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Inapproximability Results for Wavelength Assignment in WDM Optical Networks

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Abstract. We address the issue of inapproximability of the wavelength assignment problem in wavelength division multiplexing (WDM) optical networks. We prove that in an n-node WDM optical network with m lightpaths and maximum load L, if NP $\neq$ ZPP, for any constant $\delta > 0$, no polynomial time algorithm can achieve approximation ratio $n^{1/2-\delta}$ or $m^{1-\delta}$, where NP is the class of problems which can be solved by nondeterministic polynomial time algorithms, and ZPP is the class of problems that can be solved by polynomial randomized algorithms with zero probability of error. Furthermore, the above result still holds even when $L = 2$. We also prove that no algorithm can guarantee the number of wavelengths to be less than $(\sqrt{n}/2)L$ or $(m/2)L$. This is the first time inapproximability results are established for the wavelength assignment problem in WDM optical networks. We also notice the following fact, namely, there is a polynomial time algorithm for wavelength assignment which achieves approximation ratio of $O(m(\log\log m)^2/(\log m)^3)$. Therefore, the above lower bound of $m^{1-\delta}$ is nearly tight.

Keywords: approximation ratio, graph coloring, inapproximability, optical network, wavelength assignment, wavelength division multiplexing.

1. Introduction

Given a wavelength division multiplexing (WDM) optical network represented by an n-node graph $G = (V, E)$ and a set $P = \{p_1, p_2, \ldots, p_m\}$ of m lightpaths, the *wavelength assignment* problem is to assign a wavelength to each lightpath, such that no two lightpaths sharing a common link are assigned the same wavelength, and that the number of wavelengths used is minimized (Ramaswami and Sivarajan, 1998).

It has been proven that for a general WDM optical network, the wavelength assignment problem is NP-complete (Chlamtac *et al.*, 1992), even if the length of any lightpath is bounded by two (Harder, 1998). The wavelength assignment problem remains NP-complete for simple networks such as mesh and tree networks (Choi and Harder, 1998).

Extensive research has been conducted to find approximation algorithms that solve the wavelength assignment problem by producing near-optimal wavelength assignments. Virtually all these algorithms are evaluated based on the *load* L of the set of lightpaths, i.e., the maximum number of lightpaths passing through an optical link. The quantity L is an obvious lower bound for the number of wavelengths required to support a set of lightpaths. It was shown in Aggarwal *et al.* (1994) that for an arbitrary WDM optical network, the number of wavelengths required to satisfy a set of lightpaths is at most

$$\min\big((L-1)D+1,(2L-1)\sqrt{M}-L+2\big), \tag{1}$$

where D is the longest length of a lightpath and M is the number of optical links in the optical network.

Better results have been obtained for specific and simple networks. For instances, for a linear array network, a wavelength assignment which uses exactly L wavelengths can be found easily (Berge, 1973). For a ring network, it was known that $2L-1$ wavelengths are sufficient (Tucker, 1975). For star and tree networks, it was shown that $\frac{3}{2}L$ wavelengths are sufficient (Raghavan and Upfal, 1994).

However, as pointed out in Choi and Harder (1998), the sufficient number of wavelengths required to support a set of lightpaths has no direct relation to the parameter L. In particular, there is no algorithm for the wavelength assignment problem that guarantees the number of wavelengths to be bounded by a constant times L even if $L=2$ on a mesh network. Such a result is certainly interesting; however, it does not reveal inapproximability of the wavelength assignment problem, since the minimum number of wavelengths required can be arbitrarily larger than L.

Let I be an instance of a minimization problem. We use ALG(I) to represent the solution produced by a polynomial time approximation algorithm ALG, and OPT(I) the optimal solution. If

$$\frac{\text{ALG}(I)}{\text{OPT}(I)} \leqslant \alpha,$$

for all instances I, we say that algorithm ALG achieves *approximation ratio* α.

In this paper, we prove the following inapproximability results for the wavelength assignment problem in WDM optical networks.

- If NP $\neq$ ZPP, for any constant $\delta > 0$, no polynomial time algorithm can achieve approximation ratio $n^{1/2-\delta}$ or $m^{1-\delta}$.
- The above result still holds even when $L=2$.

Note. NP is the class of problems which can be solved by nondeterministic polynomial time algorithms. ZPP is the class of problems that can be solved by polynomial randomized algorithms with zero probability of error, i.e., Las Vegas algorithms (Papadimitriou, 1994).

This is the first time inapproximability results are established for the wavelength assignment problem in WDM optical networks. We notice the following fact.

- There is a polynomial time algorithm for wavelength assignment which achieves approximation ratio of $O(m(\log\log m)^2/(\log m)^3)$.

Hence, the above lower bound of $m^{1-\delta}$ is nearly tight. We also point out the hardness of approximating L. In particular, the following result is proved.

- No algorithm can guarantee the number of wavelengths to be less than $(\sqrt{n}/2)L$ or $(m/2)L$.

Therefore, attempting to approximate L in general WDM networks is meant to be fruitless.

2. Reduction from Graph Coloring

Consider an input graph $G' = (V', E')$ for the graph coloring problem. We will construct a WDM optical network $G = (V, E)$ together with a set P of lightpaths such that G' is k-colorable if and only if the lightpaths in P can be assigned k wavelengths.

The main purpose of this section is to prove the following result.

Lemma 1. *There is a polynomial time $O((n')^2)$ algorithm, which, for every graph $G' = (V', E')$ with n' vertices $V' = \{v_1, v_2, \ldots, v_{n'}\}$, constructs a WDM optical network $G = (V, E)$ with $n = (n')^2$ nodes and a set $P = \{p_1, p_2, \ldots, p_m\}$ of $m = n'$ lightpaths, such that the n' vertices in V' can be colored with $c_1, c_2, \ldots, c_k$ if and only if the m lightpaths in P can be assigned wavelengths $\lambda_1, \lambda_2, \ldots, \lambda_k$; furthermore, v_i is colored with c_{j_i} if and only if p_i is assigned wavelength λ_{j_i}, $1 \leqslant j_i \leqslant k$, for all $1 \leqslant i \leqslant n'$.*

Proof. Assume that $V' = \{v_1, v_2, \ldots, v_{n'}\}$ contains n' vertices. Without loss of generality, each edge (v_i, v_j) in E' has $i < j$. The network G has $n = (n')^2$ nodes

$$V = \{v_{i,j} \mid 1 \leqslant i, j \leqslant n'\}.$$

The optical links in G are

$$E = \{(v_{i,j}, v_{i,j+1}) \mid 1 \leqslant i \leqslant n',\ 1 \leqslant j \leqslant n' - 1\}.$$

The network G contains a set $P = \{p_1, p_2, \ldots, p_m\}$ of $m = n'$ lightpaths, where

$$p_i\colon v_{i,1} \to v_{i,2} \to v_{i,3} \to \cdots \to v_{i,n'},$$

for all $1 \leqslant i \leqslant n'$. Therefore, each vertex v_i in G' has a corresponding lightpath p_i in G, and the problem of coloring vertex v_i is treated as assigning a wavelength to lightpath p_i.

Each edge (v_i, v_j) in E' will be represented by overlapping of lightpaths p_i and p_j. On each lightpath p_i,

$$p_i\colon v_{i,1} \to v_{i,2} \to \cdots \to v_{i,i} \to v_{i,i+1} \to v_{i,i+2} \to \cdots \to v_{i,n'},$$

the edge $(v_{i,j-1}, v_{i,j})$ is reserved for possible overlapping with lightpath p_{j-1}, where $2 \leqslant j \leqslant i$, and the edge $(v_{i,j-1}, v_{i,j})$ is reserved for possible overlapping with lightpath p_j, where $i + 1 \leqslant j \leqslant n'$.

The following changes to E and lightpath p_j are performed for each edge $(v_i, v_j) \in E'$, where $1 \leqslant i < j \leqslant n'$, namely,

- deleting link $(v_{j,i}, v_{j,i+1})$ and adding links $(v_{j,i}, v_{i,j-1})$ and $(v_{j,i+1}, v_{i,j})$, as illustrated in Fig. 1;

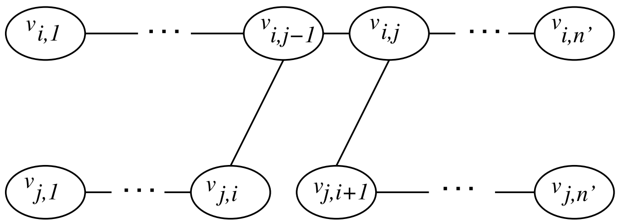

Fig. 1. Change to E in the network $G = (V, E)$ and lightpath p_j for $(v_i, v_j) \in E'$.

- modifying the lightpath p_j from

$$p_j\colon v_{j,1} \to \cdots \to v_{j,i} \to v_{j,i+1} \to \cdots \to v_{j,n'},$$

to

$$p_j\colon v_{j,1} \to \cdots \to v_{j,i} \to v_{i,j-1} \to v_{i,j} \to v_{j,i+1} \to \cdots \to v_{j,n'}.$$

The lightpath p_i remains unchanged. The above modification ensures that lightpaths p_i and p_j overlap on link $(v_{i,j-1}, v_{i,j})$.

It is clear from the above construction of $G = (V, E)$ and P that there is an edge $(v_i, v_j) \in E'$ if and only if p_i and p_j overlap. In other words, $G' = (V', E')$ is precisely the *lightpath graph* of $P = \{p_1, p_2, \ldots, p_{n'}\}$ in the reduction from the wavelength assignment problem to the graph coloring problem (Ramaswami and Sivarajan, 1998). The n' vertices in V' can be colored with $c_1, c_2, \ldots, c_k$ if and only if the m lightpaths in P can be assigned wavelengths $\lambda_1, \lambda_2, \ldots, \lambda_k$. Furthermore, v_i is colored with c_{j_i} if and only if p_i is assigned wavelength λ_{j_i}, $1 \leqslant j_i \leqslant k$, for all $1 \leqslant i \leqslant n'$.

It is easy to verify that the construction of $G = (V, E)$ and $P = \{p_1, p_2, \ldots, p_m\}$ takes polynomial time, i.e., $O((n')^2)$ time. □

Let us call the algorithm in the proof of Lemma 1 as TRANSFORM, which constructs the WDM optical network $G = (V, E)$ and the set P of lightpaths based on a graph $G' = (V', E')$.

The following result is a direct consequence of Lemma 1.

COROLLARY 1. For every graph $G' = (V', E')$, the WDM optical network $G = (V, E)$ and the set P of lightpaths constructed by algorithm TRANSFORM satisfy $\mathrm{OPT}(G') = \mathrm{OPT}(G, P)$.

3. An Example

The following example illustrates the construction performed by algorithm TRANSFORM.

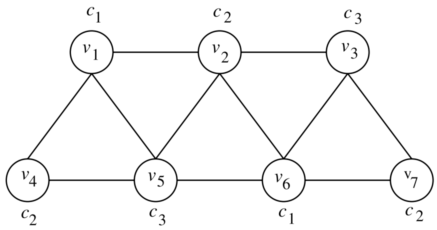

Fig. 2. A graph $G' = (V', E')$.

EXAMPLE 1. Let us consider the graph $G' = (V', E')$ with 7 vertices given in Fig. 2. The WDM network $G = (V, E)$ before and after edge and lightpath modification are shown in Figs. 3 and 4 respectively. The 11 lightly printed links in Fig. 3 are to be modified. The 7 modified lightpaths in G are

$$p_1\colon v_{1,1} \to v_{1,2} \to v_{1,3} \to v_{1,4} \to v_{1,5} \to v_{1,6} \to v_{1,7},$$
$$p_2\colon v_{2,1} \to v_{1,1} \to v_{1,2} \to v_{2,2} \to v_{2,3} \to v_{2,4} \to v_{2,5} \to v_{2,6} \to v_{2,7},$$
$$p_3\colon v_{3,1} \to v_{3,2} \to v_{2,2} \to v_{2,3} \to v_{3,3} \to v_{3,4} \to v_{3,5} \to v_{3,6} \to v_{3,7},$$
$$p_4\colon v_{4,1} \to v_{1,3} \to v_{1,4} \to v_{4,2} \to v_{4,3} \to v_{4,4} \to v_{4,5} \to v_{4,6} \to v_{4,7},$$
$$p_5\colon v_{5,1} \to v_{1,4} \to v_{1,5} \to v_{5,2} \to v_{2,4} \to v_{2,5} \to v_{5,3} \to v_{5,4} \to v_{4,4} \to v_{4,5} \to v_{5,5} \to v_{5,6} \to v_{5,7},$$
$$p_6\colon v_{6,1} \to v_{6,2} \to v_{2,5} \to v_{2,6} \to v_{6,3} \to v_{3,5} \to v_{3,6} \to v_{6,4} \to v_{6,5} \to v_{5,5} \to v_{5,6} \to v_{6,6} \to v_{6,7},$$
$$p_7\colon v_{7,1} \to v_{7,2} \to v_{7,3} \to v_{3,6} \to v_{3,7} \to v_{7,4} \to v_{7,5} \to v_{7,6} \to v_{6,6} \to v_{6,7} \to v_{7,7}.$$

An example coloring of the vertices in V' is also given in Fig. 2, where the 7 vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ are colored with $c_1, c_2, c_3, c_2, c_3, c_1, c_2$, respectively. Therefore, the 7 lightpaths $p_1, p_2, p_3, p_4, p_5, p_6, p_7$ are assigned wavelengths λ_1,λ_2,λ_3,λ_2,λ_3,λ_1,λ_2, respectively.

4. Inapproximability Results

Lemma 1 in Section 2 implies that any algorithm ALG for the wavelength assignment problem can be converted to an algorithm ALG$'$ to solve the graph coloring problem.

Lemma 2. *For every polynomial time $O(n^a)$ algorithm* ALG *for wavelength assignment, there is a polynomial time $O((n')^{2\max(a,1)})$ algorithm* ALG$'$ *for graph coloring, such that for every graph $G' = (V', E')$, algorithm* ALG$'$ *constructs a WDM optical network $G = (V, E)$ and a set P of lightpaths, with* ALG$'(G')$ = ALG(G, P) *and* OPT(G') = OPT(G, P).

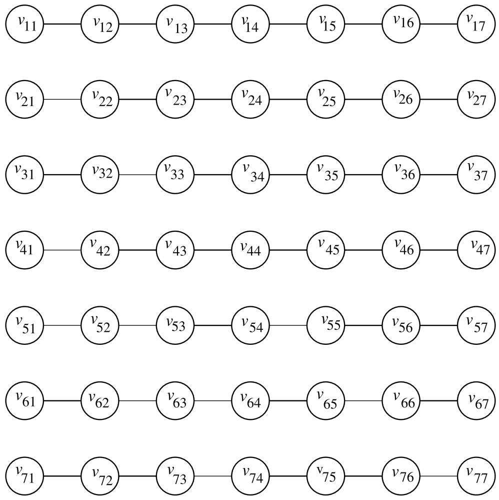

Fig. 3. A WDM network $G = (V, E)$ before edge and lightpath modification.

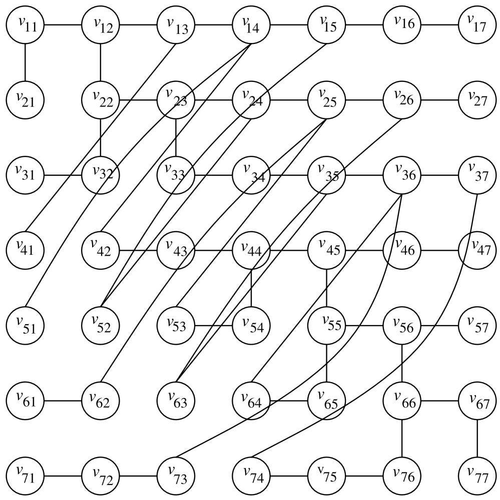

Fig. 4. A WDM network $G = (V, E)$ after edge and lightpath modification.

Proof. Given a graph $G' = (V', E')$, where $V' = \{v_1, v_2, \ldots, v_{n'}\}$, algorithm ALG$'$ performs the following three steps.

1. Construct a WDM optical network $G = (V, E)$ and a set $P = \{p_1, p_2, \ldots, p_m\}$ of lightpaths, by using algorithm TRANSFORM.
2. Call algorithm ALG to find wavelength assignment to the lightpaths in P, i.e., $\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_m}$.
3. Color vertex v_i with c_{j_i} if p_i is assigned wavelength λ_{j_i}, for all $1 \leqslant i \leqslant n'$.

The algorithm ALG$'$ is illustrated in Fig. 5.

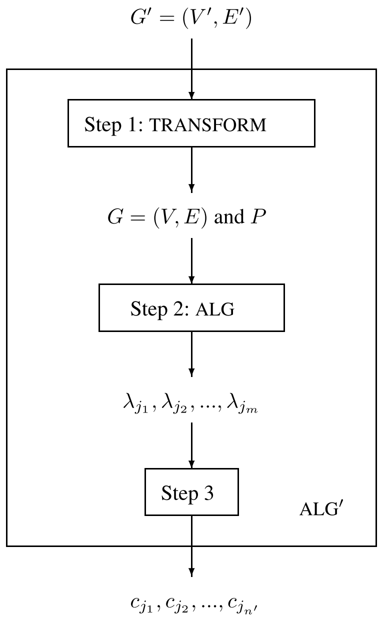

Fig. 5. Algorithm ALG′.

Step 1 takes $O((n')^2)$ time. Step 2 takes $O(n^a) = O((n')^{2a})$ time. Step 3 takes $O(n')$ time. The over time complexity of algorithm ALG′ is $O((n')^{2\max(a,1)})$.

Step 3 implies that $\text{ALG}'(G') = \text{ALG}(G, P)$. From Corollary 1, we know that $\text{OPT}(G') = \text{OPT}(G, P)$. □

Now, we are ready to prove the main result of the paper.

Theorem 1. *If* NP $\neq$ ZPP*, for any constant* $\delta > 0$*, no polynomial time algorithm can achieve approximation ratio* $n^{1/2-\delta}$ *or* $m^{1-\delta}$ *for wavelength assignment in WDM optical networks.*

Proof. Lemma 2 implies that

$$\frac{\text{ALG}'(G')}{\text{OPT}(G')} = \frac{\text{ALG}(G, P)}{\text{OPT}(G, P)}.$$

If algorithm ALG can achieve approximation ratio $n^{1/2-\delta}$ for some constant $\delta > 0$, that is,

$$\frac{\text{ALG}(G, P)}{\text{OPT}(G, P)} \leqslant n^{1/2-\delta} = m^{1-2\delta},$$

for all G and P, then

$$\frac{\text{ALG}'(G')}{\text{OPT}(G')} \leqslant (n')^{1-2\delta},$$

for all G'. If algorithm ALG can achieve approximation ratio $m^{1-\delta}$ for some constant $\delta > 0$, that is,

$$\frac{\text{ALG}(G,P)}{\text{OPT}(G,P)} \leqslant m^{1-\delta},$$

for all G and P, then

$$\frac{\text{ALG}'(G')}{\text{OPT}(G')} \leqslant (n')^{1-\delta},$$

for all G'. In either case, for some constant $\delta > 0$, the polynomial time algorithm ALG$'$ can achieve approximation ratio $(n')^{1-\delta}$ for graph coloring.

However, inapproximability of graph coloring has been studied with improved results (Bellare and Sudan, 1994; Feige and Kilian, 1996; Fürer, 1995), and the currently best result is the following fact shown in Feige and Kilian (1996).

PROPOSITION 1. If NP $\neq$ ZPP, for any constant $\delta > 0$, no polynomial time algorithm can achieve approximation ratio $(n')^{1-\delta}$ for graph coloring.

Thus, if NP $\neq$ ZPP, for any constant $\delta > 0$, no polynomial time algorithm ALG can achieve approximation ratio $n^{1/2-\delta}$ or $m^{1-\delta}$ for wavelength assignment in WDM optical networks. □

We also notice that in the reduction performed by algorithm TRANSFORM, the number of lightpaths passing through an optical link is at most 2, i.e., $L = 2$. Thus, we can claim the following.

Theorem 2. *Theorem* 1 *holds even when* $L = 2$.

To show the quality of the above inapproximability results, we notice the following reduction from the wavelength assignment problem to the graph coloring problem (Ramaswami and Sivarajan, 1998).

Lemma 3. *For every polynomial time algorithm* ALG$'$ *for graph coloring, there is a polynomial time algorithm* ALG *for wavelength assignment, such that for every WDM optical network G with a set P of m lightpaths, algorithm* ALG *constructs an m-vertex graph G' with* ALG$(G,P) =$ ALG$'(G')$ *and* OPT$(G,P) =$ OPT(G').

As for approximate graph coloring, achievable performance ratio has been improved (Berger and Rompel, 1990; Halldórsson, 1993; Wigderson, 1983), and the following fact has been proven in Halldórsson (1993).

PROPOSITION 2. There is a polynomial time algorithm ALG$'$ for graph coloring which achieves approximation ratio of $O(n'(\log\log n')^2/(\log n')^3)$.

The above fact implies the following result which says that the lower bound of $m^{1-\delta}$ is nearly tight.

Theorem 3. *There is a polynomial time algorithm* ALG *for wavelength assignment which achieves approximation ratio of* $O(m(\log\log m)^2/(\log m)^3)$.

5. Hardness of Approximating L

We now prove that on an n-node WDM optical network with m lightpaths, it is impossible for any algorithm of wavelength assignment to guarantee to use less than $(\sqrt{n}/2)L$ or $(m/2)L$ wavelengths.

Theorem 4. *No algorithm for wavelength assignment in WDM optical networks can guarantee the number of wavelengths to be less than* $(\sqrt{n}/2)L$ *or* $(m/2)L$.

Proof. The reduction performed by algorithm TRANSFORM always yields $L = 2$, even when $G' = (V', E')$ is a complete graph. In this case, the number of colors used to color the vertices in V' is n', i.e., the number of wavelengths assigned to the lightpaths in P is $n' = (\sqrt{n}/2)L$ or $m = (m/2)L$. □

6. Notes on Related Research

Reductions from the graph coloring problem to the wavelength assignment problem have been reported before (Chlamtac *et al.*, 1992; Choi and Harder, 1998). However, in all these reductions, the WDM network $G = (V, E)$ constructed contains more than $(n')^2$ nodes. For example, the WDM network in the reduction of Choi and Harder (1998) has $(n')^3$ nodes. It is clear that the reduction of this paper yields higher inapproximability results.

7. Final Remarks

For the first time, inapproximability results for the wavelength assignment problem in WDM optical networks are established. We have also pointed out the hardness of approximating L.

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Bangų ilgių paskyrimo WDM optiniuose tinkluose uždavinio sprendinių neaproksimuojamumo rezultatai

Keqin LI

Nagrinėjamas neaproksimuojamumo klausimas tokiame uždavinyje: paskirti bangų ilgius jų dalijimo multipleksiniuose (WDM) optiniuose tinkluose. Įrodoma, kad WDM optiniame tinkle joks polinominio sudėtingumo algoritmas negali užtikrinti aproksimacijos santykio $n^{1/2-\delta}$ arba $m^{1-\delta}$, jei NP $\neq$ ZPP, čia n-tinklo mazgų skaičius, m-šviesolaidžių skaičius, L – maksimalus apkrovimas, $\delta > 0$ bet kokia konstanta, NP-klasė uždavinių išsprendžiamų per polinominį laiką nedeterministiniu algoritmu, ZPP-klasė uždavinių išsprendžiamų per polinominį laiką randomizuotu algoritmu su nuline klaidos tikimybe. Toks rezultatas galioja ir specialiu atveju, kai $L = 2$. Taip pat įrodoma, kad joks algoritmas negali užtikrinti bangų ilgių skaičiaus mažesnio už $\sqrt{n/2}L$ arba $(\frac{m}{2})L$. Tai yra pirmieji neaproksimuojamumo rezultatai tokio tipo uždaviniams. Pastebėsime, kad apatinis rėžis $m^{1-\delta}$ yra beveik griežtas.