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Atomic Decompositions of Fuzzy Normed Linear Spaces for Wavelet Applications

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Abstract. Wavelet analysis is a powerful tool with modern applications as diverse as: image processing, signal processing, data compression, data mining, speech recognition, computer graphics, etc. The aim of this paper is to introduce the concept of atomic decomposition of fuzzy normed linear spaces, which play a key role in the development of fuzzy wavelet theory. Atomic decompositions appeared in applications to signal processing and sampling theory among other areas.

First we give a general definition of fuzzy normed linear spaces and we obtain decomposition theorems for fuzzy norms into a family of semi-norms, within more general settings. The results are both for Bag–Samanta fuzzy norms and for Katsaras fuzzy norms. As a consequence, we obtain locally convex topologies induced by this types of fuzzy norms.

The results established in this paper, constitute a foundation for the development of fuzzy operator theory and fuzzy wavelet theory within this more general frame.

Key words: fuzzy wavelet, atomic decomposition, fuzzy metric space, fuzzy norm, fuzzy normed linear space (FNLS).

1. Introduction

Wavelet analysis is a powerful tool with modern applications as diverse as: image processing, signal processing, data compression, data mining, speech recognition, computer graphics, etc. (Prasad and Iyengar, 1997; Daubechies, 1990; Mallat, 2008; Tomic and Sersic, 2013; DeVore *et al.*, 1992). Wavelet transform is a tool that divides up functions, data, operators into different frequency components and then studies each component. Complex information such as speech, images and music can be decomposed into elementary forms and subsequently reconstructed with high precision. Wavelet transform of a function is a improved version of Fourier transform (Sifuzzaman *et al.*, 2009).

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The concept of wavelet was introduced by J. Morlet. He considered wavelets as a family of functions constructed from translations and dilations of a single function called "mother wavelet" ψ. They are defined by

$$\psi_{a,b}(x) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right), \quad a, b \in \mathbb{R},\ a \neq 0.$$

Immediately, A. Grossmann studied inverse formula for the wavelet transform. A mathematical study of wavelet transforms and their applications was made in paper Grossmann and Morlet (1984).

If a function $f \in L_2(\mathbb{R})$, then the series $\sum_{j,k\in\mathbb{Z}}\langle f, \psi_{j,k}\rangle\psi_{j,k}(x)$ is called the wavelet series of f and $\langle f, \psi_{j,k}\rangle = d_{j,k} = \int_{-\infty}^{\infty} f(x)\psi_{j,k}(x)\,dx$ is called the wavelet coefficients of f.

Unlike the Haar functions, which form an orthogonal basis, Morlet wavelets are not orthogonal and form frames. Frames are sets of function which are not necessarily orthogonal and which are not linearly independent. Frames for Hilbert spaces were introduced in Duffin and Schaeffer (1952) and up to now they have developed very much in connection to wavelet theory.

A frame for a Hilbert space H is a family of vector $\{y_i\}_{i\in\mathbb{N}^*}$ in H, for which the norms $\|x\|_H$ and $\|\{\langle x, y_i\rangle\}\|_{l^2}$ are equivalent, i.e. there exists $A, B > 0$ such that

$$A\|x\|^2 \leqslant \sum_{i\in\mathbb{N}^*}\left|\langle x, y_i\rangle\right|^2 \leqslant B\|x\|^2, \quad \forall x \in H.$$

A is called lower frame bound and B is called upper frame bound. If we define $Sx = \{\langle x, y_i\rangle\}$, then $S^*S = \sum_{i\in\mathbb{N}^*}\langle x, y_i\rangle y_i$ is a continuous invertible mapping of H into itself. If we denote by $z_i = (S^*S)^{-1}y_i$, we obtain the reconstruction formula $x = \sum_{i\in\mathbb{N}^*}\langle x, z_i\rangle y_i = \sum_{i\in\mathbb{N}^*}\langle x, y_i\rangle z_i$.

The notion of fuzzy wavelets already exists, but differently introduced from what we have in mind, which is based on atomic decompositions of fuzzy normed linear spaces. Thus, paper (Huang and Zeng, 2009) developed a fuzzy wavelet algorithm based on fuzzy transforms and wavelets, but they were used separately. Fuzzy wavelet networks (Ho *et al.*, 2001) introduced a fuzzy model into the wavelet neural network to improve the accuracy of function approximation.

In papers of Perfilieva (2006), Di Martino and Sessa (2007), Di Martino *et al.* (2008) the authors use the certain fuzzy operator constructed on $\mathcal{C}[a, b]$ by some partition like $(A_1, A_2, \ldots, A_n)$ for compression of images. The authors define a fuzzy transform which associates a suitable n-dimensional vector to a continuous function f on the interval $[a, b]$. The advantage of the inverse formula of the fuzzy transform is a single approximate representation of the original function. In addition, the inverse fuzzy transform has nice filtering properties. Based on this fuzzy transform, paper of Beg (2013) developed another concept of fuzzy wavelet.

Atomic decomposition are used to represent an arbitrary element x of a Banach space X as a series expansion involving a fixed countable $(x_i)_i$ of elements in that space such that the coefficients of the expansion of x depend in a linear and continuous way on x. The general theory of atomic decompositions was developed in papers Feichtinger and Gröchenig (1989a, 1989b). In these papers, the authors obtain atomic decompositions for a large class of Banach spaces.

After L. Zadeh introduced in his classical paper (Zadeh, 1965) the concept of fuzzy set, many authors have tried to develop the classical results within this new frame. An important problem was finding an adequate definition of a fuzzy normed space.

In studying fuzzy topological spaces, Katsaras (1984) first introduced the notion of fuzzy norm on a linear space. Since then many mathematicians have introduced several notions of fuzzy norm from different points of view. Thus, Felbin (1992) introduced an idea of fuzzy norm on a linear space by assigning a fuzzy real number to each element of linear space. Following Cheng and Mordenson (1994), T. Bag and S.K. Samanta introduced another concept of fuzzy norm, in paper Bag and Samanta (2003), and obtained a decomposition theorem of fuzzy norms into a family of crisp norms. In paper Bag and Samanta (2005), T. Bag and S.K. Samanta introduced different types of continuities and boundedness for linear operators and they established the principles of fuzzy functional analysis. A comparative study concerning T. Bag and S.K. Samanta's definitions, of A.K. Katsaras and that of C. Felbin was made in 2008, in the paper Bag and Samanta (2008). T. Bag and S.K. Samanta's definition has proven to be the most suitable one, it can be worked with, the easiest and it can be used in most diverse and various developments. But, according to T. Bag and S.K. Samanta a fuzzy norm is fuzzy set which satisfies five axioms. In order to obtain the above mentioned results, T. Bag and S.K. Samanta impose another two axioms on the fuzzy norm. Regarded together the 7 axioms are very strong and they narrow a lot the family of fuzzy normed spaces. Thus we can say that a clear definition regarding the fuzzy norm has not been reached, but after T. Bag and S.K. Samanta, almost all authors have had as a starting point their definition and, at the same time, they have tried to simplify and improve it: Saadati and Vaezpour, 2005; Miheţ, 2009; Goleţ, 2010; Alegre and Romaguera, 2010; Katsaras, 2013.

The concept of fuzzy metric space was introduced by Kramosil and Michálek (1975) and many notions and results belonging to classical metric spaces could be extended and generalized in the context of fuzzy metric spaces. To some extent, the existence of an equivalence between the probabilistic metric spaces and fuzzy metric spaces, makes it to be impossible to speak about fuzzy normed linear spaces without making reference to the concept of probabilistic normed spaces introduced by A.N. Šerstnev (1962, 1963). In a probabilistic approach the norm of a vector is a probabilistic distribution while the fuzzy norm is a fuzzy set. Although there is a good connection between the fuzzy norm and the probabilistic one, the area of applicability of the two notions is different and this is a reason enough to develop the theory of fuzzy normed linear spaces independently.

Following the ideas of T. Bag and S.K. Samanta, in this paper, we obtain decomposition theorems for fuzzy norms into a family of semi-norms, in more general settings. The results are both for Bag-Samanta fuzzy norms and for Katsaras fuzzy norms. As a consequence, we obtain locally convex topologies induced by this types of fuzzy norms. We introduce the concept of atomic decomposition of fuzzy normed linear spaces, which play a key role in the development of fuzzy wavelet theory. The results established in this paper, constitute a foundation for the development of fuzzy operator theory and fuzzy wavelet theory in this more general frame.

2. Preliminaries

Definition 1. (See Schweizer and Sklar, 1960.) A binary operation

$$*:[0,1]\times[0,1]\to[0,1]$$

is called triangular norm (t-norm) if it satisfies the following condition:

1. $a*b=b*a, \forall a,b\in[0,1]$;
2. $a*1=a, \forall a\in[0,1]$;
3. $(a*b)*c=a*(b*c), \forall a,b,c\in[0,1]$;
4. If $a\leqslant c$ and $b\leqslant d$, with $a,b,c,d\in[0,1]$, then $a*b\leqslant c*d$.

Example 1. Three basic examples of continuous t-norms are $\wedge$, $\cdot$, $*_L$, which are defined by $a\wedge b=\min\{a,b\}$, $a\cdot b=ab$ (usual multiplication in $[0,1]$) and $a*_L b=\max\{a+b-1,0\}$ (the Lukasiewicz t-norm).

Definition 2. (See Zadeh, 1965.) Let X be a nonempty set. A fuzzy set in X is a function $\mu : X\to[0,1]$.

Remark 1. The classical union and intersection of ordinary subsets of X can be extended by the following formulas, proposed by L. Zadeh:

$$\Big(\bigvee_{i\in I}\mu_I\Big)(x)=\sup\big\{\mu_i(x): i\in I\big\},\qquad \Big(\bigwedge_{i\in I}\mu_I\Big)(x)=\inf\big\{\mu_i(x): i\in I\big\}.$$

Definition 3. (See Kramosil and Michálek, 1975.) The triple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy metric, i.e. a fuzzy set in $X\times X\times[0,\infty)$ which satisfies the following conditions:

(M1) $M(x,y,0)=0, \forall x,y\in X$;
(M2) $[M(x,y,t)=1,\ \forall t>0]$ if and only if $x=y$;
(M3) $M(x,y,t)=M(y,x,t), \forall x,y\in X, \forall t\geqslant 0$;
(M4) $M(x,z,t+s)\geqslant M(x,y,t)*M(y,z,s), \forall x,y,z\in X, \forall t,s\geqslant 0$;
(M5) $\forall x,y\in X$, $M(x,y,\cdot):[0,\infty)\to[0,1]$ is left continuous and $\lim_{t\to\infty}M(x,y,t)=1$.

Remark 2. In the definition of fuzzy metric space, I. Kramosil and J. Michálek have imposed another condition: "$M(x,y,\cdot)$ is nondecreasing, for all $x,y\in X$". Cho *et al.* (2006) showed that this affirmation derives from the other axioms.

Indeed, for $0 < t < s$, we have

$$M(x, y, s) \geqslant M(x, x, s - t) * M(x, y, t) = 1 * M(x, y, t) = M(x, y, t).$$

Example 2. (See George and Veeramani, 1994.) Let (X, d) be a metric space. Let

$$M_d : X \times X \times [0, \infty), \qquad M_d(x, y, t) = \begin{cases} \frac{t}{t+d(x,y)} & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Then $(X, M_d, \wedge)$ is a fuzzy metric space. M_d is called standard fuzzy metric.

Theorem 1. *(See* George and Veeramani, 1994.*) Let $(X, M, *)$ be a fuzzy metric space. For $x \in X$, $r \in (0, 1)$, $t > 0$ we define the open ball*

$$B(x, r, t) := \{ y \in X : M(x, y, t) > 1 - r \}.$$

Let

$$\mathcal{T}_M := \{ T \subset X : x \in T \text{ iff } (\exists) t > 0,\ r \in (0, 1) : B(x, r, t) \subseteq T \}.$$

Then $\mathcal{T}_M$ is a topology on X.

Proposition 1. *(See* George and Veeramani, 1994.*) Let (X, d) be a metric space and M_d be the corresponding standard fuzzy metric on X. Then the topology $\mathcal{T}_d$ induced by the metric d and the topology $\mathcal{T}_{M_d}$ induced by the standard fuzzy metric M_d are the same.*

Definition 4. Let $(X, M, *)$ be a fuzzy metric space and (x_n) be a sequence in X. The sequence (x_n) is said to be convergent if there exists $x \in X$ such that $M(x_n, x, t) = 1$, $\forall t > 0$. In this case, x is called the limit of the sequence (x_n) and we note $\lim_{n\to\infty} x_n = x$, or $x_n \to x$.

Remark 3. Let $(X, M, *)$ be a fuzzy metric space. A sequence (x_n) is convergent to x if and only if (x_n) is convergent to x in topology $\mathcal{T}_M$.

Indeed,

$$\begin{aligned} & x_n \to x \text{ in topology } \mathcal{T}_M \\ & \quad \Leftrightarrow \quad (\forall) r \in (0, 1),\ (\forall) t > 0,\ (\exists) n_0 \in \mathbb{N} : x_n \in B(x, r, t),\ (\forall) n \geqslant n_0 \\ & \quad \Leftrightarrow \quad (\forall) r \in (0, 1),\ (\forall) t > 0,\ (\exists) n_0 \in \mathbb{N} : M(x_n, x, t) > 1 - r,\ (\forall) n \geqslant n_0 \\ & \quad \Leftrightarrow \quad \lim_{n\to\infty} M(x_n, x, t) = 1,\ (\forall) t > 0. \end{aligned}$$

Definition 5. A topological vector space X will be called fuzzy metrizable if the topology is generated by a fuzzy metric which is translation-invariant, i.e. $M(x+z, y+z, t) = M(x, y, t)$, $(\forall) x, y, z \in X$, $(\forall) t \geqslant 0$.

Theorem 2. *(See* Gregori and Romaguera, 2000.*) A topological vector space X is fuzzy metrizable if and only if it is metrizable.*

3. Bag–Samanta Fuzzy Norm

Definition 6. Let X be a vector space over a field $\mathbb{K}$ and $*$ be a continuous t-norm. A fuzzy set N in $X \times [0, \infty)$ is called a fuzzy norm on X if it satisfies:

(N1) $N(x, 0) = 0$, $(\forall)x \in X$;
(N2) $[N(x, t) = 1,\ (\forall)t > 0]$ if and only if $x = 0$;
(N3) $N(\lambda x, t) = N(x, \frac{t}{|\lambda|})$, $(\forall)x \in X$, $(\forall)t \geqslant 0$, $(\forall)\lambda \in \mathbb{K}^*$;
(N4) $N(x + y, t + s) \geqslant N(x, t) * N(y, s)$, $(\forall)x, y \in X$, $(\forall)t, s \geqslant 0$;
(N5) $(\forall)x \in X$, $N(x, \cdot)$ is left continuous and $\lim_{t\to\infty} N(x, t) = 1$.

The triple $(X, N, *)$ will be called fuzzy normed linear space (briefly FNLS).

Remark 4.

(a) Bag and Samanta (2003, 2005) gave a similar definition for $* = \wedge$, but in order to obtain some important results they assume that the fuzzy norm satisfies also the following conditions:

(N6) $N(x, t) > 0$, $(\forall)t > 0 \Rightarrow x = 0$;
(N7) $(\forall)x \neq 0$, $N(x, \cdot)$ is a continuous function and strictly increasing on the subset $\{t: 0 < N(x, t) < 1\}$ of $\mathbb{R}$.

The results obtained by T. Bag and S.K. Samanta can be found in this more general settings.

(b) Goleţ (2010), Alegre and Romaguera (2010) gave also this definition in the context of real vector spaces.

Remark 5. $N(x, \cdot)$ is nondecreasing, $(\forall)x \in X$.

Theorem 3. *If $(X, N, *)$ is a FNLS, then*

$$M : X \times X \times [0, \infty) \to [0, 1], \qquad M(x, y, t) = N(x - y, t)$$

is a fuzzy metric on X, which is called the fuzzy metric induced by the fuzzy norm N. Moreover, we have:

1. *M is a translation-invariant fuzzy metric*;
2. *$M(\lambda x, \lambda y, t) = M(x, y, \frac{t}{|\lambda|})$, $(\forall)x \in X$, $(\forall)t \geqslant 0$, $(\forall)\lambda \in \mathbb{K}^*$.*

Proof. (M1) $M(x, y, 0) = N(x - y, 0) = 0$;

(M2) $[M(x, y, t) = 1,\ (\forall)t > 0] \Leftrightarrow [N(x - y, t) = 1,\ (\forall)t > 0] \Leftrightarrow x - y = 0 \Leftrightarrow x = y$;

(M3)

$$\begin{aligned} M(x, y, t) &= N(x-y, t) = N\big((-1)(y-x), t\big) \\ &= N\left(y-x, \frac{t}{|-1|}\right) = N(y-x, t) = M(y, x, t); \end{aligned}$$

(M4)

$$\begin{aligned} M(x, z, t+s) &= N(x-z, t+s) = N\big((x-y)+(y-z), t+s\big) \\ &\geqslant N(x-y, t) * N(y-z, s) = M(x, y, t) * M(y, z, s); \end{aligned}$$

(M5) It is obvious.

Now we verify properties (1), (2).

(1) $M(x+z, y+z, t) = N((x+z)-(y+z), t) = N(x-y, t) = M(x, y, t)$;
(2) $M(\lambda x, \lambda y, t) = N(\lambda x - \lambda y, t) = N(x-y, \frac{t}{|\lambda|}) = M(x, y, \frac{t}{|\lambda|})$.

□

Corollary 1. *Let* $(X, N, *)$ *be a FNLS. For* $x \in X$, $r \in (0, 1)$, $t > 0$ *we define the open ball*

$$B(x, r, t) := \big\{y \in X : N(x-y, t) > 1-r\big\}.$$

Then

$$\mathcal{T}_N := \big\{T \subset X : x \in T \text{ iff } (\exists) t > 0,\ r \in (0, 1) : B(x, r, t) \subseteq T\big\}$$

is a topology on X.

Moreover, if the t*-norm* $*$ *satisfies* $\sup_{x \in (0,1)} x * x = 1$, *then* $(X, \mathcal{T}_N)$ *is Hausdorff.*

Proof. The first part results from the previous theorem and Theorem 1. Let $x, y \in X$, $x \neq y$. Using $(N2)$, there exists $t > 0$: $N(x-y, t) < 1$. Let $r = N(x-y, t)$. As $\sup_{x \in (0,1)} x * x = 1$, we can find $r_1 \in (0, 1) : r_1 * r_1 > r$. We have

$$B\left(x, 1-r_1, \frac{t}{2}\right) \cap B\left(y, 1-r_1, \frac{t}{2}\right) = \emptyset.$$

Indeed, if we suppose that there exists $z \in B(x, 1-r_1, \frac{t}{2}) \cap B(y, 1-r_1, \frac{t}{2})$, we obtain that

$$N\left(x-z, \frac{t}{2}\right) > r_1, \qquad N\left(y-z, \frac{t}{2}\right) > r_1.$$

Thus

$$r = N(x-y, t) \geqslant N\left(x-z, \frac{t}{2}\right) * N\left(z-y, \frac{t}{2}\right) > r_1 * r_1 > r,$$

which is a contradiction. □

REMARK 6. Previous result was obtained by Sadeqi and Kia (2009), in 2009, using $(N7)$.

Theorem 4. *Let* $(X, N, *)$ *be a FNLS. Then* $(X, \mathcal{T}_N)$ *is a metrizable topological vector space.*

Proof. First we have to show that the mappings

$$(1)\quad (x, y) \mapsto x + y,$$
$$(2)\quad (\lambda, x) \mapsto \lambda \cdot x$$

are continuous.

(1) Let $x_n \to x$, $y_n \to y$. We have

$$N\big((x_n + y_n) - (x + y), t\big) \geqslant N\left(x_n - x, \frac{t}{2}\right) * N\left(y_n - y, \frac{t}{2}\right) \to 1.$$

Thus $x_n + y_n \to x + y$.

(2) Let $x_n \to x$, $\lambda_n \to \lambda$. We have

$$\begin{aligned} N(\lambda_n x_n - \lambda x, t) &= N\big(\lambda_n(x_n - x) + x(\lambda_n - \lambda), t\big) \\ &\geqslant N\left(\lambda_n(x_n - x), \frac{t}{2}\right) * N\left(x(\lambda_n - \lambda), \frac{t}{2}\right) \\ &= N\left(x_n - x, \frac{t}{2|\lambda_n|}\right) * N\left(x, \frac{t}{2|\lambda_n - \lambda|}\right) \to 1. \end{aligned}$$

This implies that $\lambda_n x_n \to \lambda x$.

Therefore $(X, \mathcal{T}_N)$ is a topological vector space. From Theorem 3 we have that X is fuzzy metrizable. Theorem 2 tells us that X is metrizable. □

Theorem 5. *Let* $(X, N, \wedge)$ *be a FNLS. Let*

$$p_\alpha(x) := \inf\big\{t > 0 : N(x, t) > \alpha\big\}, \quad \alpha \in (0, 1).$$

Then $\mathcal{P} = \{p_\alpha\}_{\alpha \in (0,1)}$ *is an ascending family of semi-norms on* X.

Proof. (1) As $N(0, t) = 1$, $(\forall) t > 0$, we obtain that

$$p_\alpha(0) = \inf\big\{t > 0 : N(0, t) > \alpha\} = 0.$$

(2) $p_\alpha(\lambda x) = |\lambda| p_\alpha(x)$, $(\forall) x \in X$, $(\forall) \lambda \in \mathbb{K}$.

First we note that, for $\lambda = 0$, the previous equality is obvious. For $\lambda \neq 0$, we have

$$\begin{aligned} p_\alpha(\lambda x) &= \inf\big\{t > 0 : N(\lambda x, t) > \alpha\big\} = \inf\left\{t > 0 : N\left(x, \frac{t}{|\lambda|}\right) > \alpha\right\} \\ &= \inf\left\{t|\lambda| > 0 : N\left(x, \frac{t|\lambda|}{|\lambda|}\right) > \alpha\right\} = |\lambda| \inf\big\{t > 0 : N(x, t) > \alpha\big\} \\ &= |\lambda| p_\alpha(x). \end{aligned}$$

(3)

$$
\begin{aligned}
p_\alpha(x) + p_\alpha(y) &= \inf\{t > 0 : N(x,t) > \alpha\} + \inf\{s > 0 : N(y,s) > \alpha\} \\
&= \inf\{t + s > 0 : N(x,t) > \alpha,\ N(y,s) > \alpha\} \\
&= \inf\{t + s > 0 : N(x,t) \wedge N(y,s) > \alpha\} \\
&\geqslant \inf\{t + s > 0 : N(x+y, t+s) > \alpha\} = p_\alpha(x+y).
\end{aligned}
$$

It remains to be proven that $\mathcal{P} = \{p_\alpha\}_{\alpha\in(0,1)}$ is an ascending family. Let $\alpha_1 \leqslant \alpha_2$. Then

$$\{t > 0 : N(x,t) > \alpha_2\} \subseteq \{t > 0 : N(x,t) > \alpha_1\}.$$

Thus $\inf\{t > 0 : N(x,t) > \alpha_2\} \geqslant \inf\{t > 0 : N(x,t) > \alpha_1\}$, namely $p_{\alpha_2}(x) \geqslant p_{\alpha_1}(x)$, $(\forall) x \in X$. □

Remark 7. T. Bag and S.K. Samanta defined

$$p_\alpha(x) := \inf\{t > 0 : N(x,t) \geqslant \alpha\}, \quad \alpha \in (0,1).$$

They assume that the fuzzy norm satisfies $(N6)$ and they obtained that $\mathcal{P} = \{p_\alpha\}_{\alpha\in(0,1)}$ is an ascending family of norms on X. For future development it is enough to have a family of semi-norms on X, which corresponds to the fuzzy norm N.

Corollary 2. *Let $(X, N, \wedge)$ be a FNLS. Then there exists on X a least fine topology, denoted by $\mathcal{T}_\mathcal{P}$, compatible with the structure of linear space of X, with respect to which each semi-norm p_α is continuous. With this topology X becomes a locally convex space. A fundamental system of neighborhoods of* 0 *is*

$$\mathcal{C}_\mathcal{P} = \{B(p_\alpha, t) : \alpha \in (0,1),\ t > 0\},$$

where $B(p_\alpha, t) = \{x \in X : p_\alpha(x) < t\}$.

Proposition 2. *The locally convex topology $\mathcal{T}_\mathcal{P}$ is Hausdorff.*

Proof. We need to show that the family of semi-norms $\mathcal{P}$ is sufficient, i.e. $(\forall) x \in X$, $x \neq 0$, $(\exists) p_\alpha \in \mathcal{P}$ such that $p_\alpha(x) \neq 0$. Let $x \in X$, $x \neq 0$. We suppose that $p_\alpha(x) = 0$, $(\forall)\alpha \in (0,1)$. Then $\inf\{t > 0 : N(x,t) > \alpha\} = 0$, for all $\alpha \in (0,1)$. Thus $N(x,t) > \alpha$, $(\forall)\alpha \in (0,1)$, $(\forall) t > 0$. Hence $N(x,t) = 1$, $(\forall) t > 0$. Therefore, from $(N2)$, we have $x = 0$, which is a contradiction. □

Theorem 6. *Let $(X, N, \wedge)$ be a FNLS and*

$$p_\alpha(x) := \inf\{t > 0 : N(x,t) > \alpha\}, \quad \alpha \in (0,1).$$

Then, for $x \in X$, $s > 0$, $\alpha \in (0,1)$, we have: $p_\alpha(x) < s$ if and only if $N(x,s) > \alpha$.

Proof. "$\Rightarrow$" We must show that $s \in \{t > 0 : N(x,t) > \alpha\}$. We suppose that $s \notin \{t > 0 : N(x,t) > \alpha\}$. Then there exists $t_0 \in \{t > 0 : N(x,t) > \alpha\}$ such that $t_0 < s$. (Contrary, $s \leqslant t$, $(\forall) t \in \{t > 0 : N(x,t) > \alpha\}$. Hence $s \leqslant \inf\{t > 0 : N(x,t) > \alpha\}$, i.e. $s \leqslant p_\alpha(x)$, which is a contradiction.) As $t_0 \in \{t > 0 : N(x,t) > \alpha\}$, $t_0 < s$ and $N(x,\cdot)$ is nondecreasing, we obtain that $\alpha < N(x,t_0) \leqslant N(x,s)$. Hence $N(x,s) > \alpha$, which leads to a contradiction.

"$\Leftarrow$" As $N(x,s) > \alpha$, we obtain that $s \in \{t > 0 : N(x,t) > \alpha\}$. Thus $p_\alpha(x) \leqslant s$. We suppose that $p_\alpha(x) = s$. As $N(x,\cdot)$ is left continuous in s, we have $\lim_{t\to s, t<s} N(x,t) = N(x,s)$. Thus there exists $t_0 < s$ such that $N(x,t_0) > \alpha$. (Contrary, $N(x,t) \leqslant \alpha$, for all $t \leqslant s$. Therefore $\lim_{t\to s, t<s} N(x,t) \leqslant \alpha$. Hence $N(x,s) \leqslant \alpha$, which is a contradiction.) But $t_0 < s$ and $N(x,t_0) > \alpha$ are in contradiction with the fact that $s = \inf\{t > 0 : N(x,t) > \alpha\}$. Hence $p_\alpha(x) \neq s$. Thus $p_\alpha(x) < s$. □

Remark 8. T. Bag and S.K. Samanta, using $(N6)$ and $(N7)$, proved that for $x \in X$, $x \neq 0$, $s > 0$, $\alpha \in (0,1)$, we have: $p_\alpha(x) = s$ if and only if $N(x,s) = \alpha$. This is a strong result, which is not true if the conditions $(N6)$ and $(N7)$ does not hold. But we do not need this result, the previous theorem being enough.

Corollary 3. *Let $(X, N, \wedge)$ be a FNLS. Then $\mathcal{T}_N = \mathcal{T}_\mathcal{P}$.*

Proof. In topology $\mathcal{T}_N$ a fundamental system of neighborhoods of 0 is $\mathcal{S}(0) = \{B(0,r,t) : r \in (0,1),\ t > 0\}$. In topology $\mathcal{T}_\mathcal{P}$ a fundamental system of neighborhoods of 0 is $\mathcal{C}_\mathcal{P} = \{B(p_\alpha, t) : \alpha \in (0,1),\ t > 0\}$, where $B(p_\alpha,t) = \{x \in X : p_\alpha(x) < t\}$. The former theorem shows us that the two systems are identical. Thus $\mathcal{T}_N = \mathcal{T}_\mathcal{P}$. □

Corollary 4. *Let $(X, N, \wedge)$ be a FNLS. Then X is a Hausdorff metrizable locally convex space.*

Remark 9. There exists another proof of this result, made by Cho *et al.* (2006), in the context of real random normed spaces of Šerstnev.

Definition 7. An ascending family $\{p_\alpha\}_{\alpha\in(0,1)}$ of semi-norms on a linear space X is called right continuous if for any decreasing sequence (α_n) in $(0,1)$, $\alpha_n \to \alpha \in (0,1)$, we have $p_{\alpha_n}(x) \to p_\alpha(x)$, $(\forall) x \in X$.

Theorem 7. *Let $(X, N, \wedge)$ be a FNLS and*

$$p_\alpha(x) := \inf\bigl\{t > 0 : N(x,t) > \alpha\bigr\}, \quad \alpha \in (0,1).$$

Then $\mathcal{P} = \{p_\alpha\}_{\alpha\in(0,1)}$ is right continuous.

Proof. Let $x \in X$ and (α_n) a decreasing sequence in $(0,1)$, $\alpha_n \to \alpha \in (0,1)$. Let $s > p_\alpha(x)$. Then $N(x,s) > \alpha$. As (α_n) a decreasing sequence and $\alpha_n \to \alpha$, there exists $n_0 \in \mathbb{N}$ such that $\alpha_n < N(x,s)$, $(\forall) n \geqslant n_0$. Therefore $p_{\alpha_n}(x) < s$, $(\forall) n \geqslant n_0$. Thus $p_{\alpha_n}(x) \to p_\alpha(x)$. □

Theorem 8. *Let $\{q_\alpha\}_{\alpha\in(0,1)}$ be a sufficient and ascending family of semi-norms on the linear space X. Let $N' : X \times [0,\infty) \to [0,1]$, defined by*

$$N'(x,t) = \begin{cases} \sup\{\alpha \in (0,1) : q_\alpha(x) < t\} & \text{if } t > 0, \\ 0 & \text{if } t = 0 \text{ or } \{\alpha \in (0,1) : q_\alpha(x) < t\} = \emptyset. \end{cases}$$

Then $(X, N', \wedge)$ is a FNLS.

Proof. We note firstly that $N'(x,\cdot)$ is nondecreasing. Indeed, for $t_1 < t_2$, we have

$$\{\alpha \in (0,1) : q_\alpha(x) < t_1\} \subseteq \{\alpha \in (0,1) : q_\alpha(x) < t_2\}.$$

Thus

$$\sup\{\alpha \in (0,1) : q_\alpha(x) < t_1\} \leqslant \sup\{\alpha \in (0,1) : q_\alpha(x) < t_2\}.$$

Hence $N'(x,t_1) \leqslant N'(x,t_2)$.

(N1) $N'(x,0) = 0$, $(\forall) x \in X$ is obvious.

(N2) If $x = 0$, then $q_\alpha(x) = 0$, $(\forall)\alpha \in (0,1)$. Hence, for all $\alpha \in (0,1)$, we have $q_\alpha(x) < t$, $(\forall)t > 0$. Thus $\sup\{\alpha \in (0,1) : q_\alpha(x) < t\} = 1$, $(\forall)t > 0$. Therefore $N'(x,t) = 1$, $(\forall)t > 0$.

Conversely, if $N'(x,t) = 1$, $(\forall)t > 0$, then $\sup\{\alpha \in (0,1) : q_\alpha(x) < t\} = 1$, for all $t > 0$. Hence, for all $\alpha \in (0,1)$, we have $q_\alpha(x) < t$, $(\forall)t > 0$. Thus, for all $\alpha \in (0,1)$, we have $q_\alpha(x) = 0$. As the family of semi-norms $\{q_\alpha\}_{\alpha\in(0,1)}$ is sufficient, we obtain that $x = 0$.

(N3) If $t = 0$, then $N'(\lambda x, t) = 0 = N'(x, \frac{t}{|\lambda|})$. For $t > 0$, we have

$$\begin{aligned} N'(\lambda x, t) &= \sup\{\alpha \in (0,1) : q_\alpha(\lambda x) < t\} = \sup\{\alpha \in (0,1) : |\lambda| q_\alpha(x) < t\} \\ &= \sup\left\{\alpha \in (0,1) : q_\alpha(x) < \frac{t}{|\lambda|}\right\} = N'\left(x, \frac{t}{|\lambda|}\right). \end{aligned}$$

(N4) The inequality $N'(x+y, t+s) \geqslant N'(x,t) \wedge N'(y,s)$ is obvious for $t = 0$ or $s = 0$. For $t > 0$, $s > 0$, we suppose that $N'(x+y, t+s) < N'(x,t) \wedge N'(y,s)$. Then there exists $\alpha_0 \in (0,1)$ such that

$$N'(x+y, t+s) < \alpha_0 < N'(x,t) \wedge N'(y,s).$$

As $N'(x,t) > \alpha_0$, there exists $\beta_1 \in \{\alpha \in (0,1) : q_\alpha(x) < t\}$ such that $\beta_1 > \alpha_0$. (Contrary, for all $\beta \in \{\alpha \in (0,1) : q_\alpha(x) < t\}$, we have $\beta \leqslant \alpha_0$. Hence $\sup\{\alpha \in (0,1) : q_\alpha(x) < t\} \leqslant \alpha_0$, which is a contradiction.) As $N'(y,s) > \alpha_0$, there exists $\beta_2 \in \{\alpha \in (0,1) : q_\alpha(y) < s\}$ such that $\beta_2 > \alpha_0$. Let $\beta_0 = \min\{\beta_1, \beta_2\}$. Then $\beta_0 > \alpha_0$ and $q_{\beta_0}(y) \leqslant q_{\beta_2}(y) < s$, $q_{\beta_0}(x) \leqslant q_{\beta_1}(x) < t$. Thus $q_{\beta_0}(x+y) \leqslant q_{\beta_0}(x) + q_{\beta_0}(y) < t + s$. Therefore

$$\beta_0 \in \{\alpha \in (0,1) : q_\alpha(x+y) < t+s\}.$$

Thus $\sup\{\alpha \in (0,1) : q_\alpha(x+y) < t+s\} \geqslant \beta_0 > \alpha_0$, which is in contradiction with the fact that $N'(x+y,t+s) < \alpha_0$. Hence $N'(x+y,t+s) \geqslant N'(x,t) \wedge N'(y,s)$.

(N5) We prove that $\lim_{t\to\infty} N'(x,t) = 1$. Let $\alpha_0 \in (0,1)$ arbitrary. We show that there exists $t_0 > 0$ such that $N'(x,t) > \alpha_0$, $(\forall)t \geqslant t_0$. As $N(x,\cdot)$ is nondecreasing, it will be enough to show that there exists $t_0 > 0$ such that $N'(x,t_0) > \alpha_0$. Let $t_0 > q_{\alpha_1}(x)$, where $\alpha_1 = \frac{1+\alpha_0}{2} \in (\alpha_0, 1)$. Then

$$N'(x,t_0) = \sup\left\{\alpha \in (0,1) : q_\alpha(x) < t_0\right\} \geqslant \alpha_1 > \alpha_0.$$

We prove now that $N'(x,\cdot)$ is left continuous in $t > 0$.

Case 1. $N'(x,t) = 0$. Thus, for all $s \leqslant t$, as $N'(x,s) \leqslant N'(x,t)$, we have $N'(x,s) = 0$. Therefore

$$\lim_{s\to t, s<t} N'(x,s) = 0 = N'(x,t).$$

Case 2. $N'(x,t) > 0$. Let α_0 arbitrary, such that $0 < \alpha_0 < N'(x,t)$. Let (t_n) be a sequence such that $t_n \to t$, $t_n < t$. We prove that there exists $n_0 \in \mathbb{N}$ such that $N'(x,t_n) > \alpha_0$, $(\forall)n \geqslant n_0$. (As $\alpha_0 \in (0, N'(x,t))$ is arbitrary, we will obtain that $\lim_{n\to\infty} N'(x,t_n) = N'(x,t)$.) If $0 < \alpha_0 < N'(x,t)$, then there exists $\beta_0 \in \{\alpha \in (0,1) : q_\alpha(x) < t\}$ such that $\beta_0 > \alpha_0$. (Contrary, for all

$$\beta \in \left\{\alpha \in (0,1) : q_\alpha(x) < t\right\},$$

we have $\beta \leqslant \alpha_0$. Then $\sup\{\alpha \in (0,1) : q_\alpha(x) < t\} \leqslant \alpha_0$, i.e. $N'(x,t) \leqslant \alpha_0$, which is a contradiction.) As $q_{\beta_0}(x) < t$ and $t_n \to t$, $t_n < t$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, we have $t_n > q_{\beta_0}(x)$. Thus

$$N'(x,t_n) = \sup\left\{\alpha \in (0,1) : q_\alpha(x) < t_n\right\} \geqslant \beta_0 > \alpha_0, \quad (\forall)n \geqslant n_0.$$

□

Theorem 9. *Let* $(X, N, \wedge)$ *be a FNLS and*

$$p_\alpha(x) := \inf\left\{t > 0 : N(x,t) > \alpha\right\}, \quad \alpha \in (0,1).$$

Let $N' : X \times [0,\infty) \to [0,1]$*, defined by*

$$N'(x,t) = \begin{cases} \sup\{\alpha \in (0,1) : p_\alpha(x) < t\} & \text{if } t > 0, \\ 0 & \text{if } t = 0 \text{ or } \{\alpha \in (0,1) : p_\alpha(x) < t\} = \emptyset. \end{cases}$$

Then

1. $\mathcal{P} = \{p_\alpha\}_{\alpha\in(0,1)}$ *is a right continuous and ascending family of semi-norms on* X;
2. $(X, N', \wedge)$ *is a FNLS*;
3. $N' = N$.

Proof. 1. It results from Theorems 5 and 7.

2. It results from Theorem 8.

3. For $t = 0$, we have $N'(x, t) = 0 = N(x, t)$. For $t > 0$, we have

$$N'(x, t) = \sup\left\{\alpha \in (0, 1) : p_\alpha(x) < t\right\} = \sup\left\{\alpha \in (0, 1) : N(x, t) > \alpha\right\} \leqslant N(x, t).$$

We suppose that $N'(x, t) < N(x, t)$. Then there exists $\alpha_0 \in (0, 1)$ such that $N'(x, t) < \alpha_0 < N(x, t)$. But $\alpha_0 < N(x, t)$ implies that $p_{\alpha_0}(x) < t$. Thus

$$\sup\left\{\alpha \in (0, 1) : p_\alpha(x) < t\right\} \geqslant \alpha_0,$$

i.e. $N'(x, t) \geqslant \alpha_0$, which is a contradiction. Hence $N'(x, t) = N(x, t)$. □

Theorem 10. *Let X be a linear space and $\{q_\alpha\}_{\alpha\in(0,1)}$ be a sufficient and ascending family of semi-norms on X. Let $N' : X \times [0, \infty) \to [0, 1]$, defined by*

$$N'(x, t) = \begin{cases} \sup\{\alpha \in (0, 1) : q_\alpha(x) < t\} & \text{if } t > 0, \\ 0 & \text{if } t = 0 \text{ or } \{\alpha \in (0, 1) : q_\alpha(x) < t\} = \emptyset. \end{cases}$$

Let $p_\alpha : X \to [0, \infty)$ defined by

$$p_\alpha(x) := \inf\left\{t > 0 : N'(x, t) > \alpha\right\}, \quad \alpha \in (0, 1).$$

Then

1. $(X, N', \wedge)$ *is a FNLS*;
2. $\mathcal{P} = \{p_\alpha\}_{\alpha\in(0,1)}$ *is a right continuous and ascending family of semi-norms on X*;
3. $p_\alpha = q_\alpha$, $(\forall)\alpha \in (0, 1)$ *if and only if $\{q_\alpha\}_{\alpha\in(0,1)}$ is right continuous.*

Proof. 1. It results from Theorem 8.

2. It results from Theorem 5 and Theorem 7.

3. "⇒" Is obvious.

"⇐" We suppose that there exists $\alpha_0 \in (0, 1)$ such that $p_{\alpha_0} \neq q_{\alpha_0}$. Then there exists $x \in X$ such that $p_{\alpha_0}(x) < q_{\alpha_0}(x)$ or $p_{\alpha_0}(x) > q_{\alpha_0}(x)$.

Case A. $p_{\alpha_0}(x) < q_{\alpha_0}(x)$. Let $s > 0$ such that $p_{\alpha_0}(x) < s < q_{\alpha_0}(x)$. As $p_{\alpha_0}(x) < s$, we have $N'(x, s) > \alpha_0$. We suppose that $\alpha_0 < \sup\{\alpha \in (0, 1) : q_\alpha(x)<s\}$. Then there exists $\beta \in \{\alpha \in (0, 1) : q_\alpha(x) < s\} : \alpha_0 < \beta$. (Contrary, $\alpha_0 \geqslant \beta$, for all $\beta \in \{\alpha \in (0, 1) : q_\alpha(x)< s\}$. Thus $\alpha_0 \geqslant \sup\{\alpha \in (0, 1) : q_\alpha(x) < s\}$, which contradicts our assumption.) As $\beta \in \{\alpha \in (0, 1) : q_\alpha(x) < s\} : \alpha_0 < \beta$, we have $q_{\alpha_0}(x) \leqslant q_\beta(x) < s$, which contradicts the fact that $q_{\alpha_0}(x) > s$. Thus $\alpha_0 \geqslant \sup\{\alpha \in (0, 1) : q_\alpha(x) < s\}$, namely $\alpha_0 \geqslant N'(x, t)$, which is a contradiction.

Case B. $q_{\alpha_0}(x) < p_{\alpha_0}(x)$. Let $\beta \in (\alpha_0, 1)$. We will show that $p_{\alpha_0}(x) \leqslant q_\beta(x)$. We suppose that $p_{\alpha_0}(x) > q_\beta(x)$. Let $s > 0 : q_\beta(x) < s < p_{\alpha_0}(x)$. As $q_\beta(x) < s$, we have

$N'(x,s) \geqslant \beta > \alpha_0$. Thus $p_{\alpha_0}(x) < s$, which is a contradiction. Hence $p_{\alpha_0}(x) \leqslant q_\beta(x)$, $(\forall)\beta \in (\alpha_0, 1)$. Thus $p_{\alpha_0}(x) \leqslant \lim_{\beta\to\alpha_0,\beta>\alpha_0} q_\beta(x)$. Therefore $p_{\alpha_0}(x) \leqslant q_{\alpha_0}(x)$, which is a contradiction. □

4. Katsaras Fuzzy Norm

Let X be vector space over $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$).

Definition 8. (See Katsaras and Liu, 1977.) Let $\mu_1, \mu_2, \dots, \mu_n$ be fuzzy sets in X. The sum of fuzzy sets $\mu_1, \mu_2, \dots, \mu_n$ is denoted by $\mu_1 + \mu_2 + \dots + \mu_n$ and it is defined by:

$$(\mu_1 + \mu_2 + \dots + \mu_n)(x) = \sup_{x_1+x_2+\dots+x_n=x} \big[\mu_1(x_1) \wedge \mu_2(x_2) \wedge \dots \wedge \mu_n(x_n)\big].$$

If $\lambda \in \mathbb{K}$ and μ is a fuzzy set in X, the fuzzy set $\lambda\mu$ is defined by:

$$(\lambda\mu)(x) = \begin{cases} \mu\left(\frac{x}{\lambda}\right) & \text{if } \lambda \neq 0 \\ 0 & \text{if } \lambda = 0,\ x \neq 0, \\ \vee\{\mu(y) : y \in X\} & \text{if } \lambda = 0,\ x = 0. \end{cases}$$

Definition 9. (See Katsaras and Liu, 1977.) A fuzzy set ρ in X is said to be:

1. convex if $t\rho + (1-t)\rho \subseteq \rho$, $(\forall)t \in [0,1]$;
2. balanced if $\lambda\rho \subseteq \rho$, $(\forall)\lambda \in \mathbb{K}$, $|\lambda| \leqslant 1$;
3. absorbing if $\bigvee_{t>0} t\rho = 1$;
4. absolutely convex if it is both convex and balanced.

Proposition 3. *(See* Katsaras and Liu, 1977.*) Let ρ be a fuzzy set in X. Then*:

1. *ρ is convex if and only if*

$$\rho\big(tx + (1-t)y\big) \geqslant \rho(x) \wedge \rho(y), \quad (\forall)x, y \in X,\ (\forall)t \in [0,1];$$

2. *ρ is balanced if and only if $\rho(\lambda x) \geqslant \rho(x)$, $(\forall)x \in X$, $(\forall)\lambda \in \mathbb{K}$, $|\lambda| \leqslant 1$.*

Definition 10. (See Katsaras, 1984.) A fuzzy semi-norm on X is a fuzzy set ρ in X which is absolutely convex and absorbing.

Proposition 4. *(See* Krishna and Sarma, 1991.*) Let ρ be a fuzzy semi-norm on X. Let $p_\alpha(x) := \inf\{t > 0 : \rho(\frac{x}{t}) > \alpha\}$, $\alpha \in (0,1)$. Then $\mathcal{P} = \{p_\alpha\}_{\alpha\in(0,1)}$ is an ascending family of semi-norms on X.*

Definition 11. A fuzzy semi-norm ρ on X will be called Katsaras fuzzy norm if $\rho(\frac{x}{t}) = 1$, $(\forall)t > 0$ implies $x = 0$.

REMARK 10. (a) It is easy to see that

$$\left[\rho\left(\frac{x}{t}\right)=1,\ (\forall)t>0\ \Rightarrow\ x=0\right]\quad\Leftrightarrow\quad\left[\inf_{t>0}\rho\left(\frac{x}{t}\right)<1,\ \text{for}\ x\neq 0\right].$$

(b) Our condition $[\rho(\frac{x}{t})=1,\ (\forall)t>0\ \Rightarrow\ x=0]$ is much weaker than that imposed by Katsaras (1984),

$$\left[\inf_{t>0}\rho\left(\frac{x}{t}\right)=0,\ \text{for}\ x\neq 0\right].$$

Proposition 5. *Let ρ be a fuzzy semi-norm and*

$$p_\alpha(x):=\inf\left\{t>0:\rho\left(\frac{x}{t}\right)>\alpha\right\},\quad \alpha\in(0,1).$$

Then the family of semi-norms $\mathcal{P}=\{p_\alpha\}_{\alpha\in(0,1)}$ is sufficient if and only if ρ is a Katsaras fuzzy norm.

Proof. "$\Rightarrow$" We must prove that ρ is a Katsaras fuzzy norm. We suppose that there exists $x\neq 0$, such that $\rho(\frac{x}{t})=1$, $(\forall)t>0$. Thus

$$p_\alpha(x)=\inf\left\{t>0:\rho\left(\frac{x}{t}\right)>\alpha\right\}=0,\quad (\forall)\alpha\in(0,1),$$

which contradicts the fact that the family of semi-norms $\mathcal{P}=\{p_\alpha\}_{\alpha\in(0,1)}$ is sufficient.

"$\Leftarrow$" Let $x\in X$, $x\neq 0$. We will prove that there exists $p_\alpha\in\mathcal{P}$ such that $p_\alpha(x)\neq 0$. We suppose that $p_\alpha(x)=0$, $(\forall)\alpha\in(0,1)$. Thus

$$\inf\left\{t>0:\rho\left(\frac{x}{t}\right)>\alpha\}=0,\quad (\forall)\alpha\in(0,1).$$

Hence $\rho(\frac{x}{t})>\alpha$, $(\forall)\alpha\in(0,1)$, $(\forall)t>0$. (Contrary, $(\exists)\alpha_0\in(0,1)$, $(\exists)t_0>0$ such that $\rho(\frac{x}{t_0})\leqslant\alpha_0$. Then, for $0<t\leqslant t_0$, as ρ is absorbing, we obtain that $\rho(\frac{x}{t})\leqslant\rho(\frac{x}{t_0})$. Therefore, for all $t\in(0,t_0]$, we have $\rho(\frac{x}{t})\leqslant\alpha_0$. Thus

$$\inf\left\{t>0:\rho\left(\frac{x}{t}\right)>\alpha_0\right\}\geqslant t_0>0,$$

which is a contradiction.) As $\rho(\frac{x}{t})>\alpha$, $(\forall)\alpha\in(0,1)$, $(\forall)t>0$, we obtain that $\rho(\frac{x}{t})=1$, $(\forall)t>0$. As ρ is a Katsaras fuzzy norm, we have $x=0$, which is a contradiction. □

Theorem 11. *Let ρ be a Katsaras fuzzy norm and*

$$p_\alpha(x):=\inf\left\{t>0:\rho\left(\frac{x}{t}\right)>\alpha\right\},\quad \alpha\in(0,1).$$

Let $N' : X \times [0, \infty) \to [0, 1]$, *defined by*

$$N'(x, t) = \begin{cases} \sup\{\alpha \in (0, 1) : p_\alpha(x) < t\} & \text{if } t > 0, \\ 0 & \text{if } t = 0 \text{ or } \{\alpha \in (0, 1) : p_\alpha(x) < t\} = \emptyset. \end{cases}$$

Then $(X, N', \wedge)$ *is a FNLS.*

Proof. Propositions 4, 5 and Theorem 8 imply the desired result. □

Proposition 6. *(See* Bag and Samanta, 2008.*) Let* $(X, N, \wedge)$ *be a FNLS and* $\rho : X \to [0, 1]$, $\rho(x) = N(x, 1)$. *Then* ρ *is a Katsaras fuzzy norm.*

Theorem 12. *Let* $(X, N, \wedge)$ *be a FNLS and* $\rho : X \to [0, 1]$, $\rho(x) = N(x, 1)$. *Let*

$$q_\alpha(x) := \inf\left\{t > 0 : \rho\left(\frac{x}{t}\right) > \alpha\right\}, \quad \alpha \in (0, 1).$$

Let $N' : X \times [0, \infty) \to [0, 1]$, *defined by*

$$N'(x, t) = \begin{cases} \sup\{\alpha \in (0, 1) : q_\alpha(x) < t\} & \text{if } t > 0, \\ 0 & \text{if } t = 0 \text{ or } \{\alpha \in (0, 1) : q_\alpha(x) < t\} = \emptyset. \end{cases}$$

Then:

1. ρ *is a Katsaras fuzzy norm*;
2. $\mathcal{P} = \{q_\alpha\}_{\alpha \in (0,1)}$ *is a sufficient and ascending family of semi-norms on* X;
3. $(X, N', \wedge)$ *is a FNLS*;
4. $N' = N$.

Proof. (1) It follows from Proposition 6.

(2) It follows from Propositions 4 and 5.

(3) It follows from Theorem 11.

(4) As $q_\alpha(x) := \inf\{t > 0 : N(\frac{x}{t}, 1) > \alpha\} = \inf\{t > 0 : N(x, t) > \alpha\}$, Theorem 9 implies the desired result. □

5. Atomic Decompositions

In this section we present some results which will be developed in a future paper, after we have made a systematic study of bounded linear operators between fuzzy normed linear spaces and the notion of fuzzy dual space has been introduced.

Let $(X, N, \wedge)$ be a FNLS. Let X' be the topological dual of X and $\sigma(X', X)$ be the weak*-topology on X'. If $(Y, N', \wedge)$ is a FNLS and $T : X \to Y$ is a continuous linear operator, its adjoint is denoted by $T' : Y' \to X'$ and it is defined by $T'(g)(x) = g(T(x))$, $(\forall)x \in X$, $(\forall)g \in Y'$.

DEFINITION 12. A FNLS $(X, N, \wedge)$ has an atomic decomposition if there exists $\{f_i\}_{i\in\mathbb{N}} \subset X'$ and $\{x_i\}_{i\in\mathbb{N}} \subset X$ such that

$$x = \sum_{i=1}^{\infty} f_i(x)x_i, \quad (\forall)x \in X.$$

The pair $(\{f_i\}_{i\in\mathbb{N}}, \{x_i\}_{i\in\mathbb{N}})$ will be called atomic decomposition of $(X, N, \wedge)$.

Proposition 7. *Let $(X, N, \wedge)$ be a FNLS and $P : X \to X$ be a continuous linear projection. If $(\{f_i\}_{i\in\mathbb{N}}, \{x_i\}_{i\in\mathbb{N}})$ is an atomic decomposition of X then $(\{P'(f_i)\}_{i\in\mathbb{N}}, \{P(x_i)\}_{i\in\mathbb{N}})$ is an atomic decomposition of $P(X)$.*

In particular, if X is isomorphic to a complemented subspace of a FNLS with an atomic decomposition, then X has an atomic decomposition.

Proof. As $P'(f_i)(y) = f_i(P(y)) = f_i(y)$, $(\forall)y \in P(X)$, $(\forall)i \in \mathbb{N}$, we obtain that

$$y = P(y) = P\left(\sum_{i=1}^{\infty} f_i(y)x_I\right) = \sum_{i=1}^{\infty} f_i(y)P(x_i) = \sum_{i=1}^{\infty} P'(f_i)(y)P(x_i).$$

Thus $(\{P'(f_i)\}_{i\in\mathbb{N}}, \{P(x_i)\}_{i\in\mathbb{N}})$ is an atomic decomposition of $P(X)$. □

Theorem 13. *Let $(\{f_i\}_{i\in\mathbb{N}}, \{x_i\}_{i\in\mathbb{N}})$ be an atomic decomposition of the complete FNLS $(X, N, \wedge)$. Then there exists $(X_d, N', \wedge)$ an associated complete FNLS of scalar-valued sequences indexed by $\mathbb{N}$ such that*:

1. $\{f_i(x)\}_{i\in\mathbb{N}} \in X_d$, $(\forall)x \in X$;
2. *there exist constants $A, B > 0$ such that*

$$N\left(x, \frac{s}{A}\right) \leqslant N'\big(\{f_i(x)\}_{i\in\mathbb{N}}, s\big) \leqslant N(x, sB), \quad (\forall)x \in X,\ (\forall)s > 0;$$

3. *X is isomorphic to a complemented subspace of X_d.*

Proof. Let

$$X_d = \left\{\{\lambda_i\}_{i\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}} : \sum_{i=1}^{\infty} \lambda_i x_i \text{ is convergent in } X\right\}.$$

As $\sum_{i=1}^{\infty} f_i(x)x_i$ is convergent to x, we obtain that $\{f_i(x)\}_{i\in\mathbb{N}} \in X_d$, $(\forall)x \in X$.

X_d is a complete FNLS with the fuzzy norm

$$N'\big(\{\lambda_i\}_{i\in\mathbb{N}}, s\big) = \sup_{1\leqslant n<\infty} N\left(\sum_{i=1}^{n} \lambda_i x_i, s\right).$$

Let $T : X \to X_d$, $T(x) = \{f_i(x)\}_{i\in\mathbb{N}}$. Using Uniform boundedness principle we obtain that T is bounded, namely $(\exists)A > 0$ such that

$$N'(T(x), s) \geqslant N\left(x, \frac{s}{A}\right), \quad (\forall)x \in X, \ (\forall)s > 0.$$

Thus

$$N'\big(\{f_i(x)\}_{i\in\mathbb{N}}, s\big) \geqslant N\left(x, \frac{s}{A}\right), \quad (\forall)x \in X, \ (\forall)s > 0.$$

Let $S : X_d \to X$, $S(\{\lambda_i\}_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty} \lambda_i x_i$. Then S is linear and continuous. Thus $(\exists)B > 0 : N(\sum_{i=1}^{\infty} \lambda_i x_i, s) \geqslant N'(\{\lambda_i\}_{i\in\mathbb{N}}, \frac{s}{B})$, $(\forall)\{\lambda_i\}_{i\in\mathbb{N}} \in X_d$, $(\forall)s > 0$. In particular, for $\lambda_i = f_i(x)$, we have

$$N'\left(\{f_i(x)\}_{i\in\mathbb{N}}, \frac{s}{B}\right) \leqslant N(x, s), \quad (\forall)x \in X, \ (\forall)s > 0.$$

Thus

$$N'\big(\{f_i(x)\}_{i\in\mathbb{N}}, s\big) \leqslant N(x, sB), \quad (\forall)x \in X, \ (\forall)s > 0.$$

We remark that T is an isomorphism from X into its range $R(T) \subset X_d$ and $T \circ S$ is a projection of X_d onto $R(T)$. Then X is isomorphic to a complemented subspace of X_d. □

6. Conclusions and Future Works

In this paper we have introduced the concept of atomic decomposition of fuzzy normed linear spaces. We have build a fertile ground to study, in further papers, the fuzzy wavelet theory. Also, the atomic decomposition will be used in applications to signal processing and sampling theory.

Certainly, we will make a systematic study of bounded linear operators in fuzzy normed linear spaces. We intend to obtain versions of theorems: the Open mapping theorem, the Closed graph theorem and the Uniform boundedness principle.

The results obtained in this paper leave to be foreseen that there are solutions to the problems afore mentioned. The development of fuzzy operator theory in this new context can be proven to be a powerful tool for fuzzy wavelet theory.

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Fuzzy normuotų tiesinių erdvių dekompozicijos vilnelių taikymams

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Vilnelių analizė yra galingas įrankis šiuolaikiniuose taikymuose, tokiuose kaip vaizdų ir signalų apdorojimas, duomenų suspaudimas, duomenų tyryba, kalbos atpažinimas, kompiuterinė grafika. Šio straipsnio tikslas yra įvesti fuzzy normuotų tiesinių erdvių atominės dekompozicijos sampratą, kuri vaidina esminį vaidmenį kuriant fuzzy bangelių teoriją. Straipsnyje pateiktas fuzzy normuotų tiesinių erdvių apibrėžimas, suformuluotos teoremos dekompozicijai fuzzy normų į bendresnio pavidalo semi-normų šeimą. Straipsnyje gauti rezultatai yra pagrindas vystant bendresnio pobūdžio fuzzy operatorių teoriją ir fuzzy bangelių teoriją.