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A UNIFIED VIEW ON BLOCK AND SCALAR IMPULSE RESPONSES

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Abstract. The aim of this paper is to concentrate in one place and to show the relations among the matrix block impulse response, block impulse response, matrix impulse response and impulse response of linear time-varying (LTV) systems, frozen-time LTV systems, linear periodically time-varying (LPTV) systems, and linear time-invariant (LTI) systems.

Key words: scalar impulse response, block impulse response, linear time-varying systems, linear periodically time-varying systems.

Introduction. The input–output behaviour of a LTV system (Loeffler and Burrus, 1984; Huang and Aggarwal, 1980, 1983; Park and Aggarwal, 1985; Portnoff, 1980) can be characterized in the time domain by a weighting pattern, or Green's function, $g(k, k_1)$ which represents the response of the system at time k to a unit sample applied at time k_1. Equivalently, the same system can be described by a time-varying impulse response $h(k, k_1)$ defined as the response of the system at time k to a unit sample applied k_1 samples earlier, i.e., at time $(k - k_1)$. Furthermore, the time-varying impulse response $h(k, k_1)$ and the Green's function $g(k, k_1)$ are related by

$h(k,k_1) = g(k, k-k_1)$ or, equivalently, $g(k,k_1) = h(k, k-k_1)$. If $y(k)$ is the response of a system to the input $x(k_1)$, then $y(k)$ is given by

$$y(k) = \sum_{k_1=-\infty}^{\infty} g(k,k_1)\, x(k_1) \tag{1}$$

or

$$\begin{aligned} y(k) &= \sum_{k_1=-\infty}^{\infty} h(k,k_1)\, x(k-k_1) \\ &= \sum_{k_1=-\infty}^{\infty} h(k,k-k_1)\, x(k_1) \end{aligned} \tag{2}$$

In the system (of filter) theory (Zadeh and Desoer, 1970) impulse responses have fundamental importance, as they completely characterize the behaviour of the system (or filter).

On the other hand, block implementation has some advantage such as fewer computations, a possibility to use fast convolution techniques for intermediate computations, efficient implementation by parallel processors, and reduced roundoff noise. Block structures have been studied by Barnes and Shinnaka (1980); Burrus, 1972; Clark, Mitra and Parker (1981); Nikias (1985); Vaidyanathan and Mitra (1988).

The aim of this paper is to show a close connection between scalar and block impulse responses of different types of linear systems and to analyze the relations between these impulse responses and the coefficients of difference equations (ARMA models).

Block difference equation of LTV systems. Let us assume that a linear dynamical time-varying system is described by the difference equation

$$\sum_{i=0}^{M} \widehat{b}_i(k)x(k-i) = \sum_{i=0}^{N} \widehat{a}_i(k)y(k-i), \qquad \widehat{a}_0(k) \neq 0, \quad M \leqslant N, \quad k = 0,1,2,\dots \tag{3}$$

where $x(k)$ is the input signal, $y(k)$ is the output signal of the LTV system.

Define $k = mL + n$, $m = 0,1,2,\dots$, $n = 0,1,\dots,L-1$, $L = 1,2,3,\dots$ L is the length of the block. Then from equation (3) we obtain the block difference equation of the LTV system

$$\sum_{i=0}^{r} \widehat{B}_{r,i}(m)X(m-i) = \sum_{i=0}^{p} \widehat{A}_{p,i}Y(m-i), \qquad m = 0,1,2,\dots, \quad r \leqslant p, \tag{4}$$

where $Y(m)$ and $X(m)$ are the m-th output and input blocks of the length L, respectively

$$Y(m) = [y(mL),\dots,y(mL+n),\dots,y(mL+L-1)]^T,$$
$$X(m) = [x(mL),\dots,x(mL+n),\dots,x(mL+L-1)]^T,$$

$\widehat{A}_{p,i}(m)$ and $\widehat{B}_{r,i}(m)$ are the $L\times L$ matrices given by $\widehat{A}_{p,i}(m) = [\widehat{a}_{nj}]$, where $\widehat{a}_{nj} = \widehat{a}_{iL+n-j}(iL+mL+n)$, $i = 0,1,\dots,p$, $\widehat{B}_{r,i}(m) = [\widehat{b}_{nj}]$, where $\widehat{b}_{nj} = \widehat{b}_{iL+n-j}(iL+mL+n)$, $i = 0,1,\dots,r$; $n,j = 0,1,\dots,L-1$. p and r are such smallest integers that inequalities $pL \geqslant N$ and $rL \geqslant M$ are valid. If $L \geqslant N$, then $p = r = 2$.

From equation (4) we obtain

$$Y(m) = \widehat{A}_{p,0}^{-1}(m)\sum_{i=0}^{r} \widehat{B}_{r,i}(m)X(m-i) - \widehat{A}_{p,0}^{-1}(m)\sum_{i=1}^{p} \widehat{A}_{p,i}(m)Y(m-i)$$

or

$$Y(m)=\sum_{i=0}^{r} B_{r,i}(m)X(m-i)-\sum_{i=1}^{p} A_{p,i}(m)Y(m-i), \quad (5)$$

where $A_{p,i}(m)=\widehat{A}_{p,0}^{-1}(m)\widehat{A}_{p,i}(m), \quad i=0,1,\dots p,$
$B_{r,i}(m)=\widehat{A}_{p,0}^{-1}(m)\widehat{B}_{r,i}(m), \quad i=0,1,\dots r$

Block impulse response of LTV systems. According to the definition of the block impulse response, $H(m,m_1)$ is the output of block difference equation (5) at time m to a block unit pulse input $X(m-m_1)$ applied at time m_1 block earlier, i.e.,

$$\begin{aligned} Y(m)&=\sum_{i=0}^{r} B_{r,i}(m)X[(m-m_1)-i] \\ &-\sum_{i=1}^{p} A_{p,i}Y(m-i), \end{aligned} \quad (6)$$
$$r\leqslant p, \quad m\geqslant m_1, \quad m,m_1=0,1,2,\dots,$$

where we define $k_1=m_1L+j, \; j=0,1,\dots,L-1$.

According to the definition of the block unit-pulse input, $X[(m-m_1)-i]=1$, for $i=m-m_1$ and $X[(m-m_1)-i]=0$, for $i\neq m-m_1$. Then equation (6) gives the matrix block impulse response of the LTV system H_B defined as $H_B=[H_{mm_1}]$, where $H_{mm_1}=H(m,m_1)$, $m,m_1=0,1,2,\dots$ The block impulse response of the LTV system

$$H(m,m_1)=B_{r,m-m_1}(m)-\sum_{i=1}^{p} A_{p,i}(m)H(m-i,m_1), \quad (7)$$
$$m,m_1=0,1,2,\dots \quad m\geqslant m_1,$$

where $B_{r,m-m_1}(m)=0$, for $m-m_1>r$. $H(m,m_1)$ is the $L\times L$ matrix given by $H(m,m_1)=[h_{nj}]$, where $h_{nj}=h(mL+n,\; m_1L+j)$, $n,j=0,1,\dots,L-1$.

REMARK 1: If a block unit pulse input signal of the LTV system is $X(m_1) = 1$, for $m_1 = 0$ and $X(m_1) = 0$, for $m_1 \neq 0$, then equation (5) gives block Green's function $G(m, m_1) = H(m, m - m_1)$.

For the non-recursive LTV system ($A_{p,i}(m) = I$, for $i = 0$ and $A_{p,i}(m) = 0$, for $i \neq 0$) the block impulse response is

$$\begin{aligned} &H(m, m_1) = B_{r,m-m_1}(m), \\ &m, m_1 = 0, 1, 2, \ldots, \quad m \geqslant m_1, \end{aligned} \tag{8}$$

where $B_{r,m-m_1}(m) = 0$, for $m - m_1 > r$.

For the recursive LTV system ($B_{r,m-m_1}(m) = I$, for $m = m_1$ and $B_{r,m-m_1}(m) = 0$, for $m \neq m_1$) the block impulse response is as in (7).

The impulse response of a slowly varying system may be approximated by the invariant impulse response through freezing the variant difference equation at an instant of consideration (Nikolic, 1975). We obtain the block impulse response of the frozen-time LTV system $H^*(m, m_1)$ from equation (7)

$$\begin{aligned} H^*_{;}(m, m_1) &= B_{r,m-m_1}(m_1) \\ &\quad - \sum_{i=1}^{p} A_{p,i}(m_1) H(m - i, m_1), \\ &m, m_1 = 0, 1, 2, \ldots \quad m \geqslant m_1, \end{aligned} \tag{9}$$

where $B_{r,m-m_1}(m_1) = 0$, for $m - m_1 > r$.

A special case of LTV systems is LPTV systems. Consider a discrete-time linear system, whose coefficients vary periodically in time, with period L. For such systems it is valid: $\widehat{a}_i(k + L) = \widehat{a}_i(k)$, $i = 0, 1, \ldots, N$; $k = 0, 1, 2, \ldots$ and $\widehat{b}_i(k + L) = \widehat{b}_i(k)$, $i = 0, 1, \ldots, M$; $k = 0, 1, 2, \ldots$

If the length of the block is equal to L, then equation (7) gives the block impulse response of the LPTV system

$$H(m) = B_{r,m} - \sum_{i=1}^{p} A_{p,i} H(m - i), \quad m = 0, 1, 2, \ldots \tag{10}$$

where $B_{r,m} = 0$, for $0 > m > r$. $A_{p,i}$, $B_{r,m}$ and $H(m)$ are the $L \times L$ matrices given by $A_{p,i} = \widehat{A}_{p,0}^{-1}\widehat{A}_{p,i}$, $i = 0, 1, \ldots, p$; $B_{r,m} = \widehat{A}_{p,0}^{-1}\widehat{B}_{r,m}$, $m = 0, 1, \ldots, r$,

$$\begin{aligned}
\widehat{A}_{p,i} &= [\widehat{a}_{nj}], \text{ where } \widehat{a}_{nj} = \widehat{a}_{iL+n-j}(iL+n), \\
&\qquad n, j = 0, 1, \ldots, L-1, \\
\widehat{B}_{r,m} &= [\widehat{b}_{nj}], \text{ where } \widehat{b}_{nj} = \widehat{b}_{mL+n-j}(mL+n), \\
&\qquad n, j = 0, 1, \ldots, L-1, \\
H(m) &= [h_{nj}], \text{ where } h_{nj} = h(mL+n-j), \\
&\qquad n, j = 0, 1, \ldots, L-1.
\end{aligned}$$

For the non-recursive LPTV system ($A_{p,i} = I$, for $i = 0$ and $A_{p,i} = 0$, for $i \neq 0$) the block impulse response is

$$H(m) = B_{r,m}, \qquad m = 0, 1, 2, \ldots, \tag{11}$$

where $B_{r,m} = 0$, for $0 > m > r$.

For the recursive LPTV system ($B_{r,m} = I$, for $m = 0$ and $B_{r,m} = 0$, for $m \neq 0$) the block impulse response is as in (10).

A special case of LTV (or of LPTV) systems is LTI systems. From equation (7) we obtain the block impulse response of the LTI system

$$H(m) = B_{r,m} - \sum_{i=1}^{p} A_{p,i} H(m-i), \qquad m = 0, 1, 2, \ldots, \tag{12}$$

where $B_{r,m} = 0$, for $0 > m > r$. $A_{p,i}$, $B_{r,m}$ and $H(m)$ are the $L \times L$ matrices given by $A_{p,i} = \widehat{A}_{p,0}^{-1}\widehat{A}_{p,i}$, $i = 0, 1, \ldots, p$;

$B_{r,m} = \widehat{A}_{p,0}^{-1}\widehat{B}_{r,m},\ m = 0, 1, \ldots, r,$

$$\begin{aligned}
\widehat{A}_{p,i} &= [\widehat{a}_{nj}], \text{ where } \widehat{a}_{nj} = \widehat{a}_{iL+n-j}, \\
&\qquad\qquad n, j = 0, 1, \ldots, L-1, \\
\widehat{B}_{r,m} &= [\widehat{b}_{nj}], \text{ where } \widehat{b}_{nj} = \widehat{b}_{mL+n-j}, \\
&\qquad\qquad n, j = 0, 1, \ldots, L-1, \\
H(m) &= [h_{nj}], \text{ where } h_{nj} = h(mL + n - j), \\
&\qquad\qquad m = 0, 1, 2, \ldots, \\
&\qquad\qquad n, j = 0, 1, \ldots, L-1.
\end{aligned}$$

For the non-recursive LTI system ($A_{p,i} = I$, for $i = 0$ and $A_{p,i} = 0$, for $i \neq 0$) the block impulse response is

$$H(m) = B_{r,m}, \qquad m = 0, 1, 2, \ldots, \tag{13}$$

where $B_{r,m} = 0$, for $0 > m > r$.

For the recursive LTI system ($B_{r,m} = I$, for $m = 0$ and $B_{r,m} = 0$, for $m \neq 0$) the block impulse response is as in (12).

Scalar Impulse Response of LTV systems. If $L = 1$, then $n = 0$, $m = k$, $m_1 = k_1$, $p = N$, $r = M$. So from equation (5) we obtain a difference equation of the LTV system

$$\begin{aligned}
y(k) &= \sum_{i=0}^{M} b_i(k)x(k-i) - \sum_{i=1}^{N} a_i(k)y(k-i), \\
&k = 0, 1, 2, \ldots, \quad M \leqslant N,
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
b_i(k) &= \widehat{a}_0^{-1}(k)\widehat{b}_i(k), \quad i = 0, 1, \ldots, M; \\
&\qquad\qquad \widehat{a}_0(k) \neq 0, \quad k = 0, 1, 2, \ldots, \\
a_i(k) &= \widehat{a}_0^{-1}(k)\widehat{a}_i(k), \quad i = 0, 1, \ldots, N.
\end{aligned}$$

The matrix impulse response of the LTV system $H_M = [h_{kk_1}]$, where $h_{kk_1} = h(k, k_1)$ we obtain from block impulse response (7) of the LTV system

$$h(k, k_1) = b_{k-k_1}(k) - \sum_{i=1}^{N} a_i(k) h(k-i, k_1), \tag{15}$$
$$k, k_1 = 0, 1, 2, \ldots, \qquad k \geqslant k_1,$$

where $b_{k-k_1}(k) = 0$, for $k - k_1 > M$, i.e., according to the definition of impulse response of the LTV system $h(k, k_1)$ is the output of the difference equation (14) at time k to *a* unit pulse input applied k_1 samples earlier.

REMARK 2: If an input signal of the system is $x(k_1) = 1$, for $k_1 = 0$ and $x(k_1) = 0$, for $k_1 \neq 0$, then equation (14) gives Green's function $g(k, k_1) = h(k, k - k_1)$.

For the non-recursive LTV system ($a_i(k) = 1$, for $i = 0$ and $a_i(k) = 0$, for $i \neq 0$) we have scalar impulse response

$$h(k, k_1) = b_{k-k_1}(k), \quad k \geqslant k_1; \quad k, k_1 = 0, 1, 2, \ldots, \tag{16}$$

where $b_{k-k_1}(k) = 0$, for $k - k_1 > M$.

For the recursive LTV system the scalar impulse response is as in (15), where $b_{k-k_1}(k) = 1$, for $k - k_1 = 0$ and $b_{k-k_1} = 0$, for $k - k_1 \neq 0$.

The scalar impulse response of the linear frozen-time LTV system gives equation (9). For the scalar system $L = 1$, then $m = k$, $m_1 = k_1$, $p = N$, thus

$$h^*(k, k_1) = b_{k-k_1}(k_1) - \sum_{i=1}^{N} a_i(k_1) h(k - k_1, k_1), \tag{17}$$
$$k, k_1 = 0, 1, 2, \ldots, \quad k \geqslant k_1,$$

where $b_{k-k_1}(k_1) = 0$, for $k - k_1 > M$.

The scalar impulse response of the LPTV system $h(k, k_1)$ gives equation (7). As $m = k$, $m_1 = k_1$, $p = N$, so

$$h(k, k_1) = b_{k-k_1}(k) - \sum_{i=1}^{N} a_i(k)h(k-i, k_1), \qquad k = 0, 1, 2, \ldots, \quad k \geqslant k_1, \tag{18}$$

where $b_{k-k_1}(k) = 0$, for $k - k_1 > M$; $k_1 = 0, 1, \ldots, L-1$.

For the non-recursive LPTV system the scalar impulse response is

$$h(k, k_1) = b_{k-k_1}(k), \qquad k = 0, 1, 2, \ldots, \qquad k \geqslant k_1, \tag{19}$$

where $b_{k-k_1}(k) = 0$, for $k - k_1 > M$; $k_1 = 0, 1, \ldots, L-1$.

For the recursive LPTV system the scalar impulse response is

$$h(k, k_1) = b_{k-k_1}(k) - \sum_{i=1}^{N} a_i(k)h(k-i, k_1), \qquad k = 0, 1, 2, \ldots, \quad k \geqslant k_1, \tag{20}$$

where $b_{k-k_1}(k) = 1$, for $k - k_1 = 0$ and $b_{k-k_1}(k) = 0$, for $k - k_1 \neq 0$; $k_1 = 0, 1, 2, \ldots, L-1$.

A special case of the scalar impulse response of the LTV (or LPTV) system is the scalar impulse response of the LTI system. In such a case $h(k, k_1) = h(k - k_1) = h(k^*) = h(k)$. From equation (15) we have

$$h(k) = b_k - \sum_{i=1}^{N} a_i h(k-i), \qquad k = 0, 1, 2, \ldots, \tag{21}$$

where $b_k = 0$, for $k > M$.

For the non-recursive LTI system the scalar impulse response

$$h(k) = b_k, \qquad k = 0, 1, 2, \ldots, \tag{22}$$

where $b_k = 0$, for $k > M$.

For the recursive LTI system the scalar impulse response

$$h(k) = b_k - \sum_{i=1}^{N} a_i h(k-i), \qquad k = 0, 1, 2, \ldots, \tag{23}$$

where $b_k = 1$, for $k = 0$ and $b_k = 0$, for $k \neq 0$.

For all the cases discussed earlier we can form matrix impulse responses $H_M = [h_{kk_1}]$, where $h_{kk_1} = h(k, k_1)$ or matrix block impulse responses $H_B = [H_{mm_1}]$, where $H_{mm_1} = H(m, m_1)$. In the case when $L = 1$, i.e., $m = k$, $m_1 = k_1$ we have $H_B = H_M$.

Conclusions. The main aim of this paper has been to explore the theoretical relationship between scalar and block impulse responses of LTV, frozen-time LTV, LPTV and LTI systems. It has been shown that for all the cases analysed in the paper Green's function depend on the time index k_1 (m_1 – for block systems), while impulse responses depend on the time index $k - k_1$ ($m - m_1$ – for block systems). The matrices $A_{p,i}$ and $B_{r,m}$ of block LPTV systems and block LTI systems are different, however their impulse responses are the same and do not depend on the instant m_1 of the behavior of the block unit pulse input. Impulse responses of scalar LPTV, LTV and LTV with frozen-time systems are the same and depend on the instant k_1 of the behavior of the unit pulse input.

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