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Generalized Ordered Weighted Proportional Averaging Operator and Its Application to Group Decision Making

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Abstract. We present a new aggregation operator called the generalized ordered weighted proportional averaging (GOWPA) operator based on an optimal model with penalty function, which extends the ordered weighted geometric averaging (OWGA) operator. We investigate some properties and different families of the GOWPA operator. We also generalize the GOWPA operator. The key advantage of the GOWPA operator is that it is an aggregation operator with theoretic basis on aggregation, which focuses on its structure and importance of arguments. Moreover, we propose an orness measure of the GOWPA operator and indicate some properties of this orness measure. Furthermore, we introduce the generalized least squares method (GLSM) to determine the GOWPA operator weights based on its orness measure. Finally, we present a numerical example to illustrate the new approach in an investment selection decision making problem.

Key words: group decision making, aggregation operator, OWA operator, generalized least squares method.

1. Introduction

The increasing complexity of the socio-economic environment makes it less and less possible for a single expert or decision maker to consider all relevant aspects of a problem. Therefore, some complex decision making problems should be conducted by integrating a group of experts' knowledge and experiences. In general, the practice of multiple attribute group decision making is to evaluate each attribute of every alternative individually and to obtain the best solution(s) (Pérez *et al.*, 2011; Wang and Parkan, 2005). The fundamental prerequisite of decision making is how to aggregate individual experts' preference information on alternatives. Information aggregation is a process that combines individual experts' preferences into an overall one by using a proper aggregation technique. Recently, the investigation on information aggregation has received surprisingly extensive attention from practitioners and researchers due to its practical and academic significance (Ahn, 2006; Merigó, 2010, 2011; Merigó and Casanovas, 2009, 2010, 2011a, 2011b, 2011c;

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Merigó and Gil-Lafuente, 2009, 2010; Merigó *et al.*, 2010, 2011; Yager, 2003; Yager and Filev, 1999; Zhou and Chen, 2011). A very common aggregation method for aggregating the information is the ordered weighted averaging (OWA) operator introduced by Yager (Yager, 1988; Yager and Kacprzyk, 1997). It provides a parameterized family of aggregation operators that includes the maximum, the minimum and the average criteria. Since its appearance, the OWA operator has been used in a wide range of applications (Calvo *et al.*, 2002; Merigó and Gil-Lafuente, 2010; Xu, 2004, 2006b; Xu and Xia, 2011; Zhou and Chen, 2010). In the meantime, a lot of new aggregation operators and their extensions have been developed. For example, motivated by the OWA operator, in Chiclana *et al.* (2000), Xu and Da (2002), the ordered weighted geometric averaging (OWGA) operator was developed. Generally, Yager provided a generalization of the OWA operator in Yager (2004), called the generalized ordered weighted averaging (GOWA) operator. This operator added to the OWA operator an additional parameter, controlling the power to which the argument values were raised and had some special cases such as the OWA operator, the OWGA operator and the ordered weighted harmonic averaging (OWHA) operator, and so on. Further studies on these generalization are found in Ahn (2006), Merigó (2010, 2011), Merigó and Casanovas (2009, 2010, 2011a, 2011b, 2011c), Merigó and Gil-Lafuente (2009, 2010), Merigó *et al.* (2010, 2011), Xu (2006b), Xu and Xia (2011), Yager (1993, 1996, 2003, 2007, 2009a, 2009b), Yu *et al.* (2011), Zhou and Chen (2010, 2011). However, these aggregation operators may focus on their weighting patterns, not their structures of aggregation and importance of arguments. It is necessary to introduce the new aggregation operators with more effectively theoretic foundation, considering the structure of aggregation.

An other important issue in the theory of OWA aggregation is the determination of the associated weights. A number of approaches have been developed for obtaining the OWA operator weights. O'Hagan (1988) was the first to determine OWA operator weights and suggested a maximum entropy method, which formulated the OWA operator weight problem as a constrained nonlinear optimization model. Motivated by O'Hagan (1988), Wang and Parkan (2005) proposed a minimax disparity approach for obtaining OWA operator weights. In Yager (1993), was suggested an interesting way to compute the weights of the OWA operator using linguistic quantifiers. Filev and Yager brought forward a learning method based on observed data and an exponential smoothing method, which produced the exponential OWA operator and the operator weights (Filev and Yager, 1998). Moreover, in Wang *et al.* (2007), developed two models to determine the OWA operator weights called the least-squares method (LSM) and the chi-square (χ^2) method (CSM) without following a regular distribution. Numerous authors have studied other developments concerning the OWA operator weights (Amin and Emrouznejad, 2010; Emrouznejad and Amin, 2010; Fullér and Majlender, 2001; Liu, 2008; Liu and Chen, 2004; Wang *et al.*, 2007; Wang and Parkan, 2007; Xu, 2005, 2006a; Yager, 1996, 2007, 2009a, 2009b; Zhou and Chen, 2010).

The aim of this paper is to analyze the structures of aggregation operators and present a new aggregation operator called the generalized ordered weighted proportional averaging (GOWPA) operator motivated by the idea of penalty function. We study some properties of the GOWPA operator and investigate some families of the GOWPA operator. The main advantages of the GOWPA operator are not only that it is based on an optimal model, but also that the weighting vector is related to the structure and importance of aggregation arguments. Furthermore, we extend the GOWPA operator and obtain the generalized hybrid proportional averaging (GHPA) operator.

In order to determine the GOWPA operator weights, we propose an orness measure of the GOWPA operator and discuss some properties associated with this orness measure. Furthermore, we propose a generalized least squares method (GLSM) based on the generalized least squares model. We also present an application of the new approach to group decision making in an example of an investment selection problem. The prominent characteristic of GLSM is that it does not follow a regular distribution and is also applicable to different group decision making problems effectively, such as strategic management, human resource selection and financial management.

This paper is organized as follows. In Section 2, we briefly review basic concepts of aggregation operators. In Section 3, we present the GOWPA operator and study some properties of the GOWPA operator and different families of the GOWPA operator. Then we extend the GOWPA operator. Section 4 introduces the orness measure of the GOWEMA operator and some properties of its orness measure. In Section 5, we introduce the GLSM to determine the GOWPA operator weights. In Section 6, we present a method for group decision making with the GOWPA operator and Section 7 develops a numerical example of the new approach. Finally, we summarize the main conclusions of the paper in Section 8.

2. Preliminaries

In this section, we briefly describe the OWA operator, the GOWA operator and the penalty function.

The OWA operator was introduced by Yager (1988), Yager and Kacprzyk (1997) and has been used in a wide range of applications (Calvo *et al.*, 2002; Merigó and Gil-Lafuente, 2010; Xu, 2004, 2006b; Xu and Xia, 2011; Zhou and Chen, 2010). It can be defined as follows:

Definition 1. An OWA operator of dimension n is a mapping $OWA: R^n \to R$ that has an associated weighting vector w of dimension n such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, according to the following formula:

$$OWA(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{1}$$

where b_j is the jth largest of the a_i.

The fundamental aspect of the OWA operator is the reordering step, in particular, an argument a_i is not associated with a particular weight w_i, but rather the weight w_i is associated with a particular ordered position i of the arguments.

As we can see, while the OWA operator can take its arguments values from the real line, an important special case occurs when the arguments are drawn from the unit interval $I = [0, 1]$. For simplicity, aggregation operators discussed in this paper shall focus on this special case.

Actually, in the aggregation process, assume that the aggregation operator of dimension n is a mapping f, then the usual weighted averaging operator can be presented by the optimal model as follows:

$$\min z_1 = \sum_{j=1}^{n} w_j \left[f(a_1, a_2, \dots, a_n) - a_j \right]^2, \tag{2}$$

where $w = (w_1, w_2, \dots, w_n)^T$ is a weighting vector satisfying $w_j \in [0, 1]$ for all j and $\sum_{j=1}^{n} w_j = 1$. If we take the partial derivative with respect to f, then we have

$$\frac{\partial z_1}{\partial f} = 2 \sum_{j=1}^{n} w_j (f - a_j).$$

Let $\frac{\partial z_1}{\partial f} = 0$, then we obtain the usual weighted averaging operator as Eq. (3):

$$f(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j a_j. \tag{3}$$

If we reorder arguments a_j, then we can obtain the OWA operator as Eq. (1).

Furthermore, the generalized mean can be obtained by adding to the following model an additional parameter λ:

$$\min z_2 = \sum_{j=1}^{n} w_j \left[f^{\lambda} - a_j^{\lambda} \right]^2, \tag{4}$$

and taking the partial derivative with respect to f with $\partial z_2 / \partial f = 0$. Then the GOWA operator can be defined as follows:

Definition 2. A GOWA operator of dimension n is a mapping GOWA : $I^n \to I$ defined by an associated weighting vector w of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, and a parameter $\lambda \in (-\infty, +\infty)$, $\lambda \neq 0$, according to the following formula:

$$\mathrm{GOWA}(a_1, a_2, \dots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{\frac{1}{\lambda}}, \tag{5}$$

where b_j is the jth largest of the a_i.

If we consider the possible values of the parameter r in the GOWA operator, we can obtain a group of particular cases. For instance, the OWA operator, the OWGA operator, the OWHA operator and the ordered weighted quadratic averaging (OWQA) operator (Yager, 2004) can be obtained as follows:

- The OWA operator is found if $r = 1$.
- The OWQA operator is found when $r = 2$.
- The OWGA operator is obtained when $r \to 0$:

$$\mathrm{OWGA}(a_1, a_2, \dots, a_n) = \prod_{j=1}^{n} b_j^{w_j}, \tag{6}$$

- The OWHA operator is formed when $r = -1$:

$$\mathrm{OWHA}(a_1, a_2, \dots, a_n) = \frac{1}{\sum_{j=1}^{n} w_j / b_j}, \tag{7}$$

Similarly, Zhou and Chen (2010) presented the following optimal model:

$$\min z_3 = \sum_{j=1}^{n} w_j \big[(\ln f)^{\lambda} - (\ln a_j)^{\lambda}\big]^2. \tag{8}$$

Letting $\frac{\partial z_3}{\partial f} = 0$, then we have the generalized weighted logarithm averaging (GWLA) operator as follows:

$$f(a_1, a_2, \dots, a_n) = \exp\left\{\left(\sum_{j=1}^{n} w_j (\log a_j)^{\lambda}\right)^{\frac{1}{\lambda}}\right\}. \tag{9}$$

By reordering the arguments a_i, we have the generalized ordered weighted logarithm averaging (GOWLA) operator as follows:

$$\mathrm{GOWLA}(a_1, a_2, \dots, a_n) = \exp\left\{\left(\sum_{j=1}^{n} w_j (\log b_j)^{\lambda}\right)^{\frac{1}{\lambda}}\right\} \tag{10}$$

where b_j is the jth largest of the a_i and parameter $\lambda \in (-\infty, 0) \cup (0, +\infty)$. Note that the GOWLA operator is extension of the OWGA operator.

Aggregation function based on penalties has been studied in Calvo and Beliakov (2010), Calvo *et al.* (2004), Grabisch *et al.* (2011), which can be defined as follows:

DEFINITION 3. The function $P : I^{n+1} \to I$ is a penalty function if and only if it satisfies:

(1) $P(\mathbf{x}, y) \geqslant 0$ for all $\mathbf{x} \in I^n$ and $y \in I$.
(2) $P(\mathbf{x}, y) = 0$ if $\mathbf{x} = \mathbf{y}$, $\mathbf{y} = (y, y, \dots, y) \in I^n$.
(3) For every fixed $\mathbf{x}$, the set of minimizers of $P(\mathbf{x}, y)$ is either a singleton or an interval.

It can be shown that the aggregation operators can be obtained by considering different penalty functions. For example, z_1 in the OWA operator, z_2 in the GOWA operator and z_3 in the GOWLA operator. In next section, we will develop a new aggregation operator by constructing a new optimal model with penalty function.

3. The Generalized Ordered Weighted Proportional Aggregation Operators

In this section, we will present the GOWPA operator based on an optimal problem with penalty function.

3.1. *The GOWPA Operator*

Let $a_1, a_2, \ldots, a_n$ be the aggregation arguments, and $w = (w_1, w_2, \ldots, w_n)^T$ be a weighting vector satisfying $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. Assume that the aggregation operator of dimension n is a mapping f determined by the formula as follows:

$$y = f(a_1, a_2, \ldots, a_n). \tag{11}$$

In the aggregation process, we hope that the deviation between the arguments a_i and the aggregation result y is as possible as small. In order to minimize the deviation between y and a_j, we can construct the optimal model as follows:

$$\min z = \sum_{j=1}^{n} w_j \left[1 - (a_j/y)^{\lambda}\right]^2, \tag{12}$$

where λ is a parameter such that $\lambda \in (-\infty, +\infty)$ and $\lambda \neq 0$. If we take the partial derivative with respect to y, then we have

$$\frac{\partial z}{\partial y} = \sum_{j=1}^{n} 2w_j \left[1 - \left(\frac{a_j}{y}\right)^{\lambda}\right]\left[-\lambda\left(\frac{a_j}{y}\right)^{\lambda-1} \times \left(-\frac{a_j}{y^2}\right)\right].$$

Let $\frac{\partial z}{\partial y} = 0$, then we obtain the formula as follows:

$$y = \left(\sum_{j=1}^{n} w_j a_j^{2\lambda} \Big/ \sum_{j=1}^{n} w_j a_j^{\lambda}\right)^{1/\lambda}. \tag{13}$$

Thus we can define the generalized weighted proportional averaging (GWPA) operator as follows:

Definition 4. A GWPA operator of dimension n is a mapping GWPA : $I^n \to I$ that has associated with a weighting vector $w = (w_1, w_2, \dots, w_n)^T$, satisfying $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$, according to the following formula:

$$\mathrm{GWPA}(a_1, a_2, \dots, a_n) = \left(\sum_{j=1}^n w_j a_j^{2\lambda} \Big/ \sum_{j=1}^n w_j a_j^{\lambda} \right)^{1/\lambda}, \tag{14}$$

where λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$.

If $w_j = 1/n$ for all j, then the GWPA operator becomes the usual generalized proportional mean (GPM), which is expressed as the following formula:

$$\mathrm{GPM}(a_1, a_2, \dots, a_n) = \left(\sum_{j=1}^n a_j^{2\lambda} \Big/ \sum_{j=1}^n a_j^{\lambda} \right)^{1/\lambda}. \tag{15}$$

Example 1. Assume the following arguments in an aggregation process: $a_1 = 0.4$, $a_2 = 0.3$, $a_3 = 0.5$, $a_4 = 0.3$. If we assume that $w = (0.2, 0.3, 0.4, 0.1)^T$ and $\lambda = 2$, then the aggregation formula is

$$\begin{aligned}&\mathrm{GWPA}(a_1, a_2, a_3, a_4)\\ &\quad= \left(\frac{0.2 \times 0.4^{2\times2} + 0.3 \times 0.3^{2\times2} + 0.4 \times 0.5^{2\times2} + 0.1 \times 0.3^{2\times2}}{0.2 \times 0.4^2 + 0.3 \times 0.3^2 + 0.4 \times 0.5^2 + 0.1 \times 0.3^2} \right)^{1/2} = 0.4456.\end{aligned}$$

Furthermore, if we rearrange the arguments in the GWPA operator in descending order, then we can define the generalized ordered weighted proportional averaging (GOWPA) operator as follows:

Definition 5. A GOWPA operator of dimension n is a mapping GOWPA : $I^n \to I$ that has associated with a weighting vector $w = (w_1, w_2, \dots, w_n)^T$, satisfying $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$, according to the following formula:

$$\mathrm{GOWPA}(a_1, a_2, \dots, a_n) = \left(\sum_{j=1}^n w_j b_j^{2\lambda} \Big/ \sum_{j=1}^n w_j b_j^{\lambda} \right)^{1/\lambda}, \tag{16}$$

where b_j is the jth largest of the a_i and λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, +\infty)$.

Note that if we assume that

$$\tilde{w}_j = \frac{w_j b_j^{\lambda}}{\sum_{j=1}^n w_j b_j^{\lambda}}, \tag{17}$$

then Eq. (16) can be rewritten as follows:

Table 1
Aggregation result.

λ	−1	0.0001	1/2	1	2
Aggregation	0.3422	0.3622	0.3740	0.3865	0.4116

$$\mathrm{GOWPA}(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} \tilde{w}_j b_j^{\lambda}\right)^{1/\lambda}, \tag{18}$$

where $\sum_{j=1}^{n} \tilde{w}_j = 1$ and $\tilde{w}_j \in [0, 1]$ for all j.

It is obvious that the GOWPA operator can be considered as a GOWA operator, but $\tilde{w}_j$, the GOWPA operator weights, can be viewed as a combination weights including the weights w_j, depending on the decision makers' attitude, and the weights $b_j^{\lambda}/\sum_{j=1}^{n} b_j^{\lambda}$, relying on the arguments being aggregated. Furthermore, from Eq. (16), we can see that the GOWPA operator may focus on its structure and importance of arguments rather than the weighting pattern, which leads to the fact that the GOWPA operator with more profound theoretic basis is superior to other aggregation operators including the OWA operator, the OWGA operator, the OWHA operator, etc.

EXAMPLE 2. Take the collection of arguments in Example 1, then we have

$$b_1 = a_3 = 0.5, \qquad b_2 = a_1 = 0.4, \qquad b_3 = a_2 = b_4 = a_4 = 0.3.$$

With $\lambda = 1, -1, 1/2, 2$ and $\lambda = 0.0001$, respectively, and by Eq. (16) we have the aggregations which are shown in Table 1.

3.2. *Properties of the GOWPA Operator*

The GOWPA operator is monotonic, commutative, idempotent and bounded, and these properties can be proved in the following theorems.

Lemma 1. *Let $g_1(x)$ and $g_2(x)$ be monotonic positive continuous functions.*

(1) *If $g_1(x)$ and $g_2(x)$ are both monotonically increasing, then $g_1 + g_2$ and $g_1 \cdot g_2$ are also increasing monotonically.*
(2) *If $g_1(x)$ and $g_2(x)$ are both monotonically decreasing, then $g_1 + g_2$ and $g_1 \cdot g_2$ are also decreasing monotonically.*

Lemma 2. *Let $g(x)$ be monotonic continuous functions.*

(1) *If $g \geqslant 0$ for any x and $k \geqslant 0$, then monotonicity of the function $kg(x)$ is the same as function $g(x)$.*
(2) *If $g \geqslant 0$ for any x and $k < 0$, then monotonicity of the function $kg(x)$ is contrary to function $g(x)$.*

Theorem 1. (Monotonicity) *Assume that f is the GOWPA operator. If $a_i \geqslant c_i$ for $i = 1, 2, \ldots, n$, then*

$$f(a_1, a_2, \ldots, a_n) \geqslant f(c_1, c_2, \ldots, c_n). \tag{19}$$

Proof. Assume that f is the GOWPA operator, then by Eqs. (17) and (18), we have that

$$f(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} \tilde{w}_j b_j^{\lambda}\right)^{1/\lambda},$$

where $\tilde{w}_j = w_j b_j^{\lambda} / \sum_{j=1}^{n} w_j b_j^{\lambda}$ for all j. Then

$$\log f(a_1, a_2, \ldots, a_n) = \frac{1}{\lambda} \log\left(\sum_{j=1}^{n} \tilde{w}_j b_j^{\lambda}\right).$$

In the following, we will complete the proof in two cases.

Case 1. If $\lambda > 0$, on the one hand, we take the derivative of $\tilde{w}_j$ and obtain

$$\frac{\partial \tilde{w}_j}{\partial b_j} = \frac{\lambda w_j b_j^{\lambda-1} \sum_{j=1}^{n} w_j b_j^{\lambda} - w_j b_j^{\lambda} \times \lambda w_j b_j^{\lambda-1}}{(\sum_{j=1}^{n} w_j b_j^{\lambda})^2} = \frac{\lambda w_j b_j^{\lambda-1} \sum_{i \neq j} w_i b_i^{\lambda}}{(\sum_{j=1}^{n} w_j b_j^{\lambda})^2} \geqslant 0,$$

which implies that $\tilde{w}_j$ is monotonically increasing with respect to b_j. On the other hand, $\partial b_j^{\lambda} / \partial b_j = \lambda b_j^{\lambda-1} \geqslant 0$, which implies that b_j^{λ} is also increasing with respect to b_j. By Lemma 1, we get that $\sum_{j=1}^{n} \tilde{w}_j b_j^{\lambda}$ is an increasing function. Since $b_j \geqslant 0$ for all j and $\lambda > 0$, then f is monotonically increasing. That is to say,

$$f(a_1, a_2, \ldots, a_n) \geqslant f(c_1, c_2, \ldots, c_n).$$

Case 2. If $\lambda < 0$, on the one hand, we take the derivative of $\tilde{w}_j$ and get $\partial \tilde{w}_j / \partial b_j \leqslant 0$, which implies that $\tilde{w}_j$ is monotonically decreasing with respect to b_j. On the other hand, $\partial b_j^{\lambda} / \partial b_j \leqslant 0$, which implies that b_j^{λ} is also decreasing with respect to b_j. With the monotonicity of logarithmic function and by Lemma 1, we obtain that $\log(\sum_{j=1}^{n} \tilde{w}_j b_j^{\lambda})$ is a monotonically decreasing function. Since $b_j \geqslant 0$ for all j and $\lambda < 0$, then

$$\log\left(\sum_{j=1}^{n} \tilde{w}_j b_j^{\lambda}\right) \geqslant \log\left(\sum_{j=1}^{n} \left(\tilde{w}_j \times 1^{\lambda}\right)\right) = 0.$$

By Lemma 2, we have that f is monotonically increasing, i.e.,

$$f(a_1, a_2, \ldots, a_n) \geqslant f(c_1, c_2, \ldots, c_n).$$

The theorem is proved. □

Theorem 2. (Commutativity) *Let f be the GOWPA operator, then*

$$f(a_1, a_2, \dots, a_n) = f(c_1, c_2, \dots, c_n), \tag{20}$$

where $(c_1, c_2, \dots, c_n)$ is any permutation of the arguments $(a_1, a_2, \dots, a_n)$.

Proof. Let

$$\mathrm{GOWPA}(a_1, a_2, \dots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{2\lambda} \Big/ \sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda}$$

and

$$\mathrm{GOWPA}(c_1, c_2, \dots, c_n) = \left(\sum_{j=1}^{n} w_j d_j^{2\lambda} \Big/ \sum_{j=1}^{n} w_j d_j^{\lambda} \right)^{1/\lambda}.$$

Since $(c_1, c_2, \dots, c_n)$ is any permutation of the arguments $(a_1, a_2, \dots, a_n)$, we have $d_j = b_j$ for all j, thus

$$f(a_1, a_2, \dots, a_n) = f(c_1, c_2, \dots, c_n).$$

□

Theorem 3. (Idempotency) *Let f be the GOWPA operator and if $a_i = a$ for all i, then*

$$f(a_1, a_2, \dots, a_n) = a. \tag{21}$$

Proof. Let

$$f(a_1, a_2, \dots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{2\lambda} \Big/ \sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda}.$$

If $a_i = a$ for all i and $\sum_{j=1}^{n} w_j = 1$, then

$$\begin{aligned} f(a_1, a_2, \dots, a_n) &= \left(\sum_{j=1}^{n} w_j a^{2\lambda} \Big/ \sum_{j=1}^{n} w_j a^{\lambda} \right)^{1/\lambda} \\ &= \left(a^{2\lambda} \sum_{j=1}^{n} w_j \Big/ \left(a^{\lambda} \sum_{j=1}^{n} w_j \right) \right)^{1/\lambda} = \left(a^{\lambda}\right)^{1/\lambda} = a. \end{aligned}$$

□

Theorem 4. (Boundedness) *Let f be the GOWPA operator. If $\max_i a_i = a_{\max}$ and $\min_i a_i = a_{\min}$, then*

$$a_{\min} \leqslant f(a_1, a_2, \dots, a_n) \leqslant a_{\max}. \tag{22}$$

Proof. If $\max_i a_i = a_{\max}$ and $\min i a_i = a_{\min}$, then by Theorems 1 and 3, we have

$$f(a_1, a_2, \ldots, a_n) \leqslant \left(\sum_{j=1}^{n} w_j a_{\max}^{2\lambda} \Big/ \sum_{j=1}^{n} w_j a_{\max}^{\lambda} \right)^{1/\lambda} = a_{\max},$$

and

$$f(a_1, a_2, \ldots, a_n) \geqslant \left(\sum_{j=1}^{n} w_j a_{\min}^{2\lambda} \Big/ \sum_{j=1}^{n} w_j a_{\min}^{\lambda} \right)^{1/\lambda} = a_{\min}.$$

Therefore, $a_{\min} \leqslant f(a_1, a_2, \ldots, a_n) \leqslant a_{\max}$. □

Moreover, the GOWPA operator is monotonic with respect to the parameter λ. The property can be expressed as the following theorem.

Lemma 3. (Cauchy–Schwarz inequality) *Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be any real numbers, then*

$$\left(\sum_{j=1}^{n} x_j^2 \right) \left(\sum_{j=1}^{n} y_j^2 \right) \geqslant \sum_{j=1}^{n} (x_j y_j)^2. \tag{23}$$

Lemma 4. *Let $h(p)$ be a differentiable and increasing function with p, then*

$$H(p) = \frac{1}{p} \int_{p}^{2p} h(t)\, dt \tag{24}$$

is also increasing.

Proof. Take the derivative of H, we have

$$\frac{dH(p)}{dp} = \frac{1}{p^2} \left[2ph(2p) - ph(p) - \int_{p}^{2p} h(t)\, dt \right].$$

Let $\varphi(p) = 2ph(2p) - ph(p) - \int_{p}^{2p} h(t)\, dt$, then we will prove the result in two cases:

(1) If $p \geqslant 0$, then $2p \geqslant p \geqslant 0$, which implies that $h(2p) \geqslant h(p)$. Thus, by the mean value theorem of integral and the monotonicity of function h, we have that

$$\begin{aligned} \varphi(p) &= ph(2p) + ph(2p) - ph(p) - \int_{p}^{2p} h(t)\, dt \\ &\geqslant ph(2p) - \int_{p}^{2p} h(t)\, dt \\ &= ph(2p) - ph(\xi_1) \geqslant 0, \end{aligned}$$

where $p \leqslant \xi_1 \leqslant 2p$.

(2) If $p<0$, then $2p<p<0$, which implies that $h(2p)\leqslant h(p)$ and $ph(2p)\geqslant ph(p)$. Thus, by the mean value theorem of integral and the monotonicity of function h, we have that

$$\begin{aligned}\varphi(p) &= ph(2p)+ph(2p)-ph(p)-\int_p^{2p}h(t)\,dt\\ &\geqslant ph(2x)-\int_p^{2p}h(t)\,dt\\ &= ph(2p)-ph(\xi_2)\geqslant 0,\end{aligned}$$

where $2p\leqslant\xi_2\leqslant p$.

Therefore, $\varphi(p)\geqslant 0$, i.e., $dH(p)/dp\geqslant 0$, which completes the lemma. □

Theorem 5. (Monotonicity with respect to parameter λ) *Let f be the GOWPA operator. If $\lambda_1\geqslant\lambda_2$, then*

$$f(\lambda_1)\geqslant f(\lambda_2). \tag{25}$$

Proof. Let

$$f(a_1,a_2,\ldots,a_n)=\left(\sum_{j=1}^n w_jb_j^{2\lambda}\Big/\sum_{j=1}^n w_jb_j^{\lambda}\right)^{1/\lambda}$$

and $g(t)=\sum_{j=1}^n w_jb_j^t$, then by taking the natural logarithm of f, we have

$$\log f=\frac{1}{\lambda}\left(\log\left(\sum_{j=1}^n w_jb_j^{2\lambda}\right)-\log\left(\sum_{j=1}^n w_jb_j^{\lambda}\right)\right)=\frac{1}{\lambda}\int_\lambda^{2\lambda}\frac{g'(t)}{g(t)}\,dt.$$

In order to establish the monotonicity of f, we take the derivative with respect to t and obtain

$$\frac{d}{dt}\left(\frac{g'(t)}{g(t)}\right)=\frac{g''(t)g(t)-(g'(t))^2}{g^2(t)}.$$

Since $g'(t)=\sum_{j=1}^n w_jb_j^t\log b_j$ and $g''(t)=\sum_{j=1}^n w_jb_j^t(\log b_j)^2$, then by Lemma 3, we get

$$\begin{aligned}g''(t)g(t) &= \sum_{j=1}^n w_jb_j^t(\log b_j)^2\times\sum_{j=1}^n w_jb_j^t\geqslant\left(\sum_{j=1}^n (w_jb_j^t)^{1/2}(w_jb_j^t)^{1/2}\log b_j\right)^2\\ &= \left(\sum_{j=1}^n w_jb_j^t\log b_j\right)^2=\left(g'(t)\right)^2.\end{aligned}$$

Thus we have $\frac{d}{dt}(\frac{g'(t)}{g(t)}) \geqslant 0$, which implies that $\frac{g'(t)}{g(t)}$ is increasing with t. Therefore, by Lemma 4, $\log f$ is increasing with λ, i.e., f is increasing with λ. The theorem is proved. □

Similar to Xu and Da (2002), if we distinguish between the descending generalized ordered weighted proportional averaging (DGOWPA) operator and the ascending generalized ordered weighted proportional averaging (AGOWPA) operator, we would have $w_j^* = w_{n-j+1}$ $(j = 1, 2, \dots, n)$, where w_j is the jth weight of the DGOWPA operator and w_j^* is the jth weight of the AGOWPA operator.

3.3. *Families of GOWPA Operators*

By using a different cases of the parameter λ and the weighting vector w, we are able to obtain different types of GOWPA operators, including the maximum, the minimum, the step-GOWPA operator, the GOWPA median, the olympic-GOWPA operator, the s-GOWPA operator, the window-GOWPA operator and the center-GOWPA operator. Note that these results can be obtained both for the DGOWPA operator and the AGOWPA operator.

REMARK 1. If $\lambda = 1$, then we get the OWPA operator:

$$\mathrm{OWPA}(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j b_j^2 \Big/ \sum_{j=1}^{n} w_j b_j. \tag{26}$$

If $\lambda = -1$, we form the ordered weighted harmonic proportional averaging (OWHPA) operator:

$$\mathrm{OWHPA}(a_1, a_2, \dots, a_n) = \frac{1}{\sum_{j=1}^{n} w_j / b_j^2 / \sum_{j=1}^{n} w_j / b_j}. \tag{27}$$

And if we choose the parameter λ as $1/2$, then the GOWPA operator becomes the ordered weighted square root proportional averaging (OWSPA) operator:

$$\mathrm{OWSPA}(a_1, a_2, \dots, a_n) = \left(\sum_{j=1}^{n} w_j b_j \Big/ \sum_{j=1}^{n} w_j \sqrt{b_j} \right)^2. \tag{28}$$

If $\lambda \to 0$, let f be the GOWPA operator. Take the natural logarithm of f, and by the L'Hôpital's rule, we get

$$\begin{aligned} \lim_{\lambda \to 0} \log f &= \lim_{\lambda \to 0} \frac{\log(\sum_{j=1}^{n} w_j b_j^{2\lambda}) - \log(\sum_{j=1}^{n} w_j b_j^{\lambda})}{\lambda} \\ &= \lim_{\lambda \to 0} \frac{\frac{1}{\sum_{j=1}^{n} w_j b_j^{2\lambda}} \sum_{j=1}^{n} w_j b_j^{2\lambda} \log b_j^2 - \frac{1}{\sum_{j=1}^{n} w_j b_j^{\lambda}} \sum_{j=1}^{n} w_j b_j^{\lambda} \log b_j}{1} \\ &= \sum_{j=1}^{n} w_j \log b_j = \log \prod_{j=1}^{n} b_j^{w_j}. \end{aligned}$$

Hence, $\lim_{\lambda \to 0} f = \prod_{j=1}^{n} b_j^{w_j}$, which is the OWGA operator.

REMARK 2. If $\lambda \to +\infty$, then we get the maximum. And if $\lambda \to -\infty$, then we get the minimum. These properties can be illustrated as follows:

Let f be the GOWPA operator, i.e.,

$$f(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{2\lambda} \Big/ \sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda}.$$

If$\lambda \to +\infty$, then with the monotonicity and the idempotency of the GOWPA operator, we get

$$f(a_1, a_2, \ldots, a_n) \leqslant f(b_1, b_1, \ldots, b_1) = b_1.$$

Since $\lambda > 0$, then $\sum_{j=1}^{n} w_j b_j^{2\lambda} \geqslant w_1 b_1^{2\lambda}$ and $\sum_{j=1}^{n} w_j b_j^{\lambda} \leqslant \sum_{j=1}^{n} w_j b_1^{\lambda}$. It follows that

$$\sum_{j=1}^{n} w_j b_j^{2\lambda} \Big/ \sum_{j=1}^{n} w_j b_j^{\lambda} \geqslant w_1 b_1^{2\lambda} \Big/ \sum_{j=1}^{n} w_j b_1^{\lambda}.$$

Then, we obtain

$$\left(\sum_{j=1}^{n} w_j b_j^{2\lambda} \Big/ \sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \geqslant \left(w_1 b_1^{2\lambda} \Big/ \sum_{j=1}^{n} w_j b_1^{\lambda} \right)^{1/\lambda} = b_1 w_1^{1/\lambda}.$$

Thus, $f(a_1, a_2, \ldots, a_n) \geqslant b_1 w_1^{1/\lambda}$. Therefore,

$$b_1 w_1^{1/\lambda} \leqslant f(a_1, a_2, \ldots, a_n) \leqslant b_1.$$

Denoting $\lambda \to +\infty$, and with $\lim_{\lambda \to +\infty} w_1^{1/\lambda} = 1$, we get the maximum:

$$\lim_{\lambda \to +\infty} f = b_1 = \max_i a_i.$$

A similar proof can be given for the other part of Remark 3.

REMARK 3. The maximum, the minimum, the step-GOWPA operator and the usual generalized proportional mean (GPM) are obtained as follows:

- The maximum is found if $w_1 = 1$ and $w_i = 0$ for all $i \neq 1$.
- The minimum is formed when $w_n = 1$ and $w_i = 0$ for all $i \neq n$.
- Generally, if $w_k = 1$ and $w_i = 0$ for all $i \neq k$, then we get the step-GOWPA operator.
- The usual GPM is obtained when $w_j = 1/n$ for all j.

REMARK 4. Another particular case is the GOWPA median, which are expressed as follows:

- If n is odd, we assign $w_{(n+1)/2} = 1$ and $w_j = 0$ for $j \neq (n+1)/2$.
- If n is even, we assume $w_{n/2} = w_{n/2+1} = 0.5$ and $w_j = 0$ for all other values.

REMARK 5. Another group of interesting families are the olympic-GOWPA operator, the general olympic-GOWPA operator, the window-GOWPA operator, the generalized s-GOWPA operator and the centered GOWPA operator based on the OWA literature (Yager, 1993).

- The olympic-GOWPA operator ($w_1 = w_n = 0$ and $w_j = 1/(n-2)$ for all others).
- The general olympic-GOWPA operator ($w_j = 0$ for $j = 1, 2, \dots, k, n, n-1, \dots, n-k+1$; and for all others $w_j = 1/(n-2k)$, where $k < n/2$).
- The window-GOWPA operator ($w_j = 1/m$ for $k \leqslant j \leqslant k+m-1$, and $w_j = 0$ for $j \geqslant k+m$ and $j < k$).
- The generalized s-GOWPA operator ($w_k = (1-(\alpha+\beta))/n + \alpha$, $w_t = (1-(\alpha+\beta))/n + \beta$, and $w_j = (1-(\alpha+\beta))/n$ for all $j \neq k, t$, where $a_k = \max_i\{a_i\}$, $a_t = \min i\{a_i\}$ and $\alpha + \beta \leqslant 1$ with $\alpha, \beta \in [0, 1]$).
- The centered GOWPA operator (it is symmetric when $w_j = w_{j+n-1}$. It is strongly decaying when $i < j \leqslant (n+1)/2$ then $w_i < w_j$ and when $i > j \geqslant (n+1)/2$ then $w_i < w_j$. It is inclusive if $w_j > 0$).

REMARK 6. Using a similar methodology, other families of the GOWPA operator could be found following the literatures (Xu, 2006a; Yager, 1996, 2007, 2009a).

3.4. *Generalized Hybrid Proportional Aggregation Operators*

We can develop further extensions by adding the balance factor ω and get the generalized hybrid proportional averaging (GHPA) operator.

DEFINITION 6. An GHPA operator of dimension n is a mapping GHPA : $R^{+n} \to R^{+}$ that defined by an associated weighting vector $w = (w_1, w_2, \dots, w_n)^T$ of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$ and a parameter $\lambda \in (-\infty, +\infty)$ and $\lambda \neq 0$, according to the following formula:

$$\mathrm{GHPA}(a_1, a_2, \dots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{2\lambda} \Big/ \sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda}, \tag{29}$$

where b_j is the jth largest of $\hat{a}_i$ ($\hat{a}_i = n\omega_i a_i$, $i = 1, 2, \dots, n$). $\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ is the weighting vector of the a_i called the balance factor, with $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$.

EXAMPLE 3. Take the same information in Example 1, and set $\omega = (0.3, 0.2, 0.2, 0.3)^T$.
Based on the parameter vector ω, we get

$$\hat{a}_1 = 4 \times 0.3 \times 0.4 = 0.48, \qquad \hat{a}_2 = 4 \times 0.2 \times 0.3 = 0.24,$$

$$\hat{a}_3 = 4 \times 0.2 \times 0.5 = 0.4, \qquad \hat{a}_4 = 4 \times 0.3 \times 0.3 = 0.36.$$

Table 2
Aggregation result.

λ	-1	0.0001	1/2	1	2
Aggregation	0.3561	0.3779	0.3870	0.3950	0.4088

Then we have

$$b_1 = \hat{a}_1 = 0.48, \qquad b_2 = \hat{a}_3 = 0.4, \qquad b_3 = \hat{a}_4 = 0.36, \qquad b_4 = \hat{a}_2 = 0.24.$$

With $\lambda = 1, -1, 1/2, 2$ and $\lambda = 0.0001$, respectively, we obtain the aggregations which are shown in Table 2.

Especially, if the balance factor $\omega = (1/n, 1/n, \ldots, 1/n)^T$, then the GHPA operator is reduced to the GOWPA operator.

Similarly, if we consider different cases of the parameter λ and ω in the GHPA operator following the methodology explained in Section 3.3, we can obtain a group of aggregation operators, including the hybrid proportional averaging (HPA) operator, the hybrid harmonic proportional averaging (HHPA) operator, hybrid square root proportional averaging (HSPA) operator, hybrid geometric averaging (HGA) operator and the GOWPA operator. For example, we could analyze the following cases:

- The hybrid proportional averaging (HPA) operator: when $\lambda = 1$.
- The hybrid harmonic proportional averaging (HHPA) operator: when $\lambda = -1$.
- The hybrid square root proportional averaging (HSPA) operator: when $\lambda = 1/2$.
- The hybrid geometric averaging (HGA) operator: when $\lambda \to 0$.
- The GOWPA operator: when $\omega = (1/n, 1/n, \ldots, 1/n)^T$.

Note that some other interesting extensions can be investigated following (Merigó, 2011; Merigó and Casanovas, 2011a, 2011b, 2011c; Mesiar and Pap, 2008; Mesiar and Spirkova, 2006; Pereira and Ribeiro, 2003), such as the heavy GOWPA operator, the infinitary GOWPA operator, the mixture GOWPA operator, etc.

4. An Orness Measure for the GOWEPA Operator

In (1993), Yager introduced the orness measure, associated with the weighting vector w of the OWA operator, which can be defined as follows:

$$orness(w) = \sum_{j=1}^{n} \frac{n-j}{n-1} w_j. \tag{30}$$

It can be shown that when $w = (1, 0, \ldots, 0)$, $orness(w) = 1$. For $w = (0, 0, \ldots, 1)$, $orness(w) = 0$. For $w = (1/n, 1/n, \ldots, 1/n)$, $orness(w) = 0.5$.

Furthermore, the orness measure also can be regarded as the OWA aggregation of the arguments $a_j = (n-j)/(n-1)$ for $j = 1, 2, \dots, n$. By using this method, Yager (2004) defined the orness measure for the GOWA operator as follows:

$$orness(w) = \left(\sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right)^r\right)^{1/r}. \tag{31}$$

It is apparent that when $r = 1$, the orness measure of the GOWA operator reduces to the orness measure of the OWA operator.

Following (2004), we can define the orness measure of the GOWPA operator as follows:

$$orness_\lambda(w) = \left(\frac{\sum_{j=1}^{n} w_j (\frac{n-j}{n-1})^{2\lambda}}{\sum_{j=1}^{n} w_j (\frac{n-j}{n-1})^{\lambda}}\right)^{1/\lambda}. \tag{32}$$

From Theorems 1 and 3, we can get Theorem 6 as follows:

Theorem 6. $0 \leqslant orness_\lambda(w) \leqslant 1$.

Moreover, we can obtain the following theorem:

Theorem 7. $\lim_{\lambda\to+\infty} orness_\lambda(w) = 1$ *and* $\lim_{\lambda\to-\infty} orness_\lambda(w) = 0$.

Proof. If we let $a_j = (n-j)/(n-1)$ for $j = 1, 2, \dots, n$, then by Remark 2, we have that

$$\lim_{\lambda\to+\infty} orness_\lambda(w) = \max_i a_i = (n-1)/(n-1) = 1$$

and

$$\lim_{\lambda\to-\infty} orness_\lambda(w) = \min i a_i = (n-n)/(n-1) = 0. \qquad \square$$

From Theorem 5, we can also get the following theorem:

Theorem 8. *If* $\lambda_1 \geqslant \lambda_2$, *then*

$$orness_{\lambda_1}(w) \geqslant orness_{\lambda_2}(w). \tag{33}$$

It is also can be analyzed that the GOWPA operator and its orness measure are monotonic with respect to the weighting vector. The property can be illustrated by the following theorems.

Lemma 5. *Let* $x = (x_1, x_2, \dots, x_n)$ *be the ordered vector, which satisfies that* $x_1 \geqslant x_2 \geqslant \dots \geqslant x_n \geqslant 0$, *and let* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ *be the vector satisfying* $\alpha_i \geqslant 0$ *for*

$i = 1, 2, \ldots, n$. If weighting vector $w = (w_1, w_2, \ldots, w_n)$ satisfies $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_n$, then

$$\left(\frac{\sum_{j=1}^n \alpha_j w_j x_j^{2\lambda}}{\sum_{j=1}^n \alpha_j w_j x_j^{\lambda}}\right)^{1/\lambda} \geqslant \left(\frac{\sum_{j=1}^n \alpha_j x_j^{2\lambda}}{\sum_{j=1}^n \alpha_j x_j^{\lambda}}\right)^{1/\lambda}, \tag{34}$$

where $\lambda \in (-\infty, \infty)$ and $\lambda \neq 0$. And if $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$, the inequality is reversed.

Proof.

$$\begin{aligned}
&\frac{\sum_{j=1}^n \alpha_j w_j x_j^{2\lambda}}{\sum_{j=1}^n \alpha_j w_j x_j^{\lambda}} - \frac{\sum_{j=1}^n \alpha_j x_j^{2\lambda}}{\sum_{j=1}^n \alpha_j x_j^{\lambda}} \\
&\quad = \frac{\sum_{j=1}^n \alpha_j w_j x_j^{2\lambda} \times \sum_{j=1}^n \alpha_j x_j^{\lambda} - \sum_{j=1}^n \alpha_j x_j^{2\lambda} \times \sum_{j=1}^n \alpha_j w_j x_j^{\lambda}}{\sum_{j=1}^n \alpha_j w_j x_j^{\lambda} \cdot \sum_{j=1}^n \alpha_j x_j^{\lambda}} \\
&\quad = \frac{\sum_{i=1}^n \alpha_i w_i x_i^{2\lambda} \times \sum_{j=1}^n \alpha_j x_j^{\lambda} - \sum_{i=1}^n \alpha_i x_i^{2\lambda} \times \sum_{j=1}^n \alpha_j w_j x_j^{\lambda}}{\sum_{j=1}^n \alpha_j w_j x_j^{\lambda} \cdot \sum_{j=1}^n \alpha_j x_j^{\lambda}} \\
&\quad = \frac{\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j x_i^{2\lambda} x_j^{\lambda} (w_i - w_j)}{\sum_{j=1}^n \alpha_j w_j x_j^{\lambda} \cdot \sum_{j=1}^n \alpha_j x_j^{\lambda}} \\
&\quad = \frac{\sum_{i<j} \alpha_i \alpha_j x_i^{2\lambda} x_j^{\lambda} (w_i - w_j) + \sum_{i>j} \alpha_i \alpha_j x_i^{2\lambda} x_j^{\lambda} (w_i - w_j)}{\sum_{j=1}^n \alpha_j w_j x_j^{\lambda} \cdot \sum_{j=1}^n \alpha_j x_j^{\lambda}} \\
&\quad = \frac{\sum_{i<j} \alpha_i \alpha_j x_i^{2\lambda} x_j^{\lambda} (w_i - w_j) + \sum_{i<j} \alpha_j \alpha_i x_j^{2\lambda} x_i^{\lambda} (w_j - w_i)}{\sum_{j=1}^n \alpha_j w_j x_j^{\lambda} \cdot \sum_{j=1}^n \alpha_j x_j^{\lambda}} \\
&\quad = \frac{\sum_{i<j} \alpha_i \alpha_j x_i^{\lambda} x_j^{\lambda} (w_i - w_j)(x_i^{\lambda} - x_j^{\lambda})}{\sum_{j=1}^n \alpha_j w_j x_j^{\lambda} \cdot \sum_{j=1}^n \alpha_j x_j^{\lambda}}.
\end{aligned}$$

If $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_n$, then $w_i - w_j \geqslant 0$ for $i < j$. By choosing different λ, we will obtain the same results in the following two cases:

(1) If $\lambda > 0$, with the fact that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$, which leads to $x_i^{\lambda} - x_j^{\lambda} \geqslant 0$ for $i < j$, then we have

$$\frac{\sum_{j=1}^n \alpha_j w_j x_j^{2\lambda}}{\sum_{j=1}^n \alpha_j w_j x_j^{\lambda}} \geqslant \frac{\sum_{j=1}^n \alpha_j x_j^{2\lambda}}{\sum_{j=1}^n \alpha_j x_j^{\lambda}}.$$

Thus,

$$\left(\frac{\sum_{j=1}^n \alpha_j w_j x_j^{2\lambda}}{\sum_{j=1}^n \alpha_j w_j x_j^{\lambda}}\right)^{1/\lambda} \geqslant \left(\frac{\sum_{j=1}^n \alpha_j x_j^{2\lambda}}{\sum_{j=1}^n \alpha_j x_j^{\lambda}}\right)^{1/\lambda}.$$

(2) If $\lambda < 0$, with the fact that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$, which leads to $x_i^\lambda - x_j^\lambda \leqslant 0$ for $i < j$, then we have

$$0 \leqslant \frac{\sum_{j=1}^n \alpha_j w_j x_j^{2\lambda}}{\sum_{j=1}^n \alpha_j w_j x_j^\lambda} \leqslant \frac{\sum_{j=1}^n \alpha_j x_j^{2\lambda}}{\sum_{j=1}^n \alpha_j x_j^\lambda}.$$

Thus,

$$\left(\frac{\sum_{j=1}^n \alpha_j w_j x_j^{2\lambda}}{\sum_{j=1}^n \alpha_j w_j x_j^\lambda}\right)^{1/\lambda} \geqslant \left(\frac{\sum_{j=1}^n \alpha_j x_j^{2\lambda}}{\sum_{j=1}^n \alpha_j x_j^\lambda}\right)^{1/\lambda}.$$

A similar proof can be given for the other part of Lemma 5. □

If we set $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$ in Lemma 5, then we will obtain Corollary 1.

Corollary 1. *For ordered vector $x = (x_1, x_2, \ldots, x_n)$, $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$ and vector $w = (w_1, w_2, \ldots, w_n)$. If $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_n$, then*

$$\left(\frac{\sum_{j=1}^n w_j x_j^{2\lambda}}{\sum_{j=1}^n w_j x_j^\lambda}\right)^{1/\lambda} \geqslant \left(\frac{\sum_{j=1}^n x_j^{2\lambda}}{\sum_{j=1}^n x_j^\lambda}\right)^{1/\lambda}, \tag{35}$$

where $\lambda \in (-\infty, \infty)$ and $\lambda \neq 0$. And if $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$, the inequality is reversed.

Theorem 9. *Let f be the GOWPA operator, and $w = (w_1, w_2, \ldots, w_n)$ be the weighting vector satisfying $\sum_{i=1}^n w_i = 1$ and $w_i \in [0, 1]$ for $i = 1, 2, \ldots, n$. If $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_n$, then*

$$f(a_1, a_2, \ldots, a_n) \geqslant \left(\sum_{j=1}^n a_j^{2\lambda} \Big/ \sum_{j=1}^n a_j^\lambda\right)^{1/\lambda}, \tag{36}$$

and

$$orness_\lambda(w) \geqslant \left(\frac{\sum_{j=0}^{n-1} (j/(n-1))^{2\lambda}}{\sum_{j=0}^{n-1} (j/(n-1))^\lambda}\right)^{1/\lambda}. \tag{37}$$

And if $w_1 \leqslant w_2 \leqslant \cdots \leqslant w_n$, then the inequalities are reversed.

Proof. Let

$$f(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^n w_j b_j^{2\lambda} \Big/ \sum_{j=1}^n w_j b_j^\lambda\right)^{1/\lambda},$$

where b_j is the jth largest of arguments $a_1, a_2, \ldots, a_n$. Then we have $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n \geqslant 0$. If $w_1 \geqslant w_2 \geqslant \cdots \geqslant w_n$, then by Corollary 1, we get that

$$\left(\sum_{j=1}^{n} w_j b_j^{2\lambda} \Big/ \sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \geqslant \left(\sum_{j=1}^{n} b_j^{2\lambda} \Big/ \sum_{j=1}^{n} b_j^{\lambda}\right)^{1/\lambda} = \left(\sum_{j=1}^{n} a_j^{2\lambda} \Big/ \sum_{j=1}^{n} a_j^{\lambda}\right)^{1/\lambda}.$$

If we let $a_j = j/(n-1)$ for $j = 0, 1, 2, \ldots, n-1$, we can get

$$orness_{\lambda}(w) \geqslant \left(\frac{\sum_{j=0}^{n-1}(j/(n-1))^{2\lambda}}{\sum_{j=0}^{n-1}(j/(n-1))^{\lambda}}\right)^{1/\lambda}.$$

The other case of Theorem 12 can be proved in a similar way. □

Theorem 10. *For weighting vector $w = (w_1, w_2, \ldots, w_n)$ and $w' = (w'_1, w'_2, \ldots, w'_n)$, which satisfy $\sum_{i=1}^{n} w_i = 1$, $\sum_{i=1}^{n} w'_i = 1$ and $w_i \geqslant 0$, $w'_i \geqslant 0$ for $i = 1, 2, \ldots, n$. If $\frac{w_i}{w_{i+1}} \geqslant \frac{w'_i}{w'_{i+1}}$ for $i = 1, 2, \ldots, n-1$, then*

$$\text{GOWPA}_w(a_1, a_2, \ldots, a_n) \geqslant \text{GOWPA}_{w'}(a_1, a_2, \ldots, a_n), \tag{38}$$

and

$$orness_{\lambda}(w) \geqslant orness_{\lambda}\big(w'\big). \tag{39}$$

Proof. If $w_i/w_{i+1} \geqslant w'_i/w'_{i+1}$ for $i = 1, 2, \ldots, n-1$, then we get $w_i/w'_i \geqslant w_{i+1}/w'_{i+1}$ for $i = 1, 2, \ldots, n-1$. Assume that $w_i/w'_i = \beta_i$, then $w_i = \beta_i w'_i$ and $\beta_i \geqslant \beta_{i+1}$ for $i = 1, 2, \ldots, n-1$, which means that $\beta_i \geqslant \beta_j$ for $i < j$. Thus, we have that

$$\begin{aligned}
&\frac{\sum_{j=1}^{n} w_j b_j^{2\lambda}}{\sum_{j=1}^{n} w_j b_j^{\lambda}} - \frac{\sum_{j=1}^{n} w'_j b_j^{2\lambda}}{\sum_{j=1}^{n} w'_j b_j^{\lambda}} \\
&\quad = \frac{\sum_{j=1}^{n} w_j b_j^{2\lambda} \times \sum_{j=1}^{n} w'_j b_j^{\lambda} - \sum_{j=1}^{n} w'_j b_j^{2\lambda} \times \sum_{j=1}^{n} w_j b_j^{\lambda}}{\sum_{j=1}^{n} w_j b_j^{\lambda} \cdot \sum_{j=1}^{n} w'_j b_j^{\lambda}} \\
&\quad = \frac{\sum_{i=1}^{n} w_i b_i^{2\lambda} \times \sum_{j=1}^{n} w'_j b_j^{\lambda} - \sum_{j=1}^{n} w'_j b_j^{2\lambda} \times \sum_{i=1}^{n} w_i b_i^{\lambda}}{\sum_{j=1}^{n} w_j b_j^{\lambda} \cdot \sum_{j=1}^{n} w'_j b_j^{\lambda}} \\
&\quad = \frac{\sum_{i=1}^{n} \beta_i w'_i b_i^{2\lambda} \times \sum_{j=1}^{n} w'_j b_j^{\lambda} - \sum_{j=1}^{n} w'_j b_j^{2\lambda} \times \sum_{i=1}^{n} \beta_i w'_i b_i^{\lambda}}{\sum_{j=1}^{n} w_j b_j^{\lambda} \cdot \sum_{j=1}^{n} w'_j b_j^{\lambda}} \\
&\quad = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \beta_i w'_i w'_j b_i^{\lambda} b_j^{\lambda}(b_i^{\lambda} - b_j^{\lambda})}{\sum_{j=1}^{n} w_j b_j^{\lambda} \cdot \sum_{j=1}^{n} w'_j b_j^{\lambda}} \\
&\quad = \frac{\sum_{i<j} \beta_i w'_i w'_j b_i^{\lambda} b_j^{\lambda}(b_i^{\lambda} - b_j^{\lambda}) + \sum_{i>j} \beta_i w'_i w'_j b_i^{\lambda} b_j^{\lambda}(b_i^{\lambda} - b_j^{\lambda})}{\sum_{j=1}^{n} w_j b_j^{\lambda} \cdot \sum_{j=1}^{n} w'_j b_j^{\lambda}}
\end{aligned}$$

$$= \frac{\sum_{i<j} \beta_i w'_i w'_j b_i^\lambda b_j^\lambda (b_i^\lambda - b_j^\lambda) + \sum_{i<j} \beta_j w'_j w'_i b_j^\lambda b_i^\lambda (b_j^\lambda - b_i^\lambda)}{\sum_{j=1}^n w_j b_j^\lambda \cdot \sum_{j=1}^n w'_j b_j^\lambda}$$

$$= \frac{\sum_{i<j} w'_i w'_j b_i^\lambda b_j^\lambda (b_i^\lambda - b_j^\lambda)(\beta_i - \beta_j)}{\sum_{j=1}^n w_j b_j^\lambda \cdot \sum_{j=1}^n w'_j b_j^\lambda}.$$

If $\lambda > 0$, as $\beta_i \geqslant \beta_j$ and $b_i^\lambda - b_j^\lambda \geqslant 0$ for $i < j$, we get that

$$\frac{\sum_{j=1}^n w_j b_j^{2\lambda}}{\sum_{j=1}^n w_j b_j^\lambda} \geqslant \frac{\sum_{j=1}^n w'_j b_j^{2\lambda}}{\sum_{j=1}^n w'_j b_j^\lambda}.$$

Then with $\lambda > 0$, we can obtain that

$$\left(\frac{\sum_{j=1}^n w_j b_j^{2\lambda}}{\sum_{j=1}^n w_j b_j^\lambda}\right)^{1/\lambda} \geqslant \left(\frac{\sum_{j=1}^n w'_j b_j^{2\lambda}}{\sum_{j=1}^n w'_j b_j^\lambda}\right)^{1/\lambda},$$

which implies that

$$\mathrm{GOWPA}_w(a_1, a_2, \dots, a_n) \geqslant \mathrm{GOWPA}_{w'}(a_1, a_2, \dots, a_n).$$

If $\lambda < 0$, as $\beta_i \geqslant \beta_j$ and $b_i^\lambda - b_j^\lambda \leqslant 0$ for $i < j$, we get that

$$\frac{\sum_{j=1}^n w_j b_j^{2\lambda}}{\sum_{j=1}^n w_j b_j^\lambda} \leqslant \frac{\sum_{j=1}^n w'_j b_j^{2\lambda}}{\sum_{j=1}^n w'_j b_j^\lambda}.$$

Then with $\lambda < 0$, we can also obtain that

$$\left(\frac{\sum_{j=1}^n w_j b_j^{2\lambda}}{\sum_{j=1}^n w_j b_j^\lambda}\right)^{1/\lambda} \geqslant \left(\frac{\sum_{j=1}^n w'_j b_j^{2\lambda}}{\sum_{j=1}^n w'_j b_j^\lambda}\right)^{1/\lambda},$$

which also implies that

$$\mathrm{GOWPA}_w(a_1, a_2, \dots, a_n) \geqslant \mathrm{GOWPA}_{w'}(a_1, a_2, \dots, a_n).$$

If we let $a_j = (n - j)/(n - 1)$ for $j = 1, 2, \dots, n$, we can get

$$orness_\lambda(w) \geqslant orness_\lambda(w').$$

The theorem is proved. □

Theorem 10 describes the relative change relations of the GOWPA operator and its orness measure.

5. Generalized Least Squares Method for Determining GOWPA Weights

To determine the OWA operator weights, O'Hagan suggested a maximum entropy method (O'Hagan, 1988), which requires the solution of the following constrained nonlinear optimization model:

$$\begin{aligned}&\text{Maximize } \operatorname{Disp}(w) = -\sum_{i=1}^{n} w_i \ln w_i,\\&\text{s.t.} \quad orness(w) = \alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i, \quad 0 < \alpha < 1,\\&\sum_{i=1}^{n} w_i = 1, \quad 0 \leqslant w_i \leqslant 1,\ i = 1, 2, \ldots, n.\end{aligned} \tag{40}$$

Wang and Parkan proposed the following model for minimizing the maximum disparity between two adjacent weights under a given level of orness (Wang and Parkan, 2005):

$$\begin{aligned}&\text{Minimize } \underset{i\in\{1,2,\ldots,n-1\}}{\operatorname{Max}} |w_i - w_{i+1}|\\&\text{s.t.} \quad orness(w) = \alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i, \quad 0 < \alpha < 1,\\&\sum_{i=1}^{n} w_i = 1, \quad 0 \leqslant w_i \leqslant 1,\ i = 1, 2, \ldots, n.\end{aligned} \tag{41}$$

Considering the importance of the OWA weights, Wang, Luo and Liu constructed the least squares deviation (LSD) model and the chi-square (χ^2) model for determining the OWA operator weights (Wang *et al.*, 2007). The two models can be expressed as follows:

$$\begin{aligned}&\text{Minimize } J_1 = \sum_{i=1}^{n-1}(w_i - w_{i+1})^2\\&\text{s.t.} \quad orness(w) = \alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i, \quad 0 < \alpha < 1,\\&\sum_{i=1}^{n} w_i = 1, \quad 0 \leqslant w_i \leqslant 1,\ i = 1, 2, \ldots, n.\end{aligned} \tag{42}$$

$$\begin{aligned}&\text{Minimize } J_2 = \sum_{i=1}^{n-1}\left(\frac{w_i}{w_{i+1}} + \frac{w_{i+1}}{w_i} - 2\right),\\&\text{s.t.} \quad orness(w) = \alpha = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i, \quad 0 < \alpha < 1,\\&\sum_{i=1}^{n} w_i = 1, 0 \leqslant w_i \leqslant 1, i = 1, 2, \ldots, n.\end{aligned} \tag{43}$$

As it is explained in Wang *et al.* (2007), the aggregation operator weights are equally important and all the arguments can be equally aggregated. If we take the orness constraint into consideration, then we have

$$orness(w) = \alpha = \left(\frac{\sum_{j=1}^{n} w_j (\frac{n-j}{n-1})^{2\lambda}}{\sum_{j=1}^{n} w_j (\frac{n-j}{n-1})^{\lambda}}\right)^{1/\lambda}, \quad 0 \leqslant \alpha \leqslant 1,$$

which can be rewritten as

$$\sum_{j=1}^{n} w_j (n-j)^{\lambda}\left[(n-j)^{\lambda} - (n-1)\alpha^{\lambda}\right] = 0.$$

Thus, our model should be expressed as making all the weights be as close to each other as possible with a given degree of orness, and we can construct the following model to determine the GOWPA weights:

$$\begin{aligned} &\text{Minimize } J = \sum_{i=1}^{n-1} \left(w_i^{\mu} - w_j^{\mu}\right)^2, \\ &\text{s.t.} \qquad \sum_{j=1}^{n} w_j (n-j)^{\lambda}\left[(n-j)^{\lambda} - (n-1)\alpha^{\lambda}\right] = 0, \quad 0 \leqslant \alpha \leqslant 1, \\ &\sum_{i=1}^{n} w_i = 1, \quad 0 \leqslant w_i \leqslant 1, \ i = 1, 2, \ldots, n. \end{aligned} \tag{44}$$

Where μ is a parameter.

For convenience, we refer to model (44) as the generalized least squares method (GLSM), which imposes the disparity of any distinct ratios of weights rather than two adjacent weights regardless of a regular distribution. As we can see, parameter μ can be used in some particular cases, which depends on the interests of decision-maker in the specific problem. Note that model (44) is nonlinear and can be solved by using LINGO software package. Note also that the GOWPA operator weights does not follow a regular distribution which is the main advantage of the GLSM. It can be easily be shown that the GLSM could be used in the OWA operator, the GOWA operator and other aggregation operators.

Example 4. Suppose $n = 5$ and $\mu = 2$. It is necessary to determine the GOWPA operator weights satisfying different degrees of orness: $0, 0.1, \ldots, 0.9, 1$, which are provided by the decision-maker. By using LINGO software package, the GOWPA operator weights are determined in Table 3 which is also depicted in Fig. 1.

It is observed from Table 3 and Fig. 1 that w_1 increases monotonically, as the orness level, α, increases. w_2, w_3 and w_4 first increase and then decrease as α increases. But w_5 first decreases and then increases as α increases. It can be seen from Fig. 1 that the weights of the GOWPA operator are almost equal to each other with the situation where the orness

Table 3
The GOWPA operator weights determined by the GLSM.

w	$orness(w) = \alpha$										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
w_1	0	0	0	0	0	0	0.0382	0.1360	0.2760	0.4010	0.5
w_2	0	0	0	0	0	0.1043	0.1776	0.1810	0.2319	0.2937	0
w_3	0	0	0	0.0540	0.2646	0.2505	0.2370	0.2149	0.1744	0	0
w_4	0	0	0	0.4321	0.3529	0.3128	0.2677	0.2320	0.1480	0.0435	0
w_5	1	1	1	0.5139	0.3825	0.3324	0.2795	0.2361	0.1697	0.2618	0.5

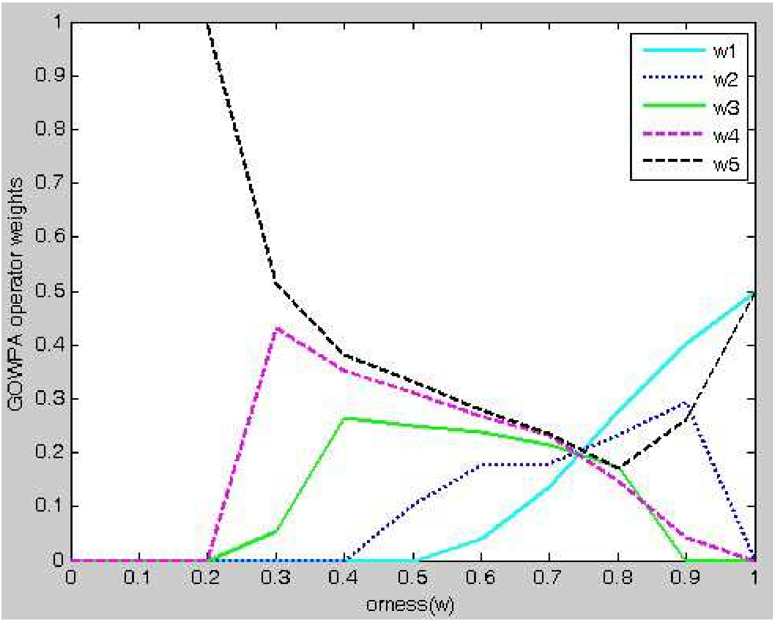

Fig. 1. Variation of the generalized least squares method for determining GOWPA operator weights with orness degree.

degree is close to 0.75, which indicates that the model to determine the GOWPA operator weights is effective.

6. Approach for Group Decision Making Problems Based on the GOWPA Operator

The GOWPA operator is applicable in a wide range of situations, such as decision making, economics, statistics and engineering. In this section, we will introduce an approach for selection of investments based on the GOWPA operator, which can be also used in strategic decision making, selection of financial products and human resource management, etc.

Consider a group decision making problem. Let $X = \{x_1, x_2, \ldots, x_m\}$ be a discrete set of m feasible alternatives, and $C = \{c_1, c_2, \ldots, c_n\}$ be a finite set of attributes. Let $D = \{d_1, d_2, \ldots, d_t\}$ be the set of decision makers, and $v = (v_1, v_2, \ldots, v_t)^T$ be the weighting vector of decision makers satisfying $v_k \in [0, 1]$ and $\sum_{k=1}^{t} v_k = 1$ but v_k is unknown completely. Assume that each decision maker provides their own decision matrix $A^{(k)} = (a_{ij}^{(k)})_{m\times n}$, in which $a_{ij}^{(k)}$ is a preference value given by the decision maker $d_k \in D$, for the alternative $x_i \in X$ with respect to the attribute $c_j \in C$. Let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of attributes which is also unknown completely satisfying $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Due to various physical dimensions corresponding to different attributes in the multiple attribute decision making problem, the standardization of attributes is indispensable in order to eliminate the variances among different attributes. For example, the profit type, which indicates that the larger the attribute value, the better the attribute, and the cost type, which indicates that the smaller the attribute value, the better the attribute, are different attribute type. They should be standardized if they are taken into consideration in the attribute index set. In this paper, we focus on the attribute of profit type and cost type. Let I_1 be the attribute index set of profit type and I_2 be the attribute index set of cost type. In order to measure all attributes in dimensionless units and to facilitate inter-attribute comparisons , we can transform each decision matrix $A^{(k)}$ into a corresponding decision matrix $R^{(k)} = (r_{ij}^{(k)})_{m\times n}$ by the following formulas:

$$r_{ij}^{(k)} = \frac{a_{ij}^{(k)} - \min_i a_{ij}^{(k)}}{\max_i a_{ij}^{(k)} - \min_i a_{ij}^{(k)}}, \quad j \in I_1, i = 1, 2, \ldots, m \tag{45}$$

$$r_{ij}^{(k)} = \frac{\max_i a_{ij}^{(k)} - a_{ij}^{(k)}}{\max_i a_{ij}^{(k)} - \min_i a_{ij}^{(k)}}, \quad j \in I_2,\ i = 1, 2, \ldots, m. \tag{46}$$

Note that standardization of other attribute types can be found in [38]. Then the process to follow in the selection of investments based on the GOWPA operator can be summarized as follows:

Step 1. Standardize each decision matrix $A^{(k)}$ into the matrix $R^{(k)}$ by Eq. (45) and Eq. (46).

Step 2. Utilize the GLSM proposed in Section 5 to calculate the weighting vector of decision makers: $v = (v_1, v_2, \ldots, v_t)^T$, which satisfies $v_k \in [0, 1]$ and $\sum_{k=1}^{t} v_k = 1$.

Step 3. Utilize the GOWPA operator

$$\tilde{r}_{ij} = \text{GOWPA}\left(r_{ij}^{(1)}, r_{ij}^{(2)}, \ldots, r_{ij}^{(t)}\right), \quad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$$

to aggregate all the standardized decision matrices $R^{(k)}$ $(k = 1, 2, \ldots, t)$ into a collective decision matrix $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$.

Step 4. Utilize the GLSM again to calculate the weighting vector of attributes: $w = (w_1, w_2, \ldots, w_n)^T$ satisfying $w_i \in [0, 1]$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$.

Step 5. Utilize the GOWPA operator

$$r_i = \text{GOWPA}(r_{i1}, r_{i2}, \ldots, r_{in}), \quad i = 1, 2, \ldots, m$$

to derive the collective overall preference value r_i of the alternative x_i.

Step 6. Rank the collective overall preference values r_i $(i = 1, 2, \ldots, m)$ in descending order.

Step 7. Rank all the alternatives x_i $(i = 1, 2, \ldots, m)$ and select the best one(s) in accordance with the collective overall preference values r_i $(i = 1, 2, \ldots, m)$.

Step 8. End.

7. Illustrative Example

In the following, we develop a brief illustrative example of the new approach in a group decision making problem. We study an investment selection problem where investor is looking for an optimal investment.

Assume that an investor wants to invest his money in an company. After analyzing the market, he considers five possible alternatives:

- x_1 is a computer company.
- x_2 is a car company.
- x_3 is a furniture company.
- x_4 is a food company.
- x_5 is a chemical company.

In order to evaluate these alternatives, the investor has brought together a group of experts. The group of company experts is constituted by four persons. After careful review of the information, they summarize the ability of companies with six attributes $C = \{c_1, c_2, c_3, c_4, c_5, c_6\}$:

- c_1: Expected benefit.
- c_2: Technical ability.
- c_3: Competitive power on market.
- c_4: Ability to bear risk.
- c_5: Management capability.
- c_6: Organizational culture.

Experts offer their own opinions regarding the results obtained with each investment, and the results are shown in Tables 4–7.

With this information, we can use the proposed decision making method to get the ranking of the companies. Note that in this example, we assume that $\lambda = 1$ in the GOWPA operator and $\mu = 2$, $\alpha = 0.8$ in the GLSM. The following steps are involved:

Step 1. Standardize each decision matrices $A^{(k)}$ into the matrices $R^{(k)}$ by Eq. (45) and Eq. (46). They are shown in Tables 8–11.

Table 4
Decision matrix $A^{(1)} - d_1$.

x_i	u_1	u_2	u_3	u_4	u_5	u_6
x_1	70	80	60	70	50	90
x_2	80	60	90	70	60	70
x_3	50	40	80	30	80	80
x_4	60	70	60	70	80	60
x_5	90	80	40	70	70	80

Table 5
Decision matrix $A^{(2)}$ provided by d_2.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	80	30	70	70	60	70
x_2	60	80	50	60	40	80
x_3	70	60	80	60	70	70
x_4	70	60	80	70	80	80
x_5	60	70	50	60	80	70

Table 6
Decision matrix $A^{(3)}$ provided by d_3.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	70	80	70	70	60	80
x_2	60	40	80	70	60	70
x_3	70	60	60	60	40	70
x_4	70	60	70	60	60	70
x_5	60	50	80	50	50	80

Table 7
Decision matrix $A^{(4)}$ provided by d_4.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	60	70	70	50	80	60
x_2	70	80	60	70	60	80
x_3	40	50	90	70	60	60
x_4	70	60	40	80	70	70
x_5	80	70	60	60	70	50

Step 2. Utilize the GLSM to calculate the weighting vector of decision makers:

$$v = (0.2887, 0.2530, 0.2266, 0.2317)^T.$$

Step 3. Utilize the GOWPA operator

$$\tilde{r}_{ij} = \text{GOWPA}\left(r_{ij}^{(1)}, r_{ij}^{(2)}, r_{ij}^{(3)}, r_{ij}^{(4)}\right), \quad i = 1, 2, 3, 4, 5;\ j = 1, 2, \ldots, 6$$

Table 8
The standardized decision matrix $R^{(1)}$.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	0.7778	1.0000	0.6667	1.0000	0.6250	1.0000
x_2	0.8889	0.7500	1.0000	1.0000	0.7500	0.7778
x_3	0.5556	0.5000	0.8889	0.4286	1.0000	0.8889
x_4	0.6667	0.8750	0.6667	1.0000	1.0000	0.6667
x_5	1.0000	1.0000	0.4444	1.0000	0.8750	0.8889

Table 9
The standardized decision matrix $R^{(2)}$.

x_i	c_1	c_2	c_3	c_4	c_5	c_6
x_1	1.0000	0.3750	0.8750	1.0000	0.7500	0.8750
x_2	0.7500	1.0000	0.6250	0.8571	0.5000	1.0000
x_3	0.8750	0.7500	1.0000	0.8571	0.8750	0.8750
x_4	0.8750	0.7500	1.0000	1.0000	1.0000	1.0000
x_5	0.7500	0.8750	0.6250	0.8571	1.0000	0.8750

Table 10
The standardized decision matrix $R^{(3)}$.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	1.0000	1.0000	0.8750	1.0000	0.7500	1.0000
x_2	0.8571	0.5000	1.0000	1.0000	0.7500	0.8750
x_3	1.0000	0.7500	0.7500	0.8571	0.5000	0.8750
x_4	1.0000	0.7500	0.8750	0.8571	0.7500	0.8750
x_5	0.8571	0.6250	1.0000	0.7143	1.0000	1.0000

Table 11
The standardized decision matrix $R^{(4)}$.

	c_1	c_2	c_3	c_4	c_5	c_6
x_1	0.7500	0.8750	0.7778	0.625	1.0000	0.7500
x_2	0.8750	1.0000	0.6667	0.8750	0.7500	1.0000
x_3	0.5000	0.6250	1.0000	0.8750	0.7500	0.7500
x_4	0.8750	0.7500	0.4444	1.0000	0.8750	0.8750
x_5	1.0000	0.8750	0.6667	0.7500	0.8750	0.625

to aggregate all the standardized decision matrices $R^{(k)}$ $(k = 1, 2, 3, 4)$ into a collective decision matrix $\tilde{R} = (\tilde{r}_{ij})_{5\times 6}$, where

$$\tilde{r}_{11} = 0.9074, \quad \tilde{r}_{12} = 0.9044, \quad \tilde{r}_{13} = 0.8137, \quad \tilde{r}_{14} = 0.9405,$$
$$\tilde{r}_{15} = 0.8182, \quad \tilde{r}_{16} = 0.9253, \quad \tilde{r}_{21} = 0.8469, \quad \tilde{r}_{22} = 0.8787,$$
$$\tilde{r}_{23} = 0.8750, \quad \tilde{r}_{24} = 0.9434, \quad \tilde{r}_{25} = 0.7082, \quad \tilde{r}_{26} = 0.9295,$$
$$\tilde{r}_{31} = 0.8117, \quad \tilde{r}_{32} = 0.6797, \quad \tilde{r}_{33} = 0.9282, \quad \tilde{r}_{34} = 0.8073,$$
$$\tilde{r}_{35} = 0.8391, \quad \tilde{r}_{36} = 0.8536, \quad \tilde{r}_{41} = 0.8795, \quad \tilde{r}_{42} = 0.7902,$$

$\tilde{r}_{43} = 0.8230,\quad \tilde{r}_{44} = 0.9706,\quad \tilde{r}_{45} = 0.9253,\quad \tilde{r}_{46} = 0.8795,$
$\tilde{r}_{51} = 0.9217,\quad \tilde{r}_{52} = 0.8749,\quad \tilde{r}_{53} = 0.7628,\quad \tilde{r}_{54} = 0.8564,$
$\tilde{r}_{55} = 0.9468,\quad \tilde{r}_{56} = 0.8785.$

Step 4. Utilize the GLSM again to calculate the weighting vector of attributes and we obtain

$$w = (0.2552, 0.2210, 0.1675, 0.1133, 0.1067, 0.1363)^T.$$

Step 5. Utilize the GOWPA operator

$$r_i = \mathrm{GOWPA}(\tilde{r}_{i1}, \tilde{r}_{i2}, \tilde{r}_{i3}, \tilde{r}_{i4}, \tilde{r}_{i5}, \tilde{r}_{i6}), \quad i = 1, 2, 3, 4, 5$$

to derive the collective overall preference value r_i of the alternative x_i and we get

$r_1 = 0.8997,\quad r_2 = 0.8859,\quad r_3 = 0.8437,$
$r_4 = 0.8989,\quad r_5 = 0.8908.$

Step 6. Rank the collective overall preference values r_i $(i = 1, 2, 3, 4, 5)$ in descending order:

$$r_1 > r_4 > r_5 > r_2 > r_3.$$

Step 7. Rank all the alternatives x_i $(i = 1, 2, 3, 4, 5)$ in accordance with the collective overall preference values r_i $(i = 1, 2, 3, 4, 5)$:

$$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3.$$

Therefore, the best one is x_1. That is to say, the computer company is the best alternative in this investment problem.

In order to analyze how the different values of parameter λ have affection for the aggregation results r_i $(i = 1, 2, \dots, 5)$, we consider the values of λ, which range from -10 to 10. The results are depicted in Fig. 2.

As we can see from Fig. 2, the collective overall preference value r_i $(i = 1, 2, \dots, 5)$ increases monotonically, as the parameter λ increases. However, the ordering of the investments is different, thus leading to different decisions.

Furthermore, we also can investigate how the different particular cases of the GOWPA operator have affection for the aggregation results, in Step 4, we consider the maximum, the minimum, the median GOWPA operator, the step GOWPA operator ($k = 3$), the olympic GOWPA operator, the window GOWPA operator ($k = 2$, $m = 3$) and the GDM. The results are shown in Tables 12.

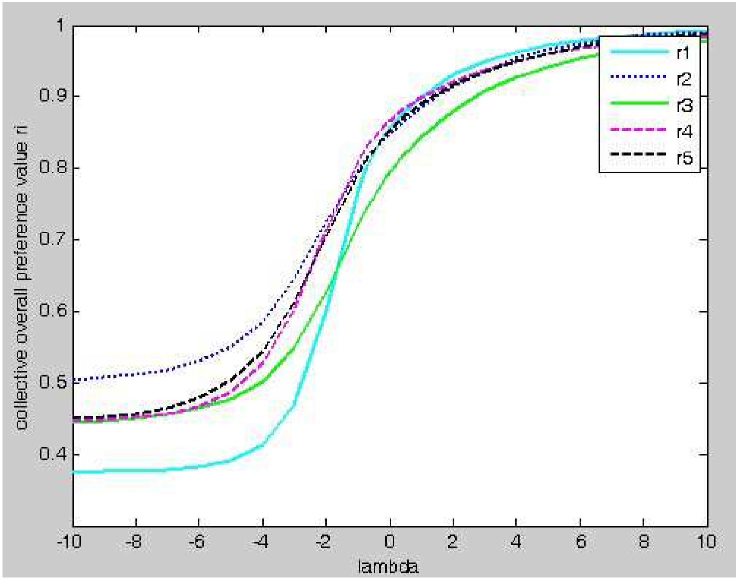

Fig. 2. Collective overall preference value r_i with different λ.

Table 12
Aggregated results.

	Maximum	Minimum	Median	Step	Olympic	Window	GPM
r_1	0.9405	0.8137	0.9059	0.9074	0.8907	0.9124	0.8878
r_2	0.9434	0.7082	0.8769	0.8787	0.8835	0.8951	0.8705
r_3	0.9282	0.6797	0.8256	0.8391	0.8284	0.8352	0.8267
r_4	0.9706	0.7902	0.8795	0.8795	0.8784	0.8953	0.8821
r_5	0.9468	0.7628	0.8767	0.8785	0.8835	0.8922	0.8774

Table 13
Ordering of the investments.

	Ordering		Ordering
Maximum	$x_4 \succ x_5 \succ x_2 \succ x_1 \succ x_3$	Minimum	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$
Nedian	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3$	Step	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3$
Olympic	$x_1 \succ x_2 \succ x_5 \succ x_4 \succ x_3$	Window	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3$
GPM	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$	GOWPA	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$

We can establish an ordering of the investments for each special case of the GOWPA operator. The results are shown in Table 13. Note that "$\succ$" means "preferred to" and "$\sim$" means "equivalent to".

As we can see, depending on the particular cases of the GOWPA operator used, the ordering of the investments is different, thus leading to different decisions. However, it seems that x_1 is the best choice for the investor as a final decision although x_4 sometimes is also the best one.

8. Concluding Remarks

In this paper, we have presented the GOWPA operator based on the optimal problem with a new penalty function. It can be seen as a generalization of the OWGA operator, but the weights depend on their aggregation arguments. With the parameter in the GOWPA operator, we have been able to generalize a wide range of the OWGA operator, including the OWPA operator, the OWHPA operator and the OWSPA operator. We have further generalized the GOWPA operator by adding a new parameter and thus we obtained the generalized hybrid proportional averaging operator. The main advantage of the GOWPA operator is that it is not only able to provide a wide range of the aggregation operators, but also it is based on an optimal model which leads to the result that the weighting vector is associated with the aggregation arguments.

In order to determine the GOWPA operator weights, we have proposed the orness measure for characterizing the weighting vector of the GOWPA operator. Furthermore, we have presented the GLSM for determining the GOWPA operator weights. We also presented an application of the new approach to group decision making in an example of an investment selection problem. The main advantage of generalized least squares model is that it does not follow a regular distribution and is also applicable to different group decision making problems effectively.

In future, we expect to develop further extensions by adding new characteristic, such as uncertainty. We will also consider other decision making problems, such as strategic decision making and product management.

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Apibendrintasis sutvarkytas svertinis proporcinis operatorius ir jo taikymas grupiniams sprendimams priimti

Ligang ZHOU, Huayou CHEN

Straipsnyje pristatomas naujas agregavimo operatorius – apibendrintasis sutvarkytas svertinis proporcinis vidurkis (GOWPA), kuris remiasi optimaliu modeliu su baudos funkcija. Šis operatorius papildo sutvarkyto svertinio geometrinio vidurkio operatorių. Aptariamos, kai kurios GOWPA operatoriaus ypatybės ir atskiri atvejai, taip pat pristatomas jo apibendrinimas. Pagrindinis GOWPA operatoriaus privalumas yra tai, kad jis paremtas teoriniais agregavimo principais, kurie atsižvelgia į agreguojamų duomenų struktūrą ir argumentų svorius. Pasiūlytas loginio operatoriaus OR taikymo laipsnio matas GOWPA operatoriui, aptartos jo savybės. Taip pat pasiūlytas apibendrintasis mažiausių kvadratų metodas GOWPA operatoriaus svorių nustatymui atsižvelgiant į operatoriaus OR taikymo laipsnį. Galiausiai, skaitinis pavyzdys iliustruoja naujojo metodo taikymą investicijų valdymo srityje.