INFORMATICA, 2015, Vol. 26, No. 2, 313–334

DOI: http://dx.doi.org/10.15388/Informatica.2015.50

Optimal Congestion Control and Routing for Multipath Networks with Random Losses

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Received: September 2012; accepted: April 2014

Abstract. In this paper we consider optimal congestion control and routing schemes for multipath networks with non-congestion related packet losses which can be caused by, for example, errors on links on the routes, and develop a relaxed multipath network utility maximization problem. In order to obtain the optimum, we present a primal algorithm which is shown to be globally stable in the absence of round-trip delays. When round-trip delays are considered, decentralized sufficient conditions for local stability of the algorithm are proposed, in both continuous-time and discrete-time forms. Finally, a window-flow control mechanism is presented which can approximate the optimum of the multipath network utility maximization model.

Key words: resource allocation, congestion control, network utility maximization, stability.

1. Introduction

Since the publication of the seminal paper (Kelly *et al.*, 1998), the single path congestion control and routing schemes which can be formulated as network utility maximization problems have been extensively studied in the past years, mainly in the context of Internet congestion control and flow control (e.g., Low and Lapsley, 1999; Kelly, 2003; Low, 2003; Chiang *et al.*, 2007).

Recently, there has been much interest in multipath congestion control and routing schemes (Li *et al.*, 2011a, 2011b, 2014b; Xu *et al.*, 2011; Lilienthal and Mandjes, 2011; Szymanski, 2013), where each source-destination pair can have several different routes along which data packet can be transmitted (e.g., Wu and Wang, 2012). In multipath schemes, maximizing aggregated user utility over the network with multipath routing under the constraint of link capacity is the objective of the *multipath network utility maximization* problems. They can be viewed as an example of cross-layer optimization (Chiang *et al.*, 2007), where additional benefits are obtained by jointly optimizing at the routing (network layer) and congestion control (transport layer). However, the optimization problems to solve in multipath cases are usually concave but not strictly concave, resulting in non-unique optimums of the primal problems and discontinuous multipath dual problems. Thus, most of researchers have to relax the multipath network

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utility maximization problems so as to make them strict concave (Han *et al.*, 2006; Voice, 2007), which means that the optimums of the network utility maximization problems are unique.

In order to solve the multipath utility maximization problems and the relaxation versions, roughly speaking, multipath congestion control and routing schemes can be classed into three categories: primal algorithms (Han *et al.*, 2003; Kelly and Voice, 2005; Peng *et al.*, 2013), dual algorithms (Wang *et al.*, 2003; Voice, 2007), and primal-dual algorithms (Lin and Shroff, 2006; Han *et al.*, 2006; Jin *et al.*, 2009; Li *et al.*, 2014a). The primal algorithms have a dynamical law for adjusting user rate and a static law for generating link price, and conversely, the dual algorithms have a dynamical law for adjusting link price and a static law for generating user rate. Then, the primal-dual algorithms have dynamical laws for adjusting both user rate and link price. The primal algorithms are based on a penalty function approach, i.e., they replace the capacity constraints by a penalty function in the optimization objective. They always tend to produce biased approximates of the optimal operating points, due to the fact that penalties are only incurred when the capacity constraints are violated. In contrast, the optimal operating point is defined to be one that satisfies the capacity constraints. As for the dual algorithms, the advantage is that they are designed to compute the exact optimal operating point when the stepsizes are driven to zero in an appropriate fashion.

It is known that packets may be lost because of congestion in networks, however, just as mentioned in communication networks with multipath routing (Lin and Shroff, 2006), some *random errors* can occur when links estimate the amount of consumed resources, such that packets may also be lost due to the link errors. We regard these non-congestion related packet losses as *random losses*, which can be caused by, for example, hardware failures in a wired network, or more frequently, errors on wireless links on the routes in a wireless network. Typically, they are modeled as random phenomena which are independent across routes or paths. They have been investigated widely in single-path congestion control and rate control schemes, and some new algorithms are proposed accordingly such that the algorithms can achieve effective network control even when there are noises (Bolot, 1993; Lakshman and Madhow, 1997; Kunniyur and Srikant, 2003; Chen *et al.*, 2005; Altman *et al.*, 2005; Sun *et al.*, 2008), however, as far as the authors' knowledge, little attention has been paid to the case in multipath scenarios.

This paper assumes that users have access to two or more different routes. This means that, for example, a user is able to choose between different Internet service providers, or initial wireless links. We consider the multipath network utility maximization problem, incorporate random losses in the utility optimization problem and present an optimal congestion control and routing scheme for multipath networks with random losses. The main contributions of this paper are summarized as follows:

(1) We present an analytical framework to study joint congestion control and routing schemes for multipath networks with random losses in terms of optimizing network utility.

(2) We propose a rate-based congestion control algorithm, which can achieve the optimum of utility maximization problem, and obtain the global stability of the algorithm in the absence of round-trip delays.

(3) We investigate the local stability of the algorithm in the presence of round-trip delays and obtain some decentralized sufficient conditions for the local stability, in both continuous-time and discrete-time forms.

(4) We present a window-flow control mechanism that approximates optimum of the multipath network utility maximization model but is more convenient to implement than the rate-based flow control mechanism.

The rest of this paper is organized as follows: in Section 2, we present the framework of joint congestion control and routing scheme for multipath networks with random losses and give a rate-based congestion control algorithm; then we analyze the local stability of the proposed algorithm with round-trip delays, and obtain decentralized sufficient conditions in Section 3; in Section 4 we present the window-based flow control scheme; in Section 5 we discuss the ECN marking scheme in the algorithm; in Section 6 we give simulation results to confirm the convergence of the proposed algorithm; finally, we conclude the paper in Section 7.

2. Multipath Congestion Control and Routing with Random Losses

In this section, we shall introduce the relaxed utility maximization problem for multipath networks with random losses, and present a primal algorithm to achieve the optimum of the optimization problem. The algorithm is globally stable in the absence of round-trip delays. In the next section, we shall give decentralized sufficient conditions for local stability in the presence of round-trip delays.

2.1. *Utility Optimization Framework*

Consider a network consisting of a set of links L, a set of routes R and a set of users N. Each link $l \in L$ has capacity C_l. Each user $n \in N$ identifies a unique source-destination pair. There are multiple routes or paths between each source-destination pair. Associated with each user $n \in N$ is a set of routes $R(n)$ where each route $r \in R(n)$ is a collection of links. Obviously, associated with a route $r \in R(n)$ is a set of routes $R(n)$ all associated with the same user, and with identical source and destination. In following analysis, if a user n transmits along a route r, then we write $r \in n$; if a route r uses a link l, then we write $l \in r$.

In this paper we make no assumptions on whether the routes $r \in n$ are disjointed or not. Obviously the ability to generate link-disjointed routes can assist in the construction of highly robust end-to-end communication for the source-destination pair which is labeled by user n, but the model also covers the case where some or all of the routes $r \in n$ share some common path segments.

For user n, assume the transmitted rate on route $r \in n$ is $x_r(t)$, then the total flow rate of user n is $y_n(t) = \sum_{r:r\in n} x_r(t)$, meanwhile the aggregated rate on link l is $z_l(t) = \sum_{r:l\in r} x_r(t)$. Denote the received rate in the absence of non-congestion related packet losses on route r by $\overline{x}_r(t)$. The received rate could be equal to or less than the transmitted rate due to congestion in the network. Denote the price associated with the link l by

$p_l(z_l(t))$, which is a function of $z_l(t)$. The price functions are assumed to be strictly increasing functions with $p_l(0) = 0$. For example, the loss probability can be chosen as one type of price function. Then similar to that in single-path case (Kunniyur and Srikant, 2003), the loss rate for user n on a link $l \in r$ is $x_r(t)p_l(z_l(t))$.

Let $q_r(t) = \sum_{l:l\in r} p_l(z_l(t))$, which is the sum of the link price along the route and can be considered to be the total price associated with the route. Then the total loss rate due to congestion for user n on route r is

$$x_r(t) - \overline{x}_r(t) = \sum_{l:l\in r} x_r(t)p_l\big(z_l(t)\big) = x_r(t)q_r(t). \tag{1}$$

Just as mentioned and investigated in the single-path end-to-end congestion control schemes (Kunniyur and Srikant, 2003; Chen *et al.*, 2005; Altman *et al.*, 2005), some random errors may occur when links estimate the amount of consumed resources, thus the rate at which data packets are received at the destination on the route is not only a function of congestion, but also a function of non-congestion related random losses. For example, in single-path networks (Kunniyur and Srikant, 2003) random losses due to hardware failure in wired networks or errors on wireless links in wireless networks are considered, and two congestion control schemes with random losses are presented. And in wireless networks (Chen *et al.*, 2005), physical channel errors related packet losses are considered in wireless networks, and two new congestion control schemes are proposed for wireless networks such that the equilibrium points of the new, extended utility maximization system can still be obtained.

Let the received rate on route r be $\widehat{x}_r(t) = \varepsilon_r\overline{x}_r(t)$ in the presence of random losses, where $1 - \varepsilon_r$ is the fraction of packets loss due to non-congestion related reasons. Thus, the total loss rate $x_r(t) - \widehat{x}_r(t)$ on route r can be given by

$$\begin{aligned} x_r(t) - \widehat{x}_r(t) &= \varepsilon_r\big(x_r(t) - \overline{x}_r(t)\big) + (1-\varepsilon_r)x_r(t) \\ &= \varepsilon_r x_r(t)q_r(t) + (1-\varepsilon_r)x_r(t). \end{aligned} \tag{2}$$

Each user has a utility function $U_n(\cdot)$, which is a continuously differentiable, strictly concave, increasing function in its interval. We are interested in the following utility functions,

$$U_n\big(y_n(t)\big) = \begin{cases} w_n \log y_n(t), & \text{if } \alpha_n = 1, \\ w_n \frac{y_n(t)^{1-\alpha_n}}{1-\alpha_n}, & \text{if } \alpha_n \neq 1, \end{cases} \tag{3}$$

where w_n is the willingness to pay of this user; α_n is the fairness parameter which can be used to achieve the known α-fair resource allocation. This family of utility functions are known to characterize a large class of fairness concepts and have been investigated extensively (Kelly, 2003; Chiang *et al.*, 2007; Li *et al.*, 2014b). In particular, if we set $\alpha_n = 0$, the optimization problem reduces to throughput maximization. If $\alpha_n = 1$, proportional fairness among competing sources is achieved; if $\alpha_n = 2$, then harmonic mean fairness; and if $\alpha_n = \infty$, then max-min fairness. Recently this kind of α-fairness has also been investigated through relational optimization (Köppen, 2013).

We consider the following total network utility problem, i.e., one where all users jointly optimize a single network performance objective

$$\mathcal{U} = \sum_{n:n\in N}\left(U_n\left(\sum_{r:r\in n} x_r(t)\right) - \eta\sum_{r:r\in n}\frac{1-\varepsilon_r}{\varepsilon_r}x_r(t)\right) - \eta\sum_{l:l\in L}\int_0^{\sum_{r:l\in r}x_r(t)} p_l(\sigma)d\sigma. \tag{4}$$

Here, the parameter η attempts to trade off between maximizing user utility and minimizing link price. The second term in the first part of (4) can be regarded as the total loss of user utility due to non-congestion related random losses. The objective function (4) is concave but not strictly concave with respect to the variables $x_r(t)$, resulting in that the equilibrium points are not unique, however, the total flow rate y_n of each user n on its available routes and the aggregated flow rate z_l on each link l are unique because of the strict concavity of the aggregated utility function (the first part of (4)) and the strict convexity of the aggregated cost function (the second part of (4)).

In the absence of random losses ($\varepsilon_r = 1, \forall r$), the objective to be maximized above can reduce to the following one

$$\mathcal{U} = \sum_{n:n\in N} U_n\left(\sum_{r:r\in n} x_r(t)\right) - \eta\sum_{l:l\in L}\int_0^{\sum_{r:l\in r}x_r(t)} p_l(\sigma)d\sigma. \tag{5}$$

Moreover, if $\eta = 1$, the objective (5) has been considered (Kelly and Voice, 2005; Han *et al.*, 2006), and fluid-flow primal algorithms of joint rate control and routing are presented accordingly. If we choose $\eta = \infty$ and $p_l(z_l(t)) = (z_l(t) - C_l)^+/z_l(t)$ ("packet loss rate" for the price function), the network utility objective is equivalent to the following multipath network utility maximization problem

$$\begin{aligned} &\text{max} && \sum_n U_n\left(\sum_{r:r\in n} x_r(t)\right)\\ &\text{subject to} && \sum_{r:l\in r} x_r(t) \leqslant C_l\\ &\text{over} && x_r(t) \geqslant 0. \end{aligned} \tag{6}$$

This network utility maximization problem has been investigated (Lin and Shroff, 2006; Voice, 2007). The objective function in (6) is not strictly concave in the primal variables $x_r(t)$ even if the utility functions $U(\cdot)$ are strictly concave, and hence the dual of (6) may not be differentiable at every point. In order to make it strictly concave, several methods are presented to relax the network utility maximization problem (6), such as adding a quadratic term onto the objective function (Lin and Shroff, 2006).

2.2. *Primal Algorithm*

We present the following primal algorithm, which is a natural generalization of the single-path primal algorithm (Kunniyur and Srikant, 2003), to achieve the optimum of utility maximization problem (4)

$$\frac{dx_r(t)}{dt} = \kappa_r x_r(t)\left(\varepsilon_r - \frac{\eta}{U_n'(\sum_{r:r\in n} x_r(t))}\frac{x_r(t)-\widehat{x}_r(t)}{x_r(t)}\right)^+_{x_r(t)}, \tag{7}$$

where $a = (b)_c^+$ means $a = b$ if $c > 0$ and $a = \max\{0, b\}$ if $c = 0$.

Substituting (2) into (7), we obtain the following rate control algorithm

$$\frac{dx_r(t)}{dt} = \kappa_r x_r(t)\left(\varepsilon_r - \eta\frac{\varepsilon_r \sum_{l:l\in r} p_l(t) + (1-\varepsilon_r)}{U_n'(\sum_{r:r\in n} x_r(t))}\right)^+_{x_r(t)}. \tag{8}$$

We motivate the proposed algorithm (7) or (8) as follows. It is a rate-based control algorithm for the flow on route r that comprises two parts: a steady increase at rate proportional to $\kappa_r\varepsilon_r x_r(t)$; and a steady decrease at a rate depending upon both the price signals arriving back from route r, the fraction of packets lost due to congestion on route r, and the total rate of acknowledgements $y_n(t) = \sum_{r:r\in n} x_r(t)$ received by user n.

The function $\mathcal{U}$ is strictly increasing with time t, unless $x(t) = x$, the equilibrium point maximizing $\mathcal{U}$. Thus, the following theorem can be obtained.

Theorem 1. *Dynamic system* (8) *is globally asymptotically stable with the Lyapunov function* (4)*. All trajectories along system* (8) *converge to the equilibrium point that maximizes* $\mathcal{U}$*.*

Proof. For proof please see Appendix A. □

Obviously, at the equilibrium of algorithm (7) or (8), the following equation holds

$$U_n'\left(\sum_{r:r\in n} x_r\right) = \eta\left(\sum_{l:l\in r} p_l + \frac{1-\varepsilon_r}{\varepsilon_r}\right) = \eta\left(q_r + \frac{1-\varepsilon_r}{\varepsilon_r}\right), \tag{9}$$

thus, the optimal flow rate of user n is

$$y_n = \sum_{r:r\in n} x_r = U_n'^{-1}\left(\eta q_r + \eta\frac{1-\varepsilon_r}{\varepsilon_r}\right). \tag{10}$$

In the absence of random losses ($\varepsilon_r = 1$, $\forall r$), the optimal total flow rate of user n is $y_n = U_n'^{-1}(\eta q_r)$, $r \in n$. Moreover, if $\eta = 1$, then $y_n = U_n'^{-1}(q_r)$, $r \in n$, which is similar to that of the case that each user has a single route (Low and Lapsley, 1999).

Obviously, we can obtain the following theorem from (9).

Theorem 2. *At the optimum of network utility problem* (4)*, the total prices associated with routes of one user, i.e., q_r, $r \in n$, are all equal in the absence of random losses, i.e., $q_r = U_n'(y_n)/\eta$, $r \in n$.*

Considering the utility functions (3), then the primal algorithm (8) reduces to

$$\frac{dx_r(t)}{dt} = \frac{\kappa_r x_r(t)}{w_r}\left(\varepsilon_r w_r - \eta\left(\sum_{r:r\in n} x_r(t)\right)^{\alpha_r}\left(\varepsilon_r \sum_{l:l\in r} p_l(t) + (1-\varepsilon_r)\right)\right)^+_{x_r(t)} \quad (11)$$

for each route of user $n \in N$, where $w_r = w_n$, $\alpha_r = \alpha_n$ for all routes r associated with user n.

2.3. *Convergence Rate*

We have obtained in Theorem 1 that the system (7) or (8) converges to the equilibrium point of (4), next we investigate the rate of convergence, by linearization about the equilibrium point x. At the equilibrium point, let $y_n = \sum_{r:r\in n} x_r$, $U_n' = U_n'(y_n)$ and $U_n'' = U_n''(y_n)$, and suppose p_l is differentiable at this point, with derivative p_l'. Let $x_r(t) = x_r + \vartheta_r(t)$, $y_n(t) = y_n + \nu_n(t)$ and $z_l(t) = z_l + \sigma_l(t)$, then, linearizing the system (8) about x, we obtain

$$\frac{d\vartheta_r(t)}{dt} = -\frac{\varepsilon_r \kappa_r x_r}{U_n'}\left(-U_n''\nu_n(t) + \eta \sum_{l:l\in r} p_l'\sigma_l(t)\right), \quad (12)$$

$$\nu_n(t) = \sum_{r:r\in n} \vartheta_r(t), \quad (13)$$

$$\sigma_l(t) = \sum_{r:l\in r} \vartheta_r(t). \quad (14)$$

We can write (12)–(14) in matrix form as

$$\frac{d}{dt}\begin{pmatrix}\nu(t)\\ \sigma(t)\end{pmatrix} = -P^{-1}R(\varepsilon)^T R(\varepsilon) P \begin{pmatrix}\nu(t)\\ \sigma(t)\end{pmatrix},$$

where P is a $(|N|+|L|)\times(|N|+|L|)$ diagonal matrix with entries $P_{nn} = 1$, $P_{ll} = \eta p_l'$, and $R(\varepsilon)$ is an $|R| \times (|N|+|L|)$ matrix with the entries

$$R_{rn} = \left(-U_n''\frac{\varepsilon_r\kappa_r x_r}{U_n'}\right)^{1/2}, \quad \text{for } r \in n,$$

$$R_{rl} = \left(\frac{\varepsilon_r\kappa_r x_r}{U_n'}\eta p_l'\right)^{1/2}, \quad \text{for } l \in r,$$

and all other entries of $R(\varepsilon)$ are zero.

Let

$$\Gamma^T \Theta \Gamma = P^{-1} R(\varepsilon)^T R(\varepsilon) P, \tag{15}$$

where Γ is an orthogonal matrix, $\Gamma^T \Gamma = I$, and $\Theta = \text{diag}\{\phi_i, i \in N \cup L\}$ is the matrix of eigenvalues of the real, symmetric, positive definite matrix (15). Then

$$\frac{d}{dt}\begin{pmatrix} \nu(t) \\ \sigma(t) \end{pmatrix} = -\Gamma^T \Theta \Gamma \begin{pmatrix} \nu(t) \\ \sigma(t) \end{pmatrix}.$$

Thus, the rate of convergence to the equilibrium point is determined by the smallest eigenvalue ϕ_i of matrix (15). Notice that the speed of convergence increases with both the gain parameter κ and the fraction of packet loss due to congestion ε.

3. Stability of the Algorithm with Round-Trip Delays

For each route $r \in n$ and link $l \in r$, we define a *forward delay* D_{rl} from r to l, and a *return delay* D_{lr} from l to r. The forward delay is the delay incurred in communication for a packet from user n to link l along route r; the return delay is the delay incurred in communication for an acknowledgement packet from link l back to user n along route r. In the protocols under consideration, a packet must reach its destination before an acknowledgement packet, which contains congestion feedback (via some form of *explicit congestion notification*), is returned to its source. In the current Internet, each route is subject to a round-trip delay. We model this delay by assuming each route has an associated delay denoted by D_r, i.e. for all $l \in r$, $D_{rl} + D_{lr} = D_r$.

3.1. *Stability of the Continuous-Time Algorithm*

In this part, we consider the local stability of the continuous form of the proposed algorithm. Thus the algorithm (8) with round-trip delay can be described as follows

$$\frac{dx_r(t)}{dt} = \kappa_r x_r(t)\left(\varepsilon_r - \frac{\eta\varepsilon_r q_r(t) + \eta(1-\varepsilon_r)}{U_n'(\sum_{r:r\in n} x_r(t))}\right)^+_{x_r(t)}, \tag{16}$$

$$q_r(t) = \sum_{l:l\in r} \mu_l(t - D_{lr}), \tag{17}$$

$$\mu_l(t) = p_l\big(z_l(t)\big), \tag{18}$$

$$z_l(t) = \sum_{r:l\in r} x_r(t - D_{rl}), \tag{19}$$

$$y_n(t) = \sum_{r:r\in n} x_r(t - D_r). \tag{20}$$

We motivate the algorithm (16)–(20) as follows. The total flow through link l is $z_l(t)$, which is the aggregated flow rate from all routes that pass the link. However, the flow that link l observes at time t on route r is the traffic sent D_{rl} time units earlier, which is given in (19). After link l adds a price $p_l(z_l(t))$ onto packets, the congestion feedback packets leaving link l experience a delay D_{lr} before returning to user n, as shown in (17). Finally, for the total rate of acknowledgements $y_n(t)$ in (20), packets on each route $r \in n$ are subject to a round-trip delay denoted by D_r. Through (16)–(20), user n adjusts $x_r(t)$ on route r to reach the target equilibrium point.

We next establish a sufficient condition for the local stability of algorithm (16)–(20) at the equilibrium point x which satisfies (9)–(10).

Theorem 3. *Let x be the equilibrium point of dynamic system* (16)–(20)*, then the proposed algorithm* (16)–(20) *is locally stable if the following sufficient condition is satisfied for each route r serving user n*

$$\frac{\varepsilon_r \kappa_r}{U_n'}\left(- U_n'' y_n + \sum_{r:l\in r} z_l p_l' \right) < \frac{\pi}{2D_r}. \tag{21}$$

Proof. The proof of this theorem, which is based on the generalized Nyquist criterion (Desoer and Yang, 1980), can be found in Appendix B. □

Then, we can obtain the following results after substituting (3) into (21).

Remark 1. From (3) the sufficient condition (21) reduces to

$$\varepsilon_r \kappa_r \left(\alpha_n + \frac{y_n^{\alpha_n}}{w_n} \sum_{r:l\in r} z_l p_l' \right) < \frac{\pi}{2D_r}, \quad r \in n,$$

for user $n \in N$.

Moreover, if we choose $p_l(z_l(t)) = (z_l(t)/C_l)^{\beta_l}$, then from (9)–(10) the sufficient condition (21) reduces to

$$\varepsilon_r \kappa_r \left(\alpha_r + \beta_r \frac{q_r}{\eta(q_r + (1-\varepsilon_r)/\varepsilon_r)} \right) < \frac{\pi}{2D_r}, \quad r \in n,$$

for user $n \in N$, where $\beta_r = \max\{\beta_l, l \in r\}$, $\alpha_r = \alpha_n$, $r \in n$.

Remark 2. If $\varepsilon_r = 1$, $r \in n$, i.e., the random losses due to non-congestion related reasons are not considered, and $\eta = 1$, i.e., the trade off between maximizing user utility and minimizing link price is 1, then the sufficient condition (21) reduces to

$$\frac{\kappa_r}{q_r}\left(- U_n'' y_n + \sum_{r:l\in r} z_l p_l' \right) < \frac{\pi}{2D_r}.$$

From the sufficient conditions for local stability of the congestion control and routing algorithm, we can observe that network stability in the presence of round-trip delays can be guaranteed by simple, decentralized conditions on each end user and its links, i.e., in order to achieve the local stability, each end user only needs knowledge of its own information, such as the round-trip delay D_r, the aggregated price q_r along any one of its routes, and so on.

3.2. *Stability of the Discrete-Time Algorithm*

Now we investigate local stability of the discrete-time algorithm corresponding to the continuous-time system (16)–(20). Consider the following delayed system analogous to the continuous one, where D_{rl} and D_{lr} are assumed to be integer values

$$x_r[t+1] = x_r[t] + \kappa_r x_r[t]\left(\varepsilon_r - \frac{\eta\varepsilon_r q_r[t] + \eta(1-\varepsilon_r)}{U_n'(\sum_{r:r\in n} x_r[t])}\right)^+_{x_r[t]}, \tag{22}$$

$$q_r[t] = \sum_{l:l\in r} \mu_l[t - D_{lr}], \tag{23}$$

$$\mu_l[t] = p_l\big(z_l[t]\big), \tag{24}$$

$$z_l[t] = \sum_{r:l\in r} x_r[t - D_{rl}], \tag{25}$$

$$y_n[t] = \sum_{r:r\in n} x_r[t - D_r]. \tag{26}$$

We motivate the algorithm (22)–(26) as follows. The packets for user n experience a delay D_{rl} before arriving at link l on route r; then packets with price for route r leaving link l experience a delay D_{lr} before returning to user n. Finally, packets of the total rate of acknowledgements $y_n(t)$ on route $r \in n$ are subject to a round-trip delay D_r.

For the discrete-time algorithm (22)–(26), we obtain a sufficient condition for the local stability of the delayed algorithm at the equilibrium point x.

Theorem 4. *The proposed algorithm* (22)–(26) *is locally stable at the equilibrium point x if the following sufficient condition is satisfied for each route r serving user n*

$$\frac{\varepsilon_r\kappa_r}{U_n'}\left(-U_n'' y_n + \sum_{r:l\in r} z_l p_l'\right) < 2\sin\left(\frac{\pi}{2(2D_r+1)}\right). \tag{27}$$

Proof. For proof please see the Appendix C. □

Remark 3. From (3) the sufficient condition (27) reduces to

$$\varepsilon_r\kappa_r\left(\alpha_n + \frac{y_n^{\alpha_n}}{w_n}\sum_{r:l\in r} z_l p_l'\right) < 2\sin\left(\frac{\pi}{2(2D_r+1)}\right),$$

where $r \in n$ for user $n \in N$.

Moreover, if $p_l(z_l(t)) = (z_l(t)/C_l)^{\beta_l}$, then from (9)–(10) the sufficient condition (27) reduces to

$$\varepsilon_r \kappa_r \left(\alpha_r + \beta_r \frac{q_r}{\eta(q_r + (1-\varepsilon_r)/\varepsilon_r)} \right) < 2 \sin \left(\frac{\pi}{2(2D_r + 1)} \right),$$

where $r \in n$ for user $n \in N$, and $\beta_r = \max\{\beta_l, l \in r\}$, $\alpha_r = \alpha_n$, $r \in n$. Obviously, the stability of the proposed algorithm can be guaranteed on each route by simple, distributed conditions.

REMARK 4. If $\varepsilon_r = 1$, $r \in n$ and $\eta = 1$, then the sufficient condition (27) reduces to

$$\frac{\kappa_r}{q_r} \left(-U_n'' y_n + \sum_{r:l \in r} z_l p_l' \right) < 2 \sin \left(\frac{\pi}{2(2D_r + 1)} \right). \tag{28}$$

Meanwhile, (28) is also equivalent to

$$\frac{\kappa_r}{q_r} \left(\alpha_r q_r + \sum_{r:l \in r} z_l p_l' \right) < 2 \sin \left(\frac{\pi}{2(2D_r + 1)} \right),$$

which is very similar to the sufficient condition for local stability of delayed congestion control algorithm in *single-path* networks (Johari and Tan, 2001). Moreover, if $p_l(z_l(t)) = (z_l(t)/C_l)^{\beta_l}$, then (28) reduces to

$$\kappa_r(\alpha_r + \beta_r) < 2 \sin \left(\frac{\pi}{2(2D_r + 1)} \right), \quad r \in n.$$

The discrete form of sufficient conditions can give us some useful guidelines for effective implementation of multipath transmission protocols in Internet in order to achieve the stability of network in the presence of round-trip delays.

Actually, the sufficient condition given by (27) for local stability of the discrete-time algorithm approximates the sufficient condition given by (21) for local stability of the continuous-time one when the round-trip delay D_r on route r is large enough, i.e.,

$$2 \sin \left(\frac{\pi}{2(2D_r + 1)} \right) \approx 2 \frac{\pi}{2(2D_r + 1)} \approx \frac{\pi}{2D_r}.$$

4. Window Flow Control

Window-based flow control where the window size is increased or decreased upon receipt of acks (*positive* acknowledgements) or nacks (*negative* acknowledgements) is more convenient to implement than rate-based flow control mechanism since it is inherently self-clocking. To obtain a window flow control scheme, we discretize the system (11) and obtain

$$\frac{x_r(t+\delta)-x_r(t)}{\delta}=\frac{\kappa_r x_r(t)}{w_r}\left(\varepsilon_r w_r-\eta\left(\sum_{r:r\in n} x_r(t)\right)^{\alpha_r}\frac{x_r(t)-\widehat{x}_r(t)}{x_r(t)}\right)^{+}_{x_r(t)}$$

for each route of user $n \in N$.

Let $W_r(t)$ be the window size of user n on its route r at time t. We follow the approximation relating data transmission rate and window size (Lakshman and Madhow, 1997)

$$x_r(t)\approx\frac{W_r(t)}{D_r}.$$

Let $A_r(t,t+\delta)$ and $N_r(t,t+\delta)$ denote the numbers of acks and nacks received by user n on its route r in the time interval $[t,t+\delta)$, respectively. Thus,

$$\frac{A_r(t,t+\delta)}{\delta}\approx x_r(t)\approx\frac{W_r(t)}{D_r}$$

and

$$\big(x_r(t)-\widehat{x}_r(t)\big)\delta\approx N_r(t,t+\delta).$$

Then,

$$\frac{x_r(t+\delta)-x_r(t)}{\delta}=\frac{W_r(t+\delta)-W_r(t)}{A_r(t,t+\delta)}\frac{A_r(t,t+\delta)}{D_r\delta}=\frac{W_r(t+\delta)-W_r(t)}{A_r(t,t+\delta)}\frac{W_r(t)}{D_r^2}.$$

Using the approximations above, the window-based congestion control mechanism becomes

$$W_r(t+\delta)-W_r(t)=\varepsilon_r\kappa_r D_r A_r(t,t+\delta)-\eta\frac{\kappa_r}{w_r}D_r\left(\sum_{i:i\in n}\frac{W_i}{D_i}\right)^{\alpha_r}N_r(t,t+\delta), \tag{29}$$

for each route r of user $n \in N$.

REMARK 5. If $\varepsilon_r = 1$, $r \in n$ and $\eta = 1$, the window-based congestion control mechanism above reduces to

$$W_r(t+\delta)-W_r(t)=\kappa_r D_r A_r(t,t+\delta)-\frac{\kappa_r}{w_r}D_r\left(\sum_{i:i\in n}\frac{W_i}{D_i}\right)^{\alpha_r}N_r(t,t+\delta), \tag{30}$$

for each route r of user $n \in N$. This results can give us some interesting guidelines to achieve optimal window-based flow control of multipath transmission protocols in internet.

We can interpret the window flow control mechanisms (29)–(30) as follows: when each ack is received, the window size is increased by a fixed amount that is in proportion to D_r, which is a linear increase process; when each nack is received, the window size is deceased by a fixed amount that is in proportion to $D_r(\sum_{i:i\in n} W_i/D_i)^{\alpha_r}$ for user $n \in N$, which is a multiplicative decease process. Hence, the window flow control scheme realizes the adaptive increase/multiplicative decrease (AIMD) principle which is also used in the original TCP version and other variants.

5. ECN Marking

Congestion control algorithms that we have discussed so far rely on loss as the congestion indicator. Indeed, explicit congestion notification (ECN) marking is also a mechanism to provide early indication to users about imminent congestion of the network at very low levels of loss.

In order to recast the fluid model to incorporate ECN marking, we interpret “lost” packets as “marked” packets. For example we now interpret $\widehat{x}_r(t)$ as the rate at which “unmarked” packets are received at the receiver. Since buffer is always limited, we assume that, at each link l, a fraction of the packets are marked when the arrival rate exceeds the threshold $\widehat{C}_l$, where $\widehat{C}_l \leqslant C_l$. The fraction of packets marked is given by $p_l(z_l) = (z_l - \widehat{C}_l)^+/z_l$ where z_l is arrival rate on link l.

Then in this framework, it is possible to offer a loss-free service if the marking level $\widehat{C}_l(t)$ is chosen appropriately for each link l. In the following, we characterize the level $\widehat{C}_l$ at which marking should take place so that the total arrival rate on each link does not exceed the link capacity, and obtain the following theorem.

Theorem 5. *For each link l, if the marking level $\widehat{C}_l$ satisfies the following inequality*

$$\sum_{r:l\in r}\left(\frac{w_n}{\eta(1/\varepsilon_r - \widehat{C}_l/C_l)}\right)^{\frac{1}{\alpha_n}} \leqslant C_l, \tag{31}$$

where $r \in n$. Then the equilibrium of (4) *satisfies* $\sum_{r:l\in r} x_r \leqslant C_l$.

Proof. For proof see the Appendix D. □

Along with appropriate marking, the rate and window flow control algorithms we have proposed can be used to provide loss-free service by substituting marks for negative acknowledgments, e.g., in the window flow control implementation, the window size should be reduced upon receipt of a mark.

From the sufficient condition (31) we observe that, in order to increase the available capacity $\widehat{C}_l$, we need to increase η to ensure loss-free service. To understand this better, we consider a simple multipath network where every user $n \in N$ accesses two concurrent links l_1, l_2 with capacities $C_1 = C_2$. From (9) or (10), the equilibrium of (4) satisfies

$$\frac{w_n}{(x_{r_1}+x_{r_2})^{\alpha_n}} = \eta\left(\frac{\sum_{s_i:l_i\in s_i} x_{s_i}(t) - \widehat{C}_{l_i}}{\sum_{s_i:l\in s_i} x_{s_i}(t)} + \frac{1-\varepsilon_{r_i}}{\varepsilon_{r_i}}\right),$$

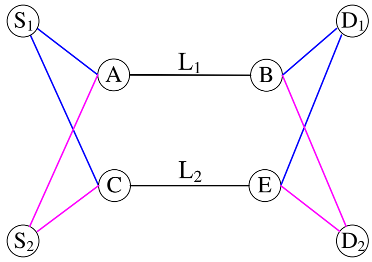


Fig. 1. Multipath network topology.

for each r_i, $i = 1, 2$. By symmetry it is clear that $x_{r_1} = x_{r_2}$ and $x_{s_i} = x_{r_i}$ for any s_i, r_i. Therefore

$$\frac{w_n}{(2x_r)^{\alpha_n}} = \eta\left(\frac{Nx_r(t) - \widehat{C}_l}{Nx_r(t)} + \frac{1-\varepsilon_r}{\varepsilon_r}\right).$$

Considering $\alpha_n = 1$ and $\varepsilon_r = 1$, then $Nx_r(t) = \widehat{C}_l + Nw_n/2\eta$. Obviously, if $\widehat{C}_l + Nw_n/2\eta \leqslant C_l$, the equilibrium of (4) results in zero loss. Notice that $\widehat{C}_l$ depends on the number N of users in the network. Thus, increasing the available capacity by increasing η is the only way to ensure loss-free service. Therefore, as N increases, we need to increase η to ensure loss-free service.

6. Numerical Examples

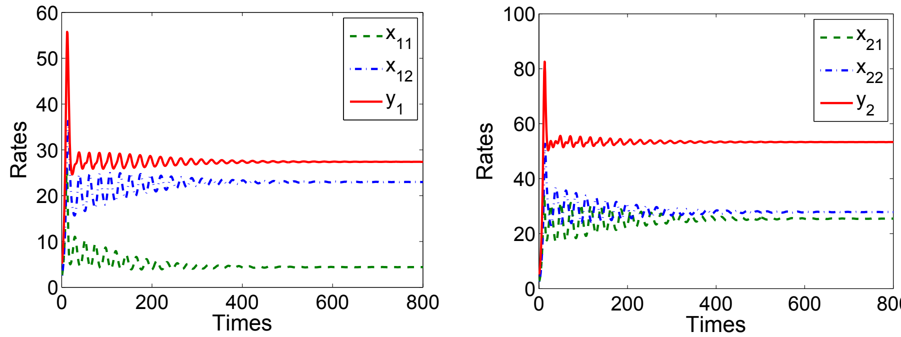
In this section we investigate the performance of the proposed algorithm. We consider the following simple multipath network consisting of two users as shown in Fig. 1. There are two available paths for each source: one is A → B, the other is C → E. We assume there is only one bottleneck link on each path, i.e., L_1 in the former and L_2 in the latter. Suppose the capacities of bottleneck links are $C = (C_1, C_2) = (30, 50)$ Mbps, the transmission delays on links are $d = (d_1, d_2) = (30\text{ ms}, 20\text{ ms})$, and the willingness to pay of users are $w = (w_1, w_2) = (5, 10)$. In the primal algorithm we choose $\eta = 1$, $\kappa = (\kappa_1, \kappa_2) = (0.05, 0.05)$, and the following price function, the "packet loss rate",

$$p_l\big(z_l(t)\big) = \frac{(z_l(t) - C_l)^+}{z_l(t)}. \tag{32}$$

Indeed, for large buffers operating with drop tail, it is a more reasonable approximation for the proportion of packets overflowing the buffer (Srikant, 2004). Here, we only consider the proportional fairness among competing users.

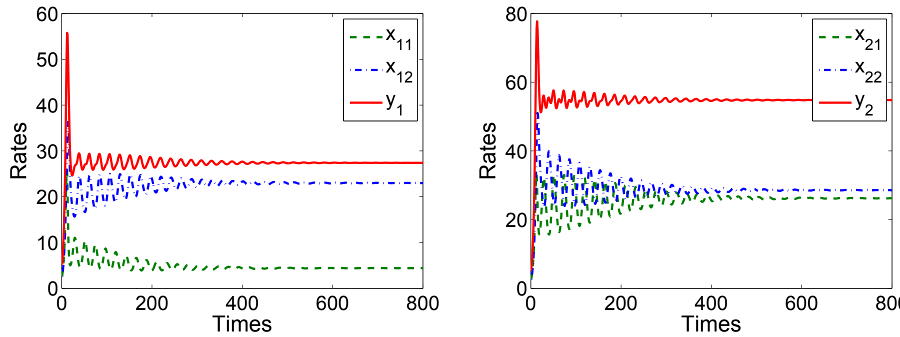
6.1. *Algorithm with Only Congestion-Related Losses*

Firstly we only consider congestion-related packet losses in the network and investigate the performance of our algorithm. The simulation results for algorithm with only



(a) Rates of user 1 (b) Rates of user 2

Fig. 2. Performance of the algorithm with only congestion-related losses.



(a) Rates of user 1 (b) Rates of user 2

Fig. 3. Performance of the algorithm with random losses.

congestion-related losses are shown in Fig. 2, where (a) is the optimal rate of user 1 on its paths and (b) is the optimal rate of user 2 on its paths, respectively. Obviously, the proposed algorithm converges to an optimal fair resource allocation within reasonable convergence times. In this case the optimal rate allocation is $x^* = (x_{11}^*, x_{12}^*, x_{21}^*, x_{22}^*) = (4.5434, 22.1212, 25.4630, 27.8724)$ Mbps.

6.2. *Algorithm with Random Losses*

Now we consider the performance of algorithm when there are non-congestion related random losses in the network along with congestion packet losses. Suppose $\varepsilon = (\varepsilon_1, \varepsilon_2) = (0.98, 0.97)$, i.e., the fraction of non-congestion related random losses in paths 1 and 2 are 0.02 and 0.03, respectively. The simulation results for algorithm with non-congestion related random losses are shown in Fig. 3, where (a) and (b) are the optimal rate of users 1 and 2 on their own paths. We can observe from the results that the algorithm is efficient to converge the optimal rate allocation, i.e., $x^* = (x_{11}^*, x_{12}^*, x_{21}^*, x_{22}^*) = (4.4030, 22.9816, 26.2116, 28.5624)$ Mbps, within reasonable convergence times.

7. Conclusions

In this paper we investigate fair, stable congestion control and routing schemes for multipath networks in the presence of non-congestion related packet losses. We develop a relaxed multipath network utility optimization problem and propose a primal algorithm to achieve the optimum of utility maximization problem. The algorithm is globally stable in the absence of round-trip delays, and when round-trip delays are considered, decentralized sufficient conditions for local stability are presented in both continuous-time and discrete-time forms. Finally, we present a window-based flow control scheme and interpret the increase or decease of window size upon receipt of acks or nacks. The proposed algorithm and sufficient conditions for local stability can apply to multipath networks with random losses which consist of arbitrary interconnections of users and links with arbitrary heterogeneous round-trip delays.

Acknowledgements. The authors would like to thank the anonymous reviewers for the useful comments on this work. S. Li and W. Sun were supported in part by the National Natural Science Foundation of China (NSFC) (Nos. 71301139, 71101124), the China Postdoctoral Science Foundation (No. 2014T70231), the Humanity and Social Science Foundation of Ministry of Education of China (No. 12YJC790101), the Research Fund for the Doctoral Program of Higher Education (No. 20131333120001), and the Natural Science Foundation of Hebei Province (Nos. F2013203136, G2012203068). Y. Zhang was supported by the NSFC (No. 71271186) and Y. Chen was supported by the NSFC (No. 71171174).

Appendix A. Proof of Theorem 1

Notice that

$$\begin{aligned}
\frac{\partial \mathcal{U}}{\partial x_r(t)} &= U_n'\Bigg(\sum_{r:r\in n} x_r(t)\Bigg) - \eta\frac{1-\varepsilon_r}{\varepsilon_r} - \eta\sum_{l:l\in r} p_l\Bigg(\sum_{j:l\in j} x_j(t)\Bigg)\\
&= U_n'\Bigg(\sum_{r:r\in n} x_r(t)\Bigg) - \eta\frac{1-\varepsilon_r}{\varepsilon_r} - \eta\frac{x_r(t)-\overline{x}_r(t)}{x_r(t)}\\
&= U_n'\Bigg(\sum_{r:r\in n} x_r(t)\Bigg) - \frac{\eta}{\varepsilon_r}\frac{x_r(t)-\widehat{x}_r(t)}{x_r(t)}\\
&= \frac{U_n'(\sum_{r:r\in n} x_r(t))}{\varepsilon_r}\Bigg(\varepsilon_r - \frac{\eta}{U_n'\big(\sum_{r:r\in n} x_r(t)\big)}\frac{x_r(t)-\widehat{x}_r(t)}{x_r(t)}\Bigg).
\end{aligned}$$

Hence, setting these derivatives to be zero identifies the maximum. Further,

$$\begin{aligned}
\frac{d\mathcal{U}}{dt} &= \sum_{n\in N}\sum_{r:r\in n}\frac{\partial \mathcal{U}}{\partial x_r(t)}\frac{dx_r(t)}{dt}\\
&= \sum_{n\in N}\sum_{r:r\in n}\frac{\kappa_r x_r(t) U_n'\big(\sum_{r:r\in n} x_r(t)\big)}{\varepsilon_r}\Bigg(\varepsilon_r - \frac{\eta}{U_n'\big(\sum_{r:r\in n} x_r(t)\big)}\frac{x_r(t)-\widehat{x}_r(t)}{x_r(t)}\Bigg)^2,
\end{aligned}$$

so that $\mathcal{U}$ is strictly increasing with time t, unless $x(t) = x$, the equilibrium point maximizing $\mathcal{U}$. The function $\mathcal{U}$ is thus a Lyapunov function for the dynamic system, and the theorem follows.

Appendix B. Proof of Theorem 3

Let $x_r(t) = x_r + \vartheta_r(t)$, $y_n(t) = y_n + \nu_n(t)$, $z_l(t) = z_l + \sigma_l(t)$. By linearizing the system (16)–(20) about the equilibrium point x, we can obtain the following equations

$$\frac{d\vartheta_r(t)}{dt} = -\frac{\varepsilon_r \kappa_r x_r}{U_n'}\Big(- U_n'' \nu_n(t) + \eta \sum_{l:l\in r} p_l' \sigma_l(t - D_{lr})\Big),$$

$$\nu_n(t) = \sum_{r:r\in n} \vartheta_r(t - D_r),$$

$$\sigma_l(t) = \sum_{r:l\in r} \vartheta_r(t - D_{rl}).$$

Taking Laplace transforms of variables $\vartheta_r(t)$, $\nu_n(t)$ and $\sigma_l(t)$, we obtain

$$\omega\vartheta_r(\omega) = -\frac{\varepsilon_r \kappa_r x_r}{U_n'}\Big(- U_n'' \nu_n(\omega) + \eta \sum_{l:l\in r} p_l' e^{-\omega D_{lr}} \sigma_l(\omega)\Big),$$

$$\nu_n(\omega) = \sum_{r:r\in n} e^{-\omega D_r} \vartheta_r(\omega),$$

$$\sigma_l(\omega) = \sum_{r:l\in r} e^{-\omega D_{rl}} \vartheta_r(\omega),$$

where $\vartheta_r(\omega) = \mathcal{L}(\vartheta_r(t))$, $\nu_n(\omega) = \mathcal{L}(\nu_n(t))$ and $\sigma_l(\omega) = \mathcal{L}(\sigma_l(t))$. Here, we assume the initial state of system is zero.

Rewrite the above equations in matrix form as

$$\begin{pmatrix} \nu(\omega) \\ \sigma(\omega) \end{pmatrix} = -P^{-1} R(-\omega)^T T(\omega) R(\omega) P \begin{pmatrix} \nu(\omega) \\ \sigma(\omega) \end{pmatrix},$$

where $T(\omega)$ is an $|R| \times |R|$ diagonal matrix with entries $T_{rr}(\omega) = e^{-\omega D_r}/\omega D_r$, P is a $(|N| + |L|) \times (|N| + |L|)$ diagonal matrix with entries $P_{nn} = 1$, $P_{ll} = \eta p_l'$, and $R(\omega)$ is an $|R| \times (|N| + |L|)$ matrix with the entries

$$R_{rn} = \Big(- U_n'' \frac{\varepsilon_r \kappa_r x_r D_r}{U_n'}\Big)^{1/2}, \quad \text{for } r \in n,$$

$$R_{rl} = e^{-\omega D_{lr}} \Big(\frac{\varepsilon_r \kappa_r x_r D_r}{U_n'} \eta p_l' \Big)^{1/2}, \quad \text{for } l \in r,$$

and all other entries of $R(\omega)$ are zero.

Let $G(\omega) = -P^{-1}R(-\omega)^T T(\omega)R(\omega)P$, which is the *return ratio* for (ν, σ). By the generalized Nyquist criterion (Desoer and Yang, 1980), the control loop is stable if the eigenvalues of $G(i\theta)$ do not encircle $(-1, i0)$ for $\theta \in (-\infty, +\infty)$.

Suppose for some $\theta \in (-\infty, +\infty)$, λ is an eigenvalue of $G(i\theta)$. Then, there exists a unit vector ξ such that $\lambda\xi = R(i\theta)^\dagger T(i\theta)R(i\theta)\xi$, where $\dagger$ denotes the matrix conjugate. Hence, $\lambda = \xi^\dagger R(i\theta)^\dagger T(i\theta)R(i\theta)\xi$.

Let $K = (2/\pi)^{1/2}R(i\theta)\xi$ with elements k_r, since $T(i\theta)$ is diagonal, then

$$\lambda = \sum_r \frac{\pi}{2}|k_r|^2 T_{rr}(i\theta) = \sum_r |k_r|^2 \frac{\pi}{2}\frac{e^{-i\theta D_r}}{i\theta D_r}.$$

Thus, $\lambda = a\varphi$, where $a = \|(2/\pi)^{1/2}R(i\theta)\xi\|^2$ and φ lies in

$$\gamma = \mathrm{Co}\left(0 \cup \left\{\frac{\pi}{2}\frac{e^{-i\theta D_r}}{i\theta D_r}\right\}\right),$$

where Co denotes the convex hull of set $\{\cdot\}$. Meanwhile, a is bounded by its spectral norm, and the spectral radius of a matrix is bounded by its maximum absolute row sum, thus

$$a = \frac{2}{\pi}\xi^\dagger R(i\theta)^\dagger R(i\theta)\xi \leqslant \frac{2}{\pi}\rho\big(R(i\theta)^\dagger R(i\theta)\big) \leqslant \frac{2}{\pi}\big\|R(i\theta)^\dagger R(i\theta)\big\|_\infty < 1,$$

where the last inequality follows from the sufficient condition (21).

It has been shown that $a\gamma$ does not contain the point $(-1, i0)$ for any given real number $0 \leqslant a < 1$ and for all $\theta \in (-\infty, +\infty)$ based on Lemma 2 (Tian and Yang, 2004a).

Therefore, when the loci of the eigenvalues of $G(i\theta)$ for $\theta \in (-\infty, +\infty)$ cross the real axis, they do so to the right of -1. So the loci of the eigenvalues of $G(i\theta)$ cannot encircle $(-1, i0)$, the generalized Nyquist stability criterion is satisfied and the system (16)–(20) is local stability at the equilibrium point. This theorem is completed.

Appendix C. Proof of Theorem 4

Let $x_r[t] = x_r + \vartheta_r[t]$, $y_n[t] = y_n + \nu_n[t]$, $z_l[t] = z_l + \sigma_l[t]$. By linearizing the system (22)–(26) about the equilibrium point x and taking the z-transforms, we can obtain the following equations

$$z\vartheta_r(z) = \vartheta_r(z) - \frac{\varepsilon_r\kappa_r x_r}{U_n'}\left(-U_n''\nu_n(z) + \eta\sum_{l:l\in r} p_l' z^{-D_{lr}}\sigma_l(z)\right),$$

$$\nu_n(z) = \sum_{r:r\in n} z^{-D_r}\vartheta_r(z),$$

$$\sigma_l(z) = \sum_{r:l\in r} z^{-D_{rl}}\vartheta_r(z),$$

where $\vartheta_r(z) = \mathcal{Z}(\vartheta_r[t])$, $\nu_n(z) = \mathcal{Z}(\nu_n[t])$ and $\sigma_l(z) = \mathcal{Z}(\sigma_l[t])$. Here, we assume the initial state of system is zero.

Rewrite the above equations in matrix form as

$$\begin{pmatrix} \nu(z) \\ \sigma(z) \end{pmatrix} = -P^{-1} R(z^{-1})^T T(z) R(z) P \begin{pmatrix} \nu(z) \\ \sigma(z) \end{pmatrix},$$

where $T(z)$ is an $|R| \times |R|$ diagonal matrix with entries $T_{rr}(z) = z^{-D_r}/(z-1)$, P is a $(|N| + |L|) \times (|N| + |L|)$ diagonal matrix with entries $P_{nn} = 1$, $P_{ll} = \eta p_l'$, and $R(z)$ is an $|R| \times (|N| + |L|)$ matrix with the entries

$$R_{rn} = \left(- U_n'' \frac{\varepsilon_r \kappa_r x_r}{U_n'} \right)^{1/2}, \quad \text{for } r \in n,$$

$$R_{rl} = z^{-D_{lr}} \left(\frac{\varepsilon_r \kappa_r x_r}{U_n'} \eta p_l' \right)^{1/2}, \quad \text{for } l \in r,$$

and all other entries of $R(z)$ are zero.

Let $G(z) = -P^{-1} R(z^{-1})^T T(z) R(z) P$, which is the *return ratio* for (ν, σ). Suppose that for some $\theta \in [-\pi, +\pi]$, λ is an eigenvalue of $G(e^{i\theta})$. Then, there exists a unit vector ζ such that $\lambda\zeta = R(e^{i\theta})^\dagger T(e^{i\theta}) R(e^{i\theta})\zeta$, where $\dagger$ denotes the matrix conjugate. Thus, $\lambda = \zeta^\dagger R(e^{i\theta})^\dagger T(e^{i\theta}) R(e^{i\theta})\zeta$.

Let $K = H^{-1/2} R(e^{i\theta})\zeta$ with elements k_r, where

$$H = \text{diag}\{h_r, r \in n\}, \quad h_r = 2 \sin\left(\frac{\pi}{2(2D_r + 1)}\right),$$

since $T(e^{i\theta})$ is diagonal, then

$$\lambda = \sum_r h_r |k_r|^2 T_{rr}\big(e^{i\theta}\big) = \sum_r |k_r|^2 h_r \frac{e^{-i\theta D_r}}{e^{i\theta} - 1}.$$

Thus, $\lambda = b\psi$, where $b = \|H^{-1/2} R(e^{i\theta})\zeta\|^2$ and ψ lies in

$$\chi = \text{Co}\left(0 \cup \left\{ h_r \frac{e^{-i\theta D_r}}{e^{i\theta} - 1} \right\}\right),$$

where Co denotes the convex hull of set $\{\cdot\}$. Mean- while,

$$\begin{aligned} b &= \zeta^\dagger R\big(e^{i\theta}\big)^\dagger H^{-1} R\big(e^{i\theta}\big)\zeta \leqslant \rho\Big(R\big(e^{i\theta}\big)^\dagger H^{-1} R\big(e^{i\theta}\big)\Big) \\ &\leqslant \Big\| R\big(e^{i\theta}\big)^\dagger H^{-1} R\big(e^{i\theta}\big) \Big\|_\infty < 1, \end{aligned}$$

where the last inequality follows from the sufficient condition (27).

It can be obtained from Lemma 2 (Tian and Yang, 2004b) that $b\chi$ does not contain the point $(-1, i0)$ for any given real number $0 \leqslant b < 1$ and for all $\theta \in [-\pi, +\pi]$.

Therefore, when the loci of the eigenvalues of $G(e^{i\theta})$ for $\theta \in [-\pi, +\pi]$ cross the real axis, they do so to the right of -1. So the loci of the eigenvalues of $G(e^{i\theta})$ cannot encircle $(-1, i0)$, thus the system (22)–(26) is local stability at the equilibrium point. This theorem is completed.

Appendix D. Proof of Theorem 5

The theorem is proved by contradiction. Suppose there exists a link l such that, at the equilibrium $\sum_{s:l\in s} x_s > C_l$.

Consider a route $r \in n$ of user $n \in N$ such that $l \in r$, at the equilibrium of (4), from (9) or (10) we can obtain

$$\begin{aligned}\frac{w_n}{\left(\sum_{r:r\in n} x_r\right)^{\alpha_n}} &= \eta\left(\sum_{l:l\in r}\frac{(\sum_{k:l\in k} x_k(t) - \widehat{C}_l)^+}{\sum_{k:l\in k} x_k(t)} + \frac{1-\varepsilon_r}{\varepsilon_r}\right)\\ &\geqslant \eta\left(\frac{\sum_{s:l\in s} x_s(t) - \widehat{C}_l}{\sum_{s:l\in s} x_s(t)} + \frac{1-\varepsilon_r}{\varepsilon_r}\right)\\ &\geqslant \eta\left(\frac{C_l - \widehat{C}_l}{C_l} + \frac{1-\varepsilon_r}{\varepsilon_r}\right).\end{aligned}$$

Then

$$\sum_{r:r\in n} x_r \leqslant \left(\frac{w_n}{\eta(1/\varepsilon_r - \widehat{C}_l/C_l)}\right)^{\frac{1}{\alpha_n}},$$

thus, for link l

$$\sum_{s:l\in s}\sum_{r:r\in n} x_r \leqslant \sum_{s:l\in s}\left(\frac{w_n}{\eta(1/\varepsilon_r - \widehat{C}_l/C_l)}\right)^{\frac{1}{\alpha_n}}.$$

From (31), for each link l the right term of the inequality above is less than or equal to C_l, meanwhile, from the assumption the left term is larger than C_l. This is a contradiction. The theorem is obtained.

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Optimali perkrovimo kontrolė ir daugiasrautinių tinklų pasiskirtymas su atsitiktiniais praradimais

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Šiame darbe nagrinėjamas optimalus perkrovimo valdymas ir daugiasrautinių tinklų skirstymo schemos su duomenų paketų praradimu. Taip pat darbe išspręsta šio tipo tinklų naudingumo problema.