

ON ONE CLASS OF DIFFERENCE SCHEMES FOR SOLUTION OF NAVIER-STOKES EQUATIONS IN VELOCITY-PRESSURE VARIABLES

Vyacheslav ABRASHIN and Sergey LAPKO

Institute of Mathematics
Belorussian Academy of Sciences
220072 Minsk, Surganova St.11, Belorussian Republic

Abstract. Difference methods in velocity-pressure variables having a number of important properties are constructed and investigated in this paper for a two-dimensional Navier-Stokes equation. Power neutral approximations of convective members and pressure gradients ensure a conservativity and absolute stability of the proposed algorithms. Their stability and convergence are investigated. The existence and uniqueness of velocity components and pressure gradients is proved.

Key words: noniterative and iterative schemes, conservativity and absolute stability, iterative process convergence, existence and uniqueness of solution.

Introduction. The construction of effective difference algorithms for solution of the Navier-Stokes system of equations is of great interest for specialists in the field of numerical methods for a boundary problem solution. Considering such a system describing non-squeezing flows there occur computational difficulties caused in the equation of continuity only by velocity components and the absence of a direct connection with pressure.

For calculation of non-squeezing flows there are usually used two approaches. The most widely spread approach describes the flows by a variable turbulence - current function. It is usually used in the two-dimensional case and allows to avoid explicit use of a continuity equation (see, e.g., Rouch, 1980; Anderson et.al., 1990; Fletcher, 1991).

The second approach is connected with the solution of the movement equation in such a form as it is written. For solution of the continuity equation different methods are used, e.g., the method of artificial compressibility (Chorin, 1967), method of markers and cells (Harlow, Welch, 1965), methods based on ε -approximation (Ladyjenskaya, 1969; Temam, 1968), etc.

The construction of computational algorithms for this problem solution is connected with the problems of boundary conditions statement, adequacy of the initial differential problem and the utilized model.

In this paper we propose another version of computational solution of the Navier–Stokes system of equations based on the approximation of movement and continuity equations on the displaced grids owing to which we manage to get difference Neuman problems on "oblique" derivatives for definition of P . And for these problems the conditions for the existence of their solutions are carried out automatically.

1. General statement of the problem. Let $\Omega^{(n)}(0 < x_\alpha < l_\alpha)$, $\alpha = \overline{1, n}$ be an n -dimensional parallelepiped with the boundary Γ . In the domain $\Omega_T^{(n)} = \Omega^{(n)} \times]0, T]$ with the boundary $S_T^{(n)} = \Gamma \times]0, T]$ consider the problem for a system of Navier–Stokes equations describing viscous non-squeezing flows

$$\frac{\partial u_k}{\partial t} - \nu \sum_{\alpha=1}^n \frac{\partial^2 u_k}{\partial x_\alpha^2} + \sum_{\alpha=1}^n u_\alpha \frac{\partial u_k}{\partial x_\alpha} = -\frac{\partial P}{\partial x_k}, \quad \nu > 0, \quad k = \overline{1, n}, \quad (1.1)$$

$$\sum_{\alpha=1}^n \frac{\partial u_\alpha}{\partial x_\alpha} = 0, \quad (1.2)$$

with initial and boundary conditions of the form

$$\begin{aligned} u_i(x, 0) &= \phi_i(x), & x \in \Omega^{(n)}, & & i = \overline{1, n}, \\ u_i(x, t) &= 0, & (x, t) \in S_T^{(n)}, & & i = \overline{1, n}. \end{aligned} \quad (1.3)$$

It is necessary to find unknown functions $u_i = i = \overline{1, n}$ and pressure P . Following the widely used in literature designations it is necessary to note that the parameter ν in (1.1) is the value $\frac{1}{Re}$, when Re is

the Reynold's number. The problems of existence and uniqueness of a problem (1.1) – (1.3) solution were considered in the paper of Lions (1972), so later on we'll assume that for problem (1.1) – (1.3) there exists a sufficiently smooth solution.

2. Noniterative scheme for two-dimensional linearized problems. Consider system (1.1) – (1.3) when $n = 2$ with linearized convective members:

$$\frac{\partial u_k}{\partial t} - \nu \sum_{\alpha=1}^2 \frac{\partial^2 u_k}{\partial x_\alpha^2} + a \frac{\partial u_k}{\partial x_1} + b \frac{\partial u_k}{\partial x_2} = -\frac{\partial P}{\partial x_k}, \quad k = \overline{1, 2}, \quad (2.1)$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \quad \nu > 0, \quad a, b \text{ are bounded constants} \quad (2.2)$$

with initial and boundary conditions analogous to (1.3).

In the domain $\Omega_T^{(2)}$ we introduce two space-time grids

$$\begin{aligned} \bar{w}_{h\tau} &= \bar{w}_h \times w_\tau, \quad \bar{w}_h = \left\{ x = (x_{1i}, x_{2j}) = (ih, jh), \quad i = \overline{0, N_1}, \right. \\ &\quad \left. j = \overline{0, N_2}, \quad h = \frac{l_1}{N_1} = \frac{l_2}{N_2} \right\}, \\ \gamma_h &= \bar{w}_h \setminus w_h \times w_\tau, \quad w_\tau = \{t_j = j\tau, \quad j = 0, 1, \dots, n\}, \\ \Omega_{h\tau} &= \Omega_h \times w_\tau, \quad \Omega_h = \left\{ X = (x_{1i-1/2}, x_{2j-1/2}) = \left(\left(i - \frac{1}{2} \right) h, \left(j - \frac{1}{2} \right) h \right), \right. \\ &\quad \left. i = \overline{1, N_1}, \quad j = \overline{1, N_2} \right\}. \end{aligned}$$

Decompose the set Ω_h into two subsets of nodes:

$$\begin{aligned} \Omega_h &= Q_h \cup G_h, \quad Q_h = \left\{ (x_{1i-\frac{1}{2}}, x_{2j-\frac{1}{2}}), \quad i = \overline{2, N_1-1}, \quad j = \overline{2, N_2-1} \right\}, \\ G_h &= \left\{ (x_{1i-\frac{1}{2}}, x_{2j-\frac{1}{2}}), \quad i = 1, \quad j = \overline{1, N_2}; \quad i = N_1, \quad j = \overline{1, N_2}; \right. \\ &\quad \left. j = 1, \quad i = \overline{2, N_1-1}; \quad j = N_2, \quad i = \overline{2, N_1-1} \right\}. \end{aligned}$$

In Ω_h also choose two sets of nodes arranged as a chess-board:

$$\begin{aligned} \Omega_h &= \Omega'_h \cup \Omega''_h, \quad \Omega'_h = \left\{ (x_{1i-\frac{1}{2}}, x_{2j-\frac{1}{2}}) \in \Omega_h, \quad i+j-1 \text{ is even} \right\}, \\ \Omega''_h &= \left\{ (x_{1i-\frac{1}{2}}, x_{2j-\frac{1}{2}}) \in \Omega_h, \quad i+j-1 \text{ is odd} \right\} \quad (\text{see Fig. 1}). \end{aligned}$$

$Q'_h, Q''_h(Q'_h \cup Q''_h = Q_h), G'_h, G''_h(G'_h \cup G''_h = G_h)$ are defined in an analogous way.

We shall consider our grid domain both in the main Cartesian system of coordinates $O_{x_1x_2}$ and in another system of coordinates $O_{\zeta_1\zeta_2}$ obtained from $O_{x_1x_2}$ by the shift of the centre of coordinates to the point $(\frac{h}{2}, \frac{h}{2})$ and by the turn of axis by 45° counterclockwise (see Fig. 1).

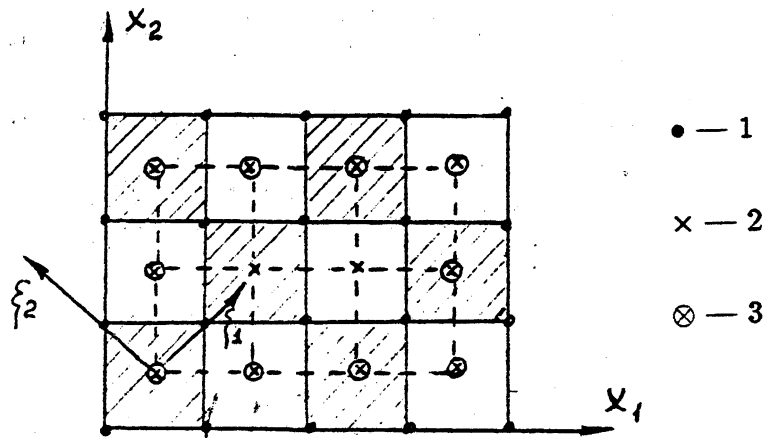


Fig. 1. 1 - ω_h , 2 - Q_h , 3 - G_h .

Replace functions u_1, u_2 and P , entering (2.1), (2.2), by their net analogues: $y_1, y_2 \approx u_1; z_1, z_2 \approx u_2$ in nodes x of the main grid $\bar{\omega}_{h\tau}$ and $q \approx P$ in nodes X of the grid $\Omega_{h\tau}$.

Later on for simplicity of definition we shall use a nonindexed form of writing in the following way: v is $v^{i,j}, v^1 = v^{i,1}, v^{(-1),1} = v^{i-1,1}, v^{-1} = v^{-1,j}, v^{(-1),(1)} = v^{i-1,j+1}$ and so on.

Basing upon the multicomponent methods of alternating direction type proposed in the papers of Abrashin (1990); Dziuba (1990); consider the following algorithm for solution of system (2.1), (2.2):

$$\begin{aligned} y_{1t} &= -\hat{q}_{x_1} + \Lambda(y_2), & x \in \omega_h, \\ z_{1t} &= -\hat{q}_{x_2} + \Lambda(z_2), & x \in \omega_h, \end{aligned}$$

$$\begin{aligned}
 y_{2t} &= -\widehat{q}_{\hat{x}_1} + \Lambda(y_2), & x \in \omega_h, \\
 z_{2t} &= -\widehat{q}_{\hat{x}_2} + \Lambda(z_2), & x \in \omega_h, \\
 y_{1\hat{x}_1} + z_{1\hat{x}_2} &= 0, & X \in \Omega_h, \\
 y_1^{(0)} = y_2^{(0)} &= \varphi_1(x), \quad z_1^{(0)} = z_2^{(0)} = \varphi_2(x), & x \in \omega_h, \\
 y_1 \Big|_{\gamma_h} &= y_2 \Big|_{\gamma_h} = z_1 \Big|_{\gamma_h} = z_2 \Big|_{\gamma_h} = 0,
 \end{aligned}
 \tag{2.3}$$

where $\Lambda(v) = \nu(v_{x_1x_1} + v_{x_2x_2}) - av_{x_1} - bv_{x_2}$, (v is y_2 or z_2) and derivatives $v_{\hat{x}_1}, v_{\hat{x}_2}$ are defined so that

$$\begin{aligned}
 v_{\hat{x}_1} &= \frac{1}{2h} \left(v\left(\frac{1}{2}, \frac{1}{2}\right) + v\left(\frac{1}{2}, -\frac{1}{2}\right) - v\left(-\frac{1}{2}, \frac{1}{2}\right) - v\left(-\frac{1}{2}, -\frac{1}{2}\right) \right), \\
 & \quad x \in \omega_h \quad \text{or} \quad X \in \Omega_h, \\
 v_{\hat{x}_2} &= \frac{1}{2h} \left(v\left(\frac{1}{2}, \frac{1}{2}\right) + v\left(-\frac{1}{2}, \frac{1}{2}\right) - v\left(\frac{1}{2}, -\frac{1}{2}\right) - v\left(-\frac{1}{2}, -\frac{1}{2}\right) \right), \\
 & \quad x \in \omega_h \quad \text{or} \quad X \in \Omega_h,
 \end{aligned}$$

(here v is y_1, z_1 or q). The rest difference derivatives are defined as in the paper of Samarsky (1989).

It is obvious that the difference equation of continuity in (2.3) will be also valid for y_{1t}, z_{1t} . Such an approximation of the equation of continuity on grid cells $\bar{\omega}_h$ and pressure gradients on grid cells Ω_h with the help of elementary algebraic operations from the first two equations and the difference equation of continuity in (2.3) allows to obtain the following two problems for \widehat{q} :

$$\begin{aligned}
 \widehat{q}_{\bar{\zeta}_1\zeta_1} + \widehat{q}_{\bar{\zeta}_2\zeta_2} &= F(y_2, z_2), & X \in Q'_h, \\
 \widehat{q}_n &= f(y_2, z_2), & X \in G'_h,
 \end{aligned}
 \tag{2.4}$$

and also

$$\begin{aligned}
 \widehat{q}_{\bar{\zeta}_1\zeta_1} + \widehat{q}_{\bar{\zeta}_2\zeta_2} &= F(y_2, z_2), & X \in Q''_h, \\
 \widehat{q}_n &= f(y_2, z_2), & X \in G''_h,
 \end{aligned}
 \tag{2.5}$$

where

$$\begin{aligned}\widehat{q}_{\zeta_1}^{(-\frac{1}{2}),(-\frac{1}{2})} &= \frac{1}{h\sqrt{2}} \left(\widehat{q}^{(\frac{1}{2}),(\frac{1}{2})} - \widehat{q}^{(-\frac{1}{2}),(-\frac{1}{2})} \right), \\ \widehat{q}_{\zeta_1}^{(-\frac{1}{2}),(-\frac{1}{2})} &= \frac{1}{h\sqrt{2}} \left(\widehat{q}^{(-\frac{1}{2}),(-\frac{1}{2})} - \widehat{q}^{(-\frac{3}{2}),(-\frac{3}{2})} \right), \\ \widehat{q}_{\zeta_2}^{(-\frac{1}{2}),(-\frac{1}{2})} &= \frac{1}{h\sqrt{2}} \left(\widehat{q}^{(-\frac{3}{2}),(\frac{1}{2})} - \widehat{q}^{(-\frac{1}{2}),(-\frac{1}{2})} \right), \\ \widehat{q}_{\zeta_2}^{(-\frac{1}{2}),(-\frac{1}{2})} &= \frac{1}{h\sqrt{2}} \left(\widehat{q}^{(-\frac{1}{2}),(-\frac{1}{2})} - \widehat{q}^{(\frac{1}{2}),(-\frac{3}{2})} \right),\end{aligned}$$

\widehat{q}_n is a net analogue of the inner normal derivative of the boundary G in the system of coordinates $0_1\zeta_1\zeta_1$ (by G we assume line connecting the points of G_h themselves) this derivative has the form

$$\frac{\partial \widehat{P}}{\partial \widehat{n}_G} = \frac{\partial \widehat{P}}{\partial \zeta_1} \cos(\widehat{\zeta_1}, \widehat{n}_G) + \frac{\partial \widehat{P}}{\partial \zeta_2} \cos(\widehat{\zeta_2}, \widehat{n}_G),$$

where the angle $(\widehat{\zeta_\alpha}, \widehat{n}_G)$, $\alpha = 1, 2$ is counted from the positive direction of axis ζ_α counterclockwise, we shall define the inner normal derivative at angle points as a semisum $\frac{\partial \widehat{P}}{\partial \widehat{n}_G}$ of two interperpendicular sides of a rectangular bound. F has the form $F = (\Lambda(y_2))_{\hat{x}_1} + (\Lambda(z_2))_{\hat{x}_2}$, $X \in Q_h$, and it is easy to obtain f from the continuity equations at the points $X \in G_h$ and the corresponding movement equations from (2.3).

Note that (2.4), (2.5) are none other than difference analogues on "oblique" derivatives of some Neiman problem

$$\begin{aligned}\frac{\partial^2 \widehat{P}}{\partial \zeta_1^2} + \frac{\partial^2 \widehat{P}}{\partial \zeta_2^2} &= \Phi, & (\zeta_1, \zeta_2) \in Q, \\ \frac{\partial \widehat{P}}{\partial \widehat{n}_G} &= f_0, & (\zeta_1, \zeta_2) \in G,\end{aligned}\tag{2.6}$$

where Q is the domain bounded by the G line.

It is known that the condition providing the existence (but not the uniqueness) of the problem (2.6) solution is the following equality

$$\iint_Q \Phi d\zeta_1 d\zeta_2 + \int_G f_0 dG = 0.\tag{2.7}$$

Formulate the lemma.

Lemma 1.1. For difference problems (2.4), (2.5) the following equalities

$$\sum_{Q'_k} h^2 F + \sum_{G'_k} h f = 0, \quad (2.8)$$

$$\sum_{Q''_k} h^2 F + \sum_{G''_k} h f = 0, \quad (2.9)$$

are valid.

The validity of (2.8), (2.9) may be verified putting directly in these equalities the expressions for F and f from (2.4), (2.5) and summing up in all corresponding nodes.

The sum of equalities (2.8), (2.9) is a discrete analogue of integral condition (2.7). If we take into account that problems (2.4), (2.5) are some systems of linear algebraic equations of the form $A_1 Y_1 = B_1$, $A_2 Y_2 = B_2$, where matrices A_1, A_2 are singular and they have the following spectra of eigenvalues $0 = \lambda_1 < \lambda_2 < \dots < \lambda_n$ then fulfilment of (2.8), (2.9) practically means that the right sides B_1, B_2 satisfy the solvability conditions of the given systems. The solutions of the given systems exist to within the constant summand. Hence it follows the existence of solution to within the constant summand of difference problems (2.4), (2.5). But from the construction (2.4), (2.5) it follows that their solutions at the same time are the solutions of the first two movement equations in (2.3). So we obtain the existence and uniqueness of the velocity components and pressure gradients in scheme (2.3).

Many papers (see, e.g., Molchanov and Nikolenko, 1973; Molchanov, 1968; Marchuk, 1989; Buzbee et al, 1970) are devoted to the solution of Neuman problem. That is why we shall not dwell on this problem considering that the solution \tilde{q} of problem (2.4), (2.5) to within the constant summand may be found in the well-known way. And a combination of the corresponding pressure gradients found in (2.4), (2.5) uniquely defines the pressure gradients of problem (2.3) in the initial system of coordinates. The third and fourth

equations in (2.3) are five-point difference schemes for definition of \hat{y}_2, \hat{z}_2 taken as an approximate solution to components u_1, u_2 . In the constructed algorithm (2.3) each difference equation approximates the corresponding differential equation of problem (2.1), (2.2) with accuracy of order $O(\tau + h^2)$.

Let us dwell on the questions of the stability and convergence of (2.3).

Theorem 2.1. *Difference scheme (2.3) is absolutely stable with respect to initial data and for its solution the estimation*

$$\begin{aligned} \nu \|\hat{y}_2\|_{(1)}^2 + \nu \|\hat{z}_2\|_{(1)}^2 + 2T \|\bar{q}_{\hat{x}_1} + \Lambda(\hat{y}_2)\|^2 + 2T \|\bar{q}_{\hat{x}_2} + \Lambda(\hat{z}_2)\|^2 \\ \leq 2\nu \|\varphi_1\|_{(1)}^2 + 2\nu \|\varphi_2\|_{(1)}^2 + 2T \|\Lambda(\varphi_1)\|^2 + 2T \|\Lambda(\varphi_2)\|^2, \end{aligned} \quad (2.10)$$

is valid where $\|v\|_{(1)}^2 = \|v_{x_1}\|_1^2 + \|v_{x_2}\|_2^2$ and $\|\cdot\|$ is the grid analogue of L_2 norm (see Samarsky, 1989; Dzjuba, 1990).

Proof. After a scalar multiplication of the first equation (2.3) by $-\tau q_{t\hat{x}_1}$, the second - by $-\tau q_{t\hat{x}_2}$, the third - by $\tau(\Lambda(y_2))_t$, the fourth - by $\tau(\Lambda(z_2))_t$, sum up the received equalities and then the sum up in all time layers $j = \overline{1, n}$.

We'll have

$$\begin{aligned} \sum_{j=1}^n \tau \left(\nu \|\hat{y}_2\|_{(1)}^2 + \nu \|\hat{z}_2\|_{(1)}^2 \right) + \|\bar{q}_{\hat{x}_1} + \Lambda(\hat{y}_2)\|^2 + \|\bar{q}_{\hat{x}_2} + \Lambda(\hat{z}_2)\|^2 \\ \leq \|\Lambda(\varphi_1)\|^2 + \|\Lambda(\varphi_2)\|^2. \end{aligned} \quad (2.11)$$

Applying the difference analogue of imbedding theorem on t to the first summand in (2.11) we obtain estimation (2.10).

The theorem is proved.

Denote $w_\alpha = u_{1h} - y_\alpha$, $v_\alpha = u_{2h} - z_\alpha$, $\alpha = 1, 2$, $r = P_h - q$ - the errors of the corresponding approximate solutions obtained in scheme (2.3). Then for the error function we may receive the following problem:

$$\begin{aligned} w_{1t} &= -\hat{r}_{\hat{x}_1} + \Lambda(w_2) + \psi_1, \\ v_{1t} &= -\hat{r}_{\hat{x}_2} + \Lambda(v_2) + \psi_2, \end{aligned}$$

$$\begin{aligned}
 w_{2t} &= -\widehat{r}_{\hat{x}_1} + \Lambda(\widehat{w}_2) + \psi_3, \\
 v_{2t} &= -\widehat{r}_{\hat{x}_2} + \Lambda(\widehat{v}_2) + \psi_4, \\
 w_{1t\hat{x}_1} + v_{1t\hat{x}_2} &= \psi_5, \\
 w_\alpha^{(0)} = v_\alpha^{(0)} &= 0, \quad w_\alpha \Big|_{\gamma_h} = v_\alpha \Big|_{\gamma_h} = 0, \quad \alpha = 1, 2.
 \end{aligned}
 \tag{2.12}$$

The errors are the values of order $O(\tau + h^2)$ under sufficient smoothness of the solution of the initial differential problem.

Theorem 2.2. *If the solution of initial problem (2.1), (2.2) is sufficiently smooth, then as $\tau, h \rightarrow 0$ a convergence of the solution of difference problem (2.3) to the solution of the initial problem takes place and for the error of the method the estimation*

$$Q(w_2, v_2, r) \leq C(\tau^{1/2} + h^2)
 \tag{2.13}$$

is valid, where $C > 0$ is a restricted value which does not depend on the stepsize of the net, and

$$\begin{aligned}
 Q = & \left(\nu \| \widehat{w}_2 \|_{(1)}^2 + \nu \| \widehat{v}_2 \|_{(1)}^2 + 2T \| \right. \\
 & \left. - \widehat{r}_{\hat{x}_1} + \Lambda(\widehat{w}_2) \|^2 + 2T \| - \widehat{r}_{\hat{x}_2} + \Lambda(\widehat{v}_2) \|^2 \right)^{\frac{1}{2}}.
 \end{aligned}$$

The proof of this theorem is analogous in many respects to the proof of the previous one, therefore we omit it.

REMARK 2.1. Following the Dzjuba paper (1990), with additional requirement for the smoothness of P it is not difficult to obtain an optimal estimate $Q(w_2, v_2, r) \leq C(\tau + h^2)$.

3. Economical scheme. For solution of problem (2.1), (2.2) consider also the following difference algorithm

$$\begin{aligned}
 y_{1t} &= -\widehat{q}_{\hat{x}_1} + \Lambda_1(y_2) + \Lambda_2(y_3), & x \in w_h, \\
 z_{1t} &= -\widehat{q}_{\hat{x}_2} + \Lambda_1(z_2) + \Lambda_2(z_3), & x \in w_h, \\
 y_{2t} &= -\widehat{q}_{\hat{x}_1} + \Lambda_1(\widehat{y}_2) + \Lambda_2(y_3), & x \in w_h, \\
 z_{2t} &= -\widehat{q}_{\hat{x}_2} + \Lambda_1(\widehat{z}_2) + \Lambda_2(z_3), & x \in w_h,
 \end{aligned}$$

$$\begin{aligned}
y_{3t} &= -\hat{q}_{\hat{x}_1} + \Lambda_1(\hat{y}_2) + \Lambda_2(\hat{y}_3), & x \in w_h, \\
z_{3t} &= -\hat{q}_{\hat{x}_2} + \Lambda_1(\hat{z}_2) + \Lambda_2(\hat{z}_3), & x \in w_h, \\
y_{1\hat{x}_1} + z_{1\hat{x}_2} &= 0, & X \in \Omega_h, \\
y_1^{(0)} = y_2^{(0)} = y_3^{(0)} &= \varphi_1(x), \quad z_1^{(0)} = z_2^{(0)} = z_3^{(0)} = \varphi_2(x), & x \in w_h, \\
y_1|_{\gamma_h} = y_2|_{\gamma_h} = y_3|_{\gamma_h} &= z_1|_{\gamma_h} = z_2|_{\gamma_h} = z_3|_{\gamma_h} = 0,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
\Lambda_1(y_2) &= \nu y_{2x_1x_1} - a y_{2x_1}, & \Lambda_1(z_2) &= \nu z_{2x_1x_1} - a z_{2x_1}, \\
\Lambda_2(y_3) &= \nu y_{3x_2x_2} - b y_{3x_2}, & \Lambda_2(z_3) &= \nu z_{3x_2x_2} - b z_{3x_2}.
\end{aligned}$$

The first two equations and the continuity equation in (3.1) define two difference Neuman problems of (2.4), (2.5) type for which there exists a solution. Having solved the Neuman problem we'll define the pressure gradients uniquely $\hat{q}_{\hat{x}_i}$, $i = 1, 2$. The following four equations in (3.1) are economical three-point schemes from which difference components of velocity are defined uniquely.

For (3.1) the theorems analogous to theorems 2.1 and 2.2 will be valid.

Theorem 3.1. *Difference scheme (3.1) is absolutely stable with respect to initial data and for its solution the estimation*

$$\begin{aligned}
&\nu \left[\|y_{2x_1}^{(n)}\|_1^2 + \|y_{3x_2}^{(n)}\|_2^2 + \|z_{2x_1}^{(n)}\|_1^2 + \|z_{3x_2}^{(n)}\|_2^2 \right. \\
&\quad \left. + 2T \left\| -\hat{q}_{\hat{x}_1} + \Lambda(y_2) + \Lambda(y_3) \right\|^2 + 2T \left\| -\hat{q}_{\hat{x}_2} + \Lambda(z_2) + \Lambda(z_3) \right\|^2 \right] \\
&\leq 2\nu \left[\|\varphi_{1x_1}\|_1^2 + \|\varphi_{1x_2}\|_2^2 + 2\nu \|\varphi_{2x_1}\|_1^2 + 2\nu \|\varphi_{2x_2}\|_2^2 \right. \\
&\quad \left. + 4T \|\Lambda(\varphi_1)\| + 4T \|\Lambda(\varphi_2)\|^2 \right]
\end{aligned}$$

is valid.

Theorem 3.2. *If the solution of problem (2.1), (2.2) is sufficiently smooth, then as $\tau, h \rightarrow 0$ a convergence of the solution of difference problem (3.1) to the solution of the initial problem takes place and for the error of the method the estimation*

$$Q(w_2, w_3, v_2, v_3, r) \leq C_1(\tau^{1/2} + h^2)$$

is valid, where $C_1 > 0$ is a restricted value which does not depend on the stepsize and

$$Q^2 = \nu \| \hat{w}_{2x_1}^{(n)} \|_1^2 + \nu \| \hat{w}_{3x_2}^{(n)} \|_2^2 + \| \hat{v}_{2x_1}^{(n)} \|_1^2 + \nu \| \hat{v}_{2x_2}^{(n)} \|_2^2 + 2T \| -\hat{r}_{\hat{x}_1}^{(n)} + \Lambda_1(\hat{w}_2) + \Lambda_2(\hat{w}_3) \|^2 + 2T \| -\hat{r}_{\hat{x}_2}^{(n)} + \Lambda_1(\hat{v}_2) + \Lambda_2(\hat{v}_3) \|^2.$$

For theorem 3.2 the remark analogous to remark 2.1 will be valid.

4. Iterative scheme for problem (2.1.), (2.2). For problem (2.1), (2.2) consider a purely implicit absolutely stable scheme

$$\begin{aligned} y_t &= -\hat{q}_{\hat{x}_1} + \Lambda(\hat{y}), & x \in w_h, \\ z_t &= -\hat{q}_{\hat{x}_2} + \Lambda(\hat{z}), & x \in w_h, \\ y_{\hat{x}_1} + z_{\hat{x}_2} &= 0, & X \in \Omega_h, \end{aligned} \tag{4.1}$$

for the solution of which we shall use the following iterative process:

$$\begin{aligned} \tau^{-1}(\hat{y}_1^{k+1} - y) &= \Lambda(\hat{y}_1^k) - \hat{q}_{\hat{x}_1}^k, \\ \tau^{-1}(\hat{z}_1^{k+1} - z) &= \Lambda(\hat{z}_1^k) - \hat{q}_{\hat{x}_2}^k, \\ \tau^{-1}(\hat{y}_2^{k+1} - y) &= \Lambda(\hat{y}_1^k) - \hat{q}_{\hat{x}_2}^{k+1}, \\ \tau^{-1}(\hat{z}_2^{k+1} - z) &= \Lambda(\hat{z}_1^k) - \hat{q}_{\hat{x}_1}^{k+1}, \\ \hat{y}_{2\hat{x}_1}^{k+1} + \hat{z}_{2\hat{x}_1}^{k+1} &= 0. \end{aligned} \tag{4.2}$$

In (4.1), (4.2) all the approximations are the same as in (2.3) and initial and boundary conditions are defined in an analogous way. The first two equations in (4.2) are the five-point schemes for definition $\hat{y}_1^{k+1}, \hat{z}_1^{k+1}$. The following three equations define difference Neuman problems

$$\begin{aligned} \hat{q}_{\hat{c}_1\hat{c}_1}^{k+1} + \hat{q}_{\hat{c}_2\hat{c}_2}^{k+1} &= F(\hat{y}_1^{k+1}, \hat{z}_1^{k+1}), & X \in Q'_h, \\ \hat{q}_n^{k+1} &= f(\hat{y}_1^{k+1}, \hat{z}_1^{k+1}), & X \in G'_h, \end{aligned} \tag{4.3}$$

$$\begin{aligned} \widehat{q}_{\bar{c}_1 c_1}^{k+1} + \widehat{q}_{\bar{c}_2 c_2}^{k+1} &= F(\widehat{y}_1^{k+1}, \widehat{z}_1^{k+1}), \quad X \in Q_h'', \\ \widehat{q}_n^{k+1} &= f(\widehat{y}_1^{k+1}, \widehat{z}_1^{k+1}), \quad X \in G_h'', \end{aligned} \quad (4.4)$$

for which there exist solutions, as shown above. Solving problems (4.3), (4.4) we find \widehat{q}^{k+1} . As a zero iteration on the upper layer we may take the solution received on the previous layer. For solution of the zero iteration for q on the first time layer it is necessary to solve the following Neuman problems

$$\begin{aligned} q_{\bar{c}_1 c_1}^0 + q_{\bar{c}_2 c_2}^0 &= F(\varphi_{1h}, \varphi_{2h}), \quad X \in Q_h', \\ q_n^0 &= f(\varphi_{1h}, \varphi_{2h}), \quad X \in G_h', \\ q_{\bar{c}_1 c_1}^0 + q_{\bar{c}_2 c_2}^0 &= F(\varphi_{1h}, \varphi_{2h}), \quad X \in Q_h'', \\ q_n^0 &= f(\varphi_{1h}, \varphi_{2h}), \quad X \in G_h''. \end{aligned}$$

The iterative process should be continued until the chosen condition of termination is satisfied by several consequent iterations. As a criterion of iterative process termination we may take, for example, the condition of closeness of the neighbouring iterations or some analogue of the relative error.

Introduce the following notation: $\widehat{\rho}_\alpha^{k+1} = \widehat{y}_\alpha^{k+1} - \widehat{y}_\alpha^k$, $\widehat{g}_\alpha^{k+1} = \widehat{z}_\alpha^{k+1} - \widehat{z}_\alpha^k$, $\alpha = 1, 2$, $d_1^{k+1} = \widehat{q}^{k+1} - \widehat{q}^k$, $d^{k+1} = -d_1^{k+1}$. Then the errors of iterative process will satisfy the following problem

$$\begin{aligned} \widehat{\rho}_1^{k+1} &= \tau \Lambda(\widehat{\rho}_1^{k+1}) + \tau d_{\widehat{x}_1}^k, & \widehat{g}_1^{k+1} &= \tau \Lambda(\widehat{g}_1^{k+1}) + \tau d_{\widehat{x}_2}^k, \\ \widehat{\rho}_2^{k+1} &= \tau \Lambda(\widehat{\rho}_2^{k+1}) + \tau d_{\widehat{x}_1}^{k+1}, & \widehat{g}_2^{k+1} &= \tau \Lambda(\widehat{g}_2^{k+1}) + \tau d_{\widehat{x}_2}^{k+1}, \\ \widehat{\rho}_{2\widehat{x}_1}^{k+1} + \widehat{g}_{2\widehat{x}_2}^{k+1} &= 0. \end{aligned} \quad (4.5)$$

If in the movement equations there are no convective members, i.e., $\Lambda(v) = \nu(v_{\widehat{x}_1 x_1} + v_{\widehat{x}_2 x_2})$, then for (4.2) the following theorem will be valid.

Theorem 4.1. *A convergence of iterative process (4.2) to the solution of implicit scheme (4.1) takes place and the following recursive inequality holds*

$$Q^{k+1} \leq q \cdot Q^k, \quad (4.6)$$

where $q = \frac{1}{1+h^{\kappa}D}$, $D = 1 - 2\tau h^{\kappa-2}\nu > 0$, $\kappa > 0$,

$$Q^2 = \frac{\nu}{2} \|\rho_1^k\|_{(1)}^2 + \frac{\nu}{2} \|g_1^k\|_{(1)}^2 + \tau \|\Lambda(\rho_1^k) + d_{x_1}^k\|^2 + \tau \|\Lambda(g_1^k) + d_{x_2}^k\|^2.$$

Inequality (4.6) means the convergence of iterative process (4.2) to the solution of implicit scheme (4.1) with the rate of geometrical progression with ratio q . At small ν , for example $\nu < 0.5$, having taken $\kappa = 1$, $\tau = h$, we'll receive that the rate of iterative process will be defined by the value $q = 1/(1 + h(1 - 2\nu))$.

If in (4.1), (4.2) operators Λ have linearized convective members, then the following theorem will be valid.

Theorem 4.2. *Iterative process (4.2) converges to the solution of implicit scheme (4.1) with the speed of geometrical progression with ratio $q_1 < 1$ and the following recurrent inequality holds*

$$Q^{k+1} \leq q_1 Q^k, \tag{4.7}$$

where $q_1 = \frac{1}{1+h^{\kappa}D_1}$,

$$D_1 = \min \left\{ 1 - 8\tau h^{\kappa-2}\nu, 1 - (4 + a^2 + b^2)^2 \tau h^{\kappa-1}/2 \right\} > 0,$$

and $Q^2 = \|\rho_1^k\|^2 + \|g_1^k\|^2 + h\nu/2(\|\rho_1^k\|_{(1)}^2 + \|g_1^k\|_{(1)}^2) + \tau h(\|\Lambda(\rho_1^k) + d_{x_1}^k\|^2 + \|\Lambda(g_1^k) + d_{x_2}^k\|^2)$ is such as in the previous theorem.

From (4.7) we see that the strict condition necessary for $q_1 < 1$ is $\tau < 2h^{\kappa+1}(4 + a^2 + b^2)^2$.

5. Nonlinear problem. Write down the problem (1.1)-(1.3) when $n = 2$ in the divergent form

$$\frac{\partial u_k}{\partial t} - \nu \sum_{\alpha=1}^2 \frac{\partial^2 u_k}{\partial x_{\alpha}^2} + \sum_{\alpha=1}^2 \frac{\partial}{\partial x_{\alpha}} (u_{\alpha} \cdot u_k) = -\frac{\partial P}{\partial x_k}, \quad k = 1, 2, \tag{5.1}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \tag{5.2}$$

with the initial and boundary conditions analogous to (1.3).

Put in accordance to problem (5.1), (5.2) the following difference scheme

$$\begin{aligned} y_t &= -\widehat{q}_{\widehat{x}_1} + L(\widehat{y}) - K_1(y, \widehat{y}) - K(z, \widehat{y}), & x \in \omega_h, \\ z_t &= -\widehat{q}_{\widehat{x}_2} + L(\widehat{z}) - K_1(y, \widehat{z}) - K(z, \widehat{z}), & x \in \omega_h, \\ y_{\widehat{x}_1} + z_{\widehat{x}_2} &= 0, & X \in \Omega_h, \end{aligned} \quad (5.3)$$

where $L(v) = \nu(v_{x_1 x_1} + v_{x_2 x_2})$ and convective members are approximated in the following way:

$$\begin{aligned} K_1(u, v) &= (uv)_{\widehat{x}_1} = \left(\widetilde{u}^{(-\frac{1}{2})}, \frac{(v + v^{(-1)})}{2} \right)_{x_1}, \\ \widetilde{u}^{(-\frac{1}{2})} &= \frac{1}{2} \left(\overline{u}^{(-\frac{1}{2}), (\frac{1}{2})} + \overline{u}^{(-\frac{1}{2}), (-\frac{1}{2})} \right) \\ &= \frac{1}{8} \left(u^{(1)} + u^{(-1), (1)} + 2u + 2u^{(-1)} + u^{(-1)} + u^{(-1), (-1)} \right), \\ K_2(u, v) &= (uv)_{\widehat{x}_2} = \left(\widetilde{u}^{(-\frac{1}{2})}, \frac{(v + v^{(-1)})}{2} \right)_{x_2}, \\ \widetilde{u}^{(-\frac{1}{2})} &= \frac{1}{2} \left(\overline{u}^{(\frac{1}{2}), (-\frac{1}{2})} + \overline{u}^{(-\frac{1}{2}), (-\frac{1}{2})} \right) \\ &= \frac{1}{8} \left(u^{(1)} + u^{(1), (-1)} + 2u + 2u^{(-1)} + u^{(-1)} + u^{(-1), (-1)} \right). \end{aligned}$$

From the given approximation the following equalities $(\widehat{y}, K_1(y, \widehat{y})) + (\widehat{y}, K_2(z, \widehat{y})) = 0$, $(\widehat{z}, K_1(y, \widehat{z})) + (\widehat{z}, K_2(z, \widehat{z})) = 0$, $(\widehat{y}, \widehat{q}_{\widehat{x}_1}) + (\widehat{z}, \widehat{q}_{\widehat{x}_2}) = 0$ arise. They provide the stability of conservative scheme (5.3). Analogous schemes were considered in Fryazinov's (1991) paper and they are called energetically neutral because the net approximation of convective members in them does not make a contribution into the energy balance.

For solution of implicit scheme (5.3) on the base of algorithm (2.3) we propose the following iterative process:

$$\begin{aligned} \tau^{-1}(\widehat{y}_1^{k+1} - y) &= L(\widehat{y}_1^{k+1}) - K_1(y, \widehat{y}_1^{k+1}) - K_2(z, \widehat{y}_1^{k+1}) - \widehat{q}_{\widehat{x}_1}^k, \\ \tau^{-1}(\widehat{z}_1^{k+1} - z) &= L(\widehat{z}_1^{k+1}) - K_1(y, \widehat{z}_1^{k+1}) - K_2(z, \widehat{z}_1^{k+1}) - \widehat{q}_{\widehat{x}_1}^k, \\ \tau^{-1}(\widehat{y}_2^{k+1} - y) &= L(\widehat{y}_2^{k+1}) - K_1(y, \widehat{y}_2^{k+1}) - K_2(z, \widehat{y}_2^{k+1}) - \widehat{q}_{\widehat{x}_2}^{k+1}, \\ \tau^{-1}(\widehat{z}_2^{k+1} - z) &= L(\widehat{z}_2^{k+1}) - K_1(y, \widehat{z}_2^{k+1}) - K_2(z, \widehat{z}_2^{k+1}) - \widehat{q}_{\widehat{x}_2}^{k+1}, \\ \widehat{y}_{2\widehat{x}_1}^{k+1} + \widehat{z}_{2\widehat{x}_2}^{k+1} &= 0. \end{aligned} \quad (5.4)$$

Scheme (5.4) refers to the family of algorithms given above. The last three equations in (5.4) define difference Neuman problems of type (4.3), (4.4) for which there exist solutions with the accuracy up to the constant summand.

For errors of the given iterative process the problem

$$\begin{aligned} \rho_1^{k+1} &= \tau \Lambda_1^{k+1} + \tau d_{\bar{x}_1}^k, & g_1^{k+1} &= \tau \Lambda_2^{k+1} + \tau d_{\bar{x}_2}^k, \\ \rho_2^{k+1} &= \tau \Lambda_1^{k+1} + \tau d_{\bar{x}_1}^{k+1}, & g_2^{k+1} &= \tau \Lambda_2^{k+1} + \tau d_{\bar{x}_2}^{k+1}, \\ \rho_{2\bar{x}_1}^{k+1} + g_{2\bar{x}_2}^{k+1} &= 0, \end{aligned}$$

will be valid, where

$$\begin{aligned} \Lambda_1^{k+1} &= \nu(\rho_{1\bar{x}_1x_1}^{k+1} + \rho_{1\bar{x}_2x_2}^{k+1}) - (y \rho_1^{k+1})_{\bar{x}_1} - (z \rho_1^{k+1})_{\bar{x}_2}, \\ \Lambda_2^{k+1} &= \nu(g_{1\bar{x}_1x_1}^{k+1} + g_{1\bar{x}_2x_2}^{k+1}) - (y g_1^{k+1})_{\bar{x}_1} - (z g_1^{k+1})_{\bar{x}_2}. \end{aligned}$$

For (5.4) the following theorem is valid.

Theorem 5.1. Satisfying the conditions $\|y\|_C \leq M$, $\|z\|_C \leq M$, where M is a positive limited constant, iterative process (5.4) converges to the solution of scheme (5.3) and the following recurrent inequality holds

$$Q^2 \leq q_2 Q^2, \quad (5.5)$$

where $q_2 = \frac{1}{1+h^*D_2}$,

$$D_2 = \min \left\{ 1 - 8\tau h^{*2}\nu, 1 - \frac{(4 + 8M^2)^2 \tau h^{*-1}}{2} \right\} > 0, \quad * > 0,$$

and $Q^2 = \|\rho_1^k\|^2 + \|g_1^k\|^2 + h\nu/2(\|\rho_1^k\|_{(1)}^2 + \|g_1^k\|_{(1)}^2) + \tau h(\|\Lambda_1^k + d_{\bar{x}_1}^k\|^2 + \|\Lambda_2^k + d_{\bar{x}_2}^k\|^2)$.

It follows from (5.5) that the most strict condition on the step-size providing $q_2 < 1$ is $\tau < 2h^{*+1}(4 + M^2)^2$.

From the iterative scheme (5.4) and the theorem of convergence it is obvious that with the growth of k the right sides of the first and the third as well as of the second and the fourth difference

equations become equal what means \hat{y}_1^{k+1} and \hat{y}_2^{k+1} coincide, as well as \hat{z}_1^{k+1} and \hat{z}_2^{k+1} .

REMARK 5.1. For solution of problem (5.3) we may use an iterative process which for each fixed iteration is an economical scheme for the velocity component:

$$\begin{aligned}\tau^{-1}(\hat{y}_1^{k+1} - y) &= \nu \hat{y}_{1\bar{x}_1 x_1}^{k+1} - (y \hat{y}_1^{k+1})_{\bar{x}_1} + \nu \hat{y}_{2\bar{x}_2 x_2}^k - (z \hat{y}_2^k)_{\bar{x}_2} - \hat{q}_{\bar{x}_1}^k, \\ \tau^{-1}(\hat{z}_1^{k+1} - z) &= \nu \hat{z}_{1\bar{x}_1 x_1}^{k+1} - (y \hat{z}_1^{k+1})_{\bar{x}_1} + \nu \hat{z}_{2\bar{x}_2 x_2}^k - (z \hat{z}_2^k)_{\bar{x}_2} - \hat{q}_{\bar{x}_2}^k, \\ \tau^{-1}(\hat{y}_2^{k+1} - y) &= \nu \hat{y}_{1\bar{x}_1 x_1}^{k+1} - (y \hat{y}_1^{k+1})_{\bar{x}_1} + \nu \hat{y}_{2\bar{x}_2 x_2}^{k+1} - (z \hat{y}_2^{k+1})_{\bar{x}_2} - \hat{q}_{\bar{x}_1}^k, \\ \tau^{-1}(\hat{z}_2^{k+1} - z) &= \nu \hat{z}_{1\bar{x}_1 x_1}^{k+1} - (y \hat{z}_1^{k+1})_{\bar{x}_1} + \nu \hat{z}_{2\bar{x}_2 x_2}^{k+1} - (z \hat{z}_2^{k+1})_{\bar{x}_2} - \hat{q}_{\bar{x}_2}^k, \\ \tau^{-1}(\hat{y}_3^{k+1} - y) &= \nu \hat{y}_{1\bar{x}_1 x_1}^{k+1} - (y \hat{y}_1^{k+1})_{\bar{x}_1} + \nu \hat{y}_{2\bar{x}_2 x_2}^{k+1} - (z \hat{y}_2^{k+1})_{\bar{x}_2} - \hat{q}_{\bar{x}_1}^{k+1}, \\ \tau^{-1}(\hat{z}_3^{k+1} - z) &= \nu \hat{z}_{1\bar{x}_1 x_1}^{k+1} - (y \hat{z}_1^{k+1})_{\bar{x}_1} + \nu \hat{z}_{2\bar{x}_2 x_2}^{k+1} - (z \hat{z}_2^{k+1})_{\bar{x}_2} - \hat{q}_{\bar{x}_2}^{k+1}, \\ \hat{y}_{3\bar{x}_1}^{k+1} + \hat{z}_{3\bar{x}_2}^{k+1} &= 0.\end{aligned}$$

Iterative process (5.6) is implemented analogously to (3.1), (4.2).

REMARK 5.2. For solution of problem (5.1), (5.2) we may use a purely implicit energetically neutral scheme which differs from (5.3) only in that the convective members in it are taken only from the upper layer. For implementation of this implicit scheme we may propose the following iterative process

$$\begin{aligned}\tau^{-1}(\hat{y}_1^{k+1} - y) &= L(\hat{y}_1^{k+1}) - K_1(\hat{y}_2^k, \hat{y}_1^{k+1}) - K_2(\hat{z}_2^k, \hat{y}_1^{k+1}) - \hat{q}_{\bar{x}_1}^k, \\ \tau^{-1}(\hat{z}_1^{k+1} - z) &= L(\hat{z}_1^{k+1}) - K_1(\hat{y}_2^k, \hat{z}_1^{k+1}) - K_2(\hat{z}_2^k, \hat{z}_1^{k+1}) - \hat{q}_{\bar{x}_2}^k, \\ \tau^{-1}(\hat{y}_2^{k+1} - y) &= L(\hat{y}_1^{k+1}) - K_1(\hat{y}_2^k, \hat{y}_1^{k+1}) - K_2(\hat{z}_2^k, \hat{y}_1^{k+1}) - \hat{q}_{\bar{x}_1}^{k+1}, \\ \tau^{-1}(\hat{z}_2^{k+1} - z) &= L(\hat{z}_1^{k+1}) - K_1(\hat{y}_2^k, \hat{z}_1^{k+1}) - K_2(\hat{z}_2^k, \hat{z}_1^{k+1}) - \hat{q}_{\bar{x}_2}^{k+1}, \\ \hat{y}_{2\bar{x}_1}^{k+1} + \hat{z}_{2\bar{x}_2}^{k+1} &= 0,\end{aligned}$$

which belongs to the same family of algorithms given above and for which the convergence theorem analogous to theorem 5.1 will be valid.

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V. Abrashin received the Degree of Candidate of Physico-mathematical Science in 1968 at the Institute of Mathematics of the Academy of Sciences of BSSR, the Degree of Doctor of Physico-mathematical Sciences in 1979 in the Computational Centre of the Academy of Sciences of the USSR. He is head of the Laboratory of Numerical Methods of Mathematical Physics. He investigates numerical methods of solution of the problems in mathematical physics and also nonlinearity problems.

S. Lapko is the investigator of the Laboratory of Numerical Methods of Mathematical Physics at the Institute of Mathematics of the Academy of Sciences of Byelorussia. He deals with the problems of construction of effective algorithms for numerical solution of different multidimensional problems in mathematical physics.