Multi-Attribute Group Decision Making with Trapezoidal Intuitionistic Fuzzy Numbers and Application to Stock Selection

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Abstract. The fuzzy number is a special case of fuzzy set. As a generalization of the fuzzy number, trapezoidal intuitionistic fuzzy number (TrIFN) is a special intuitionistic fuzzy set defined on the real number set, which seems to suitably describe an ill-known quantity. The purpose of this paper is to propose a new method for solving the multi-attribute group decision making problems, in which the attribute values are TrIFNs and the attribute weight information are incomplete. The concepts, such as the weighted lower and upper possibility means, the weighted possibility means and variances of TIFNs, are introduced. Hereby, a new lexicographic method is developed to rank the TrIFNs. In the proposed method, the weights of experts are determined in terms of the voting model of intuitionistic fuzzy set. The attribute weights are objectively derived through constructing the bi-objective programming model, which is transformed into the single objective quadratic programming model to solve. The ranking order of alternatives is generated by the collective overall attribute values of alternatives. The stock selection example and comparison analyzes show the validity and applicability of the method proposed in this paper.

Key words: multiattribute group decision making, trapezoidal intuitionistic fuzzy number, weighted possibility mean, weighted possibility variance, stock investment selection.

1. Introduction

In many real-life decision making problems, such as stock investment selection, decision maker (DM) does not know exactly the attribute values of alternative due to the complexity and uncertainty involved in the decision problems, the fuzzy sets (FSs) (Zadeh, 1965) can be used to describe an uncertain environment with vagueness, ambiguity or some other type of fuzziness. Thus, the fuzzy decision making analysis appears (Stanujkic *et al.*, 2012; Celen, 2014; Zhang and Chen, 2014; Zeng *et al.*, 2013b). However, the decision making

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problems often involve many incomplete information and relate to many complex factors, such as economy, politics, psychological behavior, ideology and so on. Therefore, there often exist some hesitation degrees in the judgments of DMs. For example, in stock investment selection, because of the incompleteness and uncertainty of information in the evaluation of the listed company's solvency indicator, the evaluation value can be expressed by trapezoidal intuitionistic fuzzy number (TrIFN) (Wang, 2008; Wang and Zhang, 2009; Wei, 2010a, 2010b, Du and Liu, 2011; Wu and Cao, 2013; Wan and Dong, 2010; Wan, 2012, 2013b; Ye, 2011, 2012; Zhang and Xu, 2012) ([4, 5, 6, 7]; 0.6, 0.3), which means that the minimum value of solvency is 4, the maximum value is 7, and the most possible value is between 5 and 6 (i.e., interval [5, 6]). Meanwhile, the maximum membership degree for the most possible value interval [5, 6] is 0.6, the minimum non-membership degree for the most possible value interval [5, 6] is 0.3, and the indeterminacy is 0.1. That is to say, the DM has some hesitation degree for the estimation on this judgment, this hesitation influences the decision making on the stock selection.

The intuitionistic fuzzy (IF) set (IFS) (Atanassov, 1986) and interval-valued intuitionistic fuzzy set (IVIFS) (Atanassov and Gargov, 1989) are just the suitable and powerful tools to represent the uncertain information with hesitation degrees. At present, both IFS and IVIFS have been widely applied to the fields of multiattribute decision making (MADM) and multiattribute group decision making (MAGDM). These researches on IFS and IVIFSs are mainly focused on the aggregation operators (Li, 2010d, 2011a, 2011b; Liu and Wang, 2007; Li *et al.*, 2010a, 2010b; Yu *et al.*, 2012; Wei and Merigó, 2012; Wan and Dong, 2014a, 2014b), similarity (or distance) measures and entropy (Huang *et al.*, 2012; Xia *et al.*, 2012; Ye, 2010; Xu and Yager, 2009; Xu, 2007a), extension of classic decision making methods (Li, 2008a, 2010a, 2010b, 2011a, 2011b; Park *et al.*, 2007, 2011; Xu, 2007a; Li and Wan, 2013; Zeng *et al.*, 2013a), new decision making methods (Chen, 2011; Chen and Yang, 2011, 2012; Guo and Liu, 2012; Li, 2007; Zhang and Xu, 2012). At the same time, the researches on the intuitionistic fuzzy numbers (IFNs) also receive a little attention (Dubois and Prade, 1980; Shu *et al.*, 2006; Li, 2010a, 2010b, 2010c, 2010d; Li *et al.*, 2010a, 2010b, 2010c; Nan *et al.*, 2010). Fuzzy numbers are a special case of fuzzy sets. As a generalization of fuzzy numbers (Dubois and Prade, 1980), IFN is a special IFS defined on the real number set, which seems to suitably describe an ill-known quantity (Li, 2008a, 2008b). Currently, there are three kinds of typical IFNs: triangular IFN (TIFN) (Shu *et al.*, 2006; Li, 2008b, 2010c; Li *et al.*, 2010a, 2010b; Nan *et al.*, 2010; Wan *et al.*, 2013a, 2013b; Wan and Dong, 2014a, 2014b; Wan and Li, 2013b), trapezoidal IFN (TrIFN) (Zhang and Xu, 2012; Ye, 2011, 2012; Wan, 2012, 2013b; Wan and Dong, 2010; Wu et al., 2012; Du and Liu, 2011; Wei, 2010a, 2010b; Wang and Zhang, 2009; Wang, 2008) and interval-valued trapezoidal IFN (IVTrIFN) (Wan, 2011, 2012).

Shu *et al.* (2006) defined the concept of a TIFN in a similar way to that of the fuzzy number (Dubois and Prade, 1980) and developed an algorithm for intuitionistic fuzzy fault tree analysis. Li (2008b) pointed out and corrected some errors in the definition of the four arithmetic operations over the TIFNs in Shu *et al.* (2006). Li (2010c) discussed the concept of the TIFN and ranking method on the basis of the concept of a ratio of the value index to the ambiguity index as well as applications to MADM problems in depth. Li *et al.* (2010c)

defined the values and ambiguities of the membership degree and the non-membership degree for TIFNs as well as the value-index and ambiguity-index. Hereby a value and ambiguity based method is developed to rank TIFNs and applied to solve MADM problems in which the ratings of alternatives on attributes are expressed using TIFNs. Nan *et al.* (2010) defined the ranking order relations of TIFNs and solved the fuzzy matrix games with payoffs of TIFNs. Wan et al. (2013a, 2013b) proposed the extended VIKOR method for MAGDM with TIFNs. As the extensions of the TIFNs, Wang (2008) defined the trapezoidal IFN (TrIFN) and interval-valued trapezoidal IFN (IVTrIFN). Wang and Zhang (2009) investigated the weighted arithmetic averaging operator and weighted geometric averaging operator on TrIFNs and their applications to MADM problems. Wei (2010a, 2010b) investigated some arithmetic aggregation operators with TrIFNs and their applications to MAGDM problems. Du and Liu (2011) extended fuzzy VIKOR method with TrIFNs. Wu and Cao (2013) developed some families of geometric aggregation operators with TrIFNs and applied to MAGDM problems. Wan and Dong (2010) defined the expectation and expectant score, ordered weighted aggregation operator and hybrid aggregation operator for TrIFNs and employed to MAGDM. Wan (2012) developed power average operators of TrIFNs and application to MAGDM. Ye (2011) developed the expected value method for intuitionistic trapezoidal fuzzy multicriteria decision-making problems. Ye (2012) proposed the MAGDM method using vector similarity measures for TrIFNs. Zhang *et al.* (2013) proposed a grey relational projection method for MAGDM based on TrIFNs. Wan (2011) firstly defined some operational laws of IVTrIFNs and developed the IVTrIFN weighted arithmetical average operator and weighted geometrical average operator. An approach to ranking IVTrIFNs is presented based on the score function and accurate function. The MAGDM method using IVTrIFNs is then proposed. Wan (2012) defined the Hamming and Euclidean distances for IVTrIFNs and proposed the fractional programming method for the MADM problems using IVTrIFNs.

The above researches about IFNs mainly focus on the operation laws (Li, 2008a, 2008b; Shu *et al.*, 2006; Wan, 2012; Zhang and Xu, 2012), aggregation operators (Wang and Zhang, 2009; Wu and Cao, 2013; Wan and Dong, 2010; Wan, 2012), ranking methods (Li *et al.*, 2010a, 2010b; Nan *et al.*, 2010; Wang, 2008; Wan, 2011, 2013a, 2013b; Wan and Dong, 2014a, 2014b), extension of classical decision making methods (Zhang and Xu, 2012; Du and Liu, 2011; Wan and Li, 2014) and new decision making methods (Ye, 2011; Wan, 2012). It is worthwhile to mention that the domains of the IFS and IVIFS are discrete sets, which are also the same as fuzzy sets. TIFNs, TrIFNs and IVTrIFNs extend the domain of IFSs from the discrete set to the continuous set. They are the extensions of fuzzy numbers (Wang and Zhang, 2009). Compared with the IFSs, TrIFNs are defined by using trapezoidal fuzzy numbers expressing their membership and non-membership functions. Hence, TrIFNs may better reflect the information of decision problems than IFSs. With ever increasing complexity in many real decision situations, there are often some challenges for the DM to provide precise and complete weight preference information due to time pressure, lack of knowledge (or data) and DM's limited expertise about the problem domain. In other words, usually attribute weights are totally unknown or partially known

a priori. Namely, weight preference information in MAGDM problems is usually incomplete (Li, 2011a, 2011b; Wan and Li, 2013a, 2014; Li and Wan, 2013, 2014; Wan and Dong, 2014a, 2014b). However, there is less investigation for the MAGDM problems in which the attribute values are in the form of TrIFNs and the weight preference information is incomplete. The existing methods about IFNs, IFSs and IVIFSs cannot be applied to MAGDM with TrIFNs and incomplete weight preference information. With the increasing complexity of modern society, continued expansion of the scale and the diversification of business, many large and important management decision optimization problems require many experts to participate in making decisions together. Therefore, the MAGDM problems with TrIFNs and incomplete weight preference information are of a great importance for scientific researches and real applications. For instance, since the real-life stock investment selection problems often involve multiple different attributes (or indices, factors) such as profit ability, debt paying ability, growth ability, market performance as well as investment income, single DM is very difficult to make judgment which stock is to be chosen for investment. Moreover, DMs are unable to accurately give these attribute weights because of various subjective and objective reasons.

The possibility theory of FSs was proposed by Zadeh (1978), its academic meaning is in building a theoretical framework of real applications for FSs. In statistics, central tendency and distribution dispersion are considered to be the important measures. For fuzzy numbers, two of the most useful measures are the mean and variance of fuzzy numbers. The possibility mean and variance are the important mathematical characteristics of fuzzy numbers. The possibilistic mean, variance and covariance of fuzzy numbers, defined by Carlsson and Fullér (2001) and Fullér and Majlender (2003) are usually used to the research of fuzzy optimal portfolio selection (Zhang *et al.*, 2009). They are similar to the mean, variance and covariance of random variance, which may quantificationally express the uncertain information implied in the fuzzy numbers. In many real decision making problems, the decision information provided by decision makers (DMs) is often imprecise or uncertain. Hence, introducing the possibility mean and variance is very importance and useful for solving fuzzy MAGDM problems. To our best knowledge, however, there is no investigation on possibility mean and variance for IFSs up to date. Hence, the aim of this paper is to define the weighted possibility means and variances of TrIFN and hereby give a new method for ranking TrIFNs. A new decision method based on the weighted possibility mean and variance is then proposed for the MAGDM problems, in which the attribute values are TrIFNs and the attribute weights are incompletely known.

The rest of this paper is structured as follows. In Section 2, we present the definition, operation laws and weighted average operator of TrIFNs. In Section 3, some concepts about TrIFNs, such as the weighted possibility means and variances of TrIFNs, are defined, respectively. A lexicographic ranking method for TrIFNs is developed. A new method for MAGDM using TrIFNs is developed in Section 4. A stock selection example and comparison analyzes are given in Section 5. Short conclusions are made in Section 6.
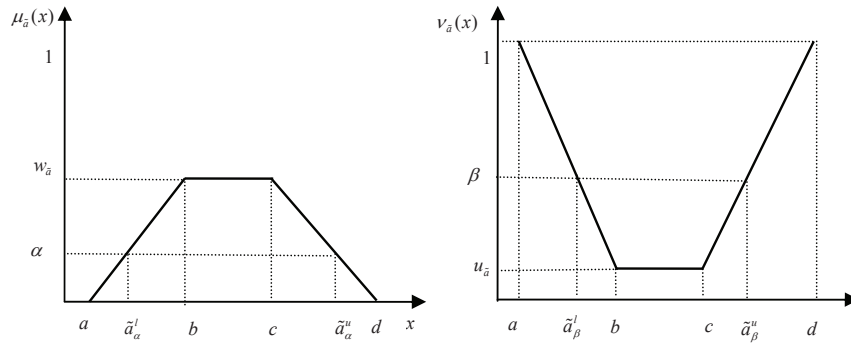
Fig. 1. α-cut set of membership function and β-cut set of non-membership function.

2. Trapezoidal Intuitionistic Fuzzy Numbers

In this section, the definition, operation laws and weighted average operator of TrIFNs are introduced.

2.1. *The Definition of TrIFNs*

DEFINITION 1. (See Wang and Zhang, 2009; Wan *et al.*, 2013a, 2013b.) Let $\tilde{a}$ be an intuitionistic fuzzy number in the set of real numbers, whose membership function and non-membership function are defined as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a}{b-a} w_{\tilde{a}}, & \text{if } a \leqslant x < b, \\ w_{\tilde{a}}, & \text{if } b \leqslant x \leqslant c, \\ \frac{d-x}{d-c} w_{\tilde{a}}, & \text{if } c < x \leqslant d, \\ 0, & \text{otherwise} \end{cases}$$

and

$$v_{\tilde{a}}(x) = \begin{cases} \frac{b-x+(x-a)u_{\tilde{a}}}{b-a}, & \text{if } a \leqslant x < b, \\ u_{\tilde{a}}, & \text{if } b \leqslant x \leqslant c, \\ \frac{x-c+(d-x)u_{\tilde{a}}}{d-c}, & \text{if } c < x \leqslant d, \\ 1, & \text{otherwise,} \end{cases}$$

respectively (see Fig. 1), where a, b, c, and d are all real numbers, the values $w_{\tilde{a}}$ and $u_{\tilde{a}}$ represent the maximum degree of membership and the minimum degree of non-membership, respectively, such that they satisfy the conditions: $0 \leqslant w_{\tilde{a}} \leqslant 1$, $0 \leqslant u_{\tilde{a}} \leqslant 1$ and $w_{\tilde{a}} + u_{\tilde{a}} \leqslant 1$. Then, the intuitionistic fuzzy number $\tilde{a}$ is called the TrIFN, denoted by $\tilde{a} = ([a, b, c, d]; w_{\tilde{a}}, u_{\tilde{a}})$.

The function $\pi_{\tilde{a}}(x) = 1 - \mu_{\tilde{a}}(x) - v_{\tilde{a}}(x)$ denotes the hesitation of $\tilde{a}$. The smaller $\pi_{\tilde{a}}(x)$, the more certain $\tilde{a}$. When $b = c$, a TrIFN reduces to a TIFN. If $a \geqslant 0$ and one of the four values b, c, and d is not equal to 0, then the TrIFN $\tilde{a} = ([a, b, c, d]; w_{\tilde{a}}, u_{\tilde{a}})$ is

called a positive TrIFN, denoted by $\tilde{a} > 0$. The TrIFNs discussed in the following are all positive TrIFNs.

For example, there is a TrIFN $\tilde{6} = ([4, 5, 7, 8]; 0.5, 0.2)$. Then, when $x = 5$, its membership degree being a TrIFN $\tilde{6}$ is 0.5, its non-membership degree not being the TrIFN $\tilde{6}$ is 0.2, and its hesitation being or not being the TrIFN $\tilde{6}$ is 0.3.

2.2. *Operation Laws and Properties for TrIFNs*

DEFINITION 2. Let $\tilde{a}_1 = ([a_1, b_1, c_1, d_1]; w_{\tilde{a}_1}, u_{\tilde{a}_1})$ and $\tilde{a}_2 = ([a_2, b_2, c_2, d_2]; w_{\tilde{a}_2}, u_{\tilde{a}_2})$ be two TrIFNs and $\lambda \geqslant 0$. Then the operation laws for TrIFNs are defined as follows:

(1) $\tilde{a}_1 + \tilde{a}_2 = ([a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; w_{\tilde{a}_1} \wedge w_{\tilde{a}_2}, u_{\tilde{a}_1} \vee u_{\tilde{a}_2})$, where the symbols "$\wedge$" and "$\vee$" are min and max operators, respectively;
(2) $\tilde{a}_1 \tilde{a}_2 = ([a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2]; w_{\tilde{a}_1} \wedge w_{\tilde{a}_2}, u_{\tilde{a}_1} \vee u_{\tilde{a}_2})$;
(3) $\lambda \tilde{a}_1 = ([\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1]; w_{\tilde{a}_1}, u_{\tilde{a}_1})$;
(4) $\tilde{a}_1^{\lambda} = ([a_1^{\lambda}, b_1^{\lambda}, c_1^{\lambda}, d_1^{\lambda}]; w_{\tilde{a}_1}, u_{\tilde{a}_1})$.

It is easy to see that if $b_i = c_i$ $(i = 1, 2)$, i.e., both $\tilde{a}_1$ and $\tilde{a}_2$ reduce to two TIFNs, then the operation laws of TrIFNs are degenerated to those of TIFNs (Li, 2010c). Therefore, the operation laws of TrIFNs defined in this paper are the natural generalization of the TIFNs (Li, 2010c).

From Definition 2, the following properties are proven:

(1) $\tilde{a}_1 + \tilde{a}_2 = \tilde{a}_2 + \tilde{a}_1$, $\tilde{a}_1 \tilde{a}_2 = \tilde{a}_2 \tilde{a}_1$;
(2) $\lambda(\tilde{a}_1 + \tilde{a}_2) = \lambda \tilde{a}_1 + \lambda \tilde{a}_2$, $\lambda_1 \tilde{a}_1 + \lambda_2 \tilde{a}_1 = (\lambda_1 + \lambda_2)\tilde{a}_1$, $\tilde{a}_1^{\lambda_1} \tilde{a}_1^{\lambda_2} = \tilde{a}_1^{\lambda_1 + \lambda_2}$;
(3) $(\tilde{a}_1^{\lambda})^k = \tilde{a}_1^{\lambda k}$.

2.3. *The Weighted Average Operator for TrIFNs*

DEFINITION 3. Assume that let $\tilde{a}_j = ([a_j, b_j, c_j, d_j]; w_{\tilde{a}_j}, u_{\tilde{a}_j})$ $(j = 1, 2, \ldots, n)$ is a collection of the TrIFNs. Let $\phi_w^A : T^n \to T$. If

$$\phi_w^A(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} w_j \tilde{a}_j, \tag{1}$$

where T is the set of all TrIFNs, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{a}_j$ $(j = 1, 2, \ldots, n)$, satisfying that $0 \leqslant w_n \leqslant 1$ and $\sum_{j=1}^{n} w_j = 1$, then the function ϕ_w^A is called the n-dimensional weighted average operator for the TrIFNs.

Proposition 1. *Let $\tilde{a}_j = ([a_j, b_j, c_j, d_j]; w_{\tilde{a}_j}, u_{\tilde{a}_j})$ $(j = 1, 2, \ldots, n)$ is a collection of the TrIFNs, then their aggregated value by using ϕ_w^A operator is also an TrIFN, and*

$$\phi_w^A(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$$
$$= \left(\left[\sum_{j=1}^{n} w_j a_j, \sum_{j=1}^{n} w_j b_j, \sum_{j=1}^{n} w_j c_j, \sum_{j=1}^{n} w_j d_j \right]; \wedge_j w_{\tilde{a}_j}, \vee_j u_{\tilde{a}_j} \right). \tag{2}$$

Proposition 1 can be easily proven by using mathematical induction on n. The ϕ_w^A operator has some desirable properties, such as idempotency, boundedness and monotonicity.

3. The Weighted Possibility Means and Variances of TrIFNs and Lexicographic Ranking Method

Since TrIFN is a special IFS, it can express the uncertain and fuzzy information inherent. In order to develop the ranking method of TrIFNs, this section is devoted to introducing the weighted lower and upper possibility means, weighted possibility means and variances of TrIFNs. Thereby, the ranking method of TrIFNs is given.

3.1. *The Weighted Possibility Means of TrIFNs*

Analogous to the cut sets of TIFNs in Li (2010c), the definitions of the cut sets for TrIFNs are given as follows.

DEFINITION 4. For a TrIFN $\tilde{a} = ([a, b, c, d]; w_{\tilde{a}}, u_{\tilde{a}})$, (α, β)-cut set, α-cut set and β-cut set are defined as $\tilde{a}_{\alpha,\beta} = \{x | \mu_{\tilde{a}}(x) \geqslant \alpha, \nu_{\tilde{a}}(x) \leqslant \beta\}$, $\tilde{a}_{\alpha} = \{x | \mu_{\tilde{a}}(x) \geqslant \alpha\}$ and $\tilde{a}_{\beta} = \{x | \nu_{\tilde{a}}(x) \leqslant \beta\}$, respectively, where $0 \leqslant \alpha \leqslant w_{\tilde{a}}$, $u_{\tilde{a}} \leqslant \beta \leqslant 1$ and $0 \leqslant \alpha + \beta \leqslant 1$.

It directly follows from Fig. 1, Definitions 1 and 4 that

$$\tilde{a}_{\alpha} = [a_{\alpha}^l, a_{\alpha}^u] = \left[a + \alpha \frac{b-a}{w_{\tilde{a}}}, d - \alpha \frac{d-c}{w_{\tilde{a}}} \right], \tag{3}$$

$$\tilde{a}_{\beta} = [a_{\beta}^l, a_{\beta}^u] = \left[\frac{b - au_{\tilde{a}} - \beta(b-a)}{1 - u_{\tilde{a}}}, \frac{c - du_{\tilde{a}} + \beta(d-c)}{1 - u_{\tilde{a}}} \right]. \tag{4}$$

Motivated by Fullér and Majlender (2003), we give the definitions of the weighted possibility mean of TrIFNs as follows.

DEFINITION 5. Let $\tilde{a}_{\alpha} = [a_{\alpha}^l, a_{\alpha}^u]$ be the α-cut set of a TrIFN $\tilde{a} = ([a, b, c, d]; w_{\tilde{a}}, u_{\tilde{a}})$ with $0 \leqslant \alpha \leqslant w_{\tilde{a}}$. A function $f : [0, w_{\tilde{a}}] \to R$ is said to be a weighting function if f is a non-negative, monotone increasing and satisfies the conditions: $\int_0^{w_{\tilde{a}}} f(\alpha) d\alpha$ and $f(0) = 0$.

The f weighted lower and upper possibility means of membership function for the TrIFN $\tilde{a}_{\alpha} = [a_{\alpha}^l, a_{\alpha}^u]$ are, respectively, defined as follows:

$$\underline{m}_{\mu}(\tilde{a}) = \int_0^{w_{\tilde{a}}} f\big(Pos\{\tilde{a} \leqslant a_{\alpha}^l\}\big) a_{\alpha}^l \, d\alpha \tag{5}$$

and

$$\overline{m}_{\mu}(\tilde{a}) = \int_0^{w_{\tilde{a}}} f\big(Pos\{\tilde{a} \geqslant a_{\alpha}^u\}\big) a_{\alpha}^u, d\alpha, \tag{6}$$

where *Pos* means possibility (Fullér and Majlender, 2003) and

$$Pos\{\tilde{a} \leqslant a_\alpha^l\} = \sup_{x \leqslant a_\alpha^l} \mu_{\tilde{a}}(x) = \alpha, \tag{7}$$

$$Pos\{\tilde{a} \geqslant a_\alpha^u\} = \sup_{x \geqslant a_\alpha^u} \mu_{\tilde{a}}(x) = \alpha. \tag{8}$$

It can be seen from Eqs. (7) and (8) that

$$\underline{m}_\mu(\tilde{a}) = \int_0^{w_{\tilde{a}}} f\big(Pos\{\tilde{a} \leqslant a_\alpha^l\}\big) a_\alpha^l \, d\alpha = \int_0^{w_{\tilde{a}}} f(\alpha) a_\alpha^l \, d\alpha,$$

$$\overline{m}_\mu(\tilde{a}) = \int_0^{w_{\tilde{a}}} f\big(Pos\{\tilde{a} \geqslant a_\alpha^u\}\big) a_\alpha^u \, d\alpha = \int_0^{w_{\tilde{a}}} f(\alpha) a_\alpha^u \, d\alpha.$$

Obviously, the f weighted lower possibility mean $\underline{m}_\mu(\tilde{a})$ of membership function is nothing else but the f weighted lower possibility weighted average of the minimum of the α-cut set. The f weighted upper possibility mean $\overline{m}_\mu(\tilde{a})$ of membership function is nothing else but the f weighted upper possibility weighted average of the maximum of the α-cut set.

DEFINITION 6. Let $\tilde{a}_\beta = [a_\beta^l, a_\beta^u]$ be the β-cut set of a TrIFN $\tilde{a} = ([a, b, c, d]; w_{\tilde{a}}, u_{\tilde{a}})$. A function $g : [u_{\tilde{a}}, 1] \to R$ is said to be a weighting function if g is a non-negative, monotone decreasing and satisfies the conditions: $\int_{u_{\tilde{a}}}^1 g(\beta) \, d\beta = 1 - u_{\tilde{a}}$ and $g(1) = 1$.

The g weighted lower and upper possibility means of non-membership function for the TrIFN $\tilde{a} = ([a, b, c, d]; w_{\tilde{a}}, u_{\tilde{a}})$ are, respectively, defined as follows:

$$\underline{m}_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^1 g\big(Pos\{\tilde{a} \leqslant a_\beta^l\}\big) a_\beta^l \, d\beta \tag{9}$$

and

$$\overline{m}_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^1 g\big(Pos\{\tilde{a} \geqslant a_\beta^u\}\big) a_\beta^u \, d\beta, \tag{10}$$

where

$$Pos\{\tilde{a} \leqslant a_\beta^l\} = \sup_{x \leqslant a_\beta^l} \mu_{\tilde{a}}(x) = \beta, \tag{11}$$

$$Pos\{\tilde{a} \geqslant a_\beta^u\} = \sup_{x \geqslant a_\beta^u} \mu_{\tilde{a}}(x) = \beta. \tag{12}$$

It yields from Eqs. (11) and (12) that

$$\underline{m}_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^1 g\big(Pos\{\tilde{a} \leqslant a_\beta^l\}\big) a_\beta^l \, d\beta = \int_{u_{\tilde{a}}}^1 g(\beta) a_\beta^l \, d\beta,$$

$$\overline{m}_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^1 g\big(Pos\{\tilde{a} \geqslant a_\beta^u\}\big) a_\beta^u \, d\beta = \int_{u_{\tilde{a}}}^1 g(\beta) a_\beta^u \, d\beta.$$

It can be seen that the g weighted lower possibility mean $\underline{m}_\nu(\tilde{a})$ of non-membership function is nothing else but the g weighted lower possibility weighted average of the minimum of the β-cut set. The g weighted upper possibility mean $\overline{m}_\nu(\tilde{a})$ of non-membership function is nothing else but the g weighted upper possibility weighted average of the maximum of the β-cut set.

DEFINITION 7. For a TrIFN $\tilde{a} = ([a, b, c, d]; w_{\tilde{a}}, u_{\tilde{a}})$, the f weighted possibility mean of membership function and g weighted possibility mean of non-membership function are, respectively, defined as follows:

$$m_\mu(\tilde{a}, \theta) = (1 - \theta)\underline{m}_\mu(\tilde{a}) + \theta \overline{m}_\mu(\tilde{a}) \tag{13}$$

and

$$m_\nu(\tilde{a}, \theta) = (1 - \theta)\underline{m}_\nu(\tilde{a}) + \theta \overline{m}_\nu(\tilde{a}), \tag{14}$$

where $\theta \in [0, 1]$ is the preference parameter of DM and can reflect different importance to the lower and upper possibility means. Different DMs have different preferences for the lower and upper possibility means. $\theta \in (0.5, 1]$ implies that DM prefers the upper possibility mean, namely DM is pessimistic; $\theta \in [0, 0.5)]$ shows that DM prefers the lower possibility mean, namely DM is optimistic; $\theta = 0.5$ indicates that DM is indifference between the lower and upper possibility means, namely DM is preference neutral.

If $\theta = 0$, then $m_\mu(\tilde{a}, 0) = \underline{m}_\mu(\tilde{a})$ and $m_\nu(\tilde{a}, 0) = \underline{m}_\nu(\tilde{a})$; If $\theta = 1$, then $m_\mu(\tilde{a}, 1) = \overline{m}_\mu(\tilde{a})$ and $m_\nu(\tilde{a}, 1) = \overline{m}_\nu(\tilde{a})$; If $\theta = 0.5$, then $m_\mu(\tilde{a}, 0.5) = \frac{1}{2}[\underline{m}_\mu(\tilde{a}) + \overline{m}_\mu(\tilde{a})]$ and $m_\nu(\tilde{a}, 0.5) = \frac{1}{2}[\underline{m}_\nu(\tilde{a}) + \overline{m}_\nu(\tilde{a})]$. Thus, if the TrIFN $\tilde{a} = ([a, b, c, d]; w_{\tilde{a}}, u_{\tilde{a}})$ degenerates to the trapezoidal fuzzy number $\tilde{a} = [a, b, c, d]$, i.e., $w_{\tilde{a}} = 1$ and $u_{\tilde{a}} = 0$, then, $m_\mu(\tilde{a}, 0.5) = \frac{1}{2}[\underline{m}_\mu(\tilde{a}) + \overline{m}_\mu(\tilde{a})]$ (or $m_\nu(\tilde{a}, 0.5) = \frac{1}{2}[\underline{m}_\nu(\tilde{a}) + \overline{m}_\nu(\tilde{a})]$ is just the f weighted possibility mean of fuzzy number defined in Definition 2 of Fullér and Majlender (2003, see pp. 365).

Obviously, $m_\mu(\tilde{a}, \theta)$ synthetically reflects the information on every membership degree, and $m_\mu(\tilde{a}, 0.5)$ may be regarded as a central value that represents from the membership function point of view. Likewise, $m_\nu(\tilde{a}, \theta)$ synthetically reflects the information on every non-membership degree, and $m_\nu(\tilde{a}, 0.5)$ may be regarded as a central value that represents from the non-membership function point of view.

EXAMPLE 1. If f and g are respectively chosen as follows:

$$f(\alpha) = 2\alpha/w_{\tilde{a}} \quad (\alpha \in [0, w_{\tilde{a}}]) \tag{15}$$

and

$$g(\beta) = 2(1 - \beta)/(1 - u_{\tilde{a}}) \quad (\beta \in [u_{\tilde{a}}, 1]), \tag{16}$$

respectively, then, according to Eqs. (5), (6), (9) and (10), we have

$$\underline{m}_\mu(\tilde{a}) = \frac{1}{3}(a + 2b)w_{\tilde{a}}, \tag{17}$$

$$\overline{m}_\mu(\tilde{a}) = \frac{1}{3}(d + 2c)w_{\tilde{a}}, \tag{18}$$

$$\underline{m}_\nu(\tilde{a}) = \frac{1}{3}(a + 2b)(1 - u_{\tilde{a}}), \tag{19}$$

$$\overline{m}_\nu(\tilde{a}) = \frac{1}{3}(d + 2c)(1 - u_{\tilde{a}}). \tag{20}$$

Further, from Eqs. (13) and (14) with $\theta = 0.5$, it yields that

$$m_\mu(\tilde{a}, 0.5) = \frac{1}{6}(a + 2b + 2c + d)w_{\tilde{a}}, \tag{21}$$

$$m_\nu(\tilde{a}, 0.5) = \frac{1}{6}(a + 2b + 2c + d)(1 - u_{\tilde{a}}). \tag{22}$$

REMARK 1. If the TrIFN $\tilde{a} = ([a, b, c, d]; w_{\tilde{a}}, u_{\tilde{a}})$ degenerates to the triangular fuzzy number $\tilde{a} = (\underline{a}, a, \overline{a})$, i.e., $w_{\tilde{a}} = 1$ and $u_{\tilde{a}} = 0$, then, it follows from Eqs. (17) and (18) (or Eqs. (19) and (20)) that the weighted lower possibility mean, weighted upper possibility mean, and weighted possibility mean of a triangular fuzzy number $\tilde{a} = (\underline{a}, a, \overline{a})$ are obtained as follows: $M_*(\tilde{a}) = (\underline{a} + 2a)/3$, $M^*(\tilde{a}) = (\overline{a} + 2a)/3$, and $\overline{M}(\tilde{a}) = (\underline{a} + 4a + \overline{a})/6$, respectively. These results of a triangular fuzzy number are the same as those of a triangular fuzzy number in Examples 2.1 of Carlsson and Fullér (2001).

EXAMPLE 2. If f and g are respectively chosen as follows:

$$f(\alpha) = (n + 1)\alpha^n / w_{\tilde{a}}^n \quad \left(\alpha \in [0, w_{\tilde{a}}]\right) \tag{23}$$

and

$$g(\beta) = (n + 1)(1 - \beta)^n / (1 - u_{\tilde{a}})^n \quad \left(\beta \in [u_{\tilde{a}}, 1]\right), \tag{24}$$

respectively, then, according to Eqs. (5), (6), (9) and (10), we have

$$\underline{m}_\mu(\tilde{a}) = \left[a + \frac{n + 1}{n + 2}(b - a)\right]w_{\tilde{a}}, \tag{25}$$

$$\overline{m}_\mu(\tilde{a}) = \left[d - \frac{n + 1}{n + 2}(d - c)\right]w_{\tilde{a}}, \tag{26}$$

$$\underline{m}_\nu(\tilde{a}) = \left[a + \frac{n + 1}{n + 2}(b - a)\right](1 - u_{\tilde{a}}), \tag{27}$$

$$\overline{m}_\nu(\tilde{a}) = \left[d - \frac{n + 1}{n + 2}(d - c)\right](1 - u_{\tilde{a}}). \tag{28}$$

Further, from Eqs. (13) and (14) with $\theta = 0.5$, it yields that

$$m_\mu(\tilde{a}, 0.5) = \frac{1}{2}\left[a + d - \frac{n+1}{n+2}(a - b + c - d)\right]w_{\tilde{a}}, \tag{29}$$

$$m_\nu(\tilde{a}, 0.5) = \frac{1}{2}\left[a + d - \frac{n+1}{n+2}(a - b + c - d)\right](1 - u_{\tilde{a}}). \tag{30}$$

REMARK 2. If the TrIFN $\tilde{a} = ([a, b, c, d]; w_{\tilde{a}}, u_{\tilde{a}})$ degenerates to the trapezoidal fuzzy number $\tilde{a} = [a, b, c, d]$, i.e., $w_{\tilde{a}} = 1$ and $u_{\tilde{a}} = 0$, then, it follows from Eqs. (25) and (26) (or Eqs. (27) and (28)) that the lower weighted possibility mean, upper weighted possibility mean, and weighted possibility mean of a trapezoidal fuzzy number $[a, b, c, d]$ are obtained as follows: $M_*(\tilde{a}) = a + \frac{n+1}{n+2}(b - a)$, $M^*(\tilde{a}) = d - \frac{n+1}{n+2}(d - c)$, and $\overline{m}(\tilde{a}) = \frac{1}{2}[a + d - \frac{n+1}{n+2}(a - b + c - d)]$, respectively. These results of a trapezoidal fuzzy number are the same as those of a trapezoidal fuzzy number in Examples 2 of Fullér and Majlender (2003, see pp. 371).

The weighted possibility means have some useful properties, which are summarized in Theorem 3.

Theorem 2. *Let* $\tilde{a}_i = ([a_i, b_i, c_i, d_i]; w_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2)$ *be two TrIFNs with* $w_{\tilde{a}_1} = w_{\tilde{a}_2}$ *and* $u_{\tilde{a}_1} = u_{\tilde{a}_2}$. *Then for any* $\gamma > 0$ *and* $\tau > 0$, *the following equalities are valid*:

$$m_\mu(\gamma\tilde{a}_1 + \tau\tilde{a}_2, \theta) = \gamma m_\mu(\tilde{a}_1, \theta) + \tau m_\mu(\tilde{a}_2, \theta), \tag{31}$$

$$m_\nu(\gamma\tilde{a}_1 + \tau\tilde{a}_2, \theta) = \gamma m_\nu(\tilde{a}_1, \theta) + \tau m_\nu(\tilde{a}_2, \theta). \tag{32}$$

Proof. Since $\gamma > 0$ and $\tau > 0$, by Definition 2, the α-cut set of TrIFN $\gamma\tilde{a}_1 + \tau\tilde{a}_2$ is as follows:

$$(\gamma\tilde{a}_1 + \tau\tilde{a}_2)_\alpha = \left[\gamma a_{1\alpha}^l + \tau a_{2\alpha}^l, \gamma a_{1\alpha}^u + \tau a_{2\alpha}^u\right].$$

By Eq. (13), we obtain

$$m_\mu(\gamma\tilde{a}_1 + \tau\tilde{a}_2, \theta)$$
$$= \int_0^{w_{\tilde{a}_1} \wedge w_{\tilde{a}_2}} f(\alpha)\left[(1 - \theta)(\gamma a_{1\alpha}^l + \tau a_{2\alpha}^l) + \theta(\gamma a_{1\alpha}^u + \tau a_{2\alpha}^u)\right]d\alpha$$
$$= \int_0^{w_{\tilde{a}_1}} \gamma f(\alpha)\left[(1 - \theta)a_{1\alpha}^l + \theta a_{1\alpha}^u\right]d\alpha + \int_0^{w_{\tilde{a}_1}} \tau f(\alpha)\left[(1 - \theta)a_{2\alpha}^l + \theta a_{2\alpha}^u\right]d\alpha$$
$$= \gamma m_\mu(\tilde{a}_1, \theta) + \tau m_\mu(\tilde{a}_2, \theta).$$

Thus, Eq. (31) holds. By the same way, Eq. (32) can be proven. Namely, Theorem 3 is proven. $\square$

It is noted that if $\gamma = \tau = 1$, then by Theorem 1 the following are valid:

$$m_\mu(\tilde{a}_1 + \tilde{a}_2, \theta) = m_\mu(\tilde{a}_1, \theta) + m_\mu(\tilde{a}_2, \theta)$$

and

$$m_\nu(\tilde{a}_1 + \tilde{a}_2, \theta) = m_\nu(\tilde{a}_1, \theta) + m_\nu(\tilde{a}_2, \theta).$$

The above two equalities are similar to Theorems 2 and 3 of Li (2010c).

3.2. *The Weighted Possibility Variances of TrIFNs*

Next, we give the definition of the weighted possibility variances of TrIFNs.

DEFINITION 8. For a TrIFN $\tilde{a} = ([a, b, c, d]; w_{\tilde{a}}, u_{\tilde{a}})$, the f weighted possibility variance of membership function is defined as follows:

$$V_\mu(\tilde{a}) = \int_0^{w_{\tilde{a}}} \left(\frac{a_\alpha^u - a_\alpha^l}{2} \right)^2 f(\alpha) \, d\alpha, \tag{33}$$

the g weighted possibility variance of non-membership function is defined as follows:

$$V_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^1 \left(\frac{a_\beta^u - a_\beta^l}{2} \right)^2 g(\beta) \, d\beta. \tag{34}$$

The f weighted possibility variance $V_\mu(\tilde{a})$ of the membership function is defined as the expected value of the squared deviations between the endpoints of α-cut set. The g weighted possibility variance $V_\nu(\tilde{a})$ of the non-membership function is defined as the expected value of the squared deviations between the endpoints of β-cut set.

It is easily seen that $a_\alpha^u - a_\alpha^l$ and $a_\beta^u - a_\beta^l$ are just about the lengths of the intervals $\tilde{a}_\alpha$ and $\tilde{a}_\beta$, respectively. Thus, $V_\mu(\tilde{a})$ and $V_\nu(\tilde{a})$ may be respectively regarded as the global spreads of the membership function $\mu_{\tilde{a}}(x)$ and non-membership function $\nu_{\tilde{a}}(x)$. Clearly, $V_\mu(\tilde{a})$ and $V_\nu(\tilde{a})$ basically measure how much there is uncertainty and vagueness in the TrIFN $\tilde{a}$.

EXAMPLE 3. The weighting functions f and g are respectively chosen as Eqs. (15) and (16). According to Eqs. (3), (4), (33) and (34), the weighted possibility variances are calculated as follows:

$$V_\mu(\tilde{a}) = \frac{1}{24} \left[6(d-a)^2 + 8(d-a)(a-b+c-d) + 3(a-b+c-d)^2 \right] w_{\tilde{a}}, \tag{35}$$

and

$$V_\nu(\tilde{a}) = \frac{1}{24} \left[6(d-a)^2 + 8(d-a)(a-b+c-d) + 3(a-b+c-d)^2 \right] (1 - u_{\tilde{a}}). \tag{36}$$

REMARK 3. If the TrIFN $\tilde{a} = ([a, b, c, d]; w_{\tilde{a}}, u_{\tilde{a}})$ degenerates to the triangular fuzzy number $\tilde{a} = (\underline{a}, a, \overline{a})$, i.e., $w_{\tilde{a}} = 1$ and $u_{\tilde{a}} = 0$, then, by Eq. (25) or Eq. (26) the possibility variance of a triangular fuzzy number $\tilde{a} = (\underline{a}, a, \overline{a})$ is obtained as $V(\tilde{a}) = (\overline{a} - \underline{a})^2/24$, which is accordance with that of a triangular fuzzy number in Carlsson and Fullér (2001, see pp. 322).

EXAMPLE 4. The weighting functions f and g are respectively chosen as Eqs. (23) and (24). According to Eqs. (3), (4), (33) and (34), the weighted possibility variances are calculated as follows:

$$
\begin{aligned}
V_\mu(\tilde{a}) = \frac{1}{4}\bigg[& 6(d-a)^2 + \frac{2(n+1)}{n+2}(d-a)(a-b+c-d) \\
& + \frac{n+1}{n+3}(a-b+c-d)^2 \bigg] w_{\tilde{a}},
\end{aligned}
\tag{37}
$$

and

$$
\begin{aligned}
V_\nu(\tilde{a}) = \frac{1}{4}\bigg[& 6(d-a)^2 + \frac{2(n+1)}{n+2}(d-a)(a-b+c-d) \\
& + \frac{n+1}{n+3}(a-b+c-d)^2 \bigg] (1-u_{\tilde{a}}).
\end{aligned}
\tag{38}
$$

REMARK 4. If the TrIFN $\tilde{a} = ([a, b, c, d]; w_{\tilde{a}}, u_{\tilde{a}})$ degenerates to the trapezoidal fuzzy number $\tilde{a} = [a, b, c, d]$, i.e., $w_{\tilde{a}} = 1$ and $u_{\tilde{a}} = 0$, then, then by Eq. (37) or Eq. (38) the possibility variance of a trapezoidal fuzzy number $\tilde{a} = [a, b, c, d]$ is obtained as $V(\tilde{a}) = \frac{1}{4}[6(d-a)^2 + \frac{2(n+1)}{n+2}(d-a)(a-b+c-d) + \frac{n+1}{n+3}(a-b+c-d)^2]$, which is accordance with that of a trapezoidal fuzzy number in Fullér and Majlender (2003, see Example 5, pp. 373).

Theorem 3. *Let $\tilde{a} = ([a, b, c, d]; w_{\tilde{a}}, u_{\tilde{a}})$ be a TrIFN. Then for any $\gamma \in R$, the following equalities are valid*:

$$V_\mu(\gamma\tilde{a}) = \gamma^2 V_\mu(\tilde{a}), \tag{39}$$

$$V_\nu(\gamma\tilde{a}) = \gamma^2 V_\nu(\tilde{a}). \tag{40}$$

Proof. We only prove $V_\mu(\gamma\tilde{a}) = \gamma^2 V_\mu(\tilde{a})$. According to Definitions 2, 3 and Eq. (33), we have

$$V_\mu(\gamma\tilde{a}) = \int_0^{w_{\tilde{a}}} \left(\frac{\gamma a_\alpha^u - \gamma a_\alpha^l}{2} \right)^2 f(\alpha)\, d\alpha = \gamma^2 V_\mu(\tilde{a}).$$

By the same way, $V_\nu(\gamma\tilde{a}) = \gamma^2 V_\nu(\tilde{a})$. Namely, Theorem 3 is proven. □

3.3. *Lexicographic Ranking Method of TrIFNs Based on Weighted Possibility Mean and Variance*

The possibility mean and variance of fuzzy numbers are similar to the mean and variance of random variables. They can be used to quantitatively characterize the values of fuzzy numbers as well as the inherent uncertainty. Obviously, the greater the possibility mean, the bigger the corresponding fuzzy numbers; the greater the possibility variance, the larger the degree of vagueness and uncertainty of the fuzzy numbers.

Let $m_\mu(\tilde{a}_i, \theta)$, $m_\nu(\tilde{a}_i, \theta)$, $V_\mu(\tilde{a}_i, \theta)$, $V_\nu(\tilde{a}_i, \theta)$ be the weighted possibility mean and variance of the membership and non-membership functions for TrIFNs $\tilde{a}_i = ([a_i, b_i, c_i, d_i]; w_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2)$, respectively. The ranking indices of the membership and non-membership functions for TrIFN $\tilde{a}_i$ are defined as follows:

$$R_\mu(\tilde{a}_i, \theta) = m_\mu(\tilde{a}_i, \theta) - \lambda V_\mu(\tilde{a}_i), \tag{41}$$

and

$$R_\nu(\tilde{a}_i, \theta) = m_\nu(\tilde{a}_i, \theta) - \lambda V_\nu(\tilde{a}_i), \tag{42}$$

respectively, where $\lambda \in [0, 1]$ is the risk-aversion parameter of DM. The larger the value of parameter λ, the greater the degree that DM hates risk.

Thereby, a lexicographic ranking method between two TrIFNs $\tilde{a}_1$ and $\tilde{a}_2$ can be summarized as follows:

(1) if $R_\mu(\tilde{a}_1, \theta) < R_\mu(\tilde{a}_2, \theta)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$;
(2) if $R_\mu(\tilde{a}_1, \theta) > R_\mu(\tilde{a}_2, \theta)$, then $\tilde{a}_1$ is bigger than $\tilde{a}_2$, denoted by $\tilde{a}_1 > \tilde{a}_2$;
(3) if $R_\mu(\tilde{a}_1, \theta) = R_\mu(\tilde{a}_2, \theta)$, then
    (a) if $R_\nu(\tilde{a}_1, \theta) < R_\nu(\tilde{a}_2, \theta)$, then $\tilde{a}_1 < \tilde{a}_2$;
    (b) if $R_\nu(\tilde{a}_1, \theta) > R_\nu(\tilde{a}_2, \theta)$, then $\tilde{a}_1 > \tilde{a}_2$;
    (c) if $R_\nu(\tilde{a}_1, \theta) = R_\nu(\tilde{a}_2, \theta)$, then $\tilde{a}_1$ and $\tilde{a}_2$ represent the same information, denoted by $\tilde{a}_1 = \tilde{a}_2$.

REMARK 5. The weighting functions f and g can be chosen as several forms. For computation convenience, the weighting functions f and g are respectively chosen as Eqs. (15) and (16) in the following. Thus, the ranking indices of the membership and non-membership functions for TrIFN $\tilde{a}$ are obtained as follows:

$$R_\mu(\tilde{a}, \theta, \lambda) = \frac{1}{3}w_{\tilde{a}}\big[(1 - \theta)(a + 2b) + \theta(c + 2d)\big] - \frac{1}{8}\lambda\big[6(d - a)^2$$
$$+ 8(d - a)(a - b + c - d) + 3(a - b + c - d)^2\big] \tag{43}$$

and

$$R_\nu(\tilde{a}, \theta, \lambda) = \frac{1}{3}(1 - u_{\tilde{a}})\big[(1 - \theta)(a + 2b) + \theta(c + 2d)\big] - \frac{1}{8}\lambda\big[6(d - a)^2$$
$$+ 8(d - a)(a - b + c - d) + 3(a - b + c - d)^2\big]. \tag{44}$$

4. MAGDM Model and Method Using TrIFNs

In this section, a new method is developed to solve the MAGDM problems with TrIFNs and incomplete weight preference information.

4.1. *Description of MAGDM Problems Using TrIFNs*

For some MAGDM problems, denote an alternative set by $A = \{A_1, A_2, \ldots, A_m\}$ and an attribute set by $C = \{C_1, C_2, \ldots, C_n\}$. Assume that there are p DMs participating in decision making, denote the set of DMs by $E = \{e_1, e_2, \ldots, e_p\}$. The weight vector of attributes given by DM e_k is $w^k = (w_1^k, w_2^k, \ldots, w_n^k)^T$ $(k = 1, 2, \ldots, p)$, satisfying that $0 \leqslant w_j^k \leqslant 1$ and $\sum_{j=1}^{n} w_j^k = 1$. The weight vector of DMs is $V = (v_1, v_2, \ldots, v_p)^T$, satisfying that $0 \leqslant v_k \leqslant 1$ and $\sum_{k=1}^{p} v_k = 1$. Both w^k and V are unknown to be determined. Suppose that the rating of an alternative A_i on an attribute C_j given by the DM e_k is a TrIFN $\tilde{a}_{ij}'^k = ([a_{ij}'^k, b_{ij}'^k, c_{ij}'^k, d_{ij}'^k]; \omega_{\tilde{a}_{ij}'^k}, u_{\tilde{a}_{ij}'^k})$, where $\omega_{\tilde{a}_{ij}'^k}$ and $u_{\tilde{a}_{ij}'^k}$ denote respectively the maximum membership degree and the minimum non-membership degree of alternative A_i on attribute C_j given by the DM e_k, satisfying $0 \leqslant \omega_{\tilde{a}_{ij}'^k} \leqslant 1$, $0 \leqslant u_{\tilde{a}_{ij}'^k} \leqslant 1$ and $\omega_{\tilde{a}_{ij}'^k} + u_{\tilde{a}_{ij}'^k} \leqslant 1$.

Hence, a MAGDM problem can be concisely expressed in matrix format as $\tilde{A}' = (\tilde{a}_{ij}'^k)_{m \times n}$ $(k = 1, 2, \ldots, p)$, which are referred to as TrIFN decision matrices usually used to represent the MAGDM problem.

To eliminate the impact of different dimensions on the decision results, the matrix $\tilde{A}'$ needs to be normalized into $\tilde{A} = (\tilde{a}_{ij}^k)_{m \times n}$, where $\tilde{a}_{ij}^k = ([a_{ij}^k, b_{ij}^k, c_{ij}^k, d_{ij}^k]; \omega_{\tilde{a}_{ij}^k}, u_{\tilde{a}_{ij}^k})$, $\omega_{\tilde{a}_{ij}^k} = \omega_{\tilde{a}_{ij}'^k}$ and $u_{\tilde{a}_{ij}^k} = u_{\tilde{a}_{ij}'^k}$. In this paper, the normalization method is chosen as follows:

For benefit attributes,

$$\tilde{a}_{ij}^k = ([a_{ij}'^k / d_j^+, b_{ij}'^k / d_j^+, c_{ij}'^k / d_j^+, d_{ij}'^k / d_j^+]; \omega_{\tilde{a}_{ij}^k}, u_{\tilde{a}_{ij}^k}). \tag{45}$$

For cost attributes,

$$\tilde{a}_{ij}^k = ([1 - d_{ij}'^k / d_j^+, 1 - c_{ij}'^k / d_j^+, 1 - b_{ij}'^k / d_j^+, 1 - a_{ij}'^k / d_j^+]; \omega_{\tilde{a}_{ij}^k}, u_{\tilde{a}_{ij}^k}), \tag{46}$$

where $d_j^+ = \max\{d_{ij}'^k \ i = 1, 2, \ldots, m; \ k = 1, 2, \ldots, p\}$ $(j = 1, 2, \ldots, n)$.

4.2. *Incomplete Weight Information Structure*

In decision making process, weights of attributes should be taken into account. The weight vector of attributes given by e_k is $w^k = (w_1^k, w_2^k, \ldots, w_n^k)^T$ $(k = 1, 2, \ldots, p)$, satisfying that $0 \leqslant w_j^k \leqslant 1$ $(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} w_j^k = 1$. Let $\Lambda_0^k = \{w^k \mid \sum_{j=1}^{n} w_j^k = 1, \ w_j^k \geqslant \varepsilon$ for $j = 1, 2, \ldots, n\}$, where ε is a sufficiently small positive number. The constraints $\varepsilon > 0$ can ensure that each weight of $w_j^k \geqslant \varepsilon$ is not smaller than a given sufficiently small positive number ε (for instance, $\varepsilon = 0.05$).

In some real decision situations, the DM e_k may specify some preference relations on weights of attributes according to his/her knowledge, experience and judgment. Such information of attribute weights is incomplete. Usually incomplete information of attribute weights can be obtained according to partial preference relations on weights given by the DM e_k and has several different structure forms. Li (2011a, 2011b) and Wan and Li (2013a, 2013b) mathematically and rigorously expressed these weight information structures in the following five basic relations among attribute weights, which are denoted by subsets Λ_h^k ($h = 1, 2, 3, 4, 5$) of weight vectors in Λ_0^k, respectively.

(1) The set of weights expressing a weak ranking: $\Lambda_1^k = \{w^k \in \Lambda_0^k | w_t^k \geqslant w_j^k$ for all $t \in T_1^k$ and $j \in J_1^k\}$, where T_1^k and J_1^k are two disjoint subsets of the subscript index set $N = \{1, 2, \ldots, n\}$. Thus, Λ_1^k is a set of all weight vectors in Λ_0^k with the property that the weight of an attribute in the set T_1^k is greater than or equal to that of an attribute in the set J_1^k.

(2) The set of weights expressing a strict ranking: $\Lambda_2^k = \{w^k \in \Lambda_0^k | \beta_{tj}^k \geqslant w_t^k - w_j^k \geqslant \alpha_{tj}^k$ for all $t \in T_2^k$ and $j \in J_2^k\}$, where $\alpha_{tj}^k > 0$ and $\beta_{tj}^k > 0$ are constants, satisfying $\beta_{tj}^k > \alpha_{tj}^k$; T_2^k and J_2^k are two disjoint subsets of N. Thus, Λ_2^k is a set of all weight vectors in Λ_0^k with the property that the weight of an attribute in the set T_2^k is greater than or equal to that of an attribute in the set J_2^k but their difference does not exceed some range, i.e., a closed interval $[\alpha_{tj}^k, \beta_{tj}^k]$.

(3) The set of weights expressing a ranking with multiples: $\Lambda_3^k = \{w^k \in \Lambda_0^k | w_t^k \geqslant \xi_{tj}^k w_j^k$ for all $t \in T_3^k$ and $j \in J_3^k\}$, where $\xi_{tj}^k > 0$ is a constant; T_3^k and J_3^k are two disjoint subsets of N. Thus, Λ_3^k is a set of all weight vectors in Λ_0^k with the property that the weight of an attribute in the set T_3^k is greater than or equal to ξ_{tj}^k multiple of that of an attribute in the set J_3^k.

(4) The set of weights expressing an interval form: $\Lambda_4^k = \{w^k \in \Lambda_0^k | \gamma_j^k \geqslant w_j^k \geqslant \eta_j^k$ for all $j \in J_4^k\}$, where $\gamma_j^k > 0$ and $\eta_j^k > 0$ are constants, satisfying $\gamma_j^k \geqslant \eta_j^k > 0$; J_4^k is a subset of N. Thus, Λ_4^k is a set of all weight vectors in Λ_0^k with the property that the weight of an attribute in the set J_4^k does not exceed some range, i.e., a closed interval $[\eta_j^k, \gamma_j^k]$.

(5) The set of weights expressing a ranking of differences: $\Lambda_5^k = \{w^k \in \Lambda_0^k | w_t^k - w_j^k \geqslant w_l^k - w_s^k$ for all $t \in T_5^k$, $j \in J_5^k$, $l \in L_5^k$, and $s \in S_5^k\}$, where T_5^k, J_5^k, L_5^k, and S_5^k are four disjoint subsets of N. Thus, Λ_5^k is a set of all weight vectors in Λ_0^k with the property that the difference between weights of attributes in the sets T_5^k and J_5^k, is greater than or equal to that of attributes in the sets L_5^k and S_5^k.

Cases 1–5 are well known types of imprecise information, and Case 5 is a ranking of differences of adjacent parameters obtained by weak rankings among the parameters, which can be subsequently constructed based on Case 1.

In reality, usually the preference information structure Λ^k of attribute importance may consist of several sets of the above basic sets Λ_h^k ($h = 1, 2, 3, 4, 5$) or may contain all the five basic sets, which depend on the characteristic and need of the real-life decision problems. For example, suppose that three attributes C_1, C_2 and C_3 are used to assess the stocks in some stock selection problem. Then, the subscript index set of all attributes is $N = \{1, 2, 3\}$. The DM e_1 may provide a preference information structure expressed as

Table 1
The relationship between linguistic variables and intuitionistic
fuzzy values for rating the importance of experts.

Linguistic variables	Intuitionistic fuzzy values
Very important	(0.90, 0.10, 0.0)
Important	(0.80, 0.10, 0.1)
Medium	(0.60, 0.30, 0.1)
Not important	(0.30, 0.60, 0.1)
Very unimportant	(0.10, 0.90, 0.0)

follows (Li, 2011a, 2011b; Wan and Li, 2013a, 2013b):

$$\Lambda^1 = \{w^1 \in \Lambda_0^1 | 0.15 \leqslant w_1^1 \leqslant 0.55, \; 0.2 \leqslant w_2^1 \leqslant 0.65, \; 0.10 \leqslant w_3^1 \leqslant 0.35,$$
$$w_2^1 \geqslant w_1^1, \; 0.02 \leqslant w_2^1 - w_3^1 \leqslant 0.45\},$$

where $\Lambda_0^1 = \{w^1 = (w_1^1, w_2^1, w_3^1)^T | w_1^1 + w_2^1 + w_3^1 = 1, \; w_j^1 \geqslant \varepsilon \; (j = 1, 2, 3)\}$. In fact, Λ^1 may be regarded as consisting of the following three basic subsets:

The set of weights expressing a strict ranking: $\Lambda_2^1 = \{w^1 \in \Lambda_0^1 | 0.02 \leqslant w_2^1 - w_3^1 \leqslant 0.45\}$, $\beta_{23}^1 = 0.45 > \alpha_{23}^1 = 0.02$, $T_2^1 = \{2\} \subseteq N$, $J_2^1 = \{3\} \subseteq N$ and $T_2^1 \cap J_2^1 = \phi$.

The set of weights expressing a ranking with multiples: $\Lambda_3^1 = \{w^1 \in \Lambda_0^1 | w_2^1 \geqslant 1.2 w_1^1\}$, where $\xi_{21}^1 = 1.2 > 0$, $T_3^1 = \{2\} \subseteq N$, $J_3^1 = \{1\} \subseteq N$ and $T_3^1 \cap J_3^1 = \phi$.

The set of weights expressing an interval form: $\Lambda_4^1 = \{w^1 \in \Lambda_0^1 | 0.15 \leqslant w_1^1 \leqslant 0.55, \; 0.2 \leqslant w_2^1 \leqslant 0.65, \; 0.10 \leqslant w_3^1 \leqslant 0.35, \}$, where $\gamma_1^1 = 0.55 > \eta_1^1 = 0.15$, $\gamma_2^1 = 0.65 > \eta_2^1 = 0.2$, $\gamma_3^1 = 0.35 > \eta_3^1 = 0.1$, $J_4^1 = \{1, 2, 3\} \subseteq N$.

In other words, the information structure Λ^1 consists of the above three sets Λ_2^1, Λ_3^1 and Λ_4^1. These parameters β_{23}^1, α_{23}^1, ξ_{21}^1, γ_1^1, η_1^1, γ_2^1, η_2^1, γ_3^1, and η_3^1 can be chosen according to the DM's knowledge, experience, preference and judgment, and thus the corresponding subsets T_2^1, J_2^1, T_3^1, and J_4^1 can be determined.

4.3. *Determining DMs' Weights on the Basis of IFS Voting Model*

In real-life decision problems, it is common that the importance of DMs are usually expressed by linguistic variables, such as "important", "medium", "not important" and so on. Assume that the linguistic variables can be transformed into intuitionistic fuzzy values. The corresponding relationship between the linguistic variables and intuitionistic fuzzy values used in this paper is listed in Table 1.

Denote the intuitionistic fuzzy value of the importance for DM e_k by $\delta_k = (\mu_k, \nu_k, \pi_k)$. According to the voting model of intuitionistic fuzzy sets, μ_k, ν_k, and π_k can be interpreted as proportions of the affirmative, dissent and abstention in a vote, respectively. Considering the possibility that in abstention group some people tend to cast affirmative votes, others are dissenters and still others tend to abstain from voting, we can divide the abstention proportion π_k into three parts: $\mu_k \pi_k$, $\nu_k \pi_k$, and $\pi_k \pi_k$, which express the proportions of the affirmative, dissent and abstention in original part of abstention (Liu and Wang, 2007). Thus, the score function of intuitionistic fuzzy value $\delta_k = (\mu_k, \nu_k, \pi_k)$ is defined

as $s_k = \mu_k + \mu_k \pi_k = \mu_k(2 - \mu_k - \nu_k)$ $(k = 1, 2, \ldots, p)$. Normalized the score functions s_k $(k = 1, 2, \ldots, p)$, the weight of DM e_k can be generated as follows:

$$v_k = \left[\mu_k(2 - \mu_k - \nu_k)\right] \Big/ \sum_{k=1}^{p} \left[\mu_k(2 - \mu_k - \nu_k)\right] \quad (k = 1, 2, \ldots, p). \tag{47}$$

4.4. Determining the Weights of Attributes Based on Bi-Objective Quadratic Programming

By using the ϕ_w^A operator to integrate all the attribute values of alternative A_i given by DM e_k, the individual overall attribute value of alternative A_i given by DM e_k is obtained as follows:

$$\begin{aligned}
\tilde{a}_i^k &= \phi_{w^k}^A\left(\tilde{a}_{i1}^k, \tilde{a}_{i2}^k, \ldots, \tilde{a}_{in}^k\right) \\
&= \left(\left[\sum_{j=1}^{n} w_j^k a_{ij}^k, \sum_{j=1}^{n} w_j^k b_{ij}^k, \sum_{j=1}^{n} w_j^k c_{ij}^k, \sum_{j=1}^{n} w_j^k d_{ij}^k\right]; \wedge_j\{\omega_{\tilde{a}_{ij}^k}\}, \vee_j\{u_{\tilde{a}_{ij}^k}\}\right), \quad \tag{48}
\end{aligned}$$

where $w^k = (w_1^k, w_2^k, \ldots, w_n^k)^T$ is the weight vector of attributes given by DM e_k.

By Eqs. (13) and (14), the weighted possibility means of the membership and non-membership functions for the individual overall attribute value $\tilde{a}_i^k$ of alternative A_i are computed as follows:

$$m_\mu(\tilde{a}_i^k, \theta_k) = \frac{1}{3} \wedge_j\{\omega_{\tilde{a}_{ij}^k}\}\left[(1 - \theta_k)\sum_{j=1}^{n} w_j^k\left(a_{ij}^k + 2b_{ij}^k\right) + \theta_k\sum_{j=1}^{n} w_j^k\left(d_{ij}^k + 2c_{ij}^k\right)\right] \tag{49}$$

and

$$\begin{aligned}
&m_\nu\left(\tilde{a}_i^k, \theta_k\right) \\
&= \frac{1}{3}\left(1 - \vee_j\{u_{\tilde{a}_{ij}^k}\}\right)\left[(1 - \theta_k)\sum_{j=1}^{n} w_j^k\left(a_{ij}^k + 2b_{ij}^k\right) + \theta_k\sum_{j=1}^{n} w_j^k\left(d_{ij}^k + 2c_{ij}^k\right)\right], \quad \tag{50}
\end{aligned}$$

respectively, where θ_k is the preference parameter for the lower and upper possibility means of DM e_k.

According to Eqs. (35) and (36), the weighted possibility variances of the membership and non-membership functions for the individual overall attribute value $\tilde{a}_i^k$ of alternative A_i are calculated as follows:

$$V_\mu(\tilde{a}_i^k) = \frac{1}{24} \wedge_j\{\omega_{\tilde{a}_{ij}^k}\}\left[6\left(\sum_{j=1}^{n} w_j^k\left(d_{ij}^k - a_{ij}^k\right)\right)^2\right]$$

$$+ 8 \sum_{j=1}^{n} w_j^k \left(d_{ij}^k - a_{ij}^k \right) \sum_{j=1}^{n} w_j^k \left(a_{ij}^k - b_{ij}^k + c_{ij}^k - d_{ij}^k \right)$$

$$+ 3 \left(\sum_{j=1}^{n} w_j^k \left(a_{ij}^k - b_{ij}^k + c_{ij}^k - d_{ij}^k \right) \right)^2 \Bigg] \tag{51}$$

and

$$V_\nu(\tilde{a}_i^k) = \frac{1}{24} \left(1 - \vee_j \{ u_{\tilde{a}_{ij}^k} \} \right) \Bigg[6 \left(\sum_{j=1}^{n} w_j^k \left(d_{ij}^k - a_{ij}^k \right) \right)^2$$

$$+ 8 \sum_{j=1}^{n} w_j^k \left(d_{ij}^k - a_{ij}^k \right) \sum_{j=1}^{n} w_j^k \left(a_{ij}^k - b_{ij}^k + c_{ij}^k - d_{ij}^k \right)$$

$$+ 3 \left(\sum_{j=1}^{n} w_j^k \left(a_{ij}^k - b_{ij}^k + c_{ij}^k - d_{ij}^k \right) \right)^2 \Bigg]. \tag{52}$$

According to Eqs. (43) and (44), the ranking indices of the membership and non-membership functions for the individual overall attribute value $\tilde{a}_i^k$ of alternative A_i are calculated as follows:

$$R_\mu(\tilde{a}_i^k, \theta_k, \lambda_k) = \frac{1}{3} \wedge_j \{ \omega_{\tilde{a}_{ij}^k} \} \Bigg\{ \Bigg[(1 - \theta_k) \sum_{j=1}^{n} w_j^k \left(a_{ij}^k + 2b_{ij}^k \right)$$

$$+ \theta_k \sum_{j=1}^{n} w_j^k \left(d_{ij}^k + 2c_{ij}^k \right) \Bigg] - \frac{1}{8} \lambda_k \Bigg[6 \left(\sum_{j=1}^{n} w_j^k \left(d_{ij}^k - a_{ij}^k \right) \right)^2$$

$$+ 8 \sum_{j=1}^{n} w_j^k \left(d_{ij}^k - a_{ij}^k \right) \sum_{j=1}^{n} w_j^k \left(a_{ij}^k - b_{ij}^k + c_{ij}^k - d_{ij}^k \right)$$

$$+ 3 \left(\sum_{j=1}^{n} w_j^k \left(a_{ij}^k - b_{ij}^k + c_{ij}^k - d_{ij}^k \right) \right)^2 \Bigg] \Bigg\} \tag{53}$$

and

$$R_\nu(\tilde{a}_i^k, \theta_k, \lambda_k) = \frac{1}{3} \left(1 - \vee_j \{ u_{\tilde{a}_{ij}^k} \} \right) \Bigg\{ \Bigg[(1 - \theta_k) \sum_{j=1}^{n} w_j^k \left(a_{ij}^k + 2b_{ij}^k \right)$$

$$+ \theta_k \sum_{j=1}^{n} w_j^k \left(d_{ij}^k + 2c_{ij}^k \right) \Bigg] - \frac{1}{8} \lambda_k \Bigg[6 \left(\sum_{j=1}^{n} w_j^k \left(d_{ij}^k - a_{ij}^k \right) \right)^2$$

$$+ 8 \sum_{j=1}^{n} w_j^k \left(d_{ij}^k - a_{ij}^k \right) \sum_{j=1}^{n} w_j^k \left(a_{ij}^k - b_{ij}^k + c_{ij}^k - d_{ij}^k \right)$$

$$+ 3 \left(\sum_{j=1}^{n} w_j^k \left(a_{ij}^k - b_{ij}^k + c_{ij}^k - d_{ij}^k \right) \right)^2 \Bigg] \Bigg\}, \tag{54}$$

where λ_k is the risk aversion parameter of DM e_k.

For each alternative, the bigger the individual overall attribute value, then the bigger the collective overall attribute value, thus the larger the possibility of the alternative becoming the best alternative. Hence, the following multi-objective fuzzy programming model for DM e_k is set up:

$$\max \left\{ \tilde{a}_i^k \right\} \quad (i = 1, 2, \dots, m)$$

$$\text{s.t. } w^k = \left(w_1^k, w_2^k, \dots, w_n^k \right)^T \in \Lambda^k. \tag{55}$$

According to the ranking method of TrIFNs in Section 2.4, the bigger the ranking indices of membership and non-membership functions, the better the alternative. Therefore, the reasonable weight vector of attributes $w^k = (w_1^k, w_2^k, \dots, w_n^k)^T$ should be obtained so that all the ranking indices of membership and non-membership functions for alternatives could be as big as possible. Consequently, Eq. (55) can be transformed into the multi-objective programming model as follows:

$$\max \left\{ R_\mu \left(\tilde{a}_i^k, \theta_k \right), R_\nu \left(\tilde{a}_i^k, \theta_k \right) \right\} \quad (i = 1, 2, \dots, m)$$

$$\text{s.t. } w^k = \left(w_1^k, w_2^k, \dots, w_n^k \right)^T \in \Lambda^k. \tag{56}$$

Since there is no any preference among the alternatives, Eq. (56) can be further transformed into the bi-objective programming model as follows:

$$\max \left\{ z_1^k = \sum_{i=1}^{m} R_\mu \left(\tilde{a}_i^k, \theta_k \right) \right\},$$

$$\max \left\{ z_2^k = \sum_{i=1}^{m} R_\nu \left(\tilde{a}_i^k, \theta_k \right) \right\}$$

$$\text{s.t. } w^k = \left(w_1^k, w_2^k, \dots, w_n^k \right)^T \in \Lambda^k. \tag{57}$$

Obviously, Eq. (57) is a bi-objective non-linear programming model on the decision variable vector $w^k = (w_1^k, w_2^k, \dots, w_n^k)^T$. There are few standard ways of defining a solution of multi-objective programming. Normally, the concept of Pareto optimal (efficient) solutions is commonly used. There exist several solution methods for them. However, in this study we focus on developing a linear sum method based on membership function to solve Eq. (56) in the sense of Pareto optimality.

The two objective functions z_1^k and z_2^k in Eq. (57) are the functions of the decision variable vector $w^k = (w_1^k, w_2^k, \ldots, w_n^k)^T$, simply denoted by $z_1^k(w^k)$ and $z_2^k(w^k)$. Let $z_t^{k\,max}$ and w_t^k respectively be the maximum objective value and the optimal solution for the following single objective quadratic programming model:

$$\max \left\{ z_t^k = z_t^k(w^k) \right\}$$
$$\text{s.t. } w^k = \left(w_1^k, w_2^k, \ldots, w_n^k \right)^T \in \Lambda^k. \tag{58}$$

Then, set $z_t^{k\,min} = \min\{z_t^k(w_1^k), z_t^k(w_2^k)\}$ $(t = 1, 2)$. The membership function of the objective function $z_t^k = z_t^k(w^k)$ $(t = 1, 2)$ can be computed as follows:

$$\mu_{z_t^k}(w^k) = \begin{cases} 0, & \text{if } x < z_t^{k\,min}, \\ \frac{z_t^k - z_t^{k\,min}}{z_t^{k\,max} - z_t^{k\,min}}, & \text{if } z_t^{k\,min} \leqslant z_t^k \leqslant z_t^{k\,max}, \\ 1, & \text{if } z_t^k > z_t^{k\,max}. \end{cases} \tag{59}$$

Thus, Eq. (57) can be solved by the following single objective quadratic programming model:

$$\max \left\{ \eta_1 \mu_{z_1^k}(w^k) + \eta_2 \mu_{z_2^k}(w^k) \right\}$$
$$\text{s.t. } w^k = \left(w_1^k, w_2^k, \ldots, w_n^k \right)^T \in \Lambda^k, \tag{60}$$

where η_t is the weight of z_t^k, satisfying that $\eta_t \geqslant 0$ $(t = 1, 2)$ and $\eta_1 + \eta_2 = 1$.

REMARK 6. Equations (58) and (60) are the quadratic programming models. When the numbers of decision variables in the models are finite, they can be solved by using the common software, such as Matlab and Lingo. When the numbers of decision variables are very large, they can be efficiently solved by using the sequential minimal optimization (SMO) algorithm (Zhang *et al.*, 2009; Keerthi *et al.*, 2001).

4.5. *MAGDM Method using TrIFNs*

In sum, an algorithm and process of the MAGDM problems with TrIFNs and incomplete weight preference information may be summarized as follows.

Step 1: Normalize the decision matrix $\tilde{A}'^k$ into $\tilde{A}^k$ according to Eqs. (45) and (46).

Step 2: Determine the weight vector of DMs $V = (v_1, v_2, \ldots, v_p)^T$ according to Eqs. (47).

Step 3: Construct the bi-objective programming model (57) to derive the attribute weight vector $w^k = (w_1^k, w_2^k, \ldots, w_n^k)^T$ given by DM e_k through solving Eq. (60).

Step 4: Calculate the individual overall attribute value $\tilde{a}_i^k$ of alternative A_i given by e_k according to Eq. (58).

Step 5: Combined the weight vector of DMs $V = (v_1, v_2, \ldots, v_p)^T$ with the ϕ_w^A operator, the collective overall attribute value of alternative A_i is calculated as follows:

$$\tilde{a}_i = \phi_V^A(\tilde{a}_i^1, \tilde{a}_i^2, \ldots, \tilde{a}_i^p) = \sum_{k=1}^{p} v_k \tilde{a}_i^k, \quad (i = 1, 2, \ldots, m). \tag{61}$$

Step 6: According to Eqs. (43) and (44), the ranking indices of the membership and non-membership functions for the collective overall attribute value $\tilde{a}_i$ of alternative A_i, $R_\mu(\tilde{a}_i, \theta, \lambda)$ and $R_\nu(\tilde{a}_i, \theta, \lambda)$, are obtained, where $\theta = \sum_{k=1}^{p} v_k \theta_k$ is the preference parameter of the decision group, $\lambda = \sum_{k=1}^{p} v_k \lambda_k$ is the risk aversion parameter of the decision group.

Step 7: Rank the alternatives in terms of $R_\mu(\tilde{a}_i, \theta, \lambda)$ and $R_\nu(\tilde{a}_i, \theta, \lambda)$ ($i = 1, 2, \ldots, m$) by the ranking method of Section 3.3.

5. An Application to a Stock Selection Problem and cOMPARISON Analysis of the Results Obtained

In this section, a stock selection problem is analyzed and the comparison analyzes are also conducted to interpret the superiority of the method proposed in this paper.

5.1. *A Stock Selection Problem and the Analysis Process*

In this subsection, the proposed MAGDM method is illustrated with a problem of stock selection. Assume that an investor desires to invest some stocks in Shanghai stock exchange. He employed four experts (i.e., DMs) e_1, e_2, e_3, and e_4 to help him to select the best stock from the four stocks $\{A_1, A_2, A_3, A_4\}$. The four experts assess the four stocks on the basis of five attributes, namely profit ability C_1, debt paying ability C_2, growth ability C_3, market performance C_4 and investment income C_5. The evaluations of importance of experts e_1, e_2, e_3, and e_4 are given in the form of linguistic variables as "very important", "important", "important" and "medium", respectively. The relationship between linguistic variables and intuitionistic fuzzy values is listed in Table 1. After statistical processing, the assessment information of each stock on attributes given by experts can be expressed as TrIFNs shown in Tables 2, 3 and 4, respectively. For example, in the fourth row and the third column of Table 2, the TrIFN ([3, 4, 5, 7]; 0.6, 0.3) indicates that expert e_1 believes that the lower and upper limit for the debt paying ability of stock A_3 are 3 and 7, respectively, the most possible value is between 4 and 5. Meanwhile, the maximum degree of membership for the most possible value [4, 5] is 0.6, the minimum degree of non-membership is 0.3, and the hesitancy degree is 0.1.

The preference information structures Λ^k of attribute importance given by the DM e^k ($k = 1, 2, 3, 4$) are given as follows:

$$\Lambda^1 = \{w^1 \in \Lambda_0^1 | w_1^1 \geqslant 2w_2^1, \ 0.1 \leqslant w_2^1 \leqslant 0.4, \ 0.18 \leqslant w_3^1 \leqslant 0.31,$$
$$w_4^1 - w_5^1 \geqslant w_2^1 - w_3^1, \ 0.2 \leqslant w_5^1 \leqslant 0.5\},$$

Table 2
TrIFN decision matrix given by expert e_1.

	C_1	C_2	C_3	C_4	C_5
A_1	([6, 7, 8, 9]; 0.6, 0.2))	([3, 4, 5, 6]; 0.6, 0.4)	([5, 6, 7, 9]; 0.3, 0.4)	([6, 7, 8, 9]; 0.5, 0.3)	([2, 3, 4, 6]; 0.6, 0.3)
A_2	([7, 8, 9, 10]; 0.7, 0.3)	([5, 6, 7, 8]; 0.5, 0.3)	([4, 5, 7, 8]; 0.7, 0.3)	([5, 7, 8, 9]; 0.4, 0.6)	([3, 4, 5, 7]; 0.4, 0.4)
A_3	([4, 5, 6, 7]; 0.4, 0.2)	([3, 4, 5, 7]; 0.6, 0.3)	([2, 4, 5, 6]; 0.5, 0.3)	([6, 7, 8, 9]; 0.5, 0.2)	([7, 8, 9, 10]; 0.3, 0.3)
A_4	([1, 2, 4, 5]; 0.8, 0.1)	([3, 4, 6, 7]; 0.2, 0.4)	([5, 6, 7, 8]; 0.6, 0.1)	([4, 5, 6, 7]; 0.7, 0.2)	([3, 6, 8, 9]; 0.2, 0.6)

Table 3
TrIFN decision matrix given by expert e_2.

	C_1	C_2	C_3	C_4	C_5
A_1	([1, 3, 4, 5]; 0.5, 0.4)	([2, 4, 6, 8]; 0.8, 0.2)	([2, 5, 6, 8]; 0.7, 0.1)	([1, 4, 5, 6]; 0.6, 0.3)	([1, 2, 3, 4]; 0.5, 0.3)
A_2	([2, 4, 6, 8]; 0.6, 0.3)	([4, 6, 8, 10]; 0.7, 0.2)	([4, 6, 8, 10]; 0.5, 0.3)	([3, 5, 6, 7]; 0.8, 0.1)	([4, 5, 6, 7]; 0.3, 0.6)
A_3	([2, 4, 6, 10]; 0.6, 0.2)	([2, 4, 6, 8]; 0.8, 0.2)	([1, 2, 6, 8]; 0.6, 0.2)	([1, 2, 3, 5]; 0.6, 0.4)	([2, 4, 6, 8]; 0.7, 0.1)
A_4	([2, 3, 4, 7]; 0.5, 0.3)	([1, 2, 3, 5]; 0.6, 0.2)	([1, 3, 5, 7]; 0.7, 0.2)	([1, 2, 4, 5]; 0.5, 0.3)	([1, 3, 4, 6]; 0.8, 0.1)

Table 4
TrIFN decision matrix given by expert e_3.

	C_1	C_2	C_3	C_4	C_5
A_1	([6, 7, 8, 9]; 0.7, 0.2)	([1, 3, 5, 6]; 0.4, 0.2)	([3, 5, 6, 7]; 0.4, 0.3)	([1, 2, 4, 5]; 0.7, 0.1)	([5, 6, 7, 9]; 0.5, 0.3)
A_2	([5, 7, 8, 9]; 0.6, 0.3)	([4, 5, 6, 8]; 0.7, 0.3)	([4, 6, 7, 8]; 0.3, 0.1)	([3, 5, 6, 8]; 0.5, 0.3)	([2, 3, 4, 5]; 0.4, 0.5)
A_3	([7, 8, 9, 10]; 0.6, 0.2)	([3, 4, 6, 7]; 0.5, 0.2)	([1, 2, 6, 8]; 0.5, 0.3)	([1, 2, 4, 5]; 0.6, 0.3)	([6, 7, 8, 9]; 0.5, 0.4)
A_4	([4, 5, 7, 9]; 0.5, 0.3)	([1, 2, 3, 4]; 0.4, 0.1)	([1, 3, 5, 6]; 0.6, 0.2)	([1, 2, 3, 5]; 0.5, 0.2)	([4, 5, 7, 9]; 0.6, 0.2)

Table 5
TrIFN decision matrix given by expert e_4.

	C_1	C_2	C_3	C_4	C_5
A_1	([4, 5, 7, 8]; 0.4, 0.5)	([4, 5, 6, 7]; 0.6, 0.4)	([5, 6, 7, 9]; 0.3, 0.4)	([4, 7, 8, 10]; 0.3, 0.6)	([1, 2, 3, 4]; 0.5, 0.3)
A_2	([5, 6, 7, 9]; 0.3, 0.5)	([5, 6, 7, 8]; 0.4, 0.3)	([4, 5, 7, 8]; 0.7, 0.3)	([5, 6, 8, 9]; 0.5, 0.6)	([2, 3, 4, 5]; 0.3, 0.4)
A_3	([3, 5, 6, 8]; 0.4, 0.2)	([2, 4, 5, 8]; 0.6, 0.2)	([2, 4, 5, 6]; 0.5, 0.3)	([3, 5, 6, 8]; 0.4, 0.2)	([4, 6, 8, 9]; 0.5, 0.4)
A_4	([1, 2, 4, 6]; 0.6, 0.3)	([3, 5, 6, 7]; 0.5, 0.1)	([5, 6, 7, 8]; 0.4, 0.3)	([2, 4, 6, 7]; 0.5, 0.1)	([3, 4, 5, 6]; 0.3, 0.5)

$$\Lambda^2 = \left\{ w^2 \in \Lambda_0^2 \middle| w_1^2 \geqslant 3w_2^2,\ 0.12 \leqslant w_2^2 \leqslant 0.45,\ 0.15 \leqslant w_3^2 \leqslant 0.35, \right.$$
$$\left. w_4^2 - w_3^2 \geqslant w_2^2 - w_1^2,\ 0.1 \leqslant w_5^2 \leqslant 0.42 \right\},$$
$$\Lambda^3 = \left\{ w^3 \in \Lambda_0^3 \middle| w_1^3 \geqslant 2.4w_2^3,\ 0.18 \leqslant w_2^3 \leqslant 0.44,\ 0.21 \leqslant w_3^3 \leqslant 0.36, \right.$$
$$\left. w_2^3 - w_3^3 \geqslant w_4^3 - w_5^3,\ 0.11 \leqslant w_4^3 \leqslant 0.27 \right\},$$

and

$$\Lambda^4 = \left\{ w^4 \in \Lambda_0^4 \middle| w_2^4 \geqslant 1.5w_3^4,\ 0.2 \leqslant w_2^4 \leqslant 0.39,\ 0.15 \leqslant w_3^4 - w_5^4 \leqslant 0.35, \right.$$
$$\left. w_2^4 - w_3^4 \geqslant w_4^4 - w_5^4,\ 0.2 \leqslant w_4^4 \leqslant 0.34 \right\},$$

respectively.

Step 1: By using Eq. (45), the fuzzy decision matrices of Tables 2–5 can be respectively normalized into the normalized decision matrices shown in Tables 6–9.

Table 6
The normalized TrIFN decision matrix given by expert e_1.

	C_1	C_2	C_3	C_4	C_5
A_1	([0.6, 0.7, 0.8, 0.9]; 0.6, 0.2)	([0.3, 0.4, 0.5, 0.6]; 0.6, 0.4)	([0.5, 0.6, 0.7, 0.9]; 0.3, 0.4)	([0.6, 0.7, 0.8, 0.9]; 0.5, 0.3)	([0.2, 0.3, 0.4, 0.6]; 0.6, 0.3)
A_2	([0.7, 0.8, 0.9, 1.0]; 0.7, 0.3)	([0.5, 0.6, 0.7, 0.8]; 0.5, 0.3)	([0.4, 0.5, 0.7, 0.8]; 0.7, 0.3)	([0.5, 0.7, 0.8, 0.9]; 0.4, 0.6)	([0.3, 0.4, 0.5, 0.7]; 0.4, 0.4)
A_3	([0.4, 0.5, 0.6, 0.7]; 0.4, 0.2)	([0.3, 0.4, 0.5, 0.7]; 0.6, 0.3)	([0.2, 0.4, 0.5, 0.6]; 0.5, 0.3)	([0.6, 0.7, 0.8, 0.9]; 0.5, 0.2)	([0.7, 0.8, 0.9, 1.0]; 0.3, 0.3)
A_4	([0.1, 0.2, 0.4, 0.5]; 0.8, 0.1)	([0.3, 0.4, 0.6, 0.7]; 0.2, 0.4)	([0.5, 0.6, 0.7, 0.8]; 0.6, 0.1)	([0.4, 0.5, 0.6, 0.7]; 0.7, 0.2)	([0.3, 0.6, 0.8, 0.9]; 0.2, 0.6)

Table 7
The normalized TrIFN decision matrix given by expert e_2.

	C_1	C_2	C_3	C_4	C_5
A_1	([0.1, 0.3, 0.4, 0.5]; 0.5, 0.4)	([0.2, 0.4, 0.6, 0.8]; 0.8, 0.2)	([0.2, 0.5, 0.6, 0.8]; 0.7, 0.1)	([0.1, 0.4, 0.5, 0.6]; 0.6, 0.3)	([0.1, 0.2, 0.3, 0.4]; 0.5, 0.3)
A_2	([0.2, 0.4, 0.6, 0.8]; 0.6, 0.3)	([0.4, 0.6, 0.8, 1.0]; 0.7, 0.2)	([0.4, 0.6, 0.8, 1.0]; 0.5, 0.3)	([0.3, 0.5, 0.6, 0.7]; 0.8, 0.1)	(0.4, 0.5, 0.6, 0.7]; 0.3, 0.6)
A_3	([0.2, 0.4, 0.6, 1.0]; 0.6, 0.2)	([0.2, 0.4, 0.6, 0.8]; 0.8, 0.2)	([0.1, 0.2, 0.6, 0.8]; 0.6, 0.2)	([0.1, 0.2, 0.3, 0.5]; 0.6, 0.4)	([0.2, 0.4, 0.6, 0.8]; 0.7, 0.1)
A_4	([0.2, 0.3, 0.4, 0.7]; 0.5, 0.3)	([0.1, 0.2, 0.3, 0.5]; 0.6, 0.2)	([0.1, 0.3, 0.5, 0.7]; 0.7, 0.2)	([0.1, 0.2, 0.4, 0.5]; 0.5, 0.3)	([0.1, 0.3, 0.4, 0.6]; 0.8, 0.1)

Table 8
The normalized TrIFN decision matrix given by expert e_3.

	C_1	C_2	C_3	C_4	C_5
A_1	([0.6, 0.7, 0.8, 0.9]; 0.7, 0.2)	([0.1, 0.3, 0.5, 0.6]; 0.4, 0.2)	([0.3, 0.5, 0.6, 0.7]; 0.4, 0.3)	([0.1, 0.2, 0.4, 0.5]; 0.7, 0.1)	([0.5, 0.6, 0.7, 0.9]; 0.5, 0.3)
A_2	([0.5, 0.7, 0.8, 0.9]; 0.6, 0.3)	([0.4, 0.5, 0.6, 0.8]; 0.7, 0.3)	([0.4, 0.6, 0.7, 0.8]; 0.3, 0.1)	([0.3, 0.5, 0.6, 0.8]; 0.5, 0.3)	([0.2, 0.3, 0.4, 0.5]; 0.4, 0.5)
A_3	([0.7, 0.8, 0.9, 1.0]; 0.6, 0.2)	([0.3, 0.4, 0.6, 0.7]; 0.5, 0.2)	([0.1, 0.2, 0.6, 0.8]; 0.5, 0.3)	([0.1, 0.2, 0.4, 0.5]; 0.6, 0.3)	([0.6, 0.7, 0.8, 0.9]; 0.5, 0.4)
A_4	([0.4, 0.5, 0.7, 0.9]; 0.5, 0.3)	([0.1, 0.2, 0.3, 0.4]; 0.4, 0.1)	([0.1, 0.3, 0.5, 0.6]; 0.6, 0.2)	([0.1, 0.2, 0.3, 0.5]; 0.5, 0.2)	([0.4, 0.5, 0.7, 0.9]; 0.6, 0.2)

Table 9
The normalized TrIFN decision matrix given by expert e_4.

	C_1	C_2	C_3	C_4	C_5
A_1	([0.4, 0.5, 0.7, 0.8]; 0.4, 0.5)	([0.4, 0.5, 0.6, 0.7]; 0.6, 0.4)	([0.5, 0.6, 0.7, 0.9]; 0.3, 0.4)	([0.4, 0.7, 0.8, 1.0]; 0.3, 0.6)	([0.1, 0.2, 0.3, 0.4]; 0.5, 0.3)
A_2	([0.5, 0.6, 0.7, 0.9]; 0.3, 0.5)	([0.5, 0.6, 0.7, 0.8]; 0.4, 0.3)	([0.4, 0.5, 0.7, 0.8]; 0.7, 0.3)	([0.5, 0.6, 0.8, 0.9]; 0.5, 0.6)	([0.2, 0.3, 0.4, 0.5]; 0.3, 0.4)
A_3	([0.3, 0.5, 0.6, 0.8]; 0.4, 0.2)	([0.2, 0.4, 0.5, 0.8]; 0.6, 0.2)	([0.2, 0.4, 0.5, 0.6]; 0.5, 0.3)	([0.3, 0.5, 0.6, 0.8]; 0.4, 0.2)	([0.4, 0.6, 0.8, 0.9]; 0.5, 0.4)
A_4	([0.1, 0.2, 0.4, 0.6]; 0.6, 0.3)	([0.3, 0.5, 0.6, 0.7]; 0.5, 0.1)	([0.5, 0.6, 0.7, 0.8]; 0.4, 0.3)	([0.2, 0.4, 0.6, 0.7]; 0.5, 0.1)	([0.3, 0.4, 0.5, 0.6]; 0.3, 0.5)

Step 2: Combined Table 1 and Eq. (47), the weight vector of experts is obtained as $V = (0.2711, 0.2651, 0.2651, 0.1988)^T$.

Step 3: Assume that the preference parameter and risk aversion parameter of expert e_1 are $\theta_1 = 0.2$ and $\lambda_1 = 0.2$, respectively. Taking the normalized matrix $\tilde{A}^1$ as an example, the bi-objective programming model for expert e_1 is constructed by Eq. (57) with $\varepsilon = 0.5$ as follows:

$$\max z_1^1$$

$$\max z_2^1$$

$$\text{s.t.} \begin{cases} w_1^1 \geqslant 2w_2^1, \ 0.1 \leqslant w_2^1 \leqslant 0.4, \ 0.18 \leqslant w_3^1 \leqslant 0.31, \\ w_4^1 - w_5^1 \geqslant w_2^1 - w_3^1, \ 0.2 \leqslant w_5^1 \leqslant 0.5 \\ \sum_{j=1}^5 w_j^1 = 1, \\ 0.05 \leqslant w_j^1 \leqslant 1 \quad (j = 1, 2, 3, 4, 5), \end{cases} \tag{62}$$

where

$$\begin{aligned}
z_1^1 = {}& 4.693w_1^1 + 3.707w_2^1 + 4.507w_3^1 + 5.2w_4^1 + 3.647w_5^1 - 0.015(3w_1^1 \\
& + w_2^1 + w_3^1 - 4w_4^1 + 2w_5^1)^2 + (0.06w_1^1 + 0.02w_2^1 + 0.04w_3^1 - 0.08w_4^1 \\
& + 0.04w_5^1)(2w_1^1 + 2w_2^1 + 3w_3^1 + 2w_4^1 + 3w_5^1) + 0.0075(2w_1^1 + 2w_2^1 + 3w_3^1 \\
& + 2w_4^1 + 3w_5^1)^2 - 0.015(3w_1^1 + w_2^1 + w_3^1 - 2w_4^1 + 2w_5^1) + (0.06w_1^1 \\
& + 0.02w_2^1 + 0.02w_3^1 - 0.04w_4^1 + 0.04w_5^1)(2w_1^1 + 2w_2^1 + 2w_3^1 + 3w_4^1 + 3w_5^1) \\
& + 0.0075(2w_1^1 + 2w_2^1 + 2w_3^1 + 2w_4^1 + 3w_5^1)^2 - 0.015(3w_1^1 + 2w_2^1 + w_3^1 \\
& - 4w_4^1 + w_5^1) + (0.06w_1^1 + 0.04w_2^1 + 0.02w_3^1 - 0.08w_4^1 + 0.02w_5^1) \\
& \times (2w_1^1 + 3w_2^1 + 4w_3^1 + 2w_4^1 + 2w_5^1) + 0.0075(2w_1^1 + 3w_2^1 + 3w_3^1 + 2w_4^1 \\
& + 2w_5^1)^2 - 0.015(4w_1^1 + w_2^1 + w_3^1 - 2w_4^1 + w_5^1) + (0.08w_1^1 + 0.02w_2^1 \\
& + 0.02w_3^1 - 0.08w_4^1 + 0.02w_5^1)(2w_1^1 + 2w_2^1 + w_3^1 + 2w_4^1 + 4w_5^1) \\
& + 0.0075(2w_1^1 + 3w_2^1 + 2w_3^1 + 2w_4^1 + 4w_5^1)^2,
\end{aligned}$$

$$\begin{aligned}
z_2^1 = {}& 10.3w_1^1 + 8.467w_2^1 + 9.44w_3^1 + 11.89w_4^1 + 9.707w_5^1 - 0.02(3w_1^1 + w_2^1 \\
& + 2w_3^1 - 4w_4^1 + 2w_5^1)^2 + (0.08002w_1^1 + 0.02667w_2^1 + 0.05344w_3^1 \\
& - 0.1067w_4^1 + 0.05344w_5^1)(2w_1^1 + 2w_2^1 + 3w_3^1 + 2w_4^1 + 3w_5^1) + 0.01(2w_1^1 \\
& + 2w_2^1 + 3w_3^1 + 2w_4^1 + 3w_5^1)^2 - 0.02(3w_1^1 + w_2^1 + w_3^1 - 2w_4^1 + 2w_5^1) \\
& + (0.08002w_1^1 + 0.02667w_2^1 + 0.02667w_3^1 - 0.05334w_4^1 + 0.05334w_5^1) \\
& \times (2w_1^1 + 2w_2^1 + 2w_3^1 + 3w_4^1 + 3w_5^1) + 0.01(2w_1^1 + 2w_2^1 + 2w_3^1 \\
& + 3w_4^1 + 3w_5^1)^2 - 0.02(3w_1^1 + 2w_2^1 + w_3^1 - 4w_4^1 + w_5^1) + (0.08002w_1^1
\end{aligned}$$

$$+ 0.05334w_2^1 + 0.02667w_3^1 - 0.05334w_4^1 + 0.02667w_5^1)(2w_1^1 + 3w_2^1$$
$$+ w_3^1 + 2w_4^1 + 2w_5^1) + 0.01(2w_1^1 + 3w_2^1 + 3w_3^1 + 2w_4^1 + 2w_5^1)^2 - 0.02(4w_1^1$$
$$+ w_2^1 + w_3^1 - 2w_4^1 + w_5^1) + (0.01067w_1^1 + 0.02667w_2^1 + 0.02667w_3^1$$
$$+ 0.02667w_5^1)(2w_1^1 + 2w_2^1 + 3w_3^1 + 2w_4^1 + 4w_5^1) + 0.01(2w_1^1 + 3w_2^1 + 2w_3^1$$
$$+ 2w_4^1 + 4w_5^1)^2.$$

Taking $\eta_1 = \eta_2 = 0.5$ and solving Eq. (62) by Eqs. (58)–(60), we can derive the weight vector of attributes given by expert e_1 as follows:

$$w^1 = (0.191, 0.235, 0.247, 0.145, 0.182)^T.$$

Assume that the preference parameter and risk aversion parameter of experts e_2, e_3 and e_4 are $\theta_2 = 0.5$, $\theta_3 = 0.7$, $\theta_4 = 0.9$, $\lambda_2 = 0.5$, $\lambda_3 = 0.7$, and $\lambda_4 = 0.9$, respectively. By the same way, the weight vectors of attributes given by other experts are respectively computed as follows:

$$w^2 = (0.206, 0.13, 0.321, 0.165, 0.178)^T,$$
$$w^3 = (0.254, 0.113, 0.317, 0.165, 0.151)^T,$$

and

$$w^4 = (0.2, 0.35, 0.2, 0.2, 0.05)^T.$$

Step 4: According to Eq. (48), the individual overall attribute values of stocks are respectively calculated as follows:

$$\tilde{a}_1^1 = ([0.4840, 0.5899, 0.6959, 0.8459]; 0.3, 0.6),$$
$$\tilde{a}_2^1 = ([0.5119, 0.6439, 0.7679, 0.8998]; 0.2, 0.5),$$
$$\tilde{a}_3^1 = ([0.5079, 0.6319, 0.7379, 0.8538]; 0.1, 0.5),$$
$$\tilde{a}_4^1 = ([0.3280, 0.4859, 0.6539, 0.7599]; 0.2, 0.6),$$
$$\tilde{a}_1^2 = ([0.1650, 0.4013, 0.5550, 0.7200]; 0.5, 0.1),$$
$$\tilde{a}_2^2 = ([0.4050, 0.6150, 0.8100, 1.0050]; 0.3, 0.1),$$
$$\tilde{a}_3^2 = ([0.2138, 0.4275, 0.6750, 0.9750]; 0.2, 0.1),$$
$$\tilde{a}_4^2 = ([0.1500, 0.3113, 0.4575, 0.7125]; 0.4, 0.1),$$
$$\tilde{a}_1^3 = ([0.1883, 0.2948, 0.3912, 0.4675]; 0.5, 0.3),$$
$$\tilde{a}_2^3 = ([0.2502, 0.3475, 0.4260, 0.5225]; 0.3, 0.5),$$
$$\tilde{a}_3^3 = ([0.1997, 0.2672, 0.4267, 0.5152]; 0.5, 0.4),$$

$$\tilde{a}_4^3 = ([0.1200, 0.2085, 0.3145, 0.4105]; 0.4, 0.5),$$

$$\tilde{a}_1^4 = ([0.4200, 0.5750, 0.7100, 0.8650]; 0.3, 0.6),$$

$$\tilde{a}_2^4 = ([0.4950, 0.6100, 0.7650, 0.9000]; 0.3, 0.6),$$

$$\tilde{a}_3^4 = ([0.3100, 0.5400, 0.6750, 0.9000]; 0.4, 0.4),$$

$$\tilde{a}_4^4 = ([0.3250, 0.4950, 0.6500, 0.7850]; 0.3, 0.5).$$

Step 5: Combined the weight vector of experts $V = (0.2711, 0.2651, 0.2651, 0.1988)^T$ with Eq. (61), the collective overall attribute values of stocks are calculated as follows:

$$\tilde{a}_1 = ([0.3083, 0.4587, 0.5806, 0.7160]; 0.3, 0.6),$$

$$\tilde{a}_2 = ([0.4109, 0.5509, 0.6879, 0.8277]; 0.2, 0.6),$$

$$\tilde{a}_3 = ([0.3089, 0.4628, 0.6262, 0.8054]; 0.1, 0.5),$$

$$\tilde{a}_4 = ([0.2251, 0.3679, 0.5111, 0.6597]; 0.2, 0.6).$$

Step 6: According to Eqs. (43) and (44), the ranking indices of the membership and non-membership functions for stocks are respectively obtained as follows:

$$R_\mu(\tilde{a}_1, \theta, \lambda) = 0.1656, \qquad R_\mu(\tilde{a}_2, \theta, \lambda) = 0.1966, \qquad R_\mu(\tilde{a}_3, \theta, \lambda) = 0.1776,$$

$$R_\mu(\tilde{a}_4, \theta, \lambda) = 0.1434, \qquad R_\nu(\tilde{a}_1, \theta, \lambda) = 0.2208, \qquad R_\nu(\tilde{a}_2, \theta, \lambda) = 0.2621,$$

$$R_\nu(\tilde{a}_3, \theta, \lambda) = 0.2368, \qquad R_\nu(\tilde{a}_4, \theta, \lambda) = 0.1912.$$

Step 7: Since $R_\mu(\tilde{a}_2, \theta, \lambda) > R_\mu(\tilde{a}_3, \theta, \lambda) > R_\mu(\tilde{a}_1, \theta, \lambda) > R_\mu(\tilde{a}_4, \theta, \lambda)$, the ranking order of stocks is generated as $A_2 \succ A_3 \succ A_1 \succ A_4$. The best selection is stock A_2.

In the sequel, we calculated the ranking indices of stocks with different preference and risk aversion parameters of experts. Some of the ranking orders of alternatives are listed in Table 10.

It is easily seen from Table 10 that the ranking orders of alternatives may be changed when the preference and risk aversion parameters of DMs differ. For instance, if all experts prefer upper possibility means and like risk extremely, i.e., $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, then the ranking order of stocks is $A_4 \succ A_2 \succ A_1 \succ A_3$; if all experts are indifferent between the lower and upper possibility means and hate risk, i.e., $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0.5$, and the risk aversion parameters are $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.5$, then the ranking order of stocks is $A_3 \succ A_2 \succ A_4 \succ A_1$; if all experts prefer lower possibility means and hate risk extremely, i.e., $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0.9$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.9$, then the ranking order of stocks is $A_3 \succ A_2 \succ A_4 \succ A_1$. This analysis shows that it is very necessarily and reasonable to consider the preference and risk aversion parameters of DMs in the MAGDM problems under intuitionistic fuzzy environment since different DM has different preference on the lower and upper weighted possibility means and has different degrees of risk aversion.

Table 10
The ranking order of alternatives with different preference and risk aversion parameters.

θ_1	λ_1	θ_2	λ_2	θ_3	λ_3	θ_4	λ_4	Ranking orders
0	0	0	0	0	0	0	0	$A_4 \succ A_2 \succ A_1 \succ A_3$
0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	$A_4 \succ A_2 \succ A_1 \succ A_3$
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	$A_2 \succ A_3 \succ A_1 \succ A_4$
0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	$A_3 \succ A_2 \succ A_4 \succ A_1$
0.2	0.2	0.5	0.5	0.7	0.7	0.9	0.9	$A_2 \succ A_3 \succ A_1 \succ A_4$
0.5	0.5	0.1	0.1	0.3	0.3	0.7	0.7	$A_4 \succ A_2 \succ A_1 \succ A_3$
0.4	0.4	0.3	0.3	0.7	0.7	0.6	0.6	$A_4 \succ A_1 \succ A_2 \succ A_3$
0.9	0.9	0.7	0.7	0.2	0.2	0.5	0.5	$A_3 \succ A_2 \succ A_4 \succ A_1$
0.8	0.8	0.6	0.2	0.5	0.2	0.2	0.6	$A_2 \succ A_3 \succ A_1 \succ A_4$
0.5	0.8	0.1	0.8	0.3	0.7	0.7	0.3	$A_3 \succ A_2 \succ A_4 \succ A_1$
0.4	0.8	0.3	0.7	0.7	0.3	0.6	0.4	$A_4 \succ A_2 \succ A_1 \succ A_3$
0.9	0.2	0.7	0.3	0.2	0.6	0.5	0.8	$A_1 \succ A_2 \succ A_3 \succ A_4$
0.8	0.3	0.6	0.2	0.5	0.8	0.2	0.7	$A_4 \succ A_3 \succ A_1 \succ A_2$

5.2. *Comparison Analysis with the Method Using TrIFNs Geometric Aggregation Operators*

Wu and Cao (2013) proposed MAGDM method based on TrIFNs geometric aggregation operators. Next, we adopt the method (Wu and Cao, 2013) to solve the above stock selection problem. Assume that the weight vector of DMs is $V = (0.2711, 0.2651, 0.2651, 0.1988)^T$ and the weight vector of attributes is $w = (\frac{1}{p} \sum_{k=1}^{p} w_1^k, \frac{1}{p} \sum_{k=1}^{p} w_2^k, \frac{1}{p} \sum_{k=1}^{p} w_n^k)^T = (0.2368, 0.2387, 0.1538, 0.1676, 0.2031)^T$. Wu and Cao (2013) used the TrIFNs weighted geometric operator and hybrid geometric operator to obtain the collective overall values and then calculated the distances between collective overall values and positive ideal solution as follows:

$$d(\tilde{r}_1, r^+) = 0.6753, \qquad d(\tilde{r}_2, r^+) = 0.6498,$$
$$d(\tilde{r}_3, r^+) = 0.6590, \qquad d(\tilde{r}_4, r^+) = 0.7256.$$

Then, the ranking order by Wu and Cao (2013) is $A_2 \succ A_3 \succ A_1 \succ A_4$, the best supplier is A_2.

Obviously, the ranking orders obtained by the methods of Wu and Cao (2013) and this paper are significantly different. Compared with the former, the latter has the following advantages:

(i) The new operation laws of TrIFNs defined in the latter take the conservative and reliable principle, which can effectively avoid losing and distorting the information, while the operation laws of the former came from Wang and Zhang (2009), which may distort the information. The normalized method proposed in the latter ensures that the normalized results of TrIFNs are still TrIFNs, while the former did not consider the normalization of data and thus only suited for the situation where the corresponding trapezoidal fuzzy numbers in TrIFNs already lie in the unit interval [0, 1].

(ii) This latter sufficiently considers the different preference for the upper and lower possibility means and the different risk preference of different DM, which makes the deci-

Table 11
IF decision matrix given by expert e_1.

	C_1	C_2	C_3	C_4	C_5
A_1	(0.6, 0.2)	(0.6, 0.4)	(0.3, 0.4)	(0.5, 0.3)	(0.6, 0.3)
A_2	(0.7, 0.3)	(0.5, 0.3)	(0.7, 0.3)	(0.4, 0.6)	(0.4, 0.4)
A_3	(0.4, 0.2)	(0.6, 0.3)	(0.5, 0.3)	(0.5, 0.2)	(0.3, 0.3)
A_4	(0.8, 0.1)	(0.2, 0.4)	(0.6, 0.1)	(0.7, 0.2)	(0.2, 0.6)

sion results more consistent with the actual situation, while the former did not consider the DM's risk preference (namely it assumes that all DMs are risk neutral). Thus, the ranking order obtained by the former, $A_2 \succ A_3 \succ A_1 \succ A_4$, is just a special case of that obtained by the latter (i.e., the case of $\theta_1 = \theta_2 = \theta_3 = \theta_4 = 0.5$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.5$ as listed in Table 10).

(iii) This latter proposes the method to determine the weights of experts on the basis of intuitionistic fuzzy set voting model and constructs the bi-objective programming to objectively derive the attribute weights, whereas the former just assumed that the weights of DMs and attributes are known in advance. That is to say, the former is only applicable to the MAGDM problems in which weights of attributes and DMs are already known *a priori*, while the latter can solve the MAGDM problems with different preference information structures between completely known and completely unknown.

(iv) The ranking method of TrIFNs adapted in the former came from Wei (2010a, 2010b), which simply calculated the distance between the TrIFNs and positive ideal solution. Such a ranking method of TrIFNs is a single-index approach, which is not always feasible and effective. The latter, however, introduces the concepts of weighted lower and upper possibility means, weighted possibility means for membership and non-membership functions of TrIFNs as well as weighted possibility variances. The ranking method of TrIFNs of the latter is a two-index approach and more reasonable than that of the former since the latter takes into consideration not only the weighted possibility means and variances of the membership functions, but also the weighted possibility means and variances of non-membership functions of TrIFNs.

5.3. *Comparison Analysis with the MAGDM Method Based on IF Power Geometric Aggregation*

In the above illustrated example, if we use IFSs to express the experts' evaluations, then Tables 2–5 can be written as Tables 11–14 through deleting the corresponding trapezoidal fuzzy numbers in TrIFNs.

Xu (2011) utilized the IF power weighted geometric (IFPWG) operator and IF weighted geometric (IFWG) operators to develop Approach I to MAGDM with IF information. To further explain the importance of using TrIFNs to represent the information of assessment, we apply Approach I proposed in Xu (2011) to solve the adapted stock selection problem (IF decision matrixes as in Tables 11–14). Assume that the attribute weight vector and DMs weight vector are $w = (0.2368, 0.2387, 0.1538, 0.1767, 0.2031)^T$

Table 12
IF decision matrix given by expert e_2.

	C_1	C_2	C_3	C_4	C_5
A_1	(0.5, 0.4)	(0.8, 0.2)	(0.7, 0.1)	(0.6, 0.3)	(0.5, 0.3)
A_2	(0.6, 0.3)	(0.7, 0.2)	(0.5, 0.3)	(0.8, 0.1)	(0.3, 0.6)
A_3	(0.6, 0.2)	(0.8, 0.2)	(0.6, 0.2)	(0.6, 0.4)	(0.7, 0.1)
A_4	(0.5, 0.3)	(0.6, 0.2)	(0.7, 0.2)	(0.5, 0.3)	(0.8, 0.1)

Table 13
IF decision matrix given by expert e_3.

	C_1	C_2	C_3	C_4	C_5
A_1	(0.7, 0.2)	(0.4, 0.2)	(0.4, 0.3)	(0.7, 0.1)	(0.5, 0.3)
A_2	(0.6, 0.3)	(0.7, 0.3)	(0.3, 0.1)	(0.5, 0.3)	(0.4, 0.5)
A_3	(0.6, 0.2)	(0.5, 0.2)	(0.5, 0.3)	(0.6, 0.3)	(0.5, 0.4)
A_4	(0.5, 0.3)	(0.4, 0.1)	(0.6, 0.2)	(0.5, 0.2)	(0.6, 0.2)

Table 14
IF decision matrix given by expert e_4.

	C_1	C_2	C_3	C_4	C_5
A_1	(0.4, 0.5)	(0.6, 0.4)	(0.3, 0.4)	(0.3, 0.6)	(0.5, 0.3)
A_2	(0.3, 0.5)	(0.4, 0.3)	(0.7, 0.3)	(0.5, 0.6)	(0.3, 0.4)
A_3	(0.4, 0.2)	(0.6, 0.2)	(0.5, 0.3)	(0.4, 0.2)	(0.5, 0.4)
A_4	(0.6, 0.3)	(0.5, 0.1)	(0.4, 0.3)	(0.5, 0.1)	(0.3, 0.5)

and $V = (0.2711, 0.2651, 0.2651, 0.1988)^T$, respectively. After computation, the overall preference values of alternatives are respectively obtained as follows:

$$r_1 = (0.5073, 0.3547, 0.1380), \qquad r_2 = (0.4570, 0.3852, 0.1578),$$

$$r_3 = (0.4637, 0.3072, 0.2292), \qquad r_4 = (0.5095, 0.2530, 0.2375).$$

The scores of r_i $(i = 1, 2, 3, 4)$ are as follows:

$$S_{r_1} = 0.5073 - 0.3547 = 0.1526, \qquad S_{r_2} = 0.4570 - 0.3852 = 0.0718,$$

$$S_{r_3} = 0.4637 - 0.3072 = 0.1565, \qquad S_{r_4} = 0.5095 - 0.2530 = 0.2565.$$

Since $S_{r_4} > S_{r_3} > S_{r_1} > S_{r_2}$, the ranking order obtained by Xu (2011) is $A_4 \succ A_3 \succ A_1 \succ A_2$, the best stock is A_4, which is just a special case of that obtained by this paper (i.e., the case of $\theta_1 = 0.8$, $\theta_2 = 0.6$, $\theta_3 = 0.5$, $\theta_4 = 0.2$ and $\lambda_1 = 0.3$, $\lambda_2 = 0.2$, $\lambda_3 = 0.8$, $\lambda_4 = 0.7$ as listed in Table 10).

Compared with Xu (2011), this paper has the following advantages:

(i) As stated in introduction, TrIFNs are defined by using trapezoidal fuzzy numbers expressing their membership and non-membership functions, which makes the membership degrees and the non-membership degrees no longer relative to a fuzzy concept "Excellent" or "Good", but relative to the trapezoidal fuzzy number. Thus, the information

given by DMs can be reflected exactly and can be expressed in different dimensions. Hence, TrIFNs may better reflect the assessment information of decision problems than IFSs through adding trapezoidal fuzzy numbers. If all trapezoidal fuzzy numbers are lost from the TrIFNs, then TrIFNs are transformed into IFSs, which weakens the representation ability of information for IFSs.

(ii) The method (Xu, 2011) did not consider the different preference for the upper and lower possibility means and the different risk preference of different DMs, while this paper can give the different decision results with different preference and risk aversion parameters. This analysis indicates that the method proposed in this paper is more flexible than that proposed in Xu (2011).

(iii) Similar to Wu and Cao (2013), the method (Xu, 2011) is also only applicable to the MAGDM problems in which weights of attributes and DMs are already known *a priori* and can not be dealt with the MAGDM problems with incomplete weight preference information.

6. Conclusions

As a special IFS on a real number set, TrIFNs are of importance for quantifying the ill-known quantities in decision data and decision making problems themselves. This paper introduces the concepts of the weighted lower and upper possibility means, the weighted possibility means and variances of TrIFN. Hereby, a lexicographic method based on the weighted possibility mean and variance is developed to rank the TrIFNs. A decision method is proposed for solving the MAGDM problems with TrIFNs and incomplete weight preference information. In this method, the expert weights are given in the form of linguistic variables, which are determined through the IFS voting model, and the attribute weights are objectively derived through constructing the bi-objective programming model, which is transformed into the single objective quadratic programming model to solve. The ranking order of alternatives is generated by the collective overall attribute values of alternatives. The proposed MAGDM method sufficiently considers the different preferences for lower and upper weighted possibility means and risk aversion degrees of different DMs, which can make the decision results more reasonable and consistent with the reality.

Although the developed method in this paper is illustrated with a stock selection problem, it is expected to be applicable to the group decision making problems in many areas, such as the supplier management, water environment assessment, threat evaluation and missile weapon system selection, warship combat plan evaluation, and so on. It is easy to see that how to construct TrIFNs (i.e., represent the assessment information of attribute values with TrIFNs) is a key problem of applying the proposed method to practical decision situations. Generating methods of TrIFNs will be investigated for future research. In addition, the weighted possibility covariance and correlation coefficient are also the important mathematical characteristics of TrIFNs, which will be introduced and employed to MAGDM with TrIFNs in the near future.

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Daugiatikslis grupinis sprendimų priėmimas su trapeciniais intuityviais neraiškiaisiais skaičiais vertybinių popierių parinkimui

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Neraiškusis skaičius yra atskiras neraiškiųjų aibių atvejis. Trapeciniai intuityvūs neraiškieji skaičiai (TrIFN) yra atskira intuityvioji neraiškioji aibė apibrėžta realiųjų skaičių aibėje, kuri yra tinkama apibrėžti blogai žinomus kiekius. Šio straipsnio tikslas – pasiūlyti naują metodą daugiatiksliams grupiniams sprendimų priėmimo uždaviniams, kuriuose rodiklių reikšmės yra TrIFN ir rodiklių svorių informacija yra nepilna. Sampratos, tokios kaip TrIFN svertinis apatinis ir viršutinis galimybių vidurkiai ir variacijos yra pateiktos. Tuo būdu, yra išplėtotas naujas metodas TrIFN surikiuoti. Šiame metode, ekspertų įtakos svoriai yra nustatyti taikant intuityviųjų neraiškiųjų aibių balsavimo modelį. Rodiklių svoriai yra nustatyti objektyviai bi-tikslo programavimo modelio pagrindu, kuris yra transformuotas į vientikslį kvadratinio programavimo modelį. Alternatyvos suranguotos pagal kolektyvinių rodiklių vidurkių reikšmes. Pateiktas akcijų pasirinkimo pavyzdys ir palyginamoji analizė patvirtinta pateikto modelio teisingumą ir tinkamumą naudoti.