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QUADRATIC 0–1 OPTIMIZATION

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Abstract. This paper briefly reviews some of the recent results on the problems and algorithms for their solution in quadratic 0-1 optimization. First, the complexity of problems is discussed. Next,some exact algorithms and heuristics are mentioned. Finally, results in the analysis of the algorithms for 0-1 quadratic problems are summarized. The papers written in Russian are considered more thoroughly here.

Key words: quadratic 0-1 optimization, graphs, analysis of algorithms.

Introduction. The general quadratic 0-1 optimization problem considered in this paper is stated as follows. Given symmetric an $n \times n$–matrix (c_{ij}) and an n–vector (c_i), find a subset $V \subseteq N = \{1, 2, \dots, n\}$ so as to maximize

$$f(V) = \sum_{i,j \in V, i<j} c_{ij} + \sum_{i \in V} c_i, \tag{1}$$

subject to

$$V \in \widetilde{H} \subseteq H, \tag{2}$$

where H is the family of all subsets of N, and $\widetilde{H}$ is a set of feasible solutions. This problem may be represented in the form involving 0-1 variables x_i

$$\max f(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij} x_i x_j + \sum_{i=1}^{n} c_i x_i, \tag{3}$$

$$\text{s.t.} \qquad \sum_{i=1}^{n} a_{ji} x_i \leqslant b_j, j = 1, \ldots, m, \tag{4}$$

$$x_i \in \{0, 1\}, i = 1, \ldots, n, \tag{5}$$

where (a_{ji}) is an $m \times n$ - matrix and (b_j) is an m-vector. With a function f we can associate the graph $G_f = (V_f, E_f)$, whose vertices are the elements of $N \cup \{n+1\}$ and whose edge set consists of all unordered pairs (i,j), such that $i < j \leqslant n$, $c_{ij} \neq 0$, or $i < j = n+1, c_i \neq 0$. To each edge $(i,j) \in E_f$ we attach a weight c_{ij} or c_i. Then (3)-(5) (or alternatively (1),(2)) may be viewed as a problem of determining in G_f an induced subgraph having maximum total sum of edge weights over the set of all induced subgraphs, containing the vertex $n+1$ and guaranteeing (4) (or (2)) to be satisfied.

A general problem (3)-(5) includes as special cases a variety of other problems, e.g., the unconstrained quadratic 0-1 optimization problem (3),(5) (or (1),(2) with $\widetilde{H} = H$) and the well known quadratic assignment problem (QAP). It is easy to see, that (3) may be rewritten as

$$\max f(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c'_{ij} (x_i - x_j)^2 + \sum_{i=1}^{n} c'_i x_i, \tag{6}$$

where c'_{ij}, c'_i are expressed through c_{ij},c_i . Thus the unconstrained problem (3), (5), in fact, coincides with that of partitioning a weighted graph into two parts with the maximum total weight of edges between them. If $c'_{ij} \geqslant 0$, $i,j = 1, \ldots, n$, $i < j$, $c'_i = 0$, $i = 1, \ldots, n$, in (6), then we obtain even more specialized case of (3)-(5)–the max-cut problem.

All the problems displayed above have a wide applicability to the modeling of many important real world situations in many diverse areas. For a discussion of most relevant applications, see the surveys by Hansen (1979) and Burkard (1984).

The purpose of this paper is to survey some recent results concerning the problems and algorithms for their solution in quadratic 0-1 optimization. Since the papers written in Russian are presumably not so well known for researchers throughout the world, they are discussed more thoroughly here. Moreover, when the list of references was drawn up, the priority was given to such papers.

Complexity of problems. It is well known, that all the problems mentioned in the introduction are NP–complete. Therefore, it is natural that the research dealing with the complexity analysis of problems is concentrated mainly in the following two directions. First rather simple special cases of the problems are tried to be declared as being NP–complete or NP–hard. Second, polynomial-time (i.e. efficient) algorithms are designed for solving nontrivial specialized versions of general problems.

We proceed with the NP–hardness results. A natural question is the following: what is the complexity status of a continuous relaxation of (3)-(5),which is obtained when constraints (5) are replaced by inequalities $0 \leqslant x_i \leqslant 1$, $i = 1, \ldots, n$? The answer is "NP-hard" even for the problem without (4) (Murty and Kabadi, 1987). The following fact has been settled by Kuzjurin (1984). For a given constant $\varepsilon > 0$, there does not exist, unless P=NP, any polynomial heuristic for obtaining ε–approximate solutions to (3)-(5). A necessary reduction is carried out from the set partitioning problem.

Let G_f^* denote a signed graph obtained from G_f by deleting the vertex $n+1$ with all incident edges and marking the remaining edges according to signs of c_{ij}. If G_f^* is balanced (no cycle has an odd number of negative edges), then the corresponding problem (3),(5) is efficiently solvable (see e.g. Hansen, 1979). Ageev and Beresnev (1988) consider a particular case of such a problem, when G_f^* is a bipartite graph having all edges negative. If the signs of edges are arbitrary, then this problem is obviously NP–hard. Other polynomially solvable cases of (3),(5) are due to the special structure of $\widetilde{G}_f$–graph obtainable from G_f^* by ignoring the signs. Such special graphs are trees (Hansen and Simeone, 1986; Matickas and Palubeckis, 1986) and series-parallel ones (Barahona, 1986). Also, an efficient (in fact, $O(n^2)$ -time) algorithm was proposed (Matickas and Palubeckis, 1986) for the problem (6),(4),(5) with $\widetilde{G}_f$ being a tree and (4) consisting of the two inequalities $a \leqslant \sum_{i=1}^{n} x_i \leqslant b$. Such a problem with the above inequalities replaced by $\sum d_i x_i \leqslant b$, however, is NP–hard. Polynomial-time algorithms are also known for the max-cut problem on some classes of graphs, e.g., planar graphs and graphs without odd cycles of length exceeding $c \log n$, c being a constant (Fleishman, 1988). Polynomially solvable cases of the QAP include some layout problems on the line, e.g., that of linear arrangement of trees.

Finding an optimal solution. In this section, we shall discuss some of the approaches to solving quadratic 0-1 optimization problems. Many of the algorithms for these problems use branch and bound techniques. The most relevant part of such algorithms is a bounding procedure. A rather straightforward approach to obtaining bounds on $\max f(x)$ is based on the transformation of (3)-(5) into an equivalent positive definite form and on a subsequent continuous relaxation of it. Such bounds were applied in the algorithms (e.g., McBride and Yormark, 1980), which have allowed to solve problems with up to 30-50 variables.

Four relaxations of the problems equivalent to (3),(5) have been investigated by Hammer, Hansen and Simeone

(1984). They have shown that the bounds associated with the four relaxations are equal.

Recently the bounding routines involving a Lagrangean relaxation have been developed for the unconstrained (Körner and Richter, 1982) and general (Shor and Davydov, 1985; Shor, 1987) quadratic problems. Suppose a function $f(x)$ given by (3) is minimized. Let the linear inequalities (4) be denoted as $g_j(x) \leqslant 0$ (in general, we may even allow functions g_j to include the quadratic terms). Then the lower bound on $\min f(x)$ is (see Shor and Davydov, 1985)

$$\begin{aligned}\psi^* = \sup_{u \in U}\{\psi(u) = \inf_{x \in X}[L(x,u) = \\ = f(x) + \sum_{j=1}^{m} u_j g_j(x) + \sum_{i=1}^{n} \bar{u}_i(x_i^2 - x_i)]\},\end{aligned} \tag{7}$$

where X is a set of feasible solutions,u is an $(m+n)$–vector of Lagrange multipliers and U is a set of such vectors satisfying $u_j \geqslant 0$, $j = 1, \ldots, m$. The function $L(x,u)$ may be rewritten as a sum of quadratic and linear parts $x'K(u)x + l(x,u)$. Let $U' = \{u \in U \mid K(u)$ be positive semidefinite $\}$ and $\varphi(u) = -\psi(u)$. It is easy to see that $\psi^* = -\inf_{u \in U'} \varphi(u)$. The problem $\min_{u \in U'} \varphi(u)$ can be solved,using the methods of nondifferentiable minimization.

A method outlined above was applied (Stetsenko and Shor, 1984) to derive the upper bounds on the maximum possible weight of an independent set in a weighted graph $G = (N, E)$. The problem is formulated as $\alpha(G) = \max\{\sum_{i=1}^{n} c_i x_i | x_i x_j = 0, (i,j) \in E,\ x_i^2 - x_i = 0, i = 1, \ldots, n\}$.

The value of (7) with sup and inf interchanged provides an upper bound on $\alpha(G)$. As shown by Stetsenko and Shor (1984), this bound is equivalent to the well-known Lovasz bound. The bounds of the type (7) were incorporated in a branch and bound algorithm for the vertex packing problem. Shor (1987) reports the computational results on weighted graphs with up to 60 vertices.

For an induced subgraph $G'_f = (V'_f, E'_f)$ of G_f, let $x(V'_f)$ denote the incidence vector of E'_f. With a quadratic function f we can associate the convex hull of the incidence vectors of all the subgraphs induced by subsets V'_f containing $n+1$, i.e., the polytope

$$P_{n+1}(G_f) = \operatorname{conv}\{x(V'_f) | V'_f \subseteq V_f, n+1 \in V'_f\}.$$

A similar polytope $\widetilde{P}_n(\widetilde{G}_f)$ related to f without a linear part can be defined. Both polytopes (called the subgraph polytopes) have been studied by Palubeckis (1985) from two different viewpoints. First, several classes of facets of $P_{n+1}(K_{n+1})$, $\widetilde{P}_n(K_n)$ have been derived. For example, the following inequalities define the facets of $P_{n+1}(K_{n+1})$

$$\begin{aligned}\sum_{i\in A}\sum_{j\in B} x_{ij} - \sum_{i,j\in A, i<j} x_{ij} - \sum_{i,j\in B, i<j} x_{ij} - \\ -2\sum_{i\in A} x_{i,n+1} + \sum_{i\in B} x_{i,n+1} \leqslant 1,\end{aligned} \tag{8}$$

$$\sum_{i\in C}\sum_{j\in D} x_{ij} - \sum_{i,j\in C, i<j} x_{ij} - \sum_{i,j\in D, i<j} x_{ij} - \sum_{i\in C} x_{i,n+1} \leqslant 0, \tag{9}$$

where $A, B, C, D \subseteq V_f \backslash \{n+1\}, A \bigcap B = \varnothing, C \bigcap D = \varnothing$, $|A| \geqslant 1, |B| \geqslant 3$ or $A = \varnothing, |B| \geqslant 2$, and $|C| \geqslant 1, |D| \geqslant 2$ or $|C| = |D| = 1$. Moreover, a complete facetial characterization of both P_n and $\widetilde{P}_n$ has been obtained for graphs coming from some restricted classes. Second, necessary and sufficient conditions for the extreme points of $P_n, \widetilde{P}_n$ to be neighboring have been provided.

A partial facetial characterization can be used to derive bounds on $f_0 = \min f(x)$. Consider, for example, the problem with $f(x) = -6x_1x_2 + 2x_1x_3 + 10x_1x_4 - 2x_2x_3 + 8x_2x_4 + 4x_3x_4 - 4x_3 - 10x_4$. Replacing x_ix_j by x_{ij} in f and adding to f the linear combination of the left hand sides of the facetial inequalities (each of the type (8),(9)) $2[(x_2 + x_4 - x_{24}) + (x_{12} + x_{23} - x_2 - x_{13})] + 4[(x_1 + x_4 - x_{14}) + (x_{12} - x_1)] + 4(x_3 + x_4 - x_{34})$, one may get a lower bound $-10 = -(2+4+4)$ on f_0 , which is equal to optimal value of f attained at a point (0,0,1,1). It is worth noting, that a 0-1 linear problem involving inequalities (9) with $|C| = |D| = 1$ only is close to the discrete Rhys form (see e.g., Hammer, Hansen and Simeone (1984) for the latter).

A polytope similar to $P_n, \widetilde{P}_n$ can be associated with the graph partitioning problem. Two classes of facetial inequalities of such a polytope, namely, $\{x_{ij} + x_{ik} - x_{jk} \geqslant 0\}$, $\{x_{ij} + x_{ik} + x_{jk} \leqslant 2\}$, have been exploited in deriving a branch and bound algorithm for this problem (Matickas and Palubeckis, 1986).

Ageev (1984) considers the problem of the form

$$\min_{x \in X} f(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} c_{ij}x_ix_j + \sum_{k=1}^{n} c_k(1 - x_k) \qquad (10)$$

with all c_{ij}, c_k being nonnegative and X consisting of all 0-1 vectors. This problem is equivalent to that of minimizing $\sum\sum c_{ij}z_{ij} + \sum c_ky_k$, subject to $y_i + y_j + z_{ij} \geqslant 1, y_k, z_{ij} \in \{0,1\}, i,j = 1,\ldots,n, i < j, k = 1,\ldots,n$. A continuous relaxation of the latter has an optimal solution with all the components equal to 0,1 or 1/2 (Ageev, 1984). Let y^0, z^0 be any such a solution. Then an optimal solution x^0 to (10) exists satisfying the property: $x_k^0 = 1 - \alpha(\alpha = 0 \text{ or } 1)$ if $y_k^0 = \alpha$. The bound obtained via a relaxation of the set covering problem given above is used in an algorithm described in (Ageev and Beresnev, 1988).

Many algorithms for solving the quadratic problems have also been developed, which differ from those of branch and bound type.Such an algorithm generating all local maximizing points for (3), (5) was suggested by Gulati, Gupta and Mittal (1984). Computational experience with this algorithm is encouraging.

A new bounding procedure for the QAP was proposed by Sergeev (1988). It is based on the following formulation of the QAP

$$\min f(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} z_{ij}x_{ij}, \tag{11}$$

$$(x_{ij}) \in \widetilde{X}, \tag{12}$$

$$z_{ij} = \sum_{k=1}^{n}\sum_{l=1}^{n} e_{ik}d_{jl}x_{kl}, i,j = 1,\dots,n, \tag{13}$$

where $\widetilde{X}$ is a set of permutation $n \times n$- matrices and $(e_{ik}), (d_{jl})$ are $n \times n$–matrices defining a QAP. The procedure resorts to solving the linear assignment problem (11), (12) and the use of a special bound improvement operation.

Heuristics. As computational results indicate, the exact solution methods can guarantee only relatively small quadratic 0-1 optimization problems (especially the QAPs) to be solved optimally within a reasonable amount of time. Therefore, heuristic methods are acceptable for producing good, but not necessarily optimal solutions.

A very simple heuristic for solving the unconstrained problem is the following (described with respect to (1),(2)).

Algorithm GREEDY

1.$V := \varnothing$.

2. If $V = N$, then go to 4. Select a vertex $k \in N\backslash V$ such that $g_k = \max\limits_{i \in N\backslash V} [g_i = f(V \bigcup \{i\}) - f(V)]$. If $g_k > 0$, then set $V = V \bigcup \{k\}$ and repeat 2; otherwise go to 3.

3. If $|V| \geqslant n-1$, then go to 4. Select a pair of vertices k,l such that $g_{kl} = \max\limits_{i,j \in N \setminus V} [g_{ij} = f(V \bigcup \{i,j\}) - f(V)]$. If $g_{kl} > 0$, then set $V = V \bigcup \{k,l\}$ and go to 2; otherwise go to 4.

4. Stop, V is a suboptimal solution to (1),(2) with $\widetilde{H} = H$.

For any $x \in X$, let $N_s(x)$ denote the set, called the neighborhood of x, consisting of all $x' \in X$ satisfying condition $\sum\limits_{i=1}^{n} |x_i - x'_i| \leqslant s$. It seems that the following heuristic algorithm should have a better performance than GREEDY. This algorithm needs an initial x to be given.

Algorithm LOCAL2

1. Check whether $h(x') := f(x) - f(x') \geqslant 0$ for all $x' \in N_1(x)$. If yes, then go to 3; otherwise choose $x' \in N_1(x)$, such that $h(x') < 0$.

2. $x := x'$. Return to 1.

3. Check whether $h(x') \geqslant 0$ for all $x' \in N_2(x) \setminus N_1(x)$. If yes, then stop; otherwise choose $x' \in N_2(x)$ such that $h(x') < 0$.

4. $x := x'$. Return to 1.

Dropping the steps 3 and 4 gives, in fact, an algorithm LOCAL described in the work by Gulati, Gupta and Mittal (1984).

The heuristic algorithms for solving the QAP are especially needed, since the size of problems amenable to exact solution methods remains to be bounded above by $n = 15$ (Burkard, 1984).The most widely used heuristic for the QAP is a pairwise exchange method which is capable to yield the solutions that are, in general, superior than those produced by constructive methods. In order to achieve even better solutions, other iterative methods would be applied. A simple way to improve a final solution is to repeat the start of the pairwise exchange heuristic from a sufficient number of different initial solutions. This approach like a similar one with "pairwise" replaced by "triple" leads to the solutions of quite a good quality (Bruijs, 1984). An algorithm being a generalization of the pairwise exchange heuristic was proposed by Metelsky and Kornienko (1983). Each iteration of it starts by selecting such a subset $N_0 \subseteq N$ that the graph $G(N_0)$ induced by edges, corresponding to $e_{ij} \neq 0$, $i, j \in N_0$, (see (11), (13)) has a simple structure, e.g., is a forest. Next, the objects from N_0 are permuted while those from $N \backslash N_0$ are retained at their locations. An optimal permutation is chosen only from the group of automorphisms of a weighted graph $G(N_0)$.. Recently an iterative algorithm based on an analogy with the annealing process has been developed (Burkard and Rendl, 1984). Computational experience shows very good behaviour of this algorithm.

The analysis of algorithms. The value of a heuristic solution to (3), (5) can be compared against $M_f = \max_{0 \leq m \leq n} M_f^m$, where M_f^m is the expected value of $f(x)$ on $X_m = \{x \in X \mid \sum x_i = m\}$. The following algorithm for any f yields a solution x satisfying $f(x) \geq M_f$. The algorithm involves first identifying l such that $M_f^l = M_f$, and then selecting l variables sequentially, which are set to 1. At the kth step, a free variable x_i, $i \in N \backslash N'$, $N' = \{s | x_s = 1\}$ is selected, giving a maximum to the function $h_j = a_j + c_j + d_j(l-k)/(n-k-1)$ with $a_j = f(N' \cup \{j\}) - f(N')$ and $d_j = f(N \backslash N') - f(N \backslash (N' \cup \{j\})) - c_j$. The index i is then included in N'.

Now we shall show that, unless P=NP, no polynomial heuristic for (3), (5) can guarantee to yield a solution x with $f(x) > M_f$ (though such a solution exists).On the contrary, let A be such a heuristic. Then the following algorithm, which is clearly polynomial for small integer c_{ij}, c_i, solves the NP-hard problem (3), (5).

1. Apply A to (3), (5). Let x' be a solution obtained:

$x'_i = 1,\ i \in K \subseteq N,\ x'_i = 0,\ i \notin K$. If $f(x') = M_f$, then stop.

2. Replace $x_i, i \in K$, by $1 - x_i$ in (3) and return to 1.

This fact is of the type of the results obtained by Lieberherr (1981).

Let $R:\ \{(V_1, V_2)\} \rightarrow \{\ ''<'', ''='', ''>''\ \}$ denote the oracle performing a comparison of $f(V_1)$ and $f(V_2)$. Let K_s stand for the class of all deterministic algorithms for (1), (2) with $\widetilde{H} = H$ exploiting the oracle R only for the pairs (V_1, V_2) such that $|(V_1 \backslash V_2) \cup (V_2 \backslash V_1)| \leqslant s$. The algorithms GREEDY, LOCAL and LOCAL2 belong to K_4, K_1 or K_2 (depending on the x' selection strategy) and K_2 or K_4, respectively.

The following assertions are proved by Palubeckis (1989). Assume $\widetilde{H} = H$ in (2). Let V_0, V^* denote, respectively, an optimal solution of (1), (2) and a solution delivered by a heuristic algorithm for (1), (2). Let $s \geqslant 1$, $n \geqslant max(4, s+1)$ and ε be any positive constant. Then the class K_s contains no algorithm having for any instance of (1), (2) (or, respectively, (1), (2) without a linear part) the performance ratio $r(f) = f(V^*)/f(V_0) \geqslant \varepsilon$ ($r(f) \geqslant \binom{s-2}{2} / \binom{n-2}{2} + \varepsilon$, respectively). For $s = 4$, the second bound is tight: a modification of GREEDY has a guaranteed performance of $1/\binom{n-2}{2}$. Also, the bounds for polynomial-time local search algorithms forming the subclass of K_n have been obtained. It should be mentioned,that not all simple local search algorithms are polynomial.For example, assume that the choice of x' in steps 1 and 3 of LOCAL2 is arbitrary among all $x' \in N_2(x)$, satisfying $h(x') < 0$. As shown by Palubeckis (1984), such a version of LOCAL2 for graph partitioning into two equally sized subgraphs can take an exponential number of iterations. Obviously, the same is true for the problem (3), (5) (since the constraint $\sum x_i = n/2$ can be brought into the objective).

For special case problem (10) an $O(n^3)$–time algorithm with $r = \max r(f) \leqslant 2$ was discovered by Ageev (1984). This algorithm is based on solving a relaxation of the set, covering problem equivalent to (10). The algorithms admitting $r = \min r(f)$ close to 1 (e.g. $r \geqslant 0.5$) are known also for the max-cut problem.

Now we look at the QAP again. Several heuristics for this problem are known that produce solutions x with $f(x)$ always not greater than the expected value M_f of f on $\widetilde{X}$. This holds for the pairwise exchange heuristic if at least one of the symmetric matrices (e_{ik}), (d_{jl}) has equal row sums (Klipker, 1978).An example constructed by Klipker (1978) shows that this condition is also necessary.

The probabilistic methods have been applied by Burkard and Fincke (1985) to evaluate the asymptotic behaviour of QAPs. The main result states that the ratio $\max f(x)/ / \min f(x)$ is arbitrarily close to 1 with probability tending to 1 as $n \to \infty$. Obviously, the same is true for the ratio $r(f) = = f(x^*)/f(x^0)$ with respect to any heuristic (x^* is a solution achieved by it and x^0 is an optimal one). However,it seems that $q(f) = [f(x^*) - f(x^0)]/[M_f - f(x^0)]$ is a more realistic measure for the evaluation of heuristic solutions to the QAP. The ratio $q(f)$, contrary to $r(f)$, is invariant under adding a constant to all elements of any matrix defining a QAP.

In the remainder of this section, we deal with an approach to an experimental investigation of heuristics based on the use of algorithms for generating test problems with a priori known optimal solutions. Recently, such generators have been developed for several combinatorial optimization problems.We shall describe an algorithm for generating unconstrained quadratic problems with known optimal 0-1 vectors. In fact, this algorithm is an extension of the test problem generator (Matickas and Palubeckas , 1985) for graph partitioning into two equally sized subgraphs.The main operation of the algorithm is related to the choice of a facet $x_{il} + x_{jl} - x_{ij} \geq 0$ of the graph bipartitioning polytope.

Given an integer $m = n/2$ expressed as a product of positive integers m_x and m_y , we denote by Q the rectangular grid of dimension $2m_x \times m_y$ with the edges of length 1 between the neighbouring points. Suppose that the set N of points is divided into two subsets N_1, N_2,corresponding to the left and right $m_x \times m_y$ subgrids, respectively.Let d_{ij} denote the length of a shortest path from point i to point j in Q , and $S(i,j)$ be a set of all shortest paths between i and j. The algorithm goes as follows.

1. Set $w_{ij} = h > 0$ and compute d_{ij} for all $i, j \in N$, $i < j$. Set $k = 0$.

2. Choose a pair (i, j), $i \in N_1$, $j \in N_2$, with a largest d_{ij} among those having $w_{ij} = h$, $d_{ij} > 1$. If no such a pair exists, then go to 4. Randomly choose a point l on some path $s_{ij} \in S(i,j)$ so that $|d_{il} - d_{jl}| \leq \delta$.

3. Set $w_{ij} = 0$, $w_{il} = w_{il} + h$, $w_{jl} = w_{jl} + h$, $k = k + 1$. If $k < t = \lfloor t_0 \binom{n}{2} \rfloor$, then return to 2.

4.Randomly generate a permutation $p = (p(1), \dots, p(n))$. Renumber the points of Q according to p. This gives a function f of (6) with $c'_i = 0$, $i = 1, \dots, n$, $c'_{ij} = w_{uv}$, where u, v are such that $p(u) = i$, $p(v) = j$, $i, j = 1, \dots, n$, $i < j$.

5. Add $\lambda \left(\sum x_i - n/2\right)^2$, $\lambda > h$, to f. Changing the signs of coefficients leads to the problem (3), (5) with $c_{ij} = 2(c'_{ij} - \lambda)$ and $c_i = \lambda(n-1) - \sum\limits_{j,i<j} c'_{ij} - \sum\limits_{j,j<i} c'_{ji}$.

The optimal vector x^o to the problem generated is defined by a set $\{x_i^0 | i = p(u),\ u \in N_1\}$ of components equal to 1. The optimal value $f(x^o) = (\lambda - h)n^2/4$. The parameters of the generator are δ, t_0, h, λ, m_x, m_y. First two parameters in all the experiments were set to 1 and 0.1,respectively.

The above described generator was applied to test algorithms GREEDY, LOCAL and LOCAL2. A version of local algorithms was examined that selects first x' encountered such that $h(x') < 0$. The results are given in Tables 1 and 2. Both tables present the averages of function values for 10 test problems and, in the last row, the number z of problems (out of solved to optimality by LOCAL2 applied to initial $x = 0$. In Table 2, the second row of values for LOCAL2 corresponds to the case when initial x is delivered by GREEDY. For LOCAL, initial $x = 0$. The quality of the solutions may be measured by the error evaluators

$$r(f) = f(x^*)/f(x^0) \text{ or}$$

$$r'(f) = \left[f(x^0) - f(x^*)\right] / \left[f(x^0) - f(1/2)\right].$$

Both measures, however, are not free from drawbacks. As Table 1 indicates, using our algorithm it is possible to generate test problems that are not easy for rather good heuristics as LOCAL2 is. For $n \geqslant 60$, average $r(f)$ equals to approximately 0.7. Table 2 demonstrates the strength of test problems for various h and λ. For $h = 1$, $\lambda = 50$, only 6 (out of 10) solutions delivered by LOCAL had the objective value not less than $M_f = 44010$.

Table 1. Performance of algorithms on problems with $h = 100$, $\lambda = 101$

	n	30	40	50	60	70	80	90	100
	m_x	3	4	5	5	5	5	5	5
optimal		225	400	625	900	1225	1600	2025	2500
GREEDY, LOCAL2		223	357	582	671	825	913	1437	1740
LOCAL		215	350	569	585	684	846	1092	1539
z		8	3	6	2	2	2	2	0

Table 2. Effect of changes in h and $\lambda(n = 60,\ m_x = 5)$

	h	100	100	10	1	1	1
	λ	101	105	11	2	10	50
optimal		900	4500	900	900	8100	44100
GREEDY		671	3917	839	892	8092	44092
LOCAL		585	3857	856	894	8032	44013
LOCAL2		671	4316	884	896	8094	44097
		671	3966	854	895	8094	44094
z		2	5	6	3	5	4

An algorithm for generating QAPs with known optimal solutions was presented by Palubeckis (1988). The value of $q(f)$ on these problems appears to be approximately 0.5 and 0.2 for simple constructive algorithms and for pairwise exchange heuristic, respectively.

Conclusions. The work aimed at the development of new and improvement of existing algorithms, both exact and heuristic, for solving 0-1 quadratic problems should be continued. This should be done in regard to both NP-hard problems and their polynomially solvable partial cases.The study of some mathematical problems could stimulate the design of better algorithms.

The results in the analysis of algorithms for quadratic problems have been obtained along few lines only. Much more work in this area is needed. The results obtained may suggest new algorithmic ideas and solution techniques in quadratic 0-1 optimization.

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