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Closed-Loop System Identification with Modifications of the Instrumental Variable Method

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Abstract. Least-squares method is the most popular method for parameter estimation. It is easy applicable, but it has considerable drawback. Under well-known conditions in the presence of noise, the LS method produces asymptotically biased and inconsistent estimates. One way to overcome this drawback is the implementation of the instrumental variable method. In this paper several modifications of this method for closed-loop system identification are considered and investigated. The covariance matrix of the instrumental variable estimates is discussed. A simulation is carried out in order to illustrate the obtained results.
Keywords: closed-loop system identification, prediction error, instrumental variable method, parameter estimation, observations.

1. Introduction

In this paper several modifications of the instrumental variable (IV) method for closed-loop system identification are presented on the base of a generalized IV method. The question of the covariance matrix of the IV estimates is considered in respect to the estimate accuracy. The optimal covariance matrix of the estimates and optimal IV estimator can only be obtained if the true plant and noise models are exactly known, thus optimal accuracy cannot be achieved in practice. Due to this fact different modifications are designed for approximation of the needed information from the measured data in different ways.

In literatures (Gilson and Van den Hof, 2001, 2003; Wada *et al.*, 2001) the estimation algorithms represent the ordinary mathematical relations on the base of auto and cross correlation functions of the corresponding signals. The aim of this paper is to present the mathematical relations in an alternative vector-matrix form (Daniusis and Vaitkus, 2008). The main advantage of the proposed form is in simpler description as well as more convenient computer implementation.

The paper is organized as follows. In the next section the problem statement is presented. In Section 3, the basic and generalized closed-loop IV methods are given. In Section 4, the tailor-made IV and BELS methods are described as IV modifications. In Section 5, the optimal covariance matrix of the estimates and its corresponding optimal IV estimator are discussed. In Section 6, two known algorithms for approximate realizations of the optimal IV estimator are displayed with the mathematical relations in vector-matrix form. Section 7 presents an example and parameter estimation results. Finally, the conclusions are given in Section 8.

2. Problem Statement

Consider a linear, single-input single-output discrete-time system shown in Fig. 1. The plant, controller and noise filter are denoted by $G_0(q)$, $C(q)$ and $H_0(q)$, respectively. Here q is time-shift operator such that $q^{-i}u(k) = u(k-i)$ and $\{e_0(k)\}$ is white noise with variance σ_0^2. The plant input signal is described by $u(k)$ and the system output signal by $y(k)$. The external signals $r_2(k)$ and $r_1(k)$ may be regarded as a set point signal and an external excitation signal, which is assumed to be uncorrelated with the error $e_0(k)$.

For the sake of simplicity, the generalized input signal is introduced as

$$r(k) = r_1(k) + C(q)r_2(k). \tag{1}$$

Thus, the mathematical description of the true closed-loop system is

$$\begin{cases} y(k) = G_0(q)u(k) + H_0(q)e_0(k), \\ u(k) = r(k) - C(q)y(k). \end{cases} \tag{2}$$

It is assumed that the numerator and denominator polynomials of the real plant transfer function

$$G_0(q) = \frac{B_0(q^{-1})}{A_0(q^{-1})} \tag{3}$$

have degree n_0.

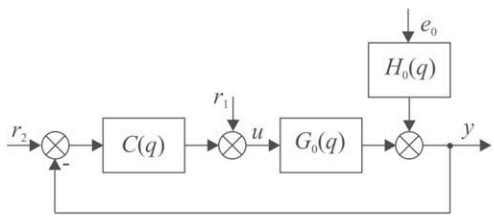

Fig. 1. A closed-loop system to be observed.

The controller is assumed to be known and described in the form

$$C(q) = \frac{S(q^{-1})}{P(q^{-1})} = \frac{s_0 + s_1 q^{-1} + \cdots + s_m q^{-m}}{1 + p_1 q^{-1} + \cdots + p_m q^{-m}}. \tag{4}$$

After substitution of expressions (3) and (4) into the system description (2) following equations are obtained

$$y(k) = \frac{G_0(q)}{1 + C(q)G_0(q)} r(k) + \frac{H_0(q)}{1 + C(q)G_0(q)} e_0(k), \tag{5}$$

$$u(k) = \frac{1}{1 + C(q)G_0(q)} r(k) - \frac{C(q)H_0(q)}{1 + C(q)G_0(q)} e_0(k), \tag{6}$$

Equations (5) and (6) present the noise-free and noise parts of the plant output and input signals. Equation (5) can be written in the form

$$y(k) = \frac{B_0^{cl}}{A_0^{cl}} r(k) + \frac{1}{A_0^{cl}} \xi(k) \tag{7}$$

with $\xi(k) = A_0 P H_0 e_0(k)$. It is clear that the polynomials B_0^{cl} and A_0^{cl} are from order $(n_0 + m)$.

The plant model can be parameterized as

$$G(q, \theta) = \frac{B(q^{-1}, \theta)}{A(q^{-1}, \theta)} = \frac{b_1 q^{-1} + \cdots + b_n q^{-n}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n}}, \tag{8}$$

with the parameter vector

$$\theta = [a_1, \ldots, a_n, b_1, \ldots, b_n]^T \in R^{2n}. \tag{9}$$

Open-loop regressors $\mathbf{f}_{yu}(k)$ and closed-loop $\mathbf{f}_{yr}(k)$ regressors are defined as

$$\mathbf{f}_{yu}(k) = [-y(k-1) \cdots - y(k-n) \quad u(k-1) \cdots u(k-n)]^T, \tag{10}$$

$$\mathbf{f}_{yr}(k) = [-y(k-1) \cdots - y(k-n-m) \quad r(k-1) \cdots r(k-l)]^T, \tag{11}$$

where l is a parameter, chosen by the user.

The following notations are introduced

$$\mathbf{f}_r(k) = \big[r(k-1) \cdots r(k-l)\big]^T, \tag{12}$$

$$\overline{\mathbf{f}}_{yu}(k) = P(q^{-1}) \mathbf{f}_{yu}(k), \tag{13}$$

$$\bar{y}(k) = P(q^{-1}) y(k). \tag{14}$$

The identification is based on data set

$$Z^N = \big\{r(1), \ldots, r(N), u(1), \ldots, u(N), y(1), \ldots, y(N)\big\}, \tag{15}$$

which consists of measurements of the generalized input signal $\{r(k)\}$, plant input signal $\{u(k)\}$ and system output $\{y(k)\}$ for $k = 1, 2, \ldots, N$.

Since the controller $C(q)$ is known, the indirect approach for closed-loop system identification can be applied. According to this approach the closed-loop system (7) is identified and then the elements of vector (9) are determined (Pupeikis, 2000).

There are two conditions for identifiability for close-loop identification. The first one is that the input-output data cannot unique determine the order of the plant model. The second one is regarding the relationship between the orders of the plant and the controller. In the case when additional, uncorrelated with the noise $\{e_0(k)\}$, excitation input signal is used, the second condition did not need to be fulfilled. This is the case here, because the excitation signal $r_1(k)$ is uncorrelated with $\{e_0(k)\}$, thus the generalized input signal $\{r(k)\}$ is also uncorrelated with the noise. The order of the controller (m) is preliminary known. According to this order the order of the plant model (n) can be chosen, thus the order of the closed loop system is also determined.

The relation between plant parameters θ and closed-loop parameters θ^{cl} is determined by the following equation

$$\theta^{cl} = \mathbf{M}\theta + \mu, \tag{16}$$

where the vector μ consists of the coefficients of the controller denominator

$$\mu = [p_1 p_2 \ldots p_m 0 \ldots 0]^T \in R^{n+m+l}. \tag{17}$$

The full-column rank matrix $\mathbf{M}$ is partitioned into four blocks

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{P1} & \mathbf{M}_S \\ \mathbf{0} & \mathbf{M}_{P2} \end{bmatrix} \in R^{(n+m+l)\times 2n}, \tag{18}$$

where its submatrices $\mathbf{M}_{P1} \in R^{(n+m)\times n}$ and $\mathbf{M}_S \in R^{(n+m)\times n}$ are Sylvester matrices. They contain the parameters of the controller denominator and numerator respectively

$$\mathbf{M}_{P1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ p_1 & 1 & \ddots & \vdots \\ \vdots & p_1 & \ddots & 0 \\ p_m & \vdots & \ddots & 1 \\ 0 & p_m & \vdots & p_1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & p_m \end{bmatrix}; \quad \mathbf{M}_S = \begin{bmatrix} s_0 & 0 & \cdots & 0 \\ s_1 & s_0 & \ddots & \vdots \\ \vdots & s_1 & \ddots & 0 \\ s_m & \vdots & \ddots & s_0 \\ 0 & s_m & \vdots & s_1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s_m \end{bmatrix}. \tag{19}$$

The submatrix $\mathbf{M}_{P2} \in R^{l\times n}$ is given by

$$\mathbf{M}_{P2} = \begin{bmatrix} \mathbf{M}_{P1} \\ \mathbf{0}_{(l-n-m)\times n} \end{bmatrix}. \tag{20}$$

The least-squares (LS) estimates $\hat{\theta}^{cl}_{\mathrm{LS}}$ are obtained by the formula

$$\hat{\theta}^{cl}_{\mathrm{LS}} = \left[\sum_{k=1}^{N} \mathbf{f}_{yr}(k)\mathbf{f}^T_{yr}(k) \right]^{-1} \sum_{k=1}^{N} \mathbf{f}_{yr}(k)\mathbf{y}^T(k). \tag{21}$$

Equation (21) can be presented in more compact vector-matrix form (Atanasov and Pupeikis, 2009; Daniusis and Vaitkus, 2008)

$$\hat{\theta}^{cl}_{\mathrm{LS}} = \left(\mathbf{F}^T_{YR}\mathbf{F}_{YR}\right)^{-1}\mathbf{F}^T_{YR}\mathbf{y}, \tag{22}$$

where $\mathbf{F}_{YR}$ is a block matrix

$$\mathbf{F}_{YR} = [\mathbf{Y} \quad \mathbf{R}] \in R^{(N-n-m)\times(n+m+l)}. \tag{23}$$

It's blocks $\mathbf{Y} \in R^{(N-n-m)\times(n+m)}$, $\mathbf{R} \in R^{(N-n-m)\times l}$ and the vector $y \in R^{N-n-m}$ are formed in accordance with Eq. (11):

$$\mathbf{y} = \left[y(n+m+1) \quad y(n+m+2) \quad \cdots \quad y(N)\right]^T; \tag{24}$$

$$\mathbf{Y} = \begin{bmatrix} -y(n+m) & -y(n+m-1) & \cdots & -y(1) \\ -y(n+m+1) & -y(n+m) & \cdots & -y(2) \\ \vdots & \vdots & \ddots & \vdots \\ -y(N-1) & -y(N-2) & \cdots & -y(N-n-m) \end{bmatrix}; \tag{25}$$

$$\mathbf{R} = \begin{bmatrix} r(n+m) & r(n+m-1) & \cdots & r(n+m-l+1) \\ r(n+m+1) & r(n+m) & \cdots & r(n+m-l+2) \\ \vdots & \vdots & \ddots & \vdots \\ r(N-1) & r(N-2) & \cdots & r(N-l) \end{bmatrix}. \tag{26}$$

It is well known fact that the LS estimates $\hat{\theta}^{cl}_{\mathrm{LS}}$ are consistent and unbiased only when $\{\xi(k)\}$ is white noise. One way to overcome this problem is the implementation of the IV method.

3. Basic and Generalized Closed-Loop IV Methods

The basic closed-loop IV method utilizes $2n$ time-shifted values of the reference signal as instruments. Thus, the estimates are calculated according to the relation (Soderstrom *et al.*, 1987)

$$\hat{\theta}_{\mathrm{IV}} = \left[\sum_{k=1}^{N} \mathbf{z}(k)\mathbf{f}^T_{yu}(k) \right]^{-1} \sum_{k=1}^{N} \mathbf{z}(k)\mathbf{y}^T(k), \tag{27}$$

where $z(k) = \mathbf{f}_r(k)$ and $l = 2n.$

The following vector-matrix form of Eq. (27) is very useful for computer realization

$$\hat{\theta}_{\mathrm{IV}} = \left(\mathbf{R}^T \mathbf{F}_{YU}\right)^{-1} \mathbf{R}^T \mathbf{y}. \tag{28}$$

Here $\mathbf{F}_{YU}$ is a block matrix

$$F_{YU} = [Y \quad U] \in R^{(N-2n)\times 2n}. \tag{29}$$

It's blocks $\mathbf{Y} \in R^{(N-2n)\times n}$, $\mathbf{U} \in R^{(N-2n)\times n}$, the vector $\mathbf{y} \in R^{N-2n}$ and the submatrix $\mathbf{R} \in R^{(N-2n)\times 2n}$ are formed in according to Eqs. (10) and (12):

$$\mathbf{y} = [y(2n+1) \quad y(2n+2) \quad \dots \quad y(N)]^T, \tag{30}$$

$$\mathbf{Y} = \begin{bmatrix} -y(2n) & -y(2n-1) & \cdots & -y(n+1) \\ -y(2n+1) & -y(2n) & \cdots & -y(n+2) \\ \vdots & \vdots & \ddots & \vdots \\ -y(N-1) & -y(N-2) & \cdots & -y(N-2n+1) \end{bmatrix}, \tag{31}$$

$$\mathbf{U} = \begin{bmatrix} u(2n) & u(2n-1) & \cdots & u(n+1) \\ u(2n+1) & u(2n) & \cdots & u(n+2) \\ \vdots & \vdots & \ddots & \vdots \\ u(N-1) & u(N-2) & \cdots & u(N-2n+1) \end{bmatrix}, \tag{32}$$

$$\mathbf{R} = \begin{bmatrix} r(2n) & r(2n-1) & \cdots & r(1) \\ r(2n+1) & r(2n) & \cdots & r(2) \\ \vdots & \vdots & \ddots & \vdots \\ r(N-1) & r(N-2) & \cdots & r(N-2n) \end{bmatrix}. \tag{33}$$

The generalized closed-loop IV method increases the quality of the IV estimates. Its estimates are a generalization of the basic IV ones, obtained by filtering the input-output data and taking advantage of the augmented instrument$z(k) \in R^{n_z}$, where $n_z \geqslant 2n$. The estimates are calculated according to the formula

$$\hat{\theta}^G_{\mathrm{IV}}(N) = \arg \underbrace{\min}_{\theta} \left\| \left[\frac{1}{N} \sum_{k=1}^{N} \mathbf{z}(k) L\left(q^{-1}\right) \mathbf{f}^T_{yu}(k) \right] \theta - \left[\frac{1}{N} \sum_{k=1}^{N} \mathbf{z}(k) L(q^{-1}) y(k) \right] \right\|^2_{\mathbf{Q}}, \tag{34}$$

where $L(q^{-1})$ is a stable filter and $\|\mathbf{x}\|^2_{\mathbf{Q}}$ is a quadratic form $\mathbf{x}^T \mathbf{Q} \mathbf{x}$ with a positive definite weighting matrix $\mathbf{Q}$.

4. Modifications of IV Method

4.1. *Tailor-Made Method*

The tailor-made IV method is based on the closed-loop prediction error

$$\varepsilon(k,\theta) = \bar{A}^{cl}\big(q^{-1},\theta\big)y(k) - \bar{B}^{cl}\big(q^{-1},\theta\big)r(k), \tag{35}$$

where

$$\bar{B}^{cl}\big(q^{-1},\theta\big) = B\big(q^{-1},\theta\big)P\big(q^{-1}\big) \tag{36}$$

and

$$\bar{A}^{cl}\big(q^{-1},\theta\big) = A\big(q^{-1},\theta\big)P\big(q^{-1}\big) + B\big(q^{-1},\theta\big)S\big(q^{-1}\big). \tag{37}$$

Due to the fact that the controller is known the prediction error is parameterised by the chosen in advance plant parameters θ.

Alternatively, the prediction error is defined as (Gilson and Van den Hof, 2001)

$$\varepsilon(k,\theta) = \bar{y}(k) - \bar{\mathbf{f}}_{yu}^{T}(k)\theta. \tag{38}$$

The tailor-made IV estimates are obtained as solution to the system of $2n$ equations (Gilson and Van den Hof, 2003)

$$\frac{1}{N}\sum_{k=1}^{N}\varepsilon\big(k,\hat{\theta}_{\mathrm{IV}}^{TM}\big)\eta(k) = 0, \tag{39}$$

where the instrument vector is calculated by delayed samples of the reference signal

$$\eta(k) = \mathbf{F}\mathbf{f}_{r}(k), \tag{40}$$

and $\mathbf{F} \in R^{2n\times l}$ is chosen by the user matrix with rank $2n$.

This method provides unbiased estimates for the plant model $G(q,\theta)$ without estimation of the noise model.

4.2. *BELS Method*

A modification of the classical least-squares method, based on the principle of bias compensation is called BELS (bias-eliminated LS; Wada *et al.*, 2001). For closed-loop identification BELS is a particular form of the tailor-made IV method. According to the relation between the model and controller orders there are two different cases (Gilson and Van den Hof, 2001; Gilson and Van den Hof, 2003).

For $m \leqslant n$ BELS is equivalent to the tailor-made IV method with $l = 2n$ and $\mathbf{F} = \mathbf{I}_{2n}$. In case of non-singular matrix $\hat{\mathbf{R}}_{f_r \bar{f}_{yu}} \in R^{2n \times 2n}$ the estimates are given by

$$\hat{\theta}_{\mathrm{BELS}} \equiv \hat{\theta}_{\mathrm{IV}}^{TM} = \hat{\mathbf{R}}_{f_r \bar{f}_{yu}}^{-1} \hat{\mathbf{R}}_{f_r \bar{y}}, \tag{41}$$

where

$$\hat{\mathbf{R}}_{\mathbf{f}_r \bar{\mathbf{f}}_{yu}} = \frac{1}{N} \sum_{k=1}^{N} \mathbf{f}_r(k) \bar{\mathbf{f}}_{yu}^T(k), \tag{42}$$

$$\hat{\mathbf{R}}_{\mathbf{f}_r \bar{\mathbf{y}}} = \frac{1}{N} \sum_{k=1}^{N} \mathbf{f}_r(k) \bar{y}^T(k). \tag{43}$$

The matrix $\hat{\mathbf{R}}_{\mathbf{f}_r \bar{\mathbf{f}}_{yu}}$ and the vector $\hat{\mathbf{R}}_{\mathbf{f}_r \bar{y}}$ consist of estimates of the cross-correlation function values of the corresponding vectors. From Eqs. (13) and (14) can be obtained following relations (Gilson and Van den Hof, 2001)

$$\bar{\mathbf{y}}(k) = \mathbf{y}(k) - \mathbf{f}_{YR}^T(k)\mu \tag{44}$$

and

$$\bar{\mathbf{f}}_{yu}^T(k) = \mathbf{f}_{YR}^T(k)\mathbf{M}. \tag{45}$$

They allow determination of the filtered data in vector-matrix form

$$\bar{\mathbf{y}} = \mathbf{y} - \mathbf{F}_{YR}\mu, \tag{46}$$

$$\bar{\mathbf{F}}_{YU} = \mathbf{F}_{YR}\mathbf{M}. \tag{47}$$

Thus, the estimates (41) can be calculated according to formula

$$\hat{\theta}_{\mathrm{BELS}} \equiv \hat{\theta}_{\mathrm{IV}}^{TM} = \left(\mathbf{R}^T \mathbf{F}_{YR} \mathbf{M}\right)^{-1} \mathbf{R}^T (\mathbf{y} - \mathbf{F}_{YR}\mu). \tag{48}$$

For $m > n$ BELS is equivalent to the tailor-made IV method with $l = n + m$ and

$$\mathbf{F} = \mathbf{M}^{\mathbf{T}} \hat{\mathbf{R}}_{\mathbf{f}_r \mathbf{f}_{yr}}^T \left[\hat{\mathbf{R}}_{\mathbf{f}_r \mathbf{f}_{yr}} \hat{\mathbf{R}}_{\mathbf{f}_r \mathbf{f}_{yr}}^T\right]^{-1}, \tag{49}$$

where

$$\hat{\mathbf{R}}_{\mathbf{f}_r \mathbf{f}_{yr}} = \frac{1}{N} \sum_{k=1}^{N} \mathbf{f}_r(k) \mathbf{f}_{yr}(k). \tag{50}$$

In this case the estimates are obtained according to

$$\hat{\theta}_{\mathrm{BELS}} \equiv \hat{\theta}_{\mathrm{IV}}^{TM} = \hat{\mathbf{R}}_{\mathbf{F}\mathbf{f}_r \bar{\mathbf{f}}_{yu}}^{-1} \hat{\mathbf{R}}_{\mathbf{F}\mathbf{f}_r \bar{y}}, \tag{51}$$

where $\hat{\mathbf{R}}_{\mathbf{Ff}_r\bar{\mathbf{f}}_{yu}}$ and $\hat{\mathbf{R}}_{\mathbf{Ff}_r\bar{y}}$ are the product of two matrices, respectively

$$\hat{\mathbf{R}}_{\mathbf{Ff}_r\bar{\mathbf{f}}_{yu}} = \mathbf{F}\hat{\mathbf{R}}_{\mathbf{f}_r\bar{\mathbf{f}}_{yu}} \quad \text{and} \quad \hat{\mathbf{R}}_{\mathbf{Ff}_r\bar{y}} = \mathbf{F}\hat{\mathbf{R}}_{\mathbf{f}_r\bar{y}}. \tag{52}$$

Equations (23) and (44)–(47) allow Eqs. (49) and (51) to be presented respectively in the form

$$\mathbf{F} = \mathbf{M}^{\mathbf{T}}\big(\mathbf{R}^T\mathbf{F}_{YR}\big)^T\big[\big(\mathbf{R}^T\mathbf{F}_{YR}\big)\big(\mathbf{R}^T\mathbf{F}_{YR}\big)^T\big]^{-1}, \tag{53}$$

$$\hat{\theta}_{\text{BELS}} \equiv \hat{\theta}_{\text{IV}}^{TM} = \big(\mathbf{F}\mathbf{R}^T\mathbf{F}_{YR}\mathbf{M}\big)^{-1}\mathbf{F}\mathbf{R}^T(\mathbf{y} - \mathbf{F}_{YR}\mu). \tag{54}$$

Equations (41) and (51) and their vector-matrix equivalents (48) and (54) substitute comparatively complex multi-steps computational procedures of BELS method (Atanasov, 2007; Gilson and Van den Hof, 2001; Wada *et al.*, 2001).

For $m > n$ tailor-made IV estimates satisfy the relation

$$\hat{\theta}_{\text{IV}}^{TM} = \arg\underbrace{\min}_{\theta}\big\|\hat{\mathbf{R}}_{\mathbf{f}_r\bar{\mathbf{f}}_{yu}}\theta - \hat{\mathbf{R}}_{\mathbf{f}_r\bar{y}}\big\|_{\mathbf{Q}}^2, \tag{55}$$

where

$$\mathbf{Q} = \big[\hat{\mathbf{R}}_{\mathbf{f}_r\mathbf{f}_{yr}}\hat{\mathbf{R}}_{\mathbf{f}_r\mathbf{f}_{yr}}^T\big]^{-1} \in R^{(n+m)\times(n+m)}. \tag{56}$$

These estimates are generalized IV estimates if the chosen instruments are $\mathbf{z}(k) = \mathbf{f}_r(k)$, $n_z = l = n + m$, and the filter $L(q^{-1})$ is the controller denominator $P(q^{-1})$.

For $l = 2n$ and $\mathbf{F} = \mathbf{I}_{2n}$, tailor-made IV estimates are generalized IV estimates with $\mathbf{z}(k) = \mathbf{f}_r(k)$, $n_z = l = 2n$, $\mathbf{Q} = \mathbf{I}_{2n}$ and $L(q^{-1}) = P(q^{-1})$.

5. Optimal Closed-Loop IV Estimator

The covariance matrix of the estimates contains their variances and covariances and characterizes estimate accuracy. The choice of the instruments $z(k)$ from dimensionn_z, weighting matrix $\mathbf{Q}$and stable filter $L(q^{-1})$ has an important influence on the covariance matrix. It is known that the Cramer-Rao inequality gives its lower bound for any unbiased estimation process. If the closed-loop IV estimates have normal distribution, according to Cramer-Rao inequality, the lower bound of the covariance matrix is given by Forsell and Ljung (1999), Gilson and Van den Hof (2001)

$$\mathbf{P}_{\text{IV}}^{opt} = \sigma_0^2\big[p\lim \tilde{f}_{\mathbf{yu}}(k)\tilde{f}_{\mathbf{yu}}^T(k)\big]^{-1}. \tag{57}$$

Here $\tilde{f}_{\mathbf{yu}}^T(k)$ is the negative gradient of the prediction error for the true plant parameters and noise filter

$$
\begin{aligned}
\tilde{f}_{\mathbf{yu}}^{T}(k) &= -\left[\frac{d}{d\theta}\varepsilon(k,\theta)\right]^{T}\Bigg|_{\theta=\theta_0} \\
&= -\left\{\frac{d}{d\theta}\left[\frac{1}{H_0(q^{-1})}\big(y(k)-G(q,\theta)u(k)\big)\right]\right\}^{T}\Bigg|_{\theta=\theta_0} \\
&= \left[A_0\big(q^{-1}\big)H_0\big(q^{-1}\big)\right]^{-1} f_{\mathbf{yu}}^{T}(k).
\end{aligned} \tag{58}
$$

Here $\tilde{f}_{\mathbf{yu}}^{T}(k)$ denotes the noise-free part of$f_{\mathbf{yu}}(k)$. The covariance matrix $\mathbf{P}_{\mathrm{IV}}^{opt}$ is obtained from the expressions

$$\mathbf{z}(k) = \frac{1}{\sigma_0^2}\left\{\left[A_0\big(q^{-1}\big)H_0\big(q^{-1}\big)\right]^{-1}\tilde{f}_{\mathbf{yu}}^{T}(k)\right\}^{\mathbf{T}}, \tag{59}$$

$$n_z = 2n, \tag{60}$$

$$\mathbf{Q} = \mathbf{I}, \tag{61}$$

$$L\big(q^{-1}\big) = \left[A_0\big(q^{-1}\big)H_0\big(q^{-1}\big)\right]^{-1}. \tag{62}$$

Last equations confirm the statement that the optimal IV estimator is only feasible if the true plant and noise models are exactly known. That's why optimal accuracy cannot be achieved in practice.

6. Approximate Realizations of the Optimal Closed-Loop IV Estimator

There are two main requirements for approximate realization of the optimal closed-loop IV estimator. The first one is regarding to the choice of the noise model. This is important for determination of the filter $L(q^{-1})$ and the instruments $\mathbf{z}(k)$. The second one is regarding to the choice of an initial plant model, used for computation of the noise-free part of the regressor $f_{\mathbf{yu}}(k)$.

6.1. *Extension to the IV4 Method*

The extension to the IV4 method for open-loop identification is an attempt for approximation of the optimal closed-loop IV method. The key point is the usage of the instruments that are uncorrelated with the noise part of the plant input signal, but correlated with the noise-free part.

The estimation is based on the following algorithm:

1. Present the model structure as an linear regression model

$$\hat{y}(k,\theta) = f_{\mathbf{yu}}^{T}(k)\theta, \tag{63}$$

or

$$\mathbf{y} = \mathbf{F}_{YU}\theta. \tag{64}$$

2. Estimate θ by LS method and obtain an initial estimate $\hat{\theta}^1_{\text{LS}}$

$$\hat{\theta}^1_{\text{LS}} = \left[\sum_{k=1}^{N} \mathbf{f}_{yu}(k)\mathbf{f}_{yu}^T(k)\right]^{-1} \sum_{k=1}^{N} \mathbf{f}_{yu}(k)\mathbf{y}^T(k), \tag{65}$$

or

$$\hat{\theta}^1_{\text{LS}} = \left(\mathbf{F}_{YU}^T \mathbf{F}_{YU}\right)^{-1} \mathbf{F}_{YU}^T \mathbf{y}, \tag{66}$$

along with the corresponding transfer function $\hat{G}_1(q)$ from order n.

3. Generate instruments $\mathbf{z_1}(k)$ as estimates of the noise free part of the open-loop regressor $f_{\mathbf{yu}}(k)$ by relations

$$\tilde{y}_1(k) = \frac{\hat{G}_1(q)}{1 + C(q)\hat{G}_1(q)} r(k), \tag{67}$$

$$\tilde{u}_1(k) = \frac{1}{1 + C(q)\hat{G}_1(q)} r(k), \tag{68}$$

$$\mathbf{z}_1(k) = \big[-\tilde{y}_1(k-1) \cdots -\tilde{y}_1(k-n)\tilde{u}_1(k-1) \cdots \tilde{u}_1(k-n)\big]^T. \tag{69}$$

4. Form the matrix $\mathbf{F1}_{\tilde{Y}\tilde{U}}$ obtained in accordance with Eqs. (29), (31) and (32) and determine the IV estimate of θ as

$$\hat{\theta}^2_{\text{IV}} = \hat{\mathbf{R}}^{-1}_{z_1 f_{yu}} \hat{\mathbf{R}}_{z_1 y}, \tag{70}$$

or

$$\hat{\theta}^2_{\text{IV}} = \left(F\mathbf{1}^T_{\tilde{Y}\tilde{U}} F_{YU}\right)^{-1} F\mathbf{1}^T_{\tilde{Y}\tilde{U}} \mathbf{y}. \tag{71}$$

The corresponding estimated n-order transfer function is

$$\hat{G}_2(q) = \frac{\hat{B}_2(q^{-1})}{\hat{A}_2(q^{-1})}. \tag{72}$$

5. Generate residual vector $\hat{\rho}$ as

$$\hat{\rho}(k) = \hat{A}_2\big(q^{-1}\big)y(k) - \hat{B}_2\big(q^{-1}\big)u(k), \tag{73}$$

and define an AR model from order $2n$ for $\hat{\rho}$

$$L(q^{-1})\hat{\rho}(k) = \hat{e}(k), \tag{74}$$

where $\{e(k)\}$ is white noise.

6. Estimate $L(q^{-1})$ by LS method and denote the result with $\hat{L}(q^{-1})$.
7. Generate instruments $\mathbf{z_2}(k)$ as

$$\tilde{y}_2(k) = \frac{\hat{G}_2(q)}{1 + C(q)\hat{G}_2(q)} r(k), \tag{75}$$

$$\tilde{u}_2(k) = \frac{1}{1 + C(q)\hat{G}_2(q)} r(k), \tag{76}$$

$$\mathbf{z_2}(k) = [-\tilde{y}_2(k-1) \cdots - \tilde{y}_2(k-n)\tilde{u}_2(k-1) \cdots \tilde{u}_2(k-n)]^T. \tag{77}$$

8. Filter the input-output data

$$\mathbf{f}_{yuF}(k) = \hat{L}\left(q^{-1}\right)\mathbf{f}_{yu}(k), \tag{78}$$
$$y_F(k) = \hat{L}(q^{-1})y(k).$$

9. Form the matrices $\mathbf{F2}_{\tilde{Y}\tilde{U}}$, $\mathbf{F}_{Y_F U_F}$ and vector $\mathbf{y}_F$ in accordance with Eqs. (29), (31) and (32).
10. Using instruments $\mathbf{z_2}(k)$, determine the IV4 estimates as

$$\hat{\theta}_{\text{IV4}} = \hat{\mathbf{R}}^{-1}_{z_2 \mathbf{f}_{yuF}} \hat{\mathbf{R}}_{z_2 y_F}, \tag{79}$$

or

$$\hat{\theta}_{\text{IV4}} = (\mathbf{F2}^T_{\tilde{Y}\tilde{U}} \mathbf{F}_{Y_F U_F})^{-1} \mathbf{F2}^T_{\tilde{Y}\tilde{U}} \mathbf{y}_F. \tag{80}$$

6.2. *Closed-Loop Quasi-Optimal IV Method*

Noise and plant models can be estimated by the help of high-order LS estimator $(n_1 > n)$. The obtained result is obviously biased, but the bias is only in the first step and does not lead to a bias in the final model (Gilson and Van den Hof, 2003).

The estimation is based on the following algorithm:

1. Present the ARX model structure as a linear regression model by Eqs. (64) or (64) and estimate θ by a high-order LS estimator. Obtain an initial estimate $\hat{\theta}^1_{\text{LS}}$ along with the plant and noise models

$$\hat{G}_1(q) = \frac{\hat{B}_1(q^{-1})}{\hat{A}_1(q^{-1})}; \qquad \hat{H}_1(q) = \frac{1}{\hat{A}_1(q^{-1})}. \tag{81}$$

2. In this case the filter is given by

$$\hat{L}(q^{-1}) = \hat{A}_1(q^{-1})\hat{H}_1(q^{-1}) = 1. \tag{82}$$

3. Compute the noise-free part $\tilde{f}_{yu}(k)$ of the open-loop regressors

$$\tilde{f}_{yu}(k) = [-\tilde{y}_1(k-1) \cdots - \tilde{y}_1(k-n)\tilde{u}_1(k-1) \cdots \tilde{u}_1(k-n)]^T, \tag{83}$$

with $\tilde{y}_1(k)$ and $\tilde{u}_1(k)$, computed according to Eqs. (76) and (77).

4. Generate the instruments as

$$\mathbf{z}(k) = \{[A_1(q^{-1})H_1(q^{-1})]^{-1}\tilde{f}_{yu}^T(k)\}^T \equiv \tilde{f}_{yu}(k) \tag{84}$$

and form the matrix $\mathbf{F}_{\tilde{Y}\tilde{U}}$.

5. Using the instruments $\mathbf{z}(k)$ determine the quasi-optimal IV estimates of θ as

$$\hat{\theta}_{\mathrm{IV}}^{OPT} = \hat{R}_{zf_{yu}}^{-1}\hat{\mathbf{R}}_{zy}, \tag{85}$$

or

$$\hat{\theta}_{\mathrm{IV}}^{OPT} = (F_{\tilde{Y}\tilde{U}}^T F_{YU})^{-1}F_{\tilde{Y}\tilde{U}}^T\mathbf{y}. \tag{86}$$

7. Example

Consider a close-loop system, described in the form of Eqs. (2), (5) and (6) with

$$G_0(q) = \frac{0.4134q^{-1}}{1 - 0.5866q^{-1}}, \tag{87}$$

$$C(q) = \frac{0.1087q^{-1} + 0.0729q^{-2}}{1 - 1.1197q^{-1} + 0.3012q^{-2}}, \tag{88}$$

$$H_0(q) = \frac{1 + 0.05q^{-1} + 0.8q^{-2}}{1 - 1.036q^{-1} + 0.2636q^{-2}}. \tag{89}$$

The input – $\{r(k)\}$ and the noise – $\{e_0(k)\}$ are random signals with normal distribution. The standard deviation of the generalized input signal $\{r(k)\}$ is one. The desired noise to signal ratio is obtained by variation of the standard deviation of the noise $\{e_0(k)\}$.

The relative mean-squared error Q_θ with respect to the true parameters is used as an accuracy criterion. The plant parameters are estimated on the basis of the data set (15) with $N = 1000$. Monte Carlo simulations have been performed for three different noise to signal ratio values – $rel_1 = 6\%$, $rel_2 = 10\%$ and $rel_3 = 15\%$.

Four estimators are investigated – Tailor-made IV (54), extended IV4 (80), quasi-optimal IV (86), and basic closed-loop IV (28). Other two estimators are used for comparison purposes. They are the LS (66) and standard optimal IV. The last one is obtained according to Eqs. (34), (59)–(62) and the true plant and noise filter parameters from equations (88) and (89).

The true plant parameters are $a_1 = -0.5866$ and $b_1 = 0.4134$. The mean values of the parameter estimates for 150 Monte Carlo runs, their standard deviation and the mean values of the relative mean-squared errors Q_θ with respect to the true parameters are presented in Table 1.

Bar-diagrams of the mean values of the relative mean-squared errors Q_θ with respect to the true parameters for the three noise to signal ratios, are shown in Figs. 2, 3 and 4.

Table 1

Parameter estimation results

No.		$rel_1 = 6\%$		$rel_2 = 10\%$		$rel_3 = 15\%$	
1	$\hat{\theta}_{\mathrm{IV}}^{ST}$	-0.5867	0.4134	-0.5869	0.4135	-0.5865	0.4136
	std	0.0024	0.0008	0.0037	0.0014	0.0056	0.0020
	Q_θ	0.0029		0.0044		0.0069	
2	$\hat{\theta}_{\mathrm{IV}}^{TM}$	-0.5866	0.4135	-0.5860	0.4133	-0.5871	0.4138
	std	0.0033	0.0013	0.0055	0.0024	0.0076	0.0037
	Q_θ	0.0042		0.0073		0.0101	
3	$\hat{\theta}_{\mathrm{IV4}}$	-0.5910	0.4134	-0.5859	0.4134	-0.5885	0.4135
	std	0.0414	0.0034	0.0000	0.0024	0.0103	0.0032
	Q_θ	0.0138		0.0097		0.0126	
4	$\hat{\theta}_{\mathrm{IV}}^{OPT}$	-0.5872	0.4135	-0.5860	0.4134	-0.5876	0.4132
	std	0.0047	0.0016	0.0070	0.0025	0.0091	0.0038
	Q_θ	0.0057		0.0085		0.0118	
5	$\hat{\theta}_{\mathrm{IV}}$	-0.5869	0.4135	-0.5862	0.4134	-0.5870	0.4136
	std	0.0039	0.0017	0.0061	0.0026	0.0088	0.0039
	Q_θ	0.0049		0.0077		0.0111	
6	$\hat{\theta}_{\mathrm{LS}}$	-0.5965	0.4134	-0.6108	0.4123	-0.6376	0.4109
	std	0.0047	0.0016	0.0066	0.0025	0.0086	0.0039
	Q_θ	0.0142		0.0341		0.0714	

The serial number of the estimators listed in Table 1 corresponds with the serial number of the bar-diagram columns.

8. Conclusion

In this paper an alternative user-friendly representation of the ordinary mathematical IV relations is proposed. This vector-matrix form is very convenient for computer based application, especially for Matlab® implementation.

Different modifications of the IV method are discussed and a comparison between different methods for close-loop identification is performed. An experiment is carried out, based on the Monte Carlo method. As comparison criteria for those estimators, the accuracy of the estimation is used. It is seen that the accuracy of the estimates in the studied modifications of the IV method is much greater then the one obtained by the classical LS method. From Table 1, Figs. 2, 3 and 4 also can be seen that the best estimator is the standard IV estimator. Unfortunately, it cannot be implemented in practise. The second

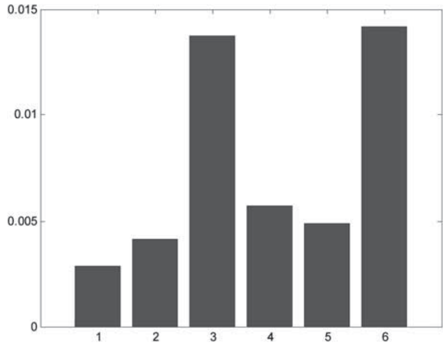

Fig. 2. Bar-diagram for $rel_1 = 6\%$.

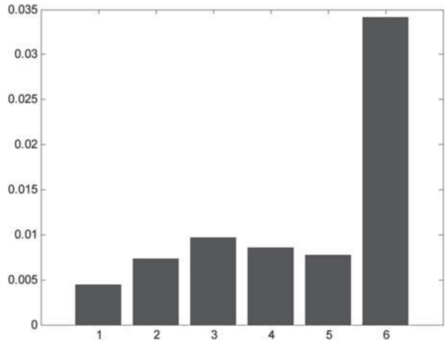

Fig. 3. Bar-diagram for $rel_2 = 10\%$.

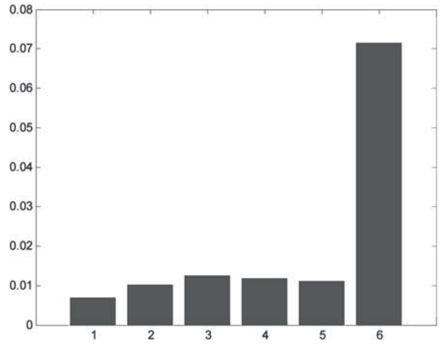

Fig. 4. Bar-diagram for $rel_3 = 15\%$.

best is the Tailor-made IV estimator, which algorithm does not incorporate intermediate LS estimates.

From the analysis of the conducted investigation it is clear that the LS method produced biased mean value of the estimate, while all IV methods provide unbiased estimates of the system parameters. The initial LS estimates and approximate determination of the noise free part of the open-loop repressors have negative impact on the accuracy of the extended IV4 and quasi-optimal IV estimators. Relatively inaccurate are also the estimates of the standard IV estimator, which did not incorporate approximate information for the plant and noise models, based on the measured data.

For future investigation the authors are planning to develop recurrent versions of the investigated algorithms in order to perform identification in real time. During the investigations, different excitation signals will be used, in order to investigate the applicability of each algorithm.

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Grįžtamojo ryšio sistemos identifikavimas, taikant instrumentinių kintamųjų metodo modifikacijas

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Šiame straipsnyje pateikiama keletas apibendrinto instrumentinių kintamųjų (IK) metodo modifikacijų grįžtamojo ryšio sistemos identifikavimui. Tiriama IK įverčių kovariacinė matrica. Nustatyta, kad optimali IK įverčių kovariacinė matrica bei optimali IK metodo procedūra gali būti gauti tik tuo atveju, kai tiksliai žinomi identifikuojamos sistemos ir triukšmo modeliai. Remiantis šiuo faktu sudarytos įvairios IK metodo modifikacijos, vienaip ar kitaip aproksimuojančios reikalingą informaciją, kuri turėtų būti gaunama iš matavimų. Pagrindinis gautų išraiškų privalumas, kad jos esti nesudėtingos matematine prasme bei patogios realizuoti praktiškai. Pateiktas grįžtamojo ryšio sistemos pavyzdys ir gauti modeliavimo kompiuteriu rezultatai.