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PREDICTION OF A STRUCTURAL INSTABILITY IN STOCHASTIC PROCESS

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Abstract. The idea of predicting the case, when the considered long-term ARMA model, fitted to the observed time series tends to become unstable because of deep changes in the structural stability of data, is developed in this paper. The aim is to predict a possible unstable regime of the process $\{X_t, t\in T\}$ τ-steps in advance before it will express itself by a high level crossing or large variance of an output variable $\tilde{X}_t$. The problem is solved here for locally stationary AR(p) sequences $\{X_t, t\in T\}$, whose estimated parameters can reach critical sets located at the boundary of the stability area. An alarm function and an alarm set are fitted here to predict catastrophic failures in systems output τ units in advance for given $\tau>0$ and a confidence level γ. The probability of false alarm is derived explicitly for AR(1) depending on τ,γ and N – the number of the last observations of $\{X_t\}$.

Key words: non-stationary AR, prediction of structural instability, τ-predictable catastrophe, alarm set, false alarm probability.

1. Introduction. A change is an universal property. Any phenomenon is changing – slowly or rapidly, continuously or abruptly. Even a rock standing immovable at the same place in the course of centuries does undergo slow changes in physical and chemical structures eventually turning into soil. A statistical model well fitted to describe the behaviour of any kind of phenomena becomes unfit after some time. A theory of various kinds of smooth behaviour becomes less and less applicable in our complicated dynamical life nowadays. A sudden change may be caused by a smooth alteration in the situation. Political and economical systems collapse, bridge and buildings fall down when some critical point in their state is reached. The probabilistic approach to predict a catastrophe, specified as a high level crossing by the sample path of a stochastic process was proposed by Jacque de Maré (1980) and Lindgren (1980) where an optimal alarm system was fitted. But the level crossing is one of the visible properties, it can be treated as a result of some structural changes in the system, generating a time series, or it can be interpreted as a rare event in the sample path of a stationary random process. The latter interpretation has been used in the papers mentioned above. Our aim is to relate such catastrophic failures in system output with deep changes in it's structural stability, since the parameters describing the system change in such a way that they reach critical sets. It is important to join analytical and probabilistic techniques in order to handle such a behaviour and to predict it in advance.

Let us prepare the probabilistic tools necessary for the solution of the mentioned problems.

2. Notation and basic facts. Let $\{X_t,\ t \in T\}$, $T = 0, \pm 1, \pm 2, \ldots$, be a time series described by the model ARMA(p,q)

$$\sum_{j=0}^{p} a_j(t)(X_{t-j} - \mu_t) = \sum_{j=0}^{q} b_j(t)\varepsilon_{t-j}, \qquad a_0(t) \equiv 1. \tag{1}$$

Here all ε's are independent identically distributed with mean zero $E\varepsilon_t = 0$ and variance $E\varepsilon_t^2 = 1$. The right hand side of equation (1) and the parameters $b_j(t)$, $j = 0 \div q$ represent non-stationary, unmeasured natural excitation while the parameters $\theta(t) = (a_1(t), \ldots, a_p(t), \mu_t)$ reflect the structure of the real object, identifiable by statistical methods on the base of the observed data – realizations $x_1, x_2, \ldots, x_{N_i}$, $i = 1, 2, \ldots, M$ of the sequence $\{X_t,\ t \in T\}$. We are interested here only in structural parameters $\theta(t)$ and their possible changes making dynamical system (1) unstable. The instability leads to "catastrophic" failures of a physical system: an upcrossing of a high level, the explosive character of output. Such unpleasant or dangerous events must be predicted in advance, say τ steps ahead on the base of the whole information available at the moment. The problems that arise in this context are the following:

(i) to choose the predictor or alarm function which carries the information on stability of a dynamical system, enabling the algorithm to make an alarm τ steps ahead;

(ii) to evaluate characteristics of the fitted alarm system, namely the probabilities of false alarm and undetected failures;

(iii) to optimize an alarm system.

The first two goals are the subject of the presented paper while the optimization will be pursued in the future investigations.

Let us formalize the problem. Necessary definitions and facts, certainly known to the reader, will be given below in order to avoid misunderstanding.

$\Theta(t)=\Theta$, the system invariant in time. Rewrite the equation (1) as:

$$\sum_{j=0}^{p} a_j(X_{t-j}-\mu)=\sum_{j=0}^{q} b_j\varepsilon_{t-j}. \tag{2}$$

DEFINITION 2.1. A stochastic system described by the equation (2) is stable iff all the roots $z_1, z_2, \ldots, z_p$ of the characteristic polynomial equation

$$z^p + a_1 z^{p-1} + \ldots + a_{p-1} z + a_p = 0 \tag{3}$$

are less than unity in magnitude

$$|z_j| < 1, \qquad j = 1 \div p. \tag{4}$$

The output of a stable system (2), i.e., time series $x_1, x_2, \ldots, x_N$, is a realization of a stationary random process or sequence $\{X_t, t \in T\}$.

In this context it is useful to mention the theorem proved in the monograph by Anderson (1971).

Theorem 2.2. *If a stationary stochastic process satisfies a stochastic difference equation for which at least one root is 1, all values of the process are the same with probability 1.*

A random process $\{X_t,\ t \in T\}$ is no longer stationary if the polynomial (3) contains a factor of the form $(z-1)^d, d \geqslant 1$, although its d-th difference would be stationary.

DEFINITION 2.3. If some root of the polynomial equation (3) is larger than 1 in absolute value, system (2) is *unstable*, its output is a non-stationary sequence and the random process $\{X_t,\ t \in T\}$ is *explosive*.

Denote by

$$Z_1 = \{z:\ |z| < 1\} \tag{5}$$

a set in a complex plane Z inside the unit circle and by $\partial Z_1 = \{z:\ |z| = 1\}$, a boundary of the stability region, according to definition 2.1.

The subset $Z_1 \subset Z$ may be mapped into a closed subset of parameters $\Theta \subset R^p$ for every fixed p, i.e., the number of characteristic roots.

Denote the mapping, known as the formulae of Viéte, relating the roots of the polynomial and its coefficients by v,

$$v:\ Z_1 \Rightarrow \Theta, \tag{6}$$

$$v:\ \partial Z_1 \Rightarrow \partial\Theta, \tag{7}$$

where

$$\Theta = \{\theta : |z_j| < 1,\ j = 1 \div p\}, \qquad \Theta \subset R^p, \tag{8}$$

and $\partial\Theta$ is the boundary of the p-dimensional stability region Θ of stochastic system (2).

Note that we consider real X_t's and only real-valued coefficients $a_1, a_2, \ldots, a_p$, so $z_1, \ldots, z_p$ are only complex conjugate and real.

The complements to Z, and Θ that are denoted here by $\bar{Z}_1$ and $\bar{\Theta}$:

$$Z_1 \cup \bar{Z}_1 = Z. \qquad \Theta \cup \bar{\Theta} = R^p, \tag{9}$$

Namely, $\bar{Z}$, and $\bar{\Theta}$ are the areas of instability in Z plane and parameter space R^p.

$\Theta(t)$ time-variant. Let us return to equation (1) and consider separately abrupt and continuous changes of θ as time goes.

(1) *Step-wise function* $\theta(t)$. Say $\theta(t)$ is piece-wise constant: the value $\theta^{(1)} \in \Theta$ changes suddenly to $\theta^{(2)}$. Two situations are possible: $\theta^{(2)} \in \Theta$ and $\theta^{(2)} \notin \Theta$. Both cases are unpredictable events if an observable sequence $\{X_t,\ t \in T\}$ is independent of θ's changes in the statistical sense. In case $\theta^{(2)} \in \Theta$ the resulting process $\{X_t\}$ satisfies the stability requirements but it is non-stationary, of course. Any change inside Θ is allowed. The processes of such a kind are known (Ozaki and Tong, 1975; Kitagawa and Akaike, 1978) as *a piece-wise stationary* or *locally stationary.* The process may consist of M independent stationary 'pieces' $\{X_t^{(i)}, t_{i-1} \leqslant t \leqslant t_i\}$ $i = 1, 2, \dots, M$, $t_o \equiv 0$, corresponding to the values $\theta^{(i)} \in \Theta$; as if it is switching from one stationary regime to another.. The result is *the switching process* that can be considered as the limiting process (Kligienė, 1976) of a locally stationary process when sudden changes are rare events and the intervals of 'stationarity' are large as compared to the value p, i.e., the order of equation (1) on the left side.

(2) *Evolutionary changing* $\theta(t)$. Here we have in mind any continuous, slow changes. In the study of a long-life phenomenon, the process $\{X_t\}$ is often non-stationary. For instance, the long-term wave effects on the offshore or the evolution of some biological system. For those phenomena one has to consider two time scales: short-term (minutes, days, say) and long-term (on the order of years). The short-term probabilistic model is then described under the conditions of local stationarity, the parameters being fixed. The long-term model is obtained on providing a long-term probabilistic description of the short-term parameters. If such a description is available, the prediction of that important change τ-steps ahead is possible. Both cases $\theta^{(2)} \in \Theta$ or $\theta^{(2)} \notin \Theta$ may be considered again, the latter being more important in our problem. It is well known (White, 1958) that an unstable process becomes explosive.

Let $\{X_t,\ t \in T\}$ be a time series defined by equation (1) with the structural parameters $\theta(t) = (a_1(t), \dots, a_p(t), \mu_t)$ for each t belonging to the stability region Θ. For each time point t let C_t be a measurable set of sample functions.

DEFINITION 2.4. A catastrophe occurs at time t if $X \in C_t$.

Such a definition of a catastrophe was introduced and used by Jacque de Maré (1980), where as an example of C_t is given:

$$C_t(x) = \{x : \ X_t \leqslant u < X_{t+s}, \quad 0 < s < \delta\}. \qquad (10)$$

As a rule, δ is a small positive number and u is a high level.
It is rather a mere occurrence of level crossing within the near future that is of interest, not the exact time of it. Therefore, we formulate the problem as a pure two-choice problem: at each time t we make one of two possible statements: either that X_{t+s} will cross the prescribed level u at least once for some $s \in (0, \delta)$, or that there will be no such crossing.

Let us restrict our investigation to a definite situation formulated here as the following

ASSUMPTIONS:

(A1) The short-term model is time invariant $\mathrm{AR}^{(i)}(p)$ model

$$\sum_{j=0}^{p} a_j^{(i)}(X_{t-j} - \mu) = \varepsilon_t, \qquad t \in T_i \qquad a_o^{(i)} \equiv 1 \qquad (11)$$

$$i = 1, 2, \ldots, M, \qquad T_i = (t_{i-1}, t_{i-1} + 1, \ldots, t_i)$$

where all the roots of the polynomial equation

$$\sum_{j=0}^{p} a_j^{(i)} z^{p-j} = 0 \qquad (12)$$

are less than 1 in absolute value.

(A2) The long-term model consists of the locally stationary processes $\mathrm{AR}^{(1)}(p), \mathrm{AR}^{(2)}(p), \ldots, \mathrm{AR}^{(M)}(p)$ defined on non-overlapping intervals $T_1, \ldots, T_M$ represented by the observed sequences $x_1^{(i)}, x_2^{(i)}, \ldots, x_{N_i}^{(i)}, \ i = 1, 2, \ldots, M$.

Theorem 2.5. *If some roots (at least one $z^{(i)}$) of the polynomial equation (12) are larger than 1 in absolute value, then a catastrophe C_t will occur at some time $t \in T_{i_0}$ with probability 1.*

Proof. Introducing the backward shift operator B^{-1} defined by

$$B^{-1}X_t = X_{t-1} \qquad \text{and} \qquad BX_t = X_{t+1}, \qquad (13)$$

where $B \cdot B^{-1} = I$, the stochastic difference equation (11) can be rewritten (we shall omit the upper index (i) what is not important here)

$$\sum_{j=0}^{p} a_j B^{-j}(X_t - \mu) = \varepsilon_t \tag{14}$$

and formally .

$$X_t - \mu = \Big(\sum_{j=o}^{p} a_j B^{-j}\Big)^{-1} \varepsilon_t = \sum_{j=0}^{\infty} \delta_j \varepsilon_{t-j} \tag{15}$$

where δ_j's are the coefficients in

$$\Big(\sum_{j=o}^{p} a_j z^j\Big)^{-1} = \sum_{j=0}^{\infty} \delta_j z^j. \tag{16}$$

Note that the roots of $\sum_{j=o}^{p} a_j x^j = 0$ are $x_i = 1/z_i$ when $a_p \neq 0$ and $|x_i| > 1$ if the condition $|z_i| < 1$ is fulfilled. Then the series

$$\frac{1}{\sum\limits_{j=o}^{p} a_j x^j} = \frac{1}{\prod\limits_{i=1}^{p}\Big(1 - \frac{x}{z_i}\Big)} = \prod_{i=1}^{p} \sum_{v=o}^{\infty} \Big(\frac{x}{z^i}\Big)^v = \sum_{r=o}^{\infty} \delta_r z^r \tag{17}$$

converges absolutely for any $x: \ |x| < \min |z_i|$.

Let us consider the case in which some roots of the polynomial equation (12) are larger than 1 in absolute values. Suppose $|x_i| > 1$, $i = 1, \dots, r$, $|x_i| < 1$, $i = r+1, \dots, p$. We write the stochastic difference equation (11) as

$$\varepsilon_t = \sum_{j=o}^{p} a_j B^{p-j} X_{t-p} = \prod_{j=o}^{p} (B - x_i) X_{t-p}. \tag{18}$$

The inverse of (18) is

$$X_{t-p} = \prod_{i=1}^{r} (B - x_i)^{-1} \prod_{u=r+1}^{p} \big[B(1 - x_i B^{-1})\big]^{-1} \varepsilon_t$$

$$= \prod_{i=1}^{r} \Big[\Big(-\frac{1}{x_i}\Big)\Big(1 - \frac{1}{x_i} B\Big)^{-1}\Big] \cdot \prod_{i=r+1}^{p} [B^{-1}(1 - x_i B^{-1})^{-1}] \varepsilon_t$$

$$= \prod_{i=1}^{r}\Big(-\frac{1}{x_i}\Big)\frac{B^r}{1-B/x_i}\cdot\prod_{i=r+1}^{p}\frac{B^{-p}}{1-B^{-1}/z_i}\cdot\varepsilon_t$$

$$= \prod_{i=1}^{r}\Big(-\frac{B^r}{x_i}\Big)\sum_{v=0}^{\infty}\Big(\frac{B}{x_i}\Big)^v\cdot\prod_{r+1}^{p}\sum_{v=0}^{\infty}\Big(\frac{B^{-1}}{z_i}\Big)^v\cdot\varepsilon_{t-p} = \sum_{t=-\infty}^{\infty}\delta_t^*\varepsilon_t, \quad (19)$$

where δ_j^*'s are the coefficients in Loran's series

$$\sum_{j=-\infty}^{\infty}\delta_j^* z^j = \prod_{i=1}^{r}\Big(-\frac{z^{-r}}{x_i}\Big)\sum_{v=0}^{\infty}\Big(\frac{z^{-1}}{x_i}\Big)^v\cdot\prod_{i=r+1}^{p}\sum_{v=0}^{\infty}\Big(\frac{z}{z_i}\Big)^v, \quad (20)$$

which is divergent because the convergence area of the essential part $|z| < \min|z_i| < 1 = \max|x_i|^{-1}$ has no common area with that of the regular part $|z^{-1}| < \min|x_i|$, i.e., $|z| > \min|x_i|^{-1}$. So the expression

$$X_t = \sum_{t=-\infty}^{\infty}\delta_t^*\varepsilon_t = \sum_{j=0}^{\infty}\delta_j\varepsilon_{t-j} \quad (21)$$

is convergent only in case $r = 0$ and divergent if $r \neq 0$. The conclusion is that $X_{t+s} > u$ for any large u with probability 1. As soon as some of the roots become larger than one: $|x_i| > 1$, the event C_t will occur with probability 1.

3. Fitting of alarm areas, probabilities of errors. On the base of Theorem 2.5 we shall focus our attention on the critical event i.e., crossing of the boundary $\partial\Theta$ of the closed area Θ by the function of the parameters $\theta(s)$ in a p-dimensional space at the moment t:

$$C_t(\theta) = \{\theta(t-1) < \partial\Theta \leqslant \theta(t)\}$$
$$\sim \{\theta(s) \in \Theta, \text{ if } s < t; \ \theta(s) \in \bar{\Theta}, \text{ if } s \geqslant t\} \quad (22)$$

In terms of the characteristic roots $z_{1t}, \ldots, z_{pt}$, considered as time dependent in a long-term model, the event analogous to (22), may be expressed as

$$C_t(z) = \{\exists i_o : \ |z_{i_o,t-1}| < 1, \ |z_{i_o,t}| > 1, \ i_o \in (1,2,\ldots,p), \ t \in T_j\} \quad (23)$$

The final outcome of the events (22) or (23) will express itself by a high level crossing (10) what follows as consequence of Theorem 2.5.

Thus, if we want to detect the failure of stability, we must detect one of the events $C_t(z)$, $C_t(\theta), C_t(X)$ not only as soon as possible but also we need to anticipate catastrophic failures, i.e., $\tau > 0$ time instants ahead to give an alarm on a catastrophe at the moment t.

Typical examples are the fatigue of some structures, the failures in nuclear plants, earthquakes, etc. One approach to forecast a catastrophe C_t τ time units ahead is to condense the information available in realization $x_1, x_2, \ldots, x_N$ of $\{X_s, s < t - \tau\}$ into a stochastic process $Y(t)$ and to give an alarm if and only if $Y(t)$ has an upcrossing of the specified alarm boundary at time t. Naturally we can consider $Y(t) = \hat{\theta}(t)$, $Y \in R^p$ if we have decided to predict the event (22). The σ-algebra generated by $Y(t)$ is denoted by $\mathcal{F}_Y^t = \sigma\{Y(t)\} = \sigma\{X_s,\ s \leqslant t - \tau\} = \mathcal{F}_X^{t-\tau}$. Define a t-indexed set by $A_t(\tau)$ which is $\mathcal{F}_Y^t$-measurable as the alarm set for a catastrophe C_t to be detected τ units in advance.

We say that there is an alarm for a catastrophe at time t if $Y \in A_t$. The alarm is false if $Y \in A_t$ but $X \notin C_t$ and there is an undetected catastrophe at time t, if $Y \notin A_t$ but $X \in C_t$. The probabilities of errors $P\{Y \in A_t \mid X \notin C_t\}$ and $P\{Y \notin A_t \mid X \in C_t\}$ are of special interest. It is not our goal to construct an optimal alarm system as it was done by Jacque de Maré (1980). At first we would like to define an alarm region $\mathcal{A}_t(\tau, P)$ for fixed τ and $P\{Y \in A_t \mid X \notin C_t\}$ – the probability of false alarm.

Definition 3.1. We shall call the catastrophe C_t τ – *predicable* if there exist such $\tau > 0$ and $Y(t)$ that

$$P\{Y \in A_t \mid C_t\} = c, \qquad c \neq 0, \qquad A_t \in \mathcal{F}_Y^t \tag{24}$$

and C_t is *unpredictable* if $\tau = 0$.

Remark 3.2. The property (24) is stated for the available level of knowledge condensed in $\mathcal{F}_Y^t$. An unpredictable catastrophe with respect to $\mathcal{F}_Y^t$ may become predictable in the sense of $\mathcal{F}_Z^t$, if $\mathcal{F}_Y^t \subset \mathcal{F}_Z^t$.

EXAMPLE 3.3. If the vector function $\theta(t) = (a_1(t), \dots, a_p(t))$ is a slowly varying in time continuous and monotonous t function (each component considered separately), then $C_t(\theta)$ is τ-predictable.

EXAMPLE 3.4. If $\theta(t)$ is a step-wise function with an unknown law of a abrupt change, independent of X, the catastrophe C_t is unpredictable. C_t will also be unpredictable if it is not related to the structure of model (1) and it is provoked by some outward effect.

DEFINITION 3.5. For every real $\varepsilon > 0$ define a subset $A(\varepsilon) \subset Z_1$ as

$$A_t(\varepsilon) = \{z : \ 1 - \varepsilon < |z| < 1\}. \tag{25}$$

Time t is attributed here to τ-upcrossing of ∂Z at t.

Note that $A(\varepsilon)$ is in the neighborhood of a boundary ∂Z of a stability region and in some sense can serve as an alarm area. The problem is to relate ε with τ and P.

Relation (6) maps a subset $A_t(\varepsilon) \subset Z_1$ into $A_t(\cdot\,,\cdot) \subset \Theta$:

$$v : \ A_t(\varepsilon) \Rightarrow A_t(\cdot\,,\cdot) \subset \Theta, \tag{26}$$

and our task is to specify the alarm boundary in concrete cases $p = 1,2$ and $Y(t)$ in terms of the observed realization $x_1, \dots, x_N$ and given values τ, P with $\tau > 0$, $0 < P < 1$.

4. Alarm boundaries for AR(1), AR(p) sequences. The mean value $\mu_t = E\{X_t\}$ is not essential in our investigation, let $\mu_t = 0$, $\forall t$, and consider AR(1) sequence defined by

$$X_t + a_1(t)X_{t-1} = b_0(t)\varepsilon_t. \tag{27}$$

Let the assumptions (A1), (A2) be valid. The short-term model admits local stationarity of the sequence $\{X_t\}$, the parameters $a_1(t) = a_1$, $|a_1| < 1$, $b_0(t) = b_0$, $\forall t$, being constant and estimated from the observed realization $x_1, x_2, \dots, x_N$ as

$$\hat{a}_1 = -\frac{1}{N-1}\sum_{t=2}^{N} x_t x_{t-1} \Big/ \frac{1}{N}\sum_{t=1}^{N} x_t^2, \tag{28}$$

$$\hat{b}_0^2 = \frac{1}{N-1}\sum_{t=2}^{N}(x_t + \hat{a}_1 x_{t-1})^2. \tag{29}$$

The long-term model will include the evolution of the parameter $a_1(t)$ from $|a_1| < 1$ to $|a_1| = 1$ and $|a_1| > 1$. We need to find the limiting distribution of $g(N)\cdot(\hat{a}_1 - a_1)$, where the function $g(N)$ is such that $g(N)(\hat{a}_1 - a_1)$ has a non-degenerate distribution for all the values of a_1. It is a usual way (White, 1958) to take $g(N) = [I(a_1)]^{\frac{1}{2}} = E^{\frac{1}{2}}(-d^2 \log f/da_1)$, where $I(a_1)$ is Fisher's information, or the value asymptotically equivalent to $[I(a_1)]^{\frac{1}{2}}$, i.e.,

$$g(N) = \begin{cases} \sqrt{\dfrac{N}{1-a_1^2}}, & \text{for } |a_1| < 1; \quad (30) \\ \dfrac{N}{\sqrt{2}}, & \text{for } |a_1| = 1; \quad (31) \\ \dfrac{|a_1|^N}{a_1^2 - 1}, & \text{for } |a_1| > 1. \quad (32) \end{cases}$$

It is well known (White, 1958) that $g(N)(\hat{a}_1 - a_1)$ is asymptotically $\mathcal{N}(0,1)$ for $|a_1| < 1$ and the limiting distribution is the Cauchy one for $|a_1| > 1$.

The confidence interval of the level $\gamma = 1 - 2P$ in the case $|a_1| < 1$ can be written out

$$P\left\{\hat{a}_1 - z_P\sqrt{\frac{1-a_1^2}{N}} \leqslant a_1 \leqslant \hat{a}_1 + z_P\sqrt{\frac{1-a_1^2}{N}}\right\} = 1 - 2P \qquad (33)$$

where z_P is the $\mathcal{N}(0,1)$ quantile of the level $1 - P$.

Relation (33) enables us to fit the alarm boundary for stability of model (27) which is stable if $|a_1(t)| < 1$ and unstable if $|a_1(t)| > 1$.

Let us fix the moment of upcrossing the unit boundary at point $s: \ |a_1(s)| = 1$. It is necessary to make an alarm not later than $s - \tau$ for fixed $\tau > 0$. Let $Y(s) = |\hat{a}_1(t)|, \ t < s - \tau$. Say we get the observations $x_{1N}^{(i)} = (x_1^{(i)}, \ldots, x_N^{(i)})$ periodically from each interval T_i and the estimates $\hat{a}^{(i)}$, of $a_1(t), \ t \in T_i$ – are derived according to (28). In such a way the continuous function $a_1(t)$ is approximated by polygon, each level $\hat{a}_1^{(i)}$ of it is provided by the confidence interval (33). Evidently, the alarm region consists of the interval $A_\varepsilon = [1 - \varepsilon, 1)$.

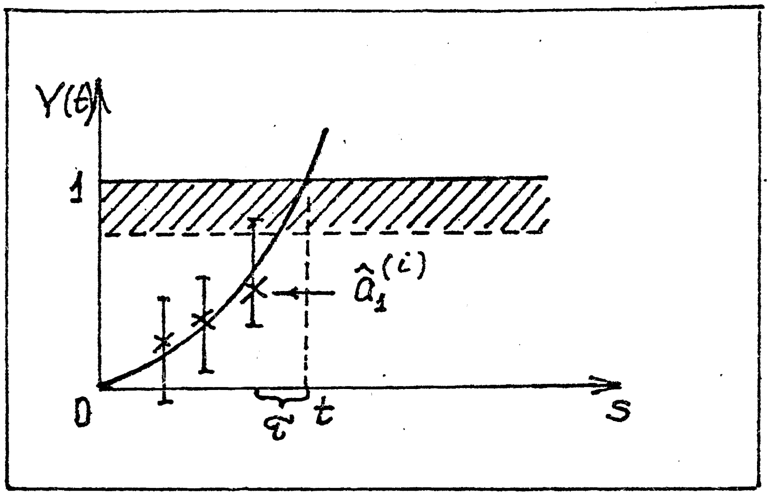

Fig. 1. The alarm region for AR(1).

It is natural to consider an exponential approaching to the boundary: $a_1(s) = e^{\alpha(s-t)}$, having the value $a_1 = 1$ at the moment $s = t$; $|a_1| < 1$ for $s < t$ and $|a_1| > 1$ for $s > t$, see Fig. 1.

Let $Y(t) = \hat{a}_1(t)$, $t < s - \tau$. Each time the estimate $\hat{a}_1^{(i)}$ of $a_1(t)$, $t \in T_i$ is derived according to (28) from the observations $x_1^{(i)}, \ldots, x_{N_i}^{(i)}$.

The alarm region on the base of expression (33) is

$$A_t(\tau, P) = \left[1 - \hat{a}_1(t-\tau) - z_P\sqrt{\frac{1-\hat{a}_1^2}{N_i}},\ 1\right), \qquad (34)$$

In the exponential case it leads to

$$A_t(\tau, P) = \left[1 - e^{-\alpha\tau} - z_P\sqrt{\frac{1-e^{-2\alpha}}{N_i}},\ 1\right), \qquad (35)$$

where α may be estimated from the value $\ln \hat{a}_1$. In addition to (A1), (A2), assume that $N_1 < N_2 < \ldots < N_M$, i.e., we are able to observe larger realizations when approaching to the boundary of stability and to get the estimates of a_1 good enough. The probability of false alarm can be calculated in the following way:

$$\begin{aligned} &P\{Y_t \in A_t \mid X_t \notin C_t;\quad a_1(t) < 1\} \\ &\quad = P\{1-\varepsilon \leqslant \hat{a}_1(t-\tau) < 1 \mid X_t \notin C_t;\quad a_1(t) < 1\}. \end{aligned} \tag{36}$$

Denote a random variable by

$$\zeta = \sqrt{N_i}(\hat{a}_1 - a_1) \sim \mathcal{N}(0, \sigma^2), \qquad \sigma = \sqrt{1-a_1^2}, \tag{37}$$

then

$$\hat{a}_1 = a_1 + \frac{\sigma}{\sqrt{N_i}}\zeta,$$

and the random variables

$$\hat{a}_1(t) = a_1(t) + \frac{\sigma}{\sqrt{N_i}}\zeta_t, \tag{38}$$

$$\hat{a}_1(t-\tau) = a_1(t-\tau) + \frac{\sigma}{\sqrt{N_i}}\zeta_{t-\tau}, \tag{39}$$

are independent because of independence of ζ_t and $\zeta_{t-\tau}$. Thus the probability (36) becomes unconditional and we have

$$\begin{aligned} &P\{Y_t \in A_t \mid X_t \notin C_t;\quad a_1(t) < 1\} \\ &= P\Big\{1-\varepsilon < \hat{a}_1(t-\tau) + \frac{\sigma}{\sqrt{N_i}}\zeta_{t-\tau} < 1\Big\} \\ &= P\Big\{\frac{\sqrt{N_i}}{\sigma}\big(1 - a_1(t-\tau) - \varepsilon\big) < \zeta_{t-\tau} < \frac{\sqrt{N_i}}{\sigma}\big(1 - a_1(t-\tau)\big)\Big\} \\ &= \Phi\Big(\frac{\sqrt{N_i}}{\sigma}\big(1 - a_1(t-\tau)\big)\Big) - \Phi\Big(\frac{\sqrt{N_i}}{\sigma}\Big(1 - 2a_1(t-\tau) - z_P\frac{\gamma}{\sqrt{N_i}}\Big)\Big), \end{aligned} \tag{40}$$

where $\Phi(\cdot)$ is a standard normal distribution function. Expression (40) relates the probability of false alarm to the length N_i of lately observed series $x_1, \dots, x_{N_i}$ and to the fixed values τ and γ.

In the general case of the AR(p) model, choosing $Y(t) = (\hat{a}_1(t), \dots, \hat{a}_p(t))$ it is possible to define an alarm region $A_t(\tau, P) \subset \Theta$ on the base of the well known (Anderson, 1971) asymptotic result

$$\sqrt{N_i}(\hat{a} - a) \sim \mathcal{N}(0,\ b_0^2 R_p^{-1}), \tag{41}$$

where $R_p = (R_{ij},\ i,j = 1 \div p),\ R_{ij} = \mathrm{cov}(X_i, X_j),\ R_p^{-1} \cdot R_p = I$.

The alarm region cannot be so simply fitted as for $p = 1$ but the idea is the same: to map an ε-environment of ∂Z_1 into Θ and to derive an alarm region there, at the stability boundary $\partial\Theta$. The case $p = 2$, given as an example in Fig. 2, illustrates the fact that the mapping in Θ has no such a simple form as its origin, namely, an ε-ring in the Z plane. That implies the thought that it depends on the location of θ in Θ how close to the boundary $\partial\Theta$ one-can approach without too much risk to hit it.

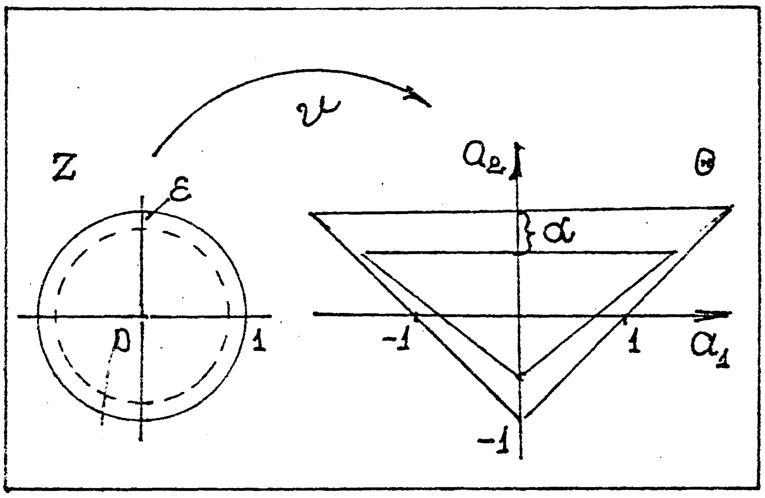

Fig. 2. The alarm region $A_t \subset R^2$ in Z and Θ planes for AR(2). Here $\alpha = 2\varepsilon - \varepsilon^2$.

5. Conclusions, comments. It was the first attempt to relate the changes in the structural parameters with catastrophic failures in the output of a stochastic system in order to predict a possible unstable regime. The idea is to focus the attention on the essence – the behaviour of characteristic roots of the stochastic difference equation, while an alarm set is made in the parametric space.

The subject deals with the non-stationary time series analysis as well as with a prediction theory, that cannot be used here in a traditional way. There is no reason to base a predictor on a criterion such as a minimal mean square prediction error or like it. Since a stability is defined through characteristic roots, it would be very natural to base a criterion on the behaviour of the roots, too. Two difficulties arise an this way: first – to estimate the roots efficiently; second – to derive their probability distribution law (at least asymptotic). First of all, we shall have to deal with complex valued variables, afterwards even a slight deviation in the estimated parameters might cause not so slight variations is the location of the roots, especially if the order of the polynomial is high. Without knowledge of statistical properties of the estimated roots we should not be able to provide them with confidence levels. That is why it is simpler to fit an alarm set in Θ, not in Z. The case $p = 1$ is a mere occurrence, when both Θ and Z is the same unit interval $(-1, 1)$.

There is no real applicator in this paper, but we all know that most of the applications are the systems governed by AR or ARMA equations and the prediction of an unstable situation is urgent in many fields. Note the paper by Popescu and Demetriu (1990) where the records of strong ground motion of the Romanian earthquake of 4 March 1977 are described by the locally stationary ARMA models whose parameters are located near the edge of stability. This is an example, convincing us that such a really catastrophic event as earthquake can be handled by means of the scheme of the τ-predictable catastrophe. The links with a conventional catastrophe theory (Poston and Stewart, 1977), as a coherent mathematical description of discontinuous changes in dynamical systems, is an open problem. It seems that a theory, analogous to the existing one, might also be developed when a stochastic difference equation describes the phenomenon instead of non-stochastic differential equations used in the catastrophe theory up till now.

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