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THREE-LEVEL STACKELBERG STRATEGIES IN LINEAR-QUADRATIC SINGULAR SYSTEMS

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Abstract. In this paper open-loop three-level Stackelberg strategies in deterministic, sequential decision-making problems for linear continuous-time singular systems and quadratic cost function are studied. Necessary conditions under which the existence of open-loop Stackelberg strategies are derived. The analytical solution of three-level open-loop Stackelberg problem is given by means of the eigenvector method. An example is given to illustrate the proposed method.

Key words: Singular systems, Stackelberg strategy, decision-making problems.

1. Introduction. A great deal of attention has been paid to methods of design and analysis of Stackelberg strategies in multi-level sequential decision-making problems, e.g., Cruz (1978), Medanic and Radojevic (1978), Basar (1981), Ho et.al.(1982), Mahmoud and Tran (1984). During the last 20 years, there is much interest in studying the singular systems (Lunberger (1977), Cobb (1984), Bender and Laub (1987)). To the best knowledge of the authors, there are no published results for multilevel sequential decision-making problems characterized by singular systems. In section 2, multi-level sequential decision-making problems characterized by quadratic cost functions and linear time-invariant continuous singular systems are considered, and necessary conditions for the existence of open-loop Stackelberg strategies are given. In section 3, by using the eigenvector method for solving the Riccati equation, the analytical solution of three-level open-loop Stackelberg problem is given. An example is given to illustrate the proposed method in section 4.

2. Problem formulation and derivation of necessary conditions. Consider a three-level Stackelberg problem for a linear singular system

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B^1u^1(t) + B^2u^2(t) + B^3u^3(t), \\ Ex(0) &= Ex_0 \end{aligned} \tag{2.1}$$

with associated cost functional for each decision maker P_i

$$\begin{aligned} J_i(u^1, u^2, u^3) = 1/2 \int_0^T \big[x(t)'Q^i x(t) + \sum_{j=1}^{3} u^j(t)'R^{ij}u^j(t)\big]\, dt \\ + 1/2 x(T)'E'Q^i(T)Ex(T), \quad i = 1, 2, 3, \end{aligned} \tag{2.2}$$

where E is a square matrix with $\operatorname{rank}(E) = r \leqslant n$, and $\det[sE - A] \not\equiv 0$, $x(t)$ is the descriptor vector of dimension n, $u^j(t)$ is an r_j – vector function controlled by player P_j, the usual positive-(semi)definiteness conditions are imposed on Q^i, $Q^i(T)$, R^{ij}, $i, j = 1, 2, 3$, as in the associated optimal control problem.

Because of the possibility of impulses in the descriptor vector trajectory $x(t)$, the existence of the cost integral must be considered, moreover the type of integral considered also must be carefully defined. We do this in the following assumption.

ASSUMPTION 2.1. The integral (2.2) is assumed to be defined in the same way as in Bender and Laub (1987); that is, as a distributional integral. This type of integral has the property that

$$\int_0^T \| \delta(t)v \|_2 \, \mathrm{dt} < \infty \quad \text{but} \quad \int_0^T \| \delta(t)v \|_2^2 \, \mathrm{dt} = \infty,$$

where $\delta(t)v$ is the impulse function along v defined by

$$< \delta(t)v, f(t) >= f(0).$$

Thus an impulse function is integrable but its square is not.

Therefore, before the necessary conditions are derived, some conditions for the existence of (2.2) are stated.

Lemma 2.1. Existence of the cost integral (Lemma 10 of Bender and Laub (1987))

Assume $T < \infty$ in (2.2). Then if (2.1) is controllable at ∞ for any player, there exists an impulse-free control $u^j(t)$ for player P_j so that (2.2) exists and is finite.

Now let us assume that the decision-making sequence is $\{P_1, P_2, P_3\}$, that is, decision maker P_3 is the leader and selects his strategy first; P_2 is the first follower and selects his strategy secondly; and P_1 is the second follower and selects his strategy last. Consequently, in making his decision, P_1 knows the controls u^2 and u^3 of the other decision makers; P_2 knows u_3, and he knows that P_1 reacts according to declared functions u^2 and u^3; P_3 knows that P_2 reacts according to his declared control u^3, and he must take into account the reaction of P_1 to declared controls u^2 and u^3. The simplest problem is solved by P_1 (an optimal control problem); a more complex problem is solved by P_2 (a two-level Stackelberg problem); and the most complex problem is solved by P_3 (a three-level Stackelberg problem). The complete solution of the problem is obtained by the solution of the leader's control problem, since the leader must solve problems faced by both P_1 and P_2 to determine their reactions to a given u^3, in order to select that control which is best with respect to J_3, taking these reactions of the followers into account.

Therefore, in order to solve three-level Stackelberg problem, we must first determine the rational reaction of the first follower P_1 to controls u^2 and u^3 which are declared by P_2 and P_3, respectively. Since the underlying information pattern is open-loop, the optimization problem faced by P_1 is reduced to an optimal control problem defined by (2.1) and (2.2), for $i = 1$, given u^2 and u^3. By using the results of Bender and Laub (1987), the necessary conditions, under which u^1 constitutes the rational reaction to given u^2 and u^3, take the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B^1u^1(t) + B^2u^2(t) + B^3u^3(t) \\ Ex(0) &= Ex_0 \qquad (2.3a) \\ E'\dot{p}^1(t) &= -Q^1x(t) - A'p^1(t) \\ E'p^1(T) &= E'Q^1(T)Ex(T) \qquad (2.3b) \\ 0 &= R^{11}u^1(t) + B^{1'}p^1(t) \qquad (2.3c) \end{aligned}$$

Now, let us consider the problem faced by P_2. In deciding the rational reaction of the second follower P_2 to u^3, the rational reaction of P_1 to u^2 and u^3 must be taken into account. Thus what P_2 must do is to minimize the cost function (2.2) for $i = 2$ subject to (2.3). By using the standard variational techniques, the necessary conditions that characterize u^2 being the rational reaction of P_2 to u^3 take the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B^1u^1(t) + B^2u^2(t) + B^3u^3(t) \\ Ex(0) &= Ex_0 \qquad (2.4a) \end{aligned}$$

$$E'\dot{p}^1(t) = -Q^1x(t) - A'p^1(t)$$
$$E'p^1(T) = E'Q^1(T)Ex(T) \tag{2.4b}$$
$$0 = R^{11}u^1(t) + B^{1'}p^1(t) \tag{2.4c}$$
$$E'\dot{p}^2(t) = -Q^2x(t) - A'p^2(t) + Q^1n^1(t)$$
$$E'p^2(T) = E'Q^2(T)Ex(T) - E'Q^1(T)En^1(T) \tag{2.4d}$$
$$E\dot{n}^1(t) = An^1(t) - B^1m^1(t), \qquad En^1(0) = 0 \tag{2.4e}$$
$$0 = R^{21}u^1(t) + B^{1'}p^2(t) + R^{11}m^1(t) \tag{2.4f}$$
$$0 = R^{22}u^2(t) + B^{2'}p^2(t) \tag{2.4g}$$

Finally, consider the problem solved by P_3. P_3 minimizes his own function (2.2) for $i = 3$, at the same time he must take into account (2.4) which characterizes the rational reactions of P_1 and P_2 to u^3. The necessary conditions for the control u^3 to constitute the open-loop Stackelberg solution of the leader p_3 take the form

$$E\dot{x}(t) = Ax(t) + B^1u^1(t) + B^2u^2(t) + B^3u^3(t)$$
$$Ex(0) = Ex_0 \tag{2.5a}$$
$$E'\dot{p}^1(t) = -Q^1x(t) - A'p^1(t)$$
$$E'p^1(T) = E'Q^1(T)Ex(T) \tag{2.5b}$$
$$0 = R^{11}u^1(t) + B^{1'}p^1(t) \tag{2.5c}$$
$$E'\dot{p}^2(t) = -Q^2x(t) - A'p^2(t) + Q^1n^1(t)$$
$$E'p^2(T) = E'Q^2(T)Ex(T) - E'Q^1(T)En^1(T) \tag{2.5d}$$
$$E\dot{n}^1(t) = An^1(t) - B^1m^1(t), \qquad En^1(0) = 0 \tag{2.5e}$$
$$0 = R^{21}u^1(t) + B^{1'}p^2(t) + R^{11}m^1(t) \tag{2.5f}$$
$$0 = R^{22}u^2(t) + B^{2'}p^2(t) \tag{2.5g}$$
$$E'\dot{p}^3(t) = -Q^3x(t) - A'p^3(t) + Q^1n^2(t) + Q^2n^3(t)$$
$$E'p^3(T) = E'Q^3(T)Ex(T) - E'Q^1(T)En^2(T) - E'Q^2(T)En^3(T) \tag{2.5h}$$

$$E\dot{n}^2(t) = An^2(t) - B^1 m^2(t), \qquad En^2(0) = 0 \tag{2.5i}$$

$$E\dot{n}^3(t) = An^3(t) - B^2 m^3(t) - B^1 w(t),$$
$$En^3(0) = 0 \tag{2.5j}$$

$$E'\dot{p}^4(t) = -Q^1 n^3(t) - A' p^4(t),$$
$$E' p^4(T) = E' Q^1(T) E n^3(T) \tag{2.5k}$$

$$0 = R^{31} u^1(t) + B^{1'} p^3(t) + R^{11} m^2(t) + R^{21} w(t) \tag{2.5l}$$

$$0 = R^{32} u^2(t) + B^{2'} p^3(t) + R^{22} m^3(t) \tag{2.5m}$$

$$0 = B^{1'} p^4(t) + R^{11} w(t) \tag{2.5n}$$

$$0 = R^{33} u^3(t) + B^{3'} p^3(t) \tag{2.5p}$$

3. Characterization of optimal solution. For any $n \times n$ matrix E with $\text{rank}(E) = r < n$, there exist $n \times n$ nonsingular matrices U and V and $r \times r$ unit matrix I such that (e.g., Liu and Zhang (1989))

$$UEV = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \tag{3.1a}$$

Therefore, for convenience in the later derivation and without loss of the generality, let us assume that E has the form (3.1a), and A, B^j and Q^j have the corresponding form

$$\{A|B^j|Q^j\} =$$
$$= \left\{ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \middle| \begin{pmatrix} B_1^j \\ B_2^j \end{pmatrix} \middle| \begin{pmatrix} Q_{11}^j & Q_{12}^j \\ (Q_{12}^j)' & Q_{22}^j \end{pmatrix} \right\} \tag{3.1b}$$

For ease in notation, we define the following matrices

$$\bar{R}^{11} = \begin{pmatrix} 0 & A_{22} & B_2^1 \\ A_{22}' & Q_{22}^1 & 0 \\ B_2^{1'} & 0 & R^{11} \end{pmatrix}$$
$$\bar{Q}_{22}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Q_{22}^2 & 0 \\ 0 & 0 & R^{21} \end{pmatrix}, \quad \bar{B}_2^2 = \begin{pmatrix} B_2^2 \\ 0 \\ 0 \end{pmatrix} \tag{3.2a}$$

$$\bar{R}^{22} = \begin{pmatrix} 0 & \bar{R}^{11} & \bar{B}_2^2 \\ \bar{R}^{11\prime} & \bar{Q}_{22}^2 & 0 \\ \bar{B}_2^{2\prime} & 0 & R^{22} \end{pmatrix}$$

$$\bar{Q}_{22}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \tilde{Q}_{22}^3 & 0 \\ 0 & 0 & R^{32} \end{pmatrix}, \quad \bar{B}_2^3 = \begin{pmatrix} B_2^3 \\ 0 \\ 0 \end{pmatrix} \tag{3.2b}$$

$$\bar{R}^{33} = \begin{pmatrix} 0 & \bar{R}^{22} & \bar{B}_2^3 \\ \bar{R}^{22\prime} & \bar{Q}_{22}^3 & 0 \\ \bar{B}_2^{3\prime} & 0 & R^{33} \end{pmatrix},$$

$$\tilde{Q}_{22}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Q_{22}^3 & 0 \\ 0 & 0 & R^{31} \end{pmatrix} \tag{3.2c}$$

$$\begin{aligned} \bar{B}_1 &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ A_{12}\ B_1^1\ B_1^2\ B_1^3) \\ \bar{B}_2 &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ A_{12}\ B_1^1\ 0\ 0\ 0\ 0\ 0) \\ \bar{B}_3 &= (0\ A_{12}\ B_1^1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\ \bar{B}_4 &= (0\ 0\ 0\ 0\ A_{12}\ B_1^1\ B_1^2\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \end{aligned} \tag{3.2d}$$

$$\begin{aligned} \bar{S}_1 &= (A_{12}'\ Q_{12}^1\ 0\ 0\ Q_{12}^2 0\ 0\ 0\ 0\ 0\ 0\ Q_{12}^3\ 0\ 0\ 0) \\ \bar{S}_2 &= (0\ 0\ 0\ A_{21}'\ Q_{12}^1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\ \bar{S}_3 &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ A_{21}'\ Q_{12}^1\ 0\ 0\ 0) \\ \bar{S}_4 &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ A_{21}'\ Q_{12}^1\ 0\ 0\ Q_{12}^2\ 0\ 0\ 0) \end{aligned} \tag{3.2e}$$

$$\begin{aligned} \bar{u}(t)' =& (p_2^3(t)',\ -n_2^2(t)',\ m^2(t)',\ -p_2^4(t)',\ -n_2^3(t)', \\ & w(t)',\ m^3(t)',\ \bar{u}^2(t)',\ u^3(t)') \\ \bar{u}^2(t)' =& (p_2^2(t)',\ -n_2^1(t)',\ m^1(t)',\ \bar{u}^1(t)',\ u^2(t)') \\ \bar{u}^1(t)' =& (p_2^1(t)',\ x^2(t)',\ u^1(t)'). \end{aligned} \tag{3.2f}$$

Using these new notation, the necessary conditions (2.5) can be rewritten as follows:

$$\dot{x}_1(t) = A_{11}x_1(t) + \bar{B}_1\bar{u}(t), \qquad x_1(0) = x_{10} \quad (3.3a)$$

$$-\dot{n}_1^1(t) = -A_{11}n_1^1(t) + \bar{B}_2\bar{u}(t), \qquad n_1^1(0) = 0 \quad (3.3b)$$

$$-\dot{n}_1^2(t) = -A_{11}n_1^2(t) + \bar{B}_3\bar{u}(t), \qquad n_1^2(0) = 0 \quad (3.3c)$$

$$-\dot{n}_1^3(t) = -A_{11}n_1^3(t) + \bar{B}_4\bar{u}(t), \qquad n_1^3(0) = 0 \quad (3.3d)$$

$$\begin{aligned} \dot{p}_1^1(t) =& - Q_{11}^1x_1(t) - A_{11}'p_1^1(t) - \bar{S}_3\bar{u}(t) \\ & p_1^1(T) = Q_{11}^1(T)x_1(T) \end{aligned} \quad (3.3e)$$

$$\begin{aligned} \dot{p}_1^2(t) =& - Q_{11}^2x_1(t) + Q_{11}^1n_1^1(t) - A_{11}'p_1^2(t) - \bar{S}_4\bar{u}(t) \\ & p_1^2(T) = Q_{11}^2(T)x_1(T) - Q_{11}^1(T)n_1^1(T) \end{aligned} \quad (3.3f)$$

$$\begin{aligned} \dot{p}_1^3(t) =& - Q_{11}^3x_1(t) + Q_{11}^1n_1^2(t) + Q_{11}^2n_1^3(t) \\ & - A_{11}'p_1^3(t) - \bar{S}_1\bar{u}(t) \\ & p_1^3(T) = Q_{11}^3(T)x_1(T) - Q_{11}^2(T)n_1^3(T) \\ & \qquad - Q_{11}^1(T)n_1^2(T) \end{aligned} \quad (3.3g)$$

$$\begin{aligned} \dot{p}_1^4(t) =& - Q_{11}^1n_1^3(t) - A_{11}'p_1^4(t) + \bar{S}_2\bar{u}(t) \\ & p_1^4(T) = Q_{11}^1n_1^3(T) \end{aligned} \quad (3.3h)$$

$$\begin{aligned} 0 =& \bar{S}_1'x_1(t) - \bar{S}_2'n_1^1(t) - \bar{S}_3'n_1^2(t) - \bar{S}_4'n_1^3(t) \\ & + \bar{B}_3'p_1^1(t) + \bar{B}_4'p_1^2(t) + \bar{B}_1'p_1^3(t) \\ & - \bar{B}_2'p_1^4(t) + \bar{R}^{33}\bar{u}(t). \end{aligned} \quad (3.3i)$$

The system (3.3) is a singular system in its own right. Morever, the matrix of this system already has the form (3.1a). In order to solve the two point boundary value problem (3.3), it is necessary for $\bar{R}^{33}$ to be invertible. Towards this end, we shall state some sufficient conditions for the invertibility of $\bar{R}(=\bar{R}^{33})$ as follows:

ASSUMPTION 3.1.

(1) $(A_{22}\ B_2^1)$ has full row rank;

(2) $Q_{22}^i > 0,\ R^{jj} > 0,\ j = 1,2,3.$

REMARKS 3.1.

(1) From Lemma 12 of Bender and Laub (1987), we can obtain that assumptions (1), $Q_{22}^1 > 0$ and $R^{11} > 0$ are one possible set of sufficient conditions for $\bar{R}^{11}$ to be nonsingular.

(2) If A is nonsingular, then the following relations is true.

$$Z = \begin{pmatrix} 0 & A & C \\ A' & B & 0 \\ C' & 0 & D \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & C'(A')^{-1} & I \end{pmatrix} \times \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -C'(A')^{-1}BA^{-1} & 0 & I \end{pmatrix} \times \begin{pmatrix} 0 & A & C \\ A' & B & 0 \\ 0 & 0 & D + C'(A')^{-1}BA^{-1}C \end{pmatrix} \qquad (3.4)$$

Thus, we can get that Z is nonsingular if A is invertable, $B \geqslant 0$ and $D > 0$. According to the usual positive-(semi)definiteness conditions which are imposed on $Q^i,\ R^{ij},\ i,j,= 1,2,3$ and assumptions (2), we have $\bar{Q}_{22}^2 \geqslant 0,\ \tilde{Q}_{22}^3 \geqslant 0$ and $\bar{Q}_{22}^3 \geqslant 0$. So from invertibility of $\bar{R}^{11}$, $R^{22} > 0$ and $R^{33} > 0$, we can conclude that $\bar{R}^{22}$ and $\bar{R}^{33}$ are nonsingular.

When introducing the linear transformations

$$-n_1^i = N_1^i x_1, \quad i = 1,2,3 \qquad (3.5a)$$

$$p_1^j = P_1^j x_1, \quad j = 1,2,3 \qquad (3.5b)$$

$$-p_1^4 = P_1^4 x_1 \qquad (3.5c)$$

Therefore $\bar{u}(t)$ can be determined by

$$\bar{u}(t) = -\bar{R}^{-1}KZ(t,t_0)x_{10} \tag{3.6a}$$

with K being given by

$$\begin{aligned} K =& \bar{S}_1' + \bar{S}_2'N_1^1 + \bar{S}_3'N_1^2 + \bar{S}_4'N_1^3 + \bar{B}_3'P_1^1 \\ &+ \bar{B}_4'P_1^2 + \bar{B}_1'P_1^3 + \bar{B}_2'P_1^4 \end{aligned} \tag{3.6b}$$

morever, the open-loop Stackelberg controls u^1, u^2 and u^3 are the 13th, 14th and 15th subvector of $\bar{u}(t)$, respectively, where $Z(t,\ t_0)$ satisfies

$$\dot{Z}(t,t_0) = [A_{11} - \bar{B}_1\bar{R}^{-1}K]Z(t,t_0), \qquad Z(t,t) = I \tag{3.6c}$$

and the N_1^1 and P_1^j matrices are obtained from

$$\begin{aligned} \dot{N}_1^i &= A_{11}N_1^i - N_1^iA_{11} + (\bar{B}_{i+1} - N_1^i\bar{B}_1)\bar{R}^{-1}K \\ &\quad N_1^i(0) = 0, \qquad i = 1,2,3 \end{aligned} \tag{3.7a}$$

$$\begin{aligned} \dot{P}_1^1 &= -Q_{11}^1 - A_{11}'P_1^1 - P_1^1A_{11} - (\bar{S}_3 + P_1^1\bar{B}_1)\bar{R}^{-1}K \\ &\quad P_1^1(T) = Q_{11}^1(T) \end{aligned} \tag{3.7b}$$

$$\begin{aligned} \dot{P}_1^2 &= -Q_{11}^2 - Q_{11}^1N_1^i - A_{11}'P_1^2 - P_1^2A_{11} - (\bar{S}_4 + P_1^2\bar{B}_1)\bar{R}^{-1}K \\ &\quad P_1^2(T) = Q_{11}^2(T) + Q_{11}^1(T)N_1^1(T) \end{aligned} \tag{3.7c}$$

$$\begin{aligned} \dot{P}_1^3 &= -Q_{11}^3 - Q_{11}^1N_1^2 - Q_{11}^2N_1^3 - A_{11}'P_1^3 - P_1^3A_{11} \\ &\qquad - (\bar{S}_1 + P_1^3\bar{B}_1)\bar{R}^{-1}K \\ &\quad P_1^3(T) = Q_{11}^3(T) + Q_{11}^1(T)N_1^2(T) + Q_{11}^2(T)N_1^3(T) \end{aligned} \tag{3.7d}$$

$$\begin{aligned} \dot{P}_1^4 &= -Q_{11}^1N_1^3 - A_{11}'P_1^4 - P_1^4A_{11} - (\bar{S}_2 + P_1^4\bar{B}_1)\bar{R}^{-1}K \\ &\quad P_1^4(T) = Q_{11}^1(T)N_1^3(T) \end{aligned} \tag{3.7e}$$

Equations (3.7a-e) represent the Riccati equations to be solved in order to obtain the open-loop Stackelberg strategy. This is not an easy task; however, under some mild conditions described in the following derivation, by using the well-known eigenvector method for solving the Riccati equation, e.g., Abou-Kandi and Bertrand (1985), Vanghan (1969), analytical expressions for the N_1^i and P_1^j matrices may be found.

If the following matrices are defined

$$\bar{A} = \mathrm{diag}(A_{11}, A_{11}, A_{11}, A_{11}), \qquad \bar{R} = \bar{R}^{33} \tag{3.8a}$$

$$\bar{B}' = (\bar{B}_1', \ \bar{B}_2', \ \bar{B}_3,' \bar{B}_4') \tag{3.8b}$$

$$\bar{S}' = (\bar{S}_1', \ \bar{S}_2', \ \bar{S}_3', \ \bar{S}_4') \tag{3.8c}$$

$$\bar{Q} = \begin{pmatrix} Q_{11}^3 & 0 & Q_{11}^1 & Q_{11}^2 \\ 0 & 0 & 0 & Q_{11}^1 \\ Q_{11}^1 & 0 & 0 & 0 \\ Q_{11}^2 & Q_{11}^1 & 0 & 0 \end{pmatrix} \tag{3.8d}$$

then the system (3.3) can be written in the compact form

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \qquad \bar{x}(0)' = (x_{10}', \ 0, \ 0, \ 0) \tag{3.9a}$$

$$\dot{\bar{p}}(t) = -\bar{Q}(t)\bar{x}(t) - \bar{A}'\bar{p}(t) - \bar{S}\bar{u}(t)$$

$$\bar{p}(T) = \bar{Q}(T)\bar{x}(T) \tag{3.9b}$$

$$0 = \bar{S}'\bar{x}(t) + \bar{B}'\bar{p}(t) + \bar{R}\bar{u}(t) \tag{3.9c}$$

where $\bar{x}(t)$ and $\bar{p}(t)$ are defined by

$$\bar{x}(t)' = (x_1(t)', \ -n_1^1(t)', \ -n_1^2(t)', \ -n_1^3(t)') \tag{3.9d}$$

$$\bar{p}(t)' = (p_1^3(t)', \ -p_1^4(t)', \ p_1^1(t)', \ p_1^2(t)'). \tag{3.9e}$$

Solving $\bar{u}(t)$ from (3.9c) and substituting it into (3.9a-b),we can get

$$\begin{pmatrix} \dot{\bar{x}}(t) \\ \dot{\bar{p}}(t) \end{pmatrix} =$$

$$= \begin{pmatrix} \bar{A} - \bar{B}\bar{R}^{-1}\bar{S}' & -\bar{B}\bar{R}^{-1}\bar{B}' \\ -(\bar{Q} - \bar{S}\bar{R}^{-1}\bar{S}') & -(\bar{A} - \bar{B}\bar{R}^{-1}\bar{S}')' \end{pmatrix} \begin{pmatrix} \bar{x}(t) \\ \bar{p}(t) \end{pmatrix} \quad (3.10)$$

Let ζ be the remaining time before T, i.e. $\zeta = T - t$, and let $\tilde{x}(\zeta)$ and $\tilde{p}(\zeta)$ be the new variables expressed in terms of ζ. Then using (3.10) one obtains the backward canonical equation

$$\begin{pmatrix} \dot{\tilde{x}}(\zeta) \\ \dot{\tilde{p}}(\zeta) \end{pmatrix} = M \begin{pmatrix} \tilde{x}(\zeta) \\ \tilde{p}(\zeta) \end{pmatrix} \quad (3.11a)$$

with M being given by

$$M = \begin{pmatrix} -(\bar{A} - \bar{B}\bar{R}^{-1}\bar{S}' & \bar{B}\bar{R}^{-1}\bar{B}' \\ (\bar{Q} - \bar{S}\bar{R}^{-1}\bar{S}') & (\bar{A} - \bar{B}\bar{R}^{-1}\bar{S}')' \end{pmatrix} \quad (3.11b)$$

while the boundary conditions become

$$\tilde{n}_1^i(T) = 0 \quad (3.11c)$$

$$\tilde{p}(0) = \bar{Q}(T)\tilde{x}(0) \quad (3.11d)$$

with $\bar{Q}(T)$ being

$$\bar{Q}(T) = \begin{pmatrix} Q_{11}^3(T) & 0 & Q_{11}^1(T) & Q_{11}^2(T) \\ 0 & 0 & 0 & Q_{11}^1(T) \\ Q_{11}^1(T) & 0 & 0 & 0 \\ Q_{11}^2(T) & Q_{11}^1(T) & 0 & 0 \end{pmatrix} \quad (3.11e)$$

It is clear that $M \in R^{8r \times 8r}$ is a Hamilton matrix, and hence its eigenvalues must be symmetric with respect to the imaginary axis of the complex plane. It will be further assumed that the eigenvalues of M are distinct; this assumption is make for the sake of clarity in the presentation and is by no means necessary.

Let D be an $8r \times 8r$ diagonal matrix having the same eigenvalues as M and arranged in such a way that

$$D = \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} \tag{3.12a}$$

with Λ being a $4r \times 4r$ diagonal matrix with positive eigenvalues. Hence, there exists a nonsingular eigenvector matrix W, so that

$$D = W^{-1}MW. \tag{3.12b}$$

Define a new vector of variables by the transformation W:

$$\begin{pmatrix} \tilde{x} \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} \bar{W}_{11} & \bar{W}_{12} \\ \bar{W}_{31} & W_{33} \end{pmatrix} \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \end{pmatrix} \tag{3.13a}$$

with

$$\hat{q}_1' = (q_1', \tilde{q}'), \ \hat{q}_2' = (q_5', q_6', q_7', q_8'), \ \tilde{q}' = (q_2', q_3', q_4') \tag{3.13b}$$

$$\bar{W}_{11} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \ \bar{W}_{12} = \begin{pmatrix} W_{13} \\ W_{23} \end{pmatrix}$$
$$\bar{W}_{31} = (W_{31} \ W_{32}) \tag{3.13c}$$

we have

$$\begin{pmatrix} \hat{q}_1(\zeta) \\ \hat{q}_2(\zeta) \end{pmatrix} = \begin{pmatrix} \exp(\Lambda\zeta) & 0 \\ 0 & \exp(-\Lambda\zeta) \end{pmatrix} \begin{pmatrix} \hat{q}_1(0) \\ \hat{q}_2(0) \end{pmatrix}. \tag{3.14}$$

Using the boundary condition (3.11d) for $p_1^j (j = 1, 2, 3, 4)$ and the coordinate transformation (3.13a), we have

$$\bar{W}_{31}\hat{q}_1(0) + W_{33}\hat{q}_2(0) = \bar{Q}(T)[\bar{W}_{11}\hat{q}_1(0) + \bar{W}_{12}\hat{q}_2(0)]. \quad (3.15a)$$

Thus, under the assumption $(\bar{Q}(T)\bar{W}_{12} - W_{33})$ is nonsingular, $\hat{q}_2(0)$ may be expressed in terms of $\hat{q}_1(0)$.

$$\hat{q}_2(0) = F\hat{q}_1(0) \quad (3.15b)$$

with

$$F = [\bar{Q}(T)\bar{W}_{12} - W_{33}]^{-1}[\bar{W}_{31} - \bar{Q}(T)\bar{W}_{11}]. \quad (3.15c)$$

Therefore, from (3.14) for any W_{11}:

$$\hat{q}_2(\zeta) = H(\zeta)\hat{q}_1(\zeta) = H_1(\zeta)q_1(\zeta) + H_2(\zeta)\tilde{q}(\zeta) \quad (3.16a)$$

with

$$H(\zeta) = [H_1(\zeta), H_2(\zeta)] = \exp(-\Lambda\zeta)F\ \exp(-\Lambda\zeta) \quad (3.16b)$$

Now using (3.13) and (3.16) with the boundary conditions $n_1^i(T) = 0 \cdot (i = 1, 2, 3)$,

$$[W_{21}+W_{23}H_1(T)]q_1(T)+[W_{22}+W_{23}H_2(T)]\tilde{q}(T) = 0 \quad (3.17a)$$

or

$$\tilde{q}(T) = G(T)q_1(T) = -[W_{22} + W_{23}H_2(T)]^{-1} \times [W_{21} + W_{23}H_1(T)]q_1(T) \quad (3.17b)$$

assuming that the above inverse exists.

From (3.14), the following relations can be gotten

$$q_1(T) = \exp(\Lambda_1 T) q_1(0) \tag{3.18a}$$
$$\tilde{q}(0) = \exp(-\tilde{\Lambda}_1 T)\tilde{q}(T) \tag{3.18b}$$

thus, (3.17b) leads to

$$\tilde{q}(0) = L(T)q_1(0) = \exp(-\tilde{\Lambda}_1 T)G(T)\exp(\Lambda_1 T)q_1(0) \tag{3.19}$$

where $\tilde{\Lambda}_1 = \mathrm{diag}(\Lambda_2, \Lambda_3, \Lambda_4)$ with $\Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$.

Using the relation between $q_1(0)$ and $q_1(\zeta)$, one finally obtains

$$\tilde{q}(\zeta) = L(\zeta)q_1(\zeta) = \exp(\tilde{\Lambda}_1\zeta)L(T)\exp(-\Lambda_1\zeta)q_1(\zeta). \tag{3.20}$$

The vector $\hat{q}_2(\zeta)$ can be expressed in terms of $q_1(\zeta)$ only, so that

$$X_1(\zeta) = M_1(\zeta)q_1(\zeta) \tag{3.21a}$$
$$N(\zeta) = M_2(\zeta)q_1(\zeta) \tag{3.21b}$$
$$P(\zeta) = M_3(\zeta)q_1(\zeta) \tag{3.21c}$$

with

$$M_i(\zeta) = W_{i1} + W_{i2}L(\zeta) + W_{13}[H_1(\zeta) + H_2(\zeta)L(\zeta)], \quad i = 1, 2, 3 \tag{3.21d}$$
$$N(\zeta)' = [N_1^1(\zeta)',\ N_1^2(\zeta)',\ N_1^3(\zeta)'] \tag{3.21e}$$
$$P(\zeta)' = [P_1^3(\zeta)',\ P_1^4(\zeta)',\ P_1^1(\zeta)',\ P_1^2(\zeta)']. \tag{3.21f}$$

Finally, with the help of the above relations, analytical expressions of the matrices in (3.7) can be deduced

$$N(\zeta) = M_2(\zeta)M_1(\zeta)^{-1}$$
$$P(\zeta) = M_3(\zeta)M_1(\zeta)^{-1}.$$

4. Illustrative example. Let the system and the cost functions for a three-level Stackelberg problem be

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$
$$+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} u^1(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u^2(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u^3(t), \qquad x_1(0) = x_{10}$$

$$J_j = \int_0^T \{1/2x(t)'x(t) + 1/2[u^j(t)]^2\}dt + 1/2[x_1(2)]^2$$
$$j = 1, 2, 3,$$

where $x(t)' = [x_1(t)\ x_2(t)]$.

Optimality conditions lead to the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 \\ 2 & -1 & -1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

whose eigenvalues are $\pm 2.528; \pm 1.959; \pm 1.595; \pm 1.493$.

And the matrix W is given by

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1.295 & -2.193 & 1.194 & .2950 \\ -2.095 & 1.355 & -.7383 & .4772 \\ 0.6180 & -1.618 & -1.618 & .6178 \\ -.7292 & .7072 & -.8072 & 1.032 \\ 0.9442 & -1.552 & -.9630 & .3043 \\ 1.528 & .9591 & .5951 & .4926 \\ -.4508 & -1.144 & 1.306 & .6378 \\ 1 & 1 & 1 & 1 \\ -1.295 & -2.193 & 1.194 & .2950 \\ -2.095 & 1.355 & -.7383 & .4772 \\ .6180 & -1.618 & -1.618 & .6178 \\ 1.684 & -2.182 & 3.520 & -5.222 \\ -2.180 & 4.788 & 4.200 & -1.540 \\ -3.528 & -2.959 & -2.595 & -2.493 \\ 1.041 & 3.531 & -5.694 & -3.228 \end{pmatrix}$$

Proceeding as explained above, we can get

$$\begin{aligned} u^1(\zeta) =& \epsilon(\zeta)^{-1}[-4.390 - 4.174\exp(-.56887\zeta) \\ & - 4.704\exp(-.9328\zeta) - 7.930\exp(-1.493\zeta) \\ & - .0475\exp(-5.056\zeta) - 3.871\exp(-4.487\zeta) \\ & - .1310\exp(-4.123\zeta) + .4158\exp(-4.020\zeta)] \\ u^2(\zeta) =& \epsilon(\zeta)^{-1}[1.295 + 4.980\exp(-.5688\zeta) \\ & - 10.32\exp(-.9328\zeta) - 10.27\exp(-1.493\zeta) \\ & + .0140\exp(-5.056\zeta) + 4.619\exp(-4.487\zeta) \\ & - .2875\exp(-4.123\zeta) + .5385\exp(-4.020\zeta)] \\ u^3(\zeta) =& \epsilon(\zeta)^{-1}[2.095 - 3.078\exp(-.5688\zeta) \\ & + 6.381\exp(-.9328\zeta) - 16.61\exp(-1.493\zeta) \\ & - .0227\exp(-5.056\zeta) - 2.854\exp(-4.487\zeta) \\ & + .1777\exp(-4.123\zeta) + .8712\exp(-4.020\zeta)] \end{aligned}$$

where $e(\zeta)$ is given by

$$\begin{aligned} e(\zeta) = 1 &+ 2.271\exp(-.5688\zeta) + 8.643\exp(-.9328\zeta) \\ &+ 34.81\exp(-1.493\zeta) + .0108\exp(-5.056\zeta) \\ &+ 2.106\exp(-4.487\zeta) + .2407\exp(-4.123\zeta) \\ &- 1.826\exp(-4.020\zeta). \end{aligned}$$

5. Conclusion. This paper develops explicit expressions for three-level open-loop Stackelberg strategies for sequential decision making problems characterized by linear continuous-time singular system and quadratic cost function. By using the eigenvector method, the Riccati equations which come from the necessary conditions of the existence of three-level open-loop Stackelberg strategies are solved. The main advantage of the proposed method is to replace a very difficult numerical integration problem which results from solving the Riccati equations. The results of the note can be straightforward extended to multilevel Stackelberg problems. But the burden of computing multilevel open-loop Stackelberg strategies will be heavy increased, so is the time of computation.

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