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# An Approach to Interval-Valued Hesitant Fuzzy Multi-Attribute Decision Making with Incomplete Weight Information Based on Hybrid Shapley Operators

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**Abstract.** With respect to interval-valued hesitant fuzzy multi-attribute decision making, this study first presents a new ranking method for interval-valued hesitant fuzzy elements. In order to obtain the comprehensive values of alternatives, two induced generalized interval-valued hesitant fuzzy hybrid operators based on the Shapley function are defined, which globally consider the importance of elements and their ordered positions as well as reflect the interactions between them. If the weight information is incompletely known, models for the optimal weight vectors on the attribute set and on the ordered set are respectively established. Furthermore, an approach to interval-valued hesitant fuzzy multi-attribute decision making with incomplete weight information and interactive characteristics is developed. Finally, an illustrative example is provided to show the concrete application of the proposed procedure.

**Key words:** multi-attribute decision making, interval-valued hesitant fuzzy set, hybrid operator, Shapley function.

## 1. Introduction

Multi-attribute decision making is one of the most common human activities (Balezentis *et al.*, 2008; Chakraborty and Zavadskas, 2014; Hasheni *et al.*, 2014; Staujkic *et al.*, 2012, 2014; Zeng *et al.*, 2013). As Torra (2010) noted, when the experts make a decision, they are usually hesitant and irresolute for one thing or another which makes it difficult to reach a final agreement. Consequently, the difficulty of establishing the membership degree is not because we have a margin of error, or some possibility distribution on the possibility values, but because we have several possible values. Hesitant fuzzy sets (HFSs) (Torra, 2010), as an extension of Zadeh's fuzzy sets, permit the membership having a set of possible values, which can well deal with inherent hesitancy and uncertainty in the human decision-making process. In order to discuss simply, Xia and

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Xu (2011) gave the concept of hesitant fuzzy elements (HFEs). Based on the relationship between HFEs and Atanassov's intuitionistic fuzzy values (AIFVs) (Atanassov, 1986; Atanassov and Gargov, 1989), Xia and Xu (2011) defined some operations on HFEs. The aggregation operators on HFSs are studied in the literature (Wei, 2012; Xia and Xu, 2011; Xia *et al.*, 2013; Zhu *et al.*, 2012a). In order to deal with the situation where the elements in a set are correlative, some Choquet integral operators are defined (Meng *et al.*, 2013a, 2014a, 2014b; Meng and Tang, 2013; Meng and Zhang, 2014; Wei *et al.*, 2012a, 2012b; Yu *et al.*, 2011; Zhu *et al.*, 2012a). More researches about HFSs can be found in the literature (Xu and Yager, 2006; Xu and Xia, 2011a, 2011b; Zhang and Wei, 2013; Zhu *et al.*, 2012b).

However, in many real life situations where due to insufficiency in information availability, it may not be easy to identify exact values. Recently, Chen *et al.* (2013) introduced the concept of interval-valued hesitant fuzzy sets (IVHFSs), which are characterized by several possible interval values in [0, 1] rather than real numbers. Such a generalization further facilitates effectively representing inherent imprecision and uncertainty in the human decision-making process. By extending the operational laws on HFSs (Xia and Xu, 2011), Chen *et al.* (2013) defined some operational laws on IVFHSs and presented some aggregation operators. Based on Einstein operations, Wei and Zhao (2012) defined some interval-valued hesitant fuzzy Einstein aggregation operators, whilst Chen *et al.* (2012) studied the correlation coefficients of IVFHSs and applied it to clustering analysis.

The purpose of this paper is to investigate interval-valued hesitant fuzzy multi-attribute decision making. A new ranking method to interval-valued hesitant fuzzy elements (IVHFEs) is introduced, which can distinguish more situations than that given by Chen *et al.* (2013). Then, two aggregation operators called the induced generalized interval-valued hesitant fuzzy hybrid Shapley weighted averaging (IG-IVHFHSWA) operator and the induced generalized interval-valued hesitant fuzzy hybrid Shapley geometric mean (IG-IVHFHSGM) operator are defined, which can be seen as an extension of some operators based on additive measures. In order to simplify the complexity of solving a fuzzy measure, we further define the induced generalized interval-valued hesitant fuzzy hybrid $\lambda$-Shapley weighted averaging (IG-IVHFH$\lambda$SWA) operator and the induced generalized interval-valued hesitant fuzzy hybrid $\lambda$-Shapley geometric mean (IG-IVHFH$\lambda$SGM) operator. In many practical situations, because of various reasons, such as time pressure and the expert's limited expertise about the problem domain, the weight information is usually incompletely known. Based on the Shapley function, models for the optimal weight vectors on the attribute set and on the ordered set are established, respectively. Then, an approach to interval-valued hesitant fuzzy multi-attribute decision making with incomplete weight information and interactive characteristics is developed. In order to do these, the rest parts of this paper are organized as follows:

In Section 2, some basic concepts related to IVHFSs and some interval-valued hesitant fuzzy aggregation operators are briefly reviewed. In Section 3, the IG-IVHFHSWA and IG-IVHFHSGM operators are defined. Meanwhile, some important cases are examined. In order to reduce the complexity of solving a fuzzy measure, we further define the IG-IVHFH$\lambda$SWA and IG-IVHFH$\lambda$SGM operators. In Section 4, we first establish models for the optimal weight vectors on the attribute set and on the ordered set. Then, an approach to interval-valued hesitant fuzzy multi-attribute decision making is developed, which considers the interactions between attributes and their ordered positions. In Section 5, an illustrative example is provided to show the effectiveness and practicality of the developed procedure.

## 2. Some Basic Concepts

### 2.1. *Interval-Valued Hesitant Fuzzy Sets*

In order to deal with the situation where the membership degree of an element has several possible values, Torra (2010) introduced the concept of hesitant fuzzy sets. Recently, Chen *et al.* (2013) further presented the concept of interval-valued hesitant fuzzy sets. Such a generalization further facilitates effectively representing inherent imprecision and uncertainty in the human decision-making process.

Definition 1. (See Chen *et al.*, 2013.) Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set, an interval-valued hesitant fuzzy set (IVHFS) $\bar{A}$ in $X$ is in terms of a function that when applied to $X$ returns a subset of $D[0, 1]$, denoted by

$$\bar{A} = \left\{\left\langle x_i, \bar{h}_{\bar{A}}(x_i)\right\rangle \,\middle|\, x_i \in X\right\},$$

where $\bar{h}_{\bar{A}}(x_i)$ is a set of all possible interval-valued membership degrees of the element $x_i \in X$ to the set $\bar{A}$ with $D[0, 1]$ being the set of all closed subintervals in [0, 1]. For convenience, Chen *et al.* (2013) called $\bar{h} = \bar{h}_{\bar{A}}(x_i)$ an interval-valued hesitant fuzzy element (IVHFE) and $\bar{H}$ the set of all IVHFEs.

If all possible interval-valued membership degrees of each element $x_i \in X$ degenerate to real numbers, then we get a HFS given by Torra (2010).

Similar to the operations on HFEs (Xia and Xu, 2011), Chen *et al.* (2013) defined the following operational laws on IVHFEs. Let $h$, $h_1$ and $h_2$ be any three IVHFEs in $\bar{H}$, then

(1) $\bar{h}^{\kappa} = \bigcup_{\bar{r}=[r^l, r^u]\in\bar{h}}\{[r^{l\kappa}, r^{u\kappa}]\}$, $\kappa > 0$,
(2) $\kappa\bar{h} = \bigcup_{\bar{r}=[r^l, r^u]\in\bar{h}}\{[1-(1-r^l)^{\kappa}, 1-(1-r^u)^{\kappa}]\}$, $\kappa > 0$,
(3) $\bar{h}_1 \oplus \bar{h}_2 = \bigcup_{\bar{r}_1=[r_1^l, r_1^u]\in\bar{h}_1, \bar{r}_2=[r_2^l, r_2^u]\in\bar{h}_2}\{[r_1^l + r_2^l - r_1^l r_2^l, r_1^u + r_2^u - r_1^u r_2^u]\}$,
(4) $\bar{h}_1 \otimes \bar{h}_2 = \bigcup_{\bar{r}_1=[r_1^l, r_1^u]\in\bar{h}_1, \bar{r}_2=[r_2^l, r_2^u]\in\bar{h}_2}\{[r_1^l r_2^l, r_1^u r_2^u]\}$.

### 2.2. *Two Generalized Interval-Valued Hesitant Fuzzy Hybrid Operators*

As we know, there are mainly three kinds of aggregation operators: the weighted average operator (Merigó, 2012; Torra, 1997; Xu and Yager, 2006; Zhang and Liu, 2010), the ordered weighted average operator (Chiclana *et al.*, 2000; Merigó and Gil-Lafuente, 2011; Merigó and Wei, 2011; Wei *et al.*, 2012a, 2012b; Yager, 1988; Zeng *et al.*, 2012;

Zhou and Chen, 2014) and the hybrid aggregation operator (Xu, 2002; Xu and Da, 2003a, 2003b). These aggregation operators are all based on the assumption that the elements in a set are independent. The weighted average operator only considers the importance of elements, while the ordered weighted average operator only gives the importance of their ordered positions. Since the hybrid aggregation operator reflects these two aspects, many researchers dedicate aggregation to the study of the hybrid operator (Lin and Jiang, 2014; Meng *et al.*, 2013b, 2014c, 2014d; Wei *et al.*, 2012a, 2012b; Xu, 2004a, 2004b; Zhou and Chen, 2011).

Let $\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n$ be a collection of IVHFEs, let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector on $\{\bar{h}_i\}_{i=1,2,\ldots,n}$ with $w_i \in [0,1]$, $\sum_{i=1}^n w_i = 1$, and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be the associated weight vector on the ordered set $N = \{1, 2, \ldots, n\}$ with $\sum_{i=1}^n \omega_i = 1$, $\omega_i \geqslant 0$.

Similar to the hybrid aggregation operators (Xu, 2002; Xu and Da, 2003a, 2003b; Zhou and Chen, 2011), Chen *et al.* (2013) defined the following generalized interval-valued hesitant fuzzy hybrid operators:

(1) The generalized interval-valued hesitant fuzzy hybrid averaging (GIVHFHA) operator

$$
\begin{aligned}
&\mathrm{GIVHFHA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n)\\
&\quad = \left(\bigoplus_{j=1}^{n} \omega_j \bar{z}_{(j)}^{\kappa}\right)^{1/\kappa}\\
&\quad = \bigcup_{\bar{\alpha}_{(1)} \in \bar{z}_{(1)}, \bar{\alpha}_{(2)} \in \bar{z}_{(2)}, \ldots, \bar{\alpha}_{(n)} \in \bar{z}_{(n)}} \left[\left(1 - \prod_{j=1}^{n}\left(1 - \left(\alpha_{(j)}^{l}\right)^{\kappa}\right)^{\omega_j}\right)^{1/\kappa},\right.\\
&\qquad \left.\left(1 - \prod_{j=1}^{n}\left(1 - \left(\alpha_{(j)}^{u}\right)^{\kappa}\right)^{\omega_j}\right)^{1/\kappa}\right],
\end{aligned}
$$

where $\kappa > 0$, $(\cdot)$ is a permutation on the weighted IVHFEs $nw_i\bar{h}_i$ $(i = 1, 2, \ldots, n)$ with $\bar{z}_{(j)} = nw_{(j)}\bar{h}_{(j)}$ being the $j$th largest value of $nw_i\bar{h}_i$ $(i = 1, 2, \ldots, n)$, and $n$ is the balancing coefficient.

(2) The generalized interval-valued hesitant fuzzy hybrid geometric (GIVHFHG) operator

$$
\begin{aligned}
&\mathrm{GIVHFHG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n)\\
&\quad = \frac{1}{\kappa}\left(\bigotimes_{j=1}^{n}\left(\kappa \bar{z}_{(j)}\right)^{\omega_j}\right)\\
&\quad = \bigcup_{\bar{\alpha}_{(1)} \in \bar{z}_{(1)}, \bar{\alpha}_{(2)} \in \bar{z}_{(2)}, \ldots, \bar{\alpha}_{(n)}} \in \bar{z}_{(n)} \left[1 - \left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \alpha_{(j)}^{l}\right)^{\kappa}\right)^{\omega_j}\right)^{1/\kappa},\right.\\
&\qquad \left. 1 - \left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \alpha_{(j)}^{u}\right)^{\kappa}\right)^{\omega_j}\right)^{1/\kappa}\right],
\end{aligned}
$$

where $\kappa > 0$, $(\cdot)$ is a permutation on the weighted IVHFEs $\bar{h}_i^{nw_i}$ $(i = 1, 2, \ldots, n)$ with $\bar{z}_{(j)} = \bar{h}_{(j)}^{nw_{(j)}}$ being the $j$th largest value of $\bar{h}_i^{nw_i}$ $(i = 1, 2, \ldots, n)$, and $n$ is the balancing coefficient.

Remark 1. If $w_i = 1/n$ $(i = 1, 2, \ldots, n)$, then the GIVHFHA operator degenerates to the interval-valued hesitant fuzzy ordered weighted averaging (IVHFOWA) operator (Chen *et al.*, 2013), and the GIVHFHG operator degenerates to interval-valued hesitant fuzzy ordered weighted geometric (IVHFOWG) operator (Chen *et al.*, 2013). However, the GIVHFHA and GIVHFHG operators are both based on the assumption that the elements in a set are independent.

## 3. Some New Interval-Valued Hesitant Fuzzy Hybrid Aggregation Operators

In some situations, the assumption that the elements in a set are independent does not hold. We give the following example: "We are to evaluate a set of different brands of cars in relation to three subjects: {security, service, price}, we want to give more importance to security than to service or price, but on the other hand we want to give some advantage to cars that are good in security and in any of service and price". In order to deal with the situations where the elements in a set are correlative and their importance is different, the fuzzy measure introduced by Sugeno (1974) seems to well cope with this issue. First, we introduce an improvement ranking method to IVHFEs.

### 3.1. *A New Ranking Method to IVHFEs*

Some basic operations on interval numbers, let $\bar{a} = [a^l, a^u]$ and $\bar{b} = [b^l, b^u]$ be any two interval numbers with $a^l \leqslant a^u$, $b^l \leqslant b^u$ and $a^l, b^l \geqslant 0$, then

(i) $\bar{a} + \bar{b} = [a^l + b^l, a^u + b^u]$,
(ii) $\kappa\bar{a} = [\kappa a^l, \kappa a^u]$, $\kappa > 0$,
(iii) $\bar{a}^\kappa = [(a^l)^\kappa, (a^u)^\kappa]$, $\kappa > 0$.

Let $\bar{a} = [a^l, a^u]$ and $\bar{b} = [b^l, b^u]$ be any two interval numbers, their order relationship is given by the following possible degree formula (Xu and Da, 2003a, 2003b):

$$p(\bar{a} \geqslant \bar{b}) = \max\left\{1 - \max\left(\frac{b^u - a^l}{d(\bar{a}) + d(\bar{b})}0, \right), 0\right\}. \tag{1}$$

If $0 \leqslant p(\bar{a} \geqslant \bar{b}) < 0.5$, then $\bar{a} < \bar{b}$; if $p(\bar{a} \geqslant \bar{b}) = 0.5$, then $\bar{a} = \bar{b}$; if $0.5 < p\ (\bar{a} \geqslant \bar{b}) \leqslant 1$, then $\bar{a} > \bar{b}$.

Similar to the score function of HFEs (Xia and Xu, 2011), Chen *et al.* (2013) gave the following definition for the score function of IVHFEs

Definition 2. (See Chen *et al.*, 2013.) For an IVHFE $\bar{h}$, $S(\bar{h}) = \sum_{\bar{r}=[r^l, r^u]\in\bar{h}}[\frac{r^l}{\#\bar{h}}, \frac{r^u}{\#\bar{h}}]$ is called the score function of $\bar{h}$ with $\#\bar{h}$ being the number of the interval values in $\bar{h}$, and $S(\bar{h})$ is an interval value in [0, 1].

Based on above possible degree formula on interval numbers, Chen *et al.* (2013) gave the following order relationship between IVHFEs. Let $\bar{h}_1$ and $\bar{h}_2$ be any two IVHFEs, if $S(\bar{h}_1) > S(\bar{h}_2)$, then $\bar{h}_1 > \bar{h}_2$; if $S(\bar{h}_1) = S(\bar{h}_2)$, then $\bar{h}_1 = \bar{h}_2$.

In some cases, the score function fails to distinguish between two distinct IVHFEs. For example, let $\bar{h}_1 = \{[0.1, 0.8], [0.3, 0.6]\}$ and $\bar{h}_2 = \{[0.2, 0.3], [0.6, 0.7]\}$, then their scores are $[0.2, 0.7]$ and $[0.4, 0.5]$. From Eq. (1), it has $p(S(\bar{h}_1) \geqslant S(\bar{h}_2)) = p(S(\bar{h}_2) \geqslant S(\bar{h}_1)) = 0.5$, therefore $\bar{h}_1 = \bar{h}_2$. But they are obviously different. In order to increase the identification of IVHFEs, we introduce an improvement method. First, we introduce the averaging deviation function, for any IVHFE $\bar{h}$, expressed by

$$D(\bar{h}) = \frac{1}{\#\bar{h}} \sum_{\bar{r}=[r^l, r^u] \in \bar{h}} \left( \left(r^l - S(\bar{h})^l\right)^2 + \left(r^u - S(\bar{h})^u\right)^2 \right), \tag{2}$$

where $S(\bar{h}) = [S(\bar{h})^l, S(\bar{h})^u]$ is the score function of $\bar{h}$, $\#\bar{h}$ is the number of the interval values in $\bar{h}$. For any two IVHFEs $\bar{h}_1$ and $\bar{h}_2$, their order relationship is defined by

$$\text{If } S(\bar{h}_1) < S(\bar{h}_2), \quad \text{then } \bar{h}_1 < \bar{h}_2.$$
$$\text{If } S(\bar{h}_1) = S(\bar{h}_2), \quad \text{then } \begin{cases} D(\bar{h}_1) > D(\bar{h}_2), & \bar{h}_1 < \bar{h}_2, \\ D(\bar{h}_1) = D(\bar{h}_2), & \bar{h}_1 = \bar{h}_2 \end{cases}$$

where $S(\bar{h}_1) < S(\bar{h}_2)$ if and only if $(S(\bar{h}_1)^l + S(\bar{h}_1)^u)/2 < (S(\bar{h}_2)^l + S(\bar{h}_2)^u)/2$ or $(S(\bar{h}_1)^l + S(\bar{h}_1)^u)/2 = (S(\bar{h}_2)^l + S(\bar{h}_2)^u)/2$ and $(S(\bar{h}_1)^l - S(\bar{h}_1)^u)/2 > (S(\bar{h}_2)^l - S(\bar{h}_2)^u)/2$, and $S(\bar{h}_1) = S(\bar{h}_2)$ if and only if $(S(\bar{h}_1)^l + S(\bar{h}_1)^u)/2 = (S(\bar{h}_2)^l + S(\bar{h}_2)^u)/2$ and $(S(\bar{h}_1)^l - S(\bar{h}_1)^u)/2 = (S(\bar{h}_2)^l - S(\bar{h}_2)^u)/2$.

In the above example, if we adopt the improvement method to rank $\bar{h}_1 = \{[0.1, 0.8], [0.3, 0.6]\}$ and $\bar{h}_2 = \{[0.2, 0.3], [0.6, 0.7]\}$, then $\bar{h}_1 < \bar{h}_2$ for $S(\bar{h}_1) < S(\bar{h}_2)$. Furthermore, if $\bar{h}_1 = \{[0.4, 0.5]\}$ and $\bar{h}_2 = \{[0.2, 0.3], [0.6, 0.7]\}$, then $\bar{h}_1 > \bar{h}_2$ for $S(\bar{h}_1) = S(\bar{h}_2)$ and $D(\bar{h}_1) < D(\bar{h}_2)$.

### 3.2. *The IG-IVHFHSWA and IG-IVHFHSGM Operators*

In a similar way to Meng *et al.* (2013b, 2014c, 2014d), the section defines two interval-valued hesitant fuzzy aggregation operators using the Shapley value with respect to fuzzy measures, which consider the importance of elements and their ordered positions as well as reflect the interactions between them. If there are no interactions, then they respectively degenerate to the hybrid weighted aggregation operators based on additive measures.

Definition 3. (See Sugeno, 1974.) A fuzzy measure on finite set $N = \{1, 2, \ldots, n\}$ is a set function $\mu : P(N) \to [0, 1]$ satisfying

(1) $\mu(\emptyset) = 0$, $\mu(N) = 1$,
(2) If $A, B \in P(N)$ and $A \subseteq B$, then $\mu(A) \leqslant \mu(B)$,

where $P(N)$ is the power set of $N$.

In game theory, the Shapley function (Shapley, 1953) provides us a reasonable payoff index, which satisfies some important properties, such as efficiency, symmetry, and additivity, expressed by

$$\varphi_i(\mu, N) = \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!} \big(\mu(S \cup i) - \mu(S)\big) \quad \forall i \in N, \tag{3}$$

where $\mu$ is a fuzzy measure on finite set $N$, $s$ and $n$ denote the cardinalities of $S$ and $N$, respectively.

From the definition of fuzzy measures, it is not difficult to know that $\varphi_i(\mu, N) \geqslant 0$ for any element $i \in N$, and $\sum_{i=1}^{n} \varphi_i(\mu, N) = 1$ by efficiency, which means that $\{\varphi_i(\mu, N)\}_{i \in N}$ is a weight vector. Furthermore, it is an expect value of the marginal contributions between the element $i$ and any subset in $N \setminus i$. When there are correlations between elements in a set, we define the following operators.

Definition 4. An IG-IVHFHSWA operator of dimension $n$ is a mapping IG-IVHFHSWA: $\bar{H}^n \to \bar{H}$ defined on the set of second arguments of two tuples $\langle u_1, \bar{h}_1\rangle, \langle u_2, \bar{h}_2\rangle, \ldots, \langle u_n, \bar{h}_n\rangle$ with a set of order-inducing variables $u_i$ $(i = 1, 2, \ldots, n)$ and a parameter $\kappa$ such that $\kappa \in (0, +\infty)$, denoted by

$$\text{IG-IVHFHSWA}_{\mu,v}\big(\langle u_1, \bar{h}_1\rangle, \langle u_2, \bar{h}_2\rangle, \ldots, \langle u_n, \bar{h}_n\rangle\big) = \left(\bigoplus_{j=1}^{n} \varphi_j(\mu, N)\bar{z}_{(j)}^{\kappa}\right)^{1/\kappa},$$

where $(\cdot)$ is a permutation on $u_i$ $(i = 1, 2, \ldots, n)$ with $u_{(j)}$ being the $j$th largest value of $u_i$ $(i = 1, 2, \ldots, n)$, $\bar{z}_i = n\varphi_{\bar{h}_i}(v, \bar{Q})\bar{h}_i$ with $\varphi_{\bar{h}_i}(v, \bar{Q})$ being the Shapley value with respect to the fuzzy measure $v$ on $\bar{Q} = \{\bar{h}_j\}_{j=1,\ldots,n}$ for $\bar{h}_j$ $(j = 1, 2, \ldots, n)$, $\varphi_j(\mu, N)$ is the Shapley value with respect to the fuzzy measure $\mu$ on the ordered set $N = \{1, 2, \ldots, n\}$ for the $j$th position, and $n$ is the balancing coefficient.

**Theorem 1.** *Let $\langle u_1, \bar{h}_1\rangle, \langle u_2, \bar{h}_2\rangle, \ldots, \langle u_n, \bar{h}_n\rangle$ be a set of two tuples with a set of order-inducing variables $u_i$ $(i = 1, 2, \ldots, n)$ and $barh_i (i = 1, 2, \ldots, n)$ being a collection of IVHFEs in $\bar{H}$, let $\mu$ be a fuzzy measure on the order set $N = \{1, 2, \ldots, n\}$, and let $v$ be the fuzzy measure on $\bar{Q} = \{\bar{h}_j\}_{j=1,\ldots,n}$. Then their aggregated value using the IG-IVHFHSWA operator is also an IVHFE, denoted by*

$$\begin{aligned}&\text{IG-IVHFHSWA}_{\mu,v}\big(\langle u_1, \bar{h}_1\rangle, \langle u_2, \bar{h}_2\rangle, \ldots, \langle u_n, \bar{h}_n\rangle\big)\\ &\quad= \bigcup_{\bar{r}_{(1)} \in \bar{h}_{(1)}, \bar{r}_{(2)} \in \bar{h}_{(2)}, \ldots, \bar{r}_{(n)} \in \bar{h}_{(n)}} \left[\left(1 - \prod_{j=1}^{n} \Big(1 - \Big(1 - \big(1 - r_{(j)}^{l}\big)^{n\varphi_{\bar{h}_{(j)}}(v,\bar{Q})}\Big)^{\kappa}\Big)^{\varphi_j(\mu,N)}\right)^{1/\kappa}\right.\\ &\qquad \left.\left(1 - \prod_{j=1}^{n} \Big(1 - \Big(1 - \big(1 - r_{(j)}^{u}\big)^{n\varphi_{\bar{h}_{(j)}}(v,\bar{Q})}\Big)^{\kappa}\Big)^{\varphi_j(\mu,N)}\right)^{1/\kappa}\right].\end{aligned}$$

From the operational laws on IVHFEs (Chen *et al.*, 2013), it is not difficult to get the conclusion.

Remark 2. If $\mu$ and $v$ are both additive, then the IG-IVHFHSWA operator degenerates to the induced generalized interval-valued hesitant fuzzy hybrid averaging (IG-IVHFHA) operator. Furthermore, if $u_i = u_j$ for all $i, j = 1, 2, \dots, n$ with $i \neq j$, then we get the GIVHFHA operator (Chen *et al.*, 2013).

In a similar way to the IG-IVHFHSWA operator, we introduce the IG-IVHFHSGM operator as follows:

Definition 5. An IG-IVHFHSGM operator of dimension $n$ is a mapping IG-IVHFHSGM: $\bar{H}^n \to \bar{H}$ defined on the set of second arguments of two tuples $\langle u_1, \bar{h}_1\rangle, \langle u_2, \bar{h}_2\rangle, \dots, \langle u_n, \bar{h}_n\rangle$ with a set of order-inducing variables $u_i$ $(i = 1, 2, \dots, n)$ and a parameter $\kappa$ such that $\kappa \in (0, +\infty)$, denoted by

$$\text{IG-IVHFHSGM}_{\mu,v}\big(\langle u_1, \bar{h}_1\rangle, \langle u_2, \bar{h}_2\rangle, \dots, \langle u_n, \bar{h}_n\rangle\big) = \frac{1}{\kappa}\left(\bigotimes_{j=1}^{n}\big(\kappa \bar{z}_{(j)}\big)^{\varphi_j(\mu,N)}\right),$$

where $(\cdot)$ is a permutation on $u_i$ $(i = 1, 2, \dots, n)$ with $u_{(j)}$ being the $j$th largest value of $u_i$ $(i = 1, 2, \dots, n)$, $\bar{z}_i = \bar{h}_i^{n\varphi_{\bar{h}_i}(v,\bar{Q})}$ with $\varphi_{\bar{h}_i}(v, \bar{Q})$ being the Shapley value with respect to the fuzzy measure $v$ on $\bar{Q} = \{\bar{h}_j\}_{j=1,\dots,n}$ for $\bar{h}_j$ $(j = 1, 2, \dots, n)$, $\varphi_j(\mu, N)$ is the Shapley value with respect to the fuzzy measure $\mu$ on the ordered set $N = \{1, 2, \dots, n\}$ for the $j$th position, and $n$ is the balancing coefficient.

**Theorem 2.** *Let $\langle u_1, \bar{h}_1\rangle, \langle u_2, \bar{h}_2\rangle, \dots, \langle u_n, \bar{h}_n\rangle$ be a set of two tuples with a set of order-inducing variables $u_i$ $(i = 1, 2, \dots, n)$ and $\bar{h}_i$ $(i = 1, 2, \dots, n)$ being a collection of IVHFEs in $\bar{H}$, let $\mu$ be a fuzzy measure on the order set $N = \{1, 2, \dots, n\}$, and let $v$ be the fuzzy measure on $\bar{Q} = \{\bar{h}_j\}_{j=1,\dots,n}$. Then their aggregated value using the IG-IVHFHSG operator is also an IVHFE, denoted by*

$$\begin{aligned}
&\text{IG-IVHFHSGM}_{\mu,v}\big(\langle u_1, \bar{h}_1\rangle, \langle u_2, \bar{h}_2\rangle, \dots, \langle u_n, \bar{h}_n\rangle\big)\\
&\quad = \bigcup_{\bar{r}_{(1)}\in\bar{h}_{(1)}, \bar{r}_{(2)}\in\bar{h}_{(2)}, \dots, \bar{r}_{(n)}\in\bar{h}_{(n)}}\left[1 - \left(1 - \prod_{j=1}^{n}\Big(1 - \Big(1 - \big(r_{(j)}^{l}\big)^{n\varphi_{\bar{h}_{(j)}}(v,\bar{Q})}\Big)^{\kappa}\Big)^{\varphi_j(\mu,N)}\right)^{1/\kappa},\right.\\
&\qquad \left. 1 - \left(1 - \prod_{j=1}^{n}\Big(1 - \Big(1 - \big(r_{(j)}^{u}\big)^{n\varphi_{\bar{h}_{(j)}}(v,\bar{Q})}\Big)^{\kappa}\Big)^{\varphi_j(\mu,N)}\right)^{1/\kappa}\right].
\end{aligned}$$

Remark 3. If $\mu$ and $v$ are both additive, then the IG-IVHFHSGM operator degenerates to the induced generalized interval-valued hesitant fuzzy hybrid geometric (IG-IVHFHG) operator. Furthermore, if $u_i = u_j$ for all $i, j = 1, 2, \dots, n$ with $i \neq j$, then we get the GIVHFHG operator (Chen *et al.*, 2013).

From Definitions 4 and 5, we know that the IG-IVHFHSWA and IG-IVHFHSGM operators do not only consider the importance of elements and their ordered positions but also reflect the interactions between them. However, the fuzzy measure is defined on the power set, which makes the problem exponentially complex. Thus, it is not easy to obtain a fuzzy measure on a set when it is large. In order to reflect the interactions between elements and simplify the complexity of solving a fuzzy measure, we further define two interval-valued hesitant fuzzy hybrid aggregation operators using the λ-fuzzy measures (Sugeno, 1974).

For a finite set $N$, the λ fuzzy measure $g_\lambda$ can be equivalently expressed by

$$g_\lambda(A) = \begin{cases} \frac{1}{\lambda}\left(\prod_{i\in A}[1+\lambda g_\lambda(i)] - 1\right) & \text{if } \lambda \neq 0, \\ \sum_{i\in A} g_\lambda(i) & \text{if } \lambda = 0, \end{cases} \tag{4}$$

where $\lambda > -1$, and $A, B \subseteq N$ with $A \cap B = \emptyset$ .

From $\mu(N) = 1$, we know that λ is determined by

$$\prod_{i\in N}\left[1+\lambda g_\lambda(i)\right] = 1 + \lambda. \tag{5}$$

So when each $g_\lambda(i)$ is given, we can obtain the value of λ. From Eq. (4), for the set $N$ with $n$ elements we only need $n$ values to get the fuzzy measure on $N$. Furthermore, if $\sum_{i=1}^{n} g_\lambda(i) = 1$, then $\lambda = 0$.

Next, we introduce an equivalent form of the Shapley function with respect to the λ-fuzzy measure, which will simplify the calculation of the Shapley value.

**Theorem 3.** *Let $g_\lambda : P(N) \to [0, 1]$ be a λ-fuzzy measure, and $\varphi(g_\lambda, N)$ be the Shapley function as given in Eq.* (3) *for $g_\lambda$. Then,*

$$\varphi_i(g_\lambda, N) = \sum_{i\in S\subseteq N} \frac{1}{s} \prod_{j\in S} \lambda^{s-1} g_\lambda(j), \quad \forall i \in N. \tag{6}$$

*Proof.* From Eqs. (3) and (4), it has

$$\begin{aligned}\varphi_i(g_\lambda, N) &= \sum_{S\subseteq N\setminus i} \frac{(n-s-1)!s!}{n!}\left(g_\lambda(S\cup i) - g_\lambda(S)\right) \\ &= \sum_{S\subseteq N\setminus i} \frac{(n-s-1)!s!}{n!} g_\lambda(i) \prod_{j\in S}\left(1+\lambda g_\lambda(j)\right) \quad \text{for any } i \in N.\end{aligned} \tag{7}$$

When $n = 1, 2$, by Eq. (7) one easily gets Eq. (6). Hypothesis, it has Eq. (6) with $n = k$, i.e.,

$$\varphi_i(g_\lambda, N) = \sum_{i\in S\subseteq N} \frac{1}{s}\lambda^{s-1} \prod_{j\in S} g_\lambda(j), \quad \forall i \in N, \tag{8}$$

where $s$ is the cardinality of $S$.

In the following, we prove Eq. (8), where $n = k+1$. Without loss of generality, suppose that $N = \{i, j_1, j_2, \ldots, j_k\}$. By Eq. (7), it gets

$$\begin{aligned}
\varphi_i(g_\lambda, N) &= \sum_{S \subseteq N\backslash i} \frac{(n-s-1)!s!}{n!} g_\lambda(i) \prod_{j\in S} \big(1+\lambda g_\lambda(j)\big) \\
&= \sum_{S \subseteq N\backslash i} \frac{(k-s)!s!}{(k+1)!} g_\lambda(i) \prod_{j\in S} \big(1+\lambda g_\lambda(j)\big) \\
&= \sum_{S \subseteq N\backslash \{i,j_1\}} \frac{(k-s)!s!}{(k+1)!} g_\lambda(i) \prod_{j\in S} \big(1+\lambda g_\lambda(j)\big) \\
&\quad + \sum_{S \subseteq N\backslash \{i,j_1\}} \frac{(k-s-1)!(s+1)!}{(k+1)!} g_\lambda(i) \prod_{j\in S} \big(1+\lambda g_\lambda(j)\big)\big(1+\lambda g_\lambda(j_1)\big) \\
&= \sum_{S \subseteq N\backslash \{i,j_1\}} \frac{(k-s)!s!}{(k+1)!} g_\lambda(i) \prod_{j\in S} \big(1+\lambda g_\lambda(j)\big) \\
&\quad + \sum_{S \subseteq N\backslash \{i,j_1\}} \frac{(k-s-1)!(s+1)!}{(k+1)!} \\
&\quad \times \left( g_\lambda(i) \prod_{j\in S} \big(1+\lambda g_\lambda(j)\big) + \lambda g_\lambda(i) g_\lambda(j_1) \prod_{j\in S} \big(1+\lambda g_\lambda(j)\big) \right) \\
&= \sum_{S \subseteq N\backslash \{i,j_1\}} \left( \frac{(k-s)!s!}{(k+1)!} + \frac{(k-s-1)!(s+1)!}{(k+1)!} \right) g_\lambda(i) \prod_{j\in S} \big(1+\lambda g_\lambda(j)\big) \\
&\quad + \sum_{S \subseteq N\backslash \{i,j_1\}} \frac{(k-s-1)!(s+1)!}{(k+1)!} \lambda g_\lambda(i) g_\lambda(j_1) \prod_{j\in S} \big(1+\lambda g_\lambda(j)\big) \\
&= \sum_{S \subseteq N\backslash \{i,j_1\}} \frac{(k-s-1)!s!}{k!} g_\lambda(i) \prod_{j\in S} \big(1+\lambda g_\lambda(j)\big) \\
&\quad + \sum_{S \subseteq N\backslash \{i,j_1\}} \frac{(k-s-1)!(s+1)!}{(k+1)!} \lambda g_\lambda(i) g_\lambda(j_1) \prod_{j\in S} \big(1+\lambda g_\lambda(j)\big).
\end{aligned}$$

Let

$$\varphi_i'(g_\lambda, N) = \sum_{S \subseteq N\backslash \{i,j_1\}} \frac{(k-s-1)!s!}{k!} g_\lambda(i) \prod_{j\in S} \big(1+\lambda g_\lambda(j)\big) \tag{9}$$

and

$$\varphi_i''(g_\lambda, N) = \sum_{S \subseteq N\backslash \{i,j_1\}} \frac{(k-s-1)!(s+1)!}{(k+1)!} \lambda g_\lambda(i) g_\lambda(j_1) \prod_{j\in S} \big(1+\lambda g_\lambda(j)\big). \tag{10}$$

For Eq. (9): from assumption, it has

$$\varphi_i'(g_\lambda, N) = \sum_{i\in S\subseteq N\backslash\{j_1\}} \frac{1}{s}\lambda^{s-1}\prod_{j\in S} g_\lambda(j). \tag{11}$$

For Eq. (10): let $q = \{i, j_1\}$, define $g_\lambda'(q) = g_\lambda(i)g_\lambda(j_1)$ and $g_\lambda'(A) = g_\lambda(A)$ for any $A \subseteq N\backslash\{i, j_1\}$. Then, $g_\lambda'$ is a λ-fuzzy measure defined on $N \setminus \{i, j_1\} \cup q$ with $k$ elements. Thus,

$$\begin{aligned}
\varphi_q''(g_\lambda', N) &= \sum_{S\subseteq N\backslash q} \frac{(k-s-1)!s!}{k!}\lambda g_\lambda'(q)\prod_{j\in S}\left(1+\lambda g_\lambda'(j)\right) \\
&= \lambda \sum_{S\subseteq N\backslash q} \frac{(k-s-1)!s!}{k!} g_\lambda'(q)\prod_{j\in S}\left(1+\lambda g_\lambda'(j)\right) \\
&= \lambda \sum_{q\in S\subseteq\{N\backslash\{i,j_1\}\cup q\}} \frac{1}{s}\lambda^{s-1}\prod_{j\in S} g_\lambda'(j) \\
&= \lambda \sum_{\{i,j_1\}\in S\subseteq\{N\backslash\{i,j_1\}\cup q\}} \frac{1}{s}\lambda^{s-1} \prod_{j\in S\backslash\{i,j_1\} g_\lambda'(j)} g_\lambda(i)g_\lambda(j_1).
\end{aligned} \tag{12}$$

By Eq. (12), it has

$$\begin{aligned}
\varphi_i''(g_\lambda, N) &= \lambda \sum_{i\in S\subseteq N} \frac{1}{s+1}\lambda^{s-1}\prod_{j\in S} g_\lambda(j)g_\lambda(j_1) \\
&= \sum_{i\in S\subseteq N\backslash j_1} \frac{1}{s+1}\lambda^{s}\prod_{j\in S} g_\lambda(j)g_\lambda(j_1).
\end{aligned} \tag{13}$$

From Eqs. (11), (13) and hypothesis, it has

$$\begin{aligned}
\varphi_i(g_\lambda, N) &= \sum_{i\in S\subseteq N\backslash\{j_1\}} \frac{1}{s}\lambda^{s-1}\prod_{j\in S} g_\lambda(j) + \sum_{i\in S\subseteq N\backslash j_1} \frac{1}{s+1}\lambda^{s}\prod_{j\in S} g_\lambda(j)g_\lambda(j_1) \\
&= \sum_{i\in S\subseteq N} \frac{1}{s}\lambda^{s-1}\prod_{j\in S} g_\lambda(j), \quad \text{where } n = k+1.
\end{aligned}$$

The result is obtained by induction. □

Because of the advantage of the λ-fuzzy measure, we only need $n$ values to get the Shapley values of $n$ elements.

Let $\langle u_1, \bar{h}_1\rangle, \langle u_2, \bar{h}_2\rangle, \ldots, \langle u_n, \bar{h}_n\rangle$ be a set of two tuples with a set of order-inducing variables $u_i$ $(i = 1, 2, \ldots, n)$ and $\bar{h}_i$ $(i = 1, 2, \ldots, n)$ being a collection of IVHFEs in $\bar{H}$, let $g_\lambda'$ be a λ-fuzzy measure on $\bar{Q} = \{\bar{h}_j\}_{j=1,\ldots,n}$, and let $g_\lambda$ be a λ-fuzzy measure on the ordered set $N = \{1, 2, \ldots, n\}$.

We define the following two operators in a similar way to the IG-IVHFHSWA and IG-IVHFHSGM operators.

(1) The induced generalized interval-valued hesitant fuzzy hybrid λ-Shapley weighted averaging (IG-IVHFHλSWA) operator

$$\text{IG-IVHFH}\lambda\text{SWA}_{g_\lambda, g'_\lambda}\big(\langle u_1, \bar{h}_1\rangle, \langle u_2, \bar{h}_2\rangle, \ldots, \langle u_n, \bar{h}_n\rangle\big) = \left(\bigoplus_{j=1}^{n} \varphi_j(g_\lambda, N)\bar{z}_{(j)}^{\kappa}\right)^{1/\kappa};$$

(2) The induced generalized interval-valued hesitant fuzzy hybrid λ-Shapley geometric mean (IG-IVHFHλSGM) operator

$$\text{IG-IVHFH}\lambda\text{SGM}_{g_\lambda, g'_\lambda}\big(\langle u_1, \bar{h}_1\rangle, \langle u_2, \bar{h}_2\rangle, \ldots, \langle u_n, \bar{h}_n\rangle\big) = \frac{1}{\kappa}\left(\bigotimes_{j=1}^{n} (\kappa \bar{x}_{(j)})^{\varphi_j(g_\lambda, N)}\right),$$

where $(\cdot)$ is a permutation on $u_i$ $(i = 1, 2, \ldots, n)$ with $u_{(j)}$ being the $j$th largest value of $u_i$ $(i = 1, 2, \ldots, n)$, $\bar{z}_i = n\varphi_{\bar{h}_i}(g'_\lambda, \bar{Q})\bar{h}_i$ and $\bar{x}_i = \bar{h}_i^{n\varphi_{\bar{h}_i}(g'_\lambda, \bar{Q})}$ being the Shapley value with respect to the λ-fuzzy measure $g_\lambda$ for $\bar{h}_j$ $(j = 1, 2, \ldots, n)$, $\varphi_j(g_\lambda, N)$ is the Shapley value with respect to the λ-fuzzy measure $g_\lambda$ for the $j$th position, and $n$ is the balancing coefficient.

## 4. An Approach to Interval-Valued Hesitant Fuzzy Multi-Attribute Decision Making

If the weight vectors on the attribute set and on the ordered set are exactly known, then we can use the associated aggregation operator to get the comprehensive values of alternatives. However, in many situations, because of various reasons, such as time pressure and the expert's limited expertise about the problem domain, we usually have incomplete information about the weight vectors. In order to deal with this situation, we first need to obtain their weight vectors.

Consider a decision-making problem, let $A = \{a_1, a_2, \ldots, a_m\}$ be the set of alternatives, and $C = \{c_1, c_2, \ldots, c_n\}$ be the set of attributes. The decision makers give their individual preferences for alternatives with respect to attributes in anonymity. If the decision makers provide several interval values for the alternative $a_i$ with respect to the attribute $c_j$, this value can be considered as an IVHFE $\bar{h}_{ij}$. By $\bar{H} = (\bar{h}_{ij})_{m\times n}$, we denote the interval-valued hesitant fuzzy decision matrix given by the decision makers.

### 4.1. *Models for the Optimal Weight Vectors*

Since the weight vector makes the comprehensive values of the alternatives the bigger the better, if the information about the weights of attributes is partly known, we establish the following models for the optimal weight vector on the attribute set $C$ with respect to the alternative $a_i$ $(i = 1, 2, \dots, m)$:

$$\max \sum_{j=1}^{n} \varphi_{c_j}(v, C)\big(S(\bar{h}_{ij})^l + S(\bar{h}_{ij})^u\big)/2$$
$$\text{s.t.} \begin{cases} v(C) = 1, \\ v(S) \leqslant v(T) \quad \forall S, T \subseteq C \quad \text{s.t. } S \subseteq T, \\ v(c_j) \in U_{c_j}, \quad v(c_j) \geqslant 0 \end{cases} \tag{14}$$

and

$$\min \sum_{j=1}^{n} \varphi_{c_j}(v, C) D(\bar{h}_{ij})$$
$$\text{s.t.} \begin{cases} v(C) = 1, \\ v(S) \leqslant v(T) \quad \forall S, T \subseteq C \quad \text{s.t. } S \subseteq T, \\ v(c_j) \in U_{c_j}, \quad v(c_j) \geqslant 0, \end{cases} \tag{15}$$

where $S(\bar{h}_{ij}) = [S(\bar{h}_{ij})^l, S(\bar{h}_{ij})^u]$ is the score value as given in Definition 2, $D(\bar{h}_{ij})$ is the averaging deviation value defined by Eq. (2), $\varphi_{c_j}(v, C)$ is the Shapley value of the attribute $c_j$ with $v$ being the fuzzy measure on the attribute set $C$, and $U_{c_j}$ is the known weight information, $j = 1, 2, \dots, n$.

Since models (14) and (15) have the same constraints, and all alternatives are non inferior, they can be combined to formulate the following linear programming

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi_{c_j}(v, C)\Big(\big(S(\bar{h}_{ij})^l + S(\bar{h}_{ij})^u\big)/2 - D(\bar{h}_{ij})\Big)$$
$$\text{s.t.} \begin{cases} v(C) = 1 \\ v(S) \leqslant v(T) \quad \forall S, T \subseteq C \quad \text{s.t. } S \subseteq T, \\ v(c_j) \in U_{c_j}, \qquad v(c_j) \geqslant 0, \quad j = 1, 2, \dots, n. \end{cases} \tag{16}$$

Now, let's consider the weight vector on the ordered set $N = \{1, 2, \dots, n\}$. For each $i = 1, 2, \dots, m$, reorder $(S(\bar{h}_{ij})^l + S(\bar{h}_{ij})^u)/2 - D(\bar{h}_{ij})$ $(j = 1, 2, \dots, n)$ with $(S(\bar{h}_{i(j)})^l + S(\bar{h}_{i(j)})^u)/2 - D(\bar{h}_{i(j)})$ being the $j$th largest value of $(S(\bar{h}_{ij})^l + S(\bar{h}_{ij})^u)/2 - D(\bar{h}_{ij})$ $(j = 1, 2, \dots, n)$. Since the optimal weight vector makes bigger comprehensive value for each alternative preferable, we build the following model for the optimal fuzzy measure on the ordered set $N$.

$$\max \sum_{j=1}^{n} \varphi_j(\mu, N)\big(\big(S(\bar{h}_{i(j)})^l + S(\bar{h}_{i(j)})^u\big)/2 - D(\bar{h}_{i(j)})\big)$$

$$\text{s.t.} \begin{cases} \mu(N) = 1, \\ \mu(S) \leqslant \mu(T) \quad \forall S, T \subseteq N \quad \text{s.t. } S \subseteq T, \\ \mu(j) \in U_j, \qquad \mu(j) \geqslant 0, \quad j = 1, 2, \ldots, n, \end{cases} \tag{17}$$

where $\varphi_j(\mu, N)$ is the Shapley value of the $j$th position with $\mu$ being the fuzzy measure on the ordered set $N$, and $U_j$ is the known weight information, $j = 1, 2, \ldots, n$.

Since all alternatives are non inferior, we further get the following model for the optimal fuzzy measure on the ordered set $N$.

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi_j(\mu, N)\big(\big(S(\bar{h}_{i(j)})^l + S(\bar{h}_{i(j)})^u\big)/2 - D(\bar{h}_{i(j)})\big)$$

$$\text{s.t.} \begin{cases} \mu(N) = 1, \\ \mu(S) \leqslant \mu(T) \quad \forall S, T \subseteq N \quad \text{s.t. } S \subseteq T, \\ \mu(j) \in U_j, \qquad \mu(j) \geqslant 0, \quad j = 1, 2, \ldots, n. \end{cases} \tag{18}$$

If $v$ is a λ-fuzzy measure $g'_\lambda$ on the attribute set $C$, and $\mu$ is a λ-fuzzy measure $g_\lambda$ on the ordered set $N$, then we obtain the following models for the optimal weight vectors on the attribute set $C$ and on the ordered set $N$, respectively.

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi_{c_j}(g'_\lambda, C)\big(\big(S(\bar{h}_{ij})^l + S(\bar{h}_{ij})^u\big)/2 - D(\bar{h}_{ij})\big)$$

$$\text{s.t.} \begin{cases} g'_\lambda(\emptyset) = 0, \qquad g'_\lambda(C) = 1, \\ g'_\lambda(c_j) \in U_{c_j}, \quad j = 1, 2, \ldots, n, \\ \lambda > -1 \end{cases} \tag{19}$$

and

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi_j(g_\lambda, N)\big(\big(S(\bar{h}_{i(j)})^l + S(\bar{h}_{i(j)})^u\big)/ - D(\bar{h}_{i(j)})\big)$$

$$\text{s.t.} \begin{cases} g_\lambda(\emptyset) = 0, \qquad g_\lambda(C) = 1, \\ g_\lambda(c_j) \in U_j, \quad j = 1, 2, \ldots, n, \\ \lambda > -1. \end{cases} \tag{20}$$

Furthermore, if there is no interaction between attributes and between their order positions, then models (16) and (18) respectively degenerate to be the following programming for the optimal weight vectors on the attribute set and on the ordered set.

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} w_{c_j} \big( \big( S(\bar{h}_{ij})^l + S(\bar{h}_{ij})^u \big)/2 - D(\bar{h}_{ij}) \big)$$

$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} w_{c_j} = 1, \\ w_{c_j} \in U_{c_j}, \qquad w_{c_j} \geqslant 0, \quad j = 1, 2, \ldots, n \end{cases} \tag{21}$$

and

$$\max \sum_{i=1}^{m} \sum_{j=1}^{n} \omega_j \big( \big( S(\bar{h}_{i(j)})^l + S(\bar{h}_{i(j)})^u \big)/2 - D(\bar{h}_{i(j)}) \big)$$

$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \omega_j = 1, \\ \omega_j \in U_j, \qquad \omega_j \geqslant 0, \quad j = 1, 2, \ldots, n, \end{cases} \tag{22}$$

where $w_{c_j} = v(c_j)$ and $\omega_j = \mu(j)$ for each $j = 1, 2, \ldots, n$.

Remark 4. In the building models, we apply the elements' Shapley values as their weights, which globally consider the interactions between them. If there are no interactions, then the elements' Shapley values equal to their own importance. When the weight information is completely unknown, we can also use the established models to obtain the optimal weight vectors, which only need to delete the range of the associated element.

### 4.2. *An Algorithm*

Based on the introduced aggregation operators and the established models for the optimal weight vectors, the section develops an approach to interval-valued hesitant fuzzy multi-attribute decision making, which address the situations where elements in a set are correlative and the weight information is not exactly known. The main decision procedure can be described as follows:

**Step 1:** Suppose that there exist $m$ alternatives $A = \{a_1, a_2, \ldots, a_m\}$ to be evaluated according to $n$ attributes $C = \{c_1, c_2, \ldots, c_n\}$ to form the interval-valued hesitant fuzzy decision matrix $\bar{H} = (\bar{h}_{ij})_{m \times n}$, where $\bar{h}_{ij}$ is an IVHFE for the alternative $a_i$ with respect to the attribute $c_j$.

**Step 2:** If all attributes $c_j$ $(j = 1, 2, \ldots, n)$ are benefits (i.e., the bigger the better), then the attribute values do not need transformation. Otherwise, we need to transform the interval-valued hesitant fuzzy decision matrix $\bar{H} = (\bar{h}_{ij})_{m \times n}$ into $\bar{H}' = (\bar{h}'_{ij})_{m \times n}$, where

$$\bar{h}'_{ij} = \begin{cases} \bar{h}_{ij} & \text{for benefit attribute } c_j, \\ \bar{h}^c_{ij} & \text{for cost attribute } c_j, \end{cases} \qquad (i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$$

with $\bar{h}^c_{ij} = \bigcup_{\bar{r} \in \bar{h}_{ij}} [1 - r^u, 1 - r^l]$.

Table 1
Interval-valued hesitant fuzzy decision matrix.

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|
| $a_1$ | {[0.2, 0.5], [0.4, 0.7]} | {[0.2, 0.3], [0.4, 0.6], [0.7, 0.8]} | {[0.5, 0.6], [0.7, 0.9]} | {[0.4, 0.7]} |
| $a_2$ | {[0.2, 0.4], [0.5, 0.6]} | {[0.1, 0.4], [0.5, 0.6]} | {[0.5, 0.7], [0.8, 0.9]} | {[0.6, 0.8]} |
| $a_3$ | {[0.3, 0.5]} | {[0.4, 0.6]} | {[0.3, 0.5], [0.7, 0.8]} | {[0.5, 0.7]} |
| $a_4$ | {[0.2, 0.4], [0.5, 0.6]} | {[0.5, 0.7]} | {[0.6, 0.7]} | {[0.4, 0.6], [0.8, 0.9]} |

**Step 3:** Utilize model (16) to solve the optimal fuzzy measure $\upsilon$ on the attribute set $C$, and calculate the Shapley value.

**Step 4:** Utilize model (18) to solve the optimal fuzzy measure $\mu$ on the ordered set $N$, and calculate the Shapley value.

**Step 5:** Let $u_j = (S(\bar{h}_{ij})^l + S(\bar{h}_{ij})^u)/2 - D(\bar{h}_{ij})$ $(j = 1, 2, \ldots, n)$ for each $i = 1, 2, \ldots, m$, utilize the IG-IVHFHSWA operator or the IG-IVHFHSGM operator to get the comprehensive IVHFE $\bar{h}_i$ $(i = 1, 2, \ldots, m)$ of the alternative $a_i$ $(i = 1, 2, \ldots, m)$.

**Step 6:** According to the comprehensive IVHFE $\bar{h}_i$ $(i = 1, 2, \ldots, m)$, calculate the score value $S(\bar{h}_i)$ and the average deviation value $D(\bar{h}_i)$. Then, to rank the comprehensive IVHFE $\bar{h}_i$ $(i = 1, 2, \ldots, m)$, and select the best alternative(s).

**Step 7:** End.

## 5. An Illustrative Example

The enterprise's board of directors, which includes five members, is to plan the development of large projects strategy initiatives for the following five years (adapted from Xia and Xu, 2011). Suppose there are four possible projects $a_i$ $(i = 1, 2, 3, 4)$ to be evaluated. It is necessary to compare these projects to select the most importance of them as well as order them from the point of view of their importance, taking into account four attributes suggested by the Balanced Scorecard methodology (it should be noted that all of them are of the maximization type): $c_1$: financial perspective, $c_2$: the customer satisfaction, $c_3$: internal business process perspective, and $c_4$: learning and growth perspective. In order to avoid influencing each other, the decision makers are required to provide their preferences in anonymity and the interval-valued hesitant fuzzy decision matrix $\bar{H} = (\bar{h}_{ij})_{4\times 4}$ is presented in Table 1, where $\bar{h}_{ij}$ $\{i, j = 1, 2, 3, 4\}$ are in the form of IVHFEs.

Assume that the weight vector on the attribute set is given by $U_C = \{[0.1, 0.3], [0.2, 0.4], [0.05, 0.25], [0.25, 0.45]\}$, and the weight vector on the ordered positions is defined by $U_N = \{[0.4, 0.5], [0.3, 0.4], [0.2, 0.3], [0.1, 0.2]\}$. To effectively solve this problem, the proposed decision procedure is followed for determining the most desirable alternative(s).

**Step 1:** Since all attributes are benefits, there is on need to transform the interval-valued hesitant fuzzy decision matrix $\bar{H}$, namely, $\bar{H} = \bar{H}'$.

Table 2
The fuzzy measure on the attribute set $C$.

| Combination | Fuzzy measure | Combination | Fuzzy measure | Combination | Fuzzy measure |
|---|---|---|---|---|---|
| $\{c_1\}$ | 0.1 | $\{c_1, c_3\}$ | 0.1 | $\{c_1, c_2, c_3\}$ | 0.2 |
| $\{c_2\}$ | 0.2 | $\{c_1, c_4\}$ | 1 | $\{c_1, c_2, c_4\}$ | 1 |
| $\{c_3\}$ | 0.05 | $\{c_2, c_3\}$ | 0.2 | $\{c_1, c_3, c_4\}$ | 1 |
| $\{c_4\}$ | 0.45 | $\{c_2, c_4\}$ | 1 | $\{c_2, c_3, c_4\}$ | 1 |
| $\{c_1, c_2\}$ | 0.2 | $\{c_3, c_4\}$ | 1 | $\{c_1, c_2, c_3, c_4\}$ | 1 |

**Step 2:** According to model (16), we get the following model for the optimal fuzzy measure on the attribute set $C$.

$$\begin{aligned}
&\max -0.0241\big(v(c_1) - v(c_2, c_3, c_4)\big) + 0.0082\big(v(c_2) - v(c_1, c_3, c_4)\big) \\
&\quad - 0.0324\big(v(c_3) - v(c_1, c_2, c_4)\big) + 0.0483\big(v(c_4) - v(c_1, c_2, c_3)\big) \\
&\quad - 0.0079\big(v(c_1, c_2) - v(c_3, c_4)\big) - 0.0282\big(v(c_1, c_3) - v(c_2, c_4)\big) \\
&\quad + 0.0121\big(v(c_1, c_4) - v(c_2, c_3)\big) + 2.1102
\end{aligned}$$

$$\text{s.t.} \begin{cases} v(c_1, c_2, c_3, c_4) = 1, \\ v(S) \leqslant v(T) \quad \forall S, T \subseteq \{c_1, c_2, c_3, c_4\} \quad \text{s.t. } S \subseteq T, \\ v(c_1) \in [0.1, 0.3], \qquad v(c_2) \in [0.2, 0.4], \\ v(c_3) \in [0.05, 0.25], \qquad v(c_4) \in [0.25, 0.45]. \end{cases}$$

After solving the above model, the fuzzy measure on the attribute set $C$ is obtained as shown in Table 2.

According to Table 2, we get the attribute Shapley values

$$\varphi_{c_1}(v, C) = 0.075, \qquad \varphi_{c_2}(v, C) = 0.125,$$
$$\varphi_{c_3}(v, C) = 0.058, \qquad \varphi_{c_4}(v, C) = 0.742.$$

**Step 3:** According to model (18), we get the following model for the optimal fuzzy measure on the ordered set $N$.

$$\begin{aligned}
&\max 0.1608\big(\mu(1) - \mu(2, 3, 4)\big) + 0.0858\big(\mu(2) - \mu(1, 3, 4)\big) - 0.0625\big(\mu(3) \\
&\quad - \mu(1, 2, 4)\big) - 0.1842\big(\mu(4) - \mu(1, 2, 3)\big) + 0.1233\big(\mu(1, 2) - \mu(3, 4)\big) \\
&\quad + 0.0492\big(\mu(1, 3) - \mu(2, 4)\big) - 0.0117\big(\mu(1, 4) - \mu(2, 3)\big) + 2.1105
\end{aligned}$$

$$\text{s.t.} \begin{cases} \mu(1, 2, 3, 4) = 1, \\ \mu(S) \leqslant \mu(T) \quad \forall S, T \subseteq \{1, 2, 3, 4\} \quad \text{s.t. } S \subseteq T, \\ \mu(1) \in [0.4, 0.5], \qquad \mu(2) \in [0.3, 0.4], \\ \mu(3) \in [0.2, 0.3], \qquad \mu(4) \in [0.1, 0.2]. \end{cases}$$

After solving the above model, the fuzzy measure on ordered set $N$ is obtained as shown in Table 3.

Table 3
The fuzzy measure on the attribute set $N$.

| Combination | Fuzzy measure | Combination | Fuzzy measure | Combination | Fuzzy measure |
|---|---|---|---|---|---|
| $\{1\}$ | 0.5 | $\{1,3\}$ | 0.5 | $\{1,2,3\}$ | 1 |
| $\{2\}$ | 0.3 | $\{1,4\}$ | 0.5 | $\{1,2,4\}$ | 1 |
| $\{3\}$ | 0.2 | $\{2,3\}$ | 0.3 | $\{1,3,4\}$ | 0.5 |
| $\{4\}$ | 0.1 | $\{2,4\}$ | 0.3 | $\{2,3,4\}$ | 0.3 |
| $\{1,2\}$ | 1 | $\{3,4\}$ | 0.2 | $\{1,2,3,4\}$ | 1 |

According to Table 3, we get the ordered position Shapley values

$$\varphi_1(\mu,N)=0.558,\qquad \varphi_2(\mu,N)=0.358,$$
$$\varphi_3(\mu,N)=0.058,\qquad \varphi_4(\mu,N)=0.025.$$

**Step 4:** Let $u_j=(S(\bar{h}_{ij})^l+S(\bar{h}_{ij})^u)/2-D(\bar{h}_{ij})$ $(j=1,2,3,4)$ for each $i=1,2,3,4$ and $\kappa=1$, utilize the IG-IVHFHSWA operator to calculate the comprehensive values for the alternatives, i.g., $i=1$,

$$\begin{aligned}
\bar{h}_1 &= \text{IG-IVHFHSWA}\big(\langle u_1,\bar{h}_{11}\rangle,\langle u_2,\bar{h}_{12}\rangle,\langle u_3,\bar{h}_{13}\rangle,\langle u_4,\bar{h}_{14}\rangle\big)\\
&= \bigcup_{\bar{r}_{13}\in\bar{h}_{13},\bar{r}_{14}\in\bar{h}_{14},\bar{r}_{11}\in\bar{h}_{11},\bar{r}_{12}\in\bar{h}_{12}} \big[1-\big(1-r_{13}^l\big)^{4\varphi_{c_3}(v,C)\varphi_1(\mu,N)}\big(1-r_{14}^l\big)^{4\varphi_{c_4}(v,C)\varphi_2(\mu,N)}\\
&\quad\times\big(1-r_{11}^l\big)^{4\varphi_{c_1}(v,C)\varphi_3(\mu,N)}\big(1-r_{12}^l\big)^{4\varphi_{c_2}(v,C)\varphi_4(\mu,N)},\\
&\quad 1-\big(1-r_{13}^u\big)^{4\varphi_{c_3}(v,C)\varphi_1(\mu,N)}\big(1-r_{14}^u\big)^{4\varphi_{c_4}(v,C)\varphi_2(\mu,N)}\\
&\quad\times\big(1-r_{11}^u\big)^{4\varphi_{c_1}(v,C)\varphi_3(\mu,N)}\big(1-r_{12}^u\big)^{4\varphi_{c_2}(v,C)\varphi_4(\mu,N)}\big]\\
&= \big\{[0.472,0.757],[0.474,0.759],[0.475,0.759],[0.479,0.761],\\
&\quad [0.477,0.761],[0.481,0.763],[0.506,0.797],[0.508,0.798],\\
&\quad [0.509,0.799],[0.512,0.8],[0.5,0.8],[0.514,0.802]\big\}.
\end{aligned}$$

**Step 5:** According to the comprehensive IVHFEs $\bar{h}_i$ $(i=1,2,3,4,5)$, the score values $S(\bar{h}_i)$ $(i=1,2,3,4,5)$ are obtained as follows:

$$S(\bar{h}_1)=[0.493,0.780],\qquad S(\bar{h}_2)=[0.802,0.941],$$
$$S(\bar{h}_3)=[0.708,0.88],\qquad S(\bar{h}_4)=[0.670,0.809].$$

From $S(\bar{h}_i)$ $(i=1,2,3,4,5)$, we get $\bar{h}_2>\bar{h}_3>\bar{h}_4>\bar{h}_1$. Thus, the project $a_2$ is the best choice.

In the above example, we only use the IG-IVHFHSWA operator with respect to $\lambda=1$ to make decision. If the IG-IVHFH$\lambda$SWA operator is applied to calculate the comprehensive values of the alternatives, then the main steps are as follows:

**Step 1:** According to model (19), we obtain the following model for the optimal fuzzy measure on the attribute set $C$.

$$\begin{aligned}
\max\, & 2.038g'_\lambda(c_1) + 2.135g'_\lambda(c_2) + 2.013g'_\lambda(c_3) + 2.255g'_\lambda(c_4) + 2.087\lambda g'_\lambda(c_1)g'_\lambda(c_2) \\
& + 2.026\lambda g'_\lambda(c_1)g'_\lambda(c_3) + 2.147\lambda g'_\lambda(c_1)g'_\lambda(c_4) + 2.195\lambda g'_\lambda(c_2)g'_\lambda(c_3) \\
& + 2.134\lambda g'_\lambda(c_2)g'_\lambda(c_4) + 2.062\lambda^2 g'_\lambda(c_1)g'_\lambda(c_2)g'_\lambda(c_3) \\
& + 2.143\lambda^2 g'_\lambda(c_1)g'_\lambda(c_2)g'_\lambda(c_4) + 2.102\lambda^2 g'_\lambda(c_1)g'_\lambda(c_3)g'_\lambda(c_4) \\
& + 2.134\lambda^2 g'_\lambda(c_2)g'_\lambda(c_3)g'_\lambda(c_4) + 2.11\lambda^3 g'_\lambda(c_1)g'_\lambda(c_2)g'_\lambda(c_3)g'_\lambda(c_4) \\
& \text{s.t.} \begin{cases} (1+\lambda g'_\lambda(c_1))(1+\lambda g'_\lambda(c_2))(1+\lambda g'_\lambda(c_3))(1+\lambda g'_\lambda(c_4)) - 1 = \lambda, \\ g'_\lambda(c_1) \in [0.1, 0.3], \qquad g'_\lambda(c_2) \in [0.2, 0.4], \qquad g'_\lambda(c_3) \in [0.05, 0.25], \\ g'_\lambda(c_4) \in [0.25, 0.45], \\ \lambda > -1. \end{cases}
\end{aligned}$$

After solving the above model, it has

$$\begin{aligned}
& \lambda = 0.9572, \qquad g'_\lambda(c_1) = 0.1, \qquad g'_\lambda(c_2) = 0.2, \\
& g'_\lambda(c_3) = 0.05, \qquad g'_\lambda(c_4) = 0.45.
\end{aligned}$$

By Eq. (6), we get the attribute Shapley values

$$\begin{aligned}
& \varphi_{c_1}(g'_\lambda, C) = 0.137, \qquad \varphi_{c_2}(g'_\lambda, C) = 0.262, \\
& \varphi_{c_3}(g'_\lambda, C) = 0.07, \qquad \varphi_{c_4}(g'_\lambda, C) = 0.53.
\end{aligned}$$

**Step 2:** According to model (20), we obtain the following model for the optimal fuzzy measure on the ordered set $N$.

$$\begin{aligned}
\max\, & 2.593g_\lambda(1) + 2.368g_\lambda(2) + 1.923g_\lambda(3) + 1.558g_\lambda(4) + 2.481\lambda g_\lambda(1)g_\lambda(2) \\
& + 2.258\lambda g_\lambda(1)g_\lambda(3) + 2.076\lambda g_\lambda(1)g_\lambda(4) + 2.146\lambda g_\lambda(2)g_\lambda(3) \\
& + 1.741\lambda g_\lambda(2)g_\lambda(4) + 2.295\lambda^2 g_\lambda(1)g_\lambda(2)g_\lambda(3) \\
& + 2.173\lambda^2 g_\lambda(1)g_\lambda(2)g_\lambda(4) + 2.025\lambda^2 g_\lambda(1)g_\lambda(3)g_\lambda(4) \\
& + 1.95\lambda^2 g_\lambda(2)g_\lambda(3)g_\lambda(4) + 2.111\lambda^3 g_\lambda(1)g_\lambda(2)g_\lambda(3)g_\lambda(4) \\
& \text{s.t.} \begin{cases} (1+\lambda g_\lambda(1))(1+\lambda g_\lambda(2))(1+\lambda g_\lambda(3))(1+\lambda g_\lambda(4)) - 1 = \lambda, \\ g_\lambda(1) \in [0.4, 0.5], \qquad g_\lambda(2) \in [0.3, 0.4], \qquad g_\lambda(3) \in [0.2, 0.3], \\ g_\lambda(4) \in [0.1, 0.2], \\ \lambda > -1. \end{cases}
\end{aligned}$$

After solving the above model, it has

$$\begin{aligned}
& \lambda = -0.438, \qquad g_\lambda(1) = 0.5, \qquad g_\lambda(2) = 0.4, \\
& g_\lambda(3) = 0.2, \qquad g_\lambda(4) = 0.1.
\end{aligned}$$

By Eq. (6), we get the ordered position Shapley values

$$\varphi_1(g_\lambda, N) = 0.428, \qquad \varphi_2(g_\lambda, N) = 0.334,$$
$$\varphi_3(g_\lambda, N) = 0.16, \qquad \varphi_4(g_\lambda, N) = 0.08.$$

**Step 3:** Let $u_j = (S(\bar{h}_{ij})^l + S(\bar{h}_{ij})^u)/2 - D(\bar{h}_{ij})$ $(j = 1, 2, 3, 4)$ for each $i = 1, 2, 3, 4$ and $\kappa = 1$, utilize the IG-IVHFH$\lambda$SWA operator to calculate the comprehensive values of the alternatives, i.g., $i = 1$,

$$\begin{aligned}
\bar{h}_1 &= \text{IG-IVHFH}\lambda\text{SWA}\big(\langle u_1, \bar{h}_{11}\rangle, \langle u_2, \bar{h}_{12}\rangle, \langle u_3, \bar{h}_{13}\rangle, \langle u_4, \bar{h}_{14}\rangle\big) \\
&= \bigcup_{\bar{r}_{13}\in\bar{h}_{13}, \bar{r}_{14}\in\bar{h}_{14}, \bar{r}_{11}\in\bar{h}_{11}, \bar{r}_{12}\in\bar{h}_{12}} \big[1 - \big(1 - r_{13}^l\big)^{4\varphi_{c_3}(g'_\lambda, C)\varphi_1(g_\lambda, N)} \\
&\quad \times \big(1 - r_{14}^l\big)^{4\varphi_{c_4}(g'_\lambda, C)\varphi_2(g_\lambda, N)} \big(1 - r_{11}^l\big)^{4\varphi_{c_1}(g'_\lambda, C)\varphi_3(g_\lambda, N)} \\
&\quad \times \big(1 - r_{12}^l\big)^{4\varphi_{c_2}(g'_\lambda, C)\varphi_4(g_\lambda, N)}, 1 - \big(1 - r_{13}^u\big)^{4\varphi_{c_3}(g'_\lambda, C)\varphi_1(g_\lambda, N)} \\
&\quad \times \big(1 - r_{14}^u\big)^{4\varphi_{c_4}(g'_\lambda, C)\varphi_2(g_\lambda, N)} \big(1 - r_{11}^u\big)^{4\varphi_{c_1}(g'_\lambda, C)\varphi_3(g_\lambda, N)} \\
&\quad \times \big(1 - r_{12}^u\big)^{4\varphi_{c_2}(g'_\lambda, C)\varphi_4(g_\lambda, N)}\big] \\
&= \big\{[0.383, 0.651], [0.398, 0.667], [0.399, 0.666], [0.432, 0.686], \\
&\quad [0.413, 0.682], [0.446, 0.670], [0.420, 0.705], [0.433, 0.718], [0.466, 0.734], \\
&\quad [0.434, 0.718], [0.448, 0.730], [0.479, 0.746]\big\}.
\end{aligned}$$

**Step 4:** According to the comprehensive IVHFEs $\bar{h}_i$ $(i = 1, 2, 3, 4, 5)$, the score values $S(\bar{h}_i)$ $(i = 1, 2, 3, 4, 5)$ are obtained as follows:

$$S(\bar{h}_1) = [0.429, 0.7], \qquad S(\bar{h}_2) = [0.636, 0.825],$$
$$S(\bar{h}_3) = [0.552, 0.749], \qquad S(\bar{h}_4) = [0.603, 0.754].$$

From $S(\bar{h}_i)$ $(i = 1, 2, 3, 4, 5)$, we get $\bar{h}_2 > \bar{h}_4 > \bar{h}_3 > \bar{h}_1$. The ranking result is slightly different to that got by the IG-IVHFHSWA operator. But the best choice is still with the project $a_2$.

Furthermore, if the GIVHFHA operator is used to calculate the comprehensive values of the alternatives, then the main steps are as follows:

**Step 1:** According to model (21), we obtain the following model for the optimal fuzzy measure on the attribute set $C$.

$$\max 2.038w_{c_1} + 2.135w_{c_2} + 2.013w_{c_3} + 2.255w_{c_4}$$
$$\text{s.t.} \begin{cases} w_{c_1} + w_{c_2} + w_{c_3} + w_{c_4} = 1, \\ w_{c_1} \in [0.1, 0.3], \qquad w_{c_2} \in [0.2, 0.4], \\ w_{c_3} \in [0.05, 0.25], \qquad w_{c_4} \in [0.25, 0.45]. \end{cases}$$

After solving the above model, it has

$$w_{c_1} = 0.1, \qquad w_{c_2} = 0.4, \qquad w_{c_3} = 0.05, \qquad w_{c_4} = 0.45.$$

**Step 2:** According to model (22), we obtain the following model for the optimal fuzzy measure on the ordered set $N$.

$$\max 2.593\omega_1 + 2.368\omega_2 + 1.923\omega_3 + 1.558\omega_4$$
$$\text{s.t.} \begin{cases} \omega_1 + \omega_2 + \omega_3 + \omega_4 = 1 \\ \omega_1 \in [0.4, 0.5], \qquad \omega_2 \in [0.3, 0.4], \\ \omega_3 \in [0.2, 0.3], \qquad \omega_4 \in [0.1, 0.2]. \end{cases}$$

After solving the above model, it has

$$\omega_1 = 0.4, \qquad \omega_2 = 0.3, \qquad \omega_3 = 0.2, \qquad \omega_4 = 0.1.$$

**Step 3:** Let $\kappa = 1$, utilize the GIVHFHA operator to calculate the comprehensive values of the alternatives, i.g., $i = 1$,

$$\begin{aligned} \bar{h}_1 &= \text{GIVHFHA}\big(\langle u_1, \bar{h}_{11}\rangle, \langle u_2, \bar{h}_{12}\rangle, \langle u_3, \bar{h}_{13}\rangle, \langle u_4, \bar{h}_{14}\rangle\big) \\ &= \bigcup_{\bar{r}_{14}\in\bar{h}_{14}, \bar{r}_{12}\in\bar{h}_{12}, \bar{r}_{11}\in\bar{h}_{11}, \bar{r}_{13}\in\bar{h}_{13}} \big[1 - \big(1 - r_{14}^l\big)^{4w_4\omega_1}\big(1 - r_{12}^l\big)^{4w_2\omega_2} \\ &\quad \times \big(1 - r_{11}^l\big)^{4w_1\omega_3}\big(1 - r_{13}^l\big)^{4w_3\omega_4}, 1 - \big(1 - r_{14}^u\big)^{4w_4\omega_1}\big(1 - r_{12}^u\big)^{4w_2\omega_2} \\ &\quad \times \big(1 - r_{11}^u\big)^{4w_1\omega_3}\big(1 - r_{13}^u\big)^{4w_3\omega_4}\big] \\ &= \big\{[0.398, 0.671], [0.404, 0.680], [0.412, 0.684], [0.417, 0.693], [0.475, 0.749], \\ &\quad [0.481, 0.755], [0.487, 0.757], [0.492, 0.765], [0.624, 0.820], [0.628, 0.825], \\ &\quad [0.632, 0.827], [0.636, 0.832]\big\}. \end{aligned}$$

**Step 4:** According to the comprehensive IVHFEs $\bar{h}_i$ $(i = 1, 2, 3, 4, 5)$, the score values $S(\bar{h}_i)$ $(i = 1, 2, 3, 4, 5)$ are obtained as follows:

$$S(\bar{h}_1) = [0.507, 0.755], \qquad S(\bar{h}_2) = [0.596, 0.797],$$
$$S(\bar{h}_3) = [0.545, 0.750], \qquad S(\bar{h}_4) = [0.659, 0.817].$$

From $S(\bar{h}_i)$ $(i = 1, 2, 3, 4, 5)$, we get $\bar{h}_4 > \bar{h}_2 > \bar{h}_3 > \bar{h}_1$. The ranking result is different to that got by the IG-IVHFHSWA and IG-IVHFHλSWA operators, and the best choice is the project $a_4$.

For the comparative convenience, the ranking results with respect to the different aggregation operators and the different values of the parameter λ are obtained as shown in Tables 4, 5, 6 and 7.

Table 4
Ranking results with respect to $\lambda = 1$.

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | Ranking orders |
|---|---|---|---|---|---|
| The GIVHFHA operator | [0.507, 0.755] | [0.596, 0.797] | [0.545, 0.750] | **[0.659, 0.817]** | $\bar{h}_4 > \bar{h}_2 > \bar{h}_3 > \bar{h}_1$ |
| The GIVHFHG operator | [0.420, 0.677] | [0.493, 0.700] | [0.417, 0.622] | **[0.615, 0.758]** | $\bar{h}_4 > \bar{h}_2 > \bar{h}_1 > \bar{h}_3$ |
| The IG-IVHFHSWA operator | [0.493, 0.780] | **[0.803, 0.941]** | [0.708, 0.880] | [0.670, 0.809] | $\bar{h}_2 > \bar{h}_3 > \bar{h}_4 > \bar{h}_1$ |
| The IG-IVHFHSGM operator | [0.341, 0.647] | [0.398, 0.664] | [0.287, 0.523] | **[0.531, 0.693]** | $\bar{h}_4 > \bar{h}_2 > \bar{h}_1 > \bar{h}_3$ |
| The IG-IVHFH$\lambda$SWA operator | [0.429, 0.700] | **[0.636, 0.825]** | [0.552, 0.749] | [0.603, 0.753] | $\bar{h}_2 > \bar{h}_4 > \bar{h}_3 > \bar{h}_1$ |
| The IG-IVHFH$\lambda$SGM operator | [0.406, 0.678] | [0.482, 0.707] | [0.404, 0.617] | **[0.548, 0.711]** | $\bar{h}_4 > \bar{h}_2 > \bar{h}_1 > \bar{h}_3$ |

Table 5
Ranking results with respect to $\lambda = 2$.

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | Ranking orders |
|---|---|---|---|---|---|
| The GIVHFHA operator | [0.537, 0.775] | [0.638, 0.818] | [0.575, 0.770] | **[0.684, 0.833]** | $\bar{h}_4 > \bar{h}_2 > \bar{h}_3 > \bar{h}_1$ |
| The GIVHFHG operator | [0.380, 0.631] | [0.449, 0.659] | [0.394, 0.598] | **[0.562, 0.720]** | $\bar{h}_4 > \bar{h}_2 > \bar{h}_1 > \bar{h}_3$ |
| The IG-IVHFHSWA operator | [0.552, 0.818] | **[0.831, 0.950]** | [0.746, 0.899] | [0.716, 0.842] | $\bar{h}_2 > \bar{h}_3 > \bar{h}_4 > \bar{h}_1$ |
| The IG-IVHFHSGM operator | [0.272, 0.568] | [0.353, 0.621] | [0.252, 0.478] | **[0.475, 0.641]** | $\bar{h}_4 > \bar{h}_2 > \bar{h}_1 > \bar{h}_3$ |
| The IG-IVHFH$\lambda$SWA operator | [0.468, 0.734] | **[0.679, 0.848]** | [0.593, 0.777] | [0.639, 0.781] | $\bar{h}_2 > \bar{h}_4 > \bar{h}_3 > \bar{h}_1$ |
| The IG-IVHFH$\lambda$SGM operator | [0.353, 0.627] | [0.442, 0.675] | [0.374, 0.588] | **[0.508, 0.675]** | $\bar{h}_4 > \bar{h}_2 > \bar{h}_1 > \bar{h}_3$ |

Table 6
Ranking results with respect to $\lambda = 5$.

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | Ranking orders |
|---|---|---|---|---|---|
| The GIVHFHA operator | [0.590, 0.813] | [0.704, 0.856] | [0.623, 0.804] | **[0.723, 0.858]** | $\bar{h}_4 > \bar{h}_2 > \bar{h}_3 > \bar{h}_1$ |
| The GIVHFHG operator | [0.307, 0.541] | [0.366, 0.561] | [0.346, 0.549] | **[0.461, 0.636]** | $\bar{h}_4 > \bar{h}_2 > \bar{h}_1 > \bar{h}_3$ |
| The IG-IVHFHSWA operator | [0.650, 0.876] | **[0.871, 0.964]** | [0.798, 0.925] | [0.788, 0.891] | $\bar{h}_2 > \bar{h}_3 > \bar{h}_4 > \bar{h}_1$ |
| The IG-IVHFHSGM operator | [0.185, 0.464] | [0.294, 0.568] | [0.201, 0.416] | **[0.394, 0.565]** | $\bar{h}_4 > \bar{h}_2 > \bar{h}_1 > \bar{h}_3$ |
| The IG-IVHFH$\lambda$SWA operator | [0.549, 0.797] | **[0.748, 0.889]** | [0.663, 0.826] | [0.704, 0.830] | $\bar{h}_2 > \bar{h}_4 > \bar{h}_3 > \bar{h}_1$ |
| The IG-IVHFH$\lambda$SGM operator | [0.273, 0.546] | [0.377, 0.622] | [0.324, 0.541] | **[0.436, 0.608]** | $\bar{h}_4 > \bar{h}_2 > \bar{h}_3 > \bar{h}_1$ |

Table 7
Ranking results with respect to $\lambda = 10$.

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | Ranking orders |
|---|---|---|---|---|---|
| The GIVHFHA operator | [0.627, 0.841] | **[0.744, 0.884]** | [0.655, 0.828] | [0.750, 0.874] | $\bar{h}_2 > \bar{h}_4 > \bar{h}_3 > \bar{h}_1$ |
| The GIVHFHG operator | [0.253, 0.476] | [0.303, 0.471] | [0.308, 0.512] | **[0.384, 0.571]** | $\bar{h}_4 > \bar{h}_3 > \bar{h}_2 > \bar{h}_1$ |
| The IG-IVHFHSWA operator | [0.706, 0.911] | **[0.894, 0.972]** | [0.828, 0.940] | [0.828, 0.920] | $\bar{h}_2 > \bar{h}_3 > \bar{h}_4 > \bar{h}_1$ |
| The IG-IVHFHSGM operator | [0.139, 0.410] | [0.262, 0.543] | [0.172, 0.384] | **[0.351, 0.526]** | $\bar{h}_4 > \bar{h}_2 > \bar{h}_3 > \bar{h}_1$ |
| The IG-IVHFH$\lambda$SWA operator | [0.602, 0.843] | **[0.792, 0.916]** | [0.709, 0.861] | [0.747, 0.863] | $\bar{h}_2 > \bar{h}_4 > \bar{h}_3 > \bar{h}_1$ |
| The IG-IVHFH$\lambda$SGM operator | [0.223, 0.493] | [0.327, 0.576] | [0.288, 0.511] | **[0.385, 0.560]** | $\bar{h}_4 > \bar{h}_2 > \bar{h}_3 > \bar{h}_1$ |

According to Tables 4, 5, 6 and 7, the ranking results show that the different optimal alternatives may be yielded using the different aggregation operators, and thus, the decision makers can properly select the aggregation operator according to the underlying interest and demanding to each problem.

## 6. Conclusion

In order to deal with the situation where the elements in a set are correlative and the weight information is not exactly known, this study first gives a new ranking method to IVHFEs.

Then, based on the Shapley function, two induced generalized interval-valued hesitant fuzzy hybrid Shapley aggregation operators are defined, which do not only globally consider the importance of elements and their ordered positions but also reflect the overall interactions between them, respectively. Because of various reasons, the information about the weight vectors is usually partly known. Based on the Shapley function, the models for the optimal fuzzy measures on the attribute set and on the ordered set are established. In the end, an approach to interval-valued hesitant fuzzy multi-attribute decision making is developed, and an illustrative example is given to show the feasibility and practicality of the proposed procedure. If there is no interaction between elements, then the introduced decision-making method degenerates to an approach based on additive measures.

Besides the application in decision making, we can also apply the introduced operators and the building models to other fields, such as digital image processing, clustering analysis and pattern recognition. Furthermore, we here only define two aggregation operators, and it will be interesting to study other interval-valued hesitant fuzzy aggregation operators.

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# Hibridiniais Shapley operatoriais grįstas neraiškusis daugiatikslis vertinimo būdas, kai rodiklių reikšmės apibrėžtos nepastoviais intervalais

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Šiame straipsnyje pateikiamas naujas nepastoviais intervalais apibrėžtų elementų surikiavimo metodas. Siekiant gauti tinkamas reikšmes alternatyvoms, aprašyti du apibendrinti intervalais matuojami nepastovieji neraiškieji hibridiniai operatoriai, kurie yra pagrįsti Shapley funkcija. Shapley funkcija globaliai apima tiek elementų svarbą ir jų (sutvarkytas) surikiuotas vietas, tiek ir atspindi tarpusavio veikas (poveikius) tarp jų. Jei informacija apie svorį yra nepilnai žinoma, tai sukuriami optimaliųjų rodiklių aibės svorių vektorių ir sutvarkytos (surikiuotos) aibės atitinkami modeliai. Vėliau yra sukuriamas neraiškusis daugiatikslis vertinimo būdas kai rodiklių reikšmės apibrėžtos nepastoviais intervalais. Pabaigoje pateiktas pavyzdys sukurtam vertinimo būdui iliustruoti.