. Thus, the best investment enterprise is A_3.

From the above calculations, it can be seen that the most desirable investment enterprise is the same, but the ranking results of the last two positions has changed. The reason for this is that the IIULWA and IIULHA operators focus on the impact of overall data, whereas the IVIULWGA and IVIULHG operators highlight the role of individual data.

6. Conclusions

In this paper, some additive operational laws of IIULNs have been defined and based on these, some new arithmetic aggregation operators, such as the IIULWA, IIULOWA and IIULHA operators have been proposed. Furthermore, various properties of these operators have been established. Moreover, it has been proved in the paper that the IIULHA operator overcomes some of the limitations of the IIULWA and IIULOWA operators. The IIULWA operator weights only the IIULNs, whereas the IIULOWA operator weights only the ordered positions of the IIULNs instead of weighting the IIULNs themselves, but the IIULHA operator weights both the given IIULNs and their ordered positions, and generalizes both the IIULWA operator and the IIULOWA operator at the same time. Furthermore, based on the IIULWA and IIULHA operators, a group decision making method with interval-valued intuitionistic uncertain linguistic information has been developed, which develops the theories of aggregation operators and linguistic multi-criteria group decision making.

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Grupinio sprendimų priėmimo požiūris, paremtas intervaliniais intuiciniais neapibrėžtaisiais lingvistiniais kintamaisiais

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Šiame straipsnyje nagrinėjamas grupinis sprendimų priėmimas, kai kriterijų reikšmės yra išreiškiamos intervaliniais intuiciniais neapibrėžtaisiais lingvistiniais skaičiais (IINLS). Apibrėžiamos sudėties operacijos su IINLS. Taip pat pasiūlyti tokie nauji aritmetiniai agregavimo operatoriai kaip intervalinis intuicinis neapibrėžtasis lingvistinis svertinis vidurkis (IINLSV), intervalinis intuicinis neapibrėžtasis lingvistinis sutvarkytasis svertinis vidurkis (IINLSSV), intervalinis intuicinis neapibrėžtasis lingvistinis hibridinis vidurkis (IINLHV). Pasiūlyti vidurkiai remiasi apibrėžtomis operacijomis. Pasiūlytas grupinio sprendimų priėmimo metodas, kuriame taikomi IINLSV ir IINLHV operatoriai. Pateikiamas metodo taikymo pavyzdys.