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COMPARISON BETWEEN SOLUTIONS OF THE MAXWELL SYSTEM OF EQUATIONS AND ITS QUASI-STATIONARY APPROXIMATION

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Abstract. Models for determining the electromagnetic fields are considered. The models consist of system of the Maxwell equations in the complete form and in the quasi-stationary approximation. Using the quasi-stationary approximation in media containing nonconducting subdomains is noncorrect from the physical point of view and give rise to a number of additional mathematical problems. The solutions of the Maxwell equations in the complete form and in the quasi-stationary approximation are compared. Initial boundary value problems are considered for conducting, nonconducting and mixed media. The conditions ensuring the closeness of solutions are established. The estimates are obtained in terms of input data of the problem. In particular, it has been proved that as the ratio of the characteristic rate to the light velocity tends to zero the strength of electric field in the conducting part and the strength of magnetic field in the entire domain, corresponding to the complete problem, converge to the ones corresponding to the approximate problem.

Key words: Maxwell equations, quasi-stationary approximation.

1. Introduction. This paper deals with mathematical modeling of electromagnetic fields in media with different electroconductivities.

As is known (Tamm, 1966; Kulikovsky and Lyubimov, 1962), the problem of determining electromagnetic fields consists in solving the Maxwell equations:

$$\operatorname{rot} \vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \qquad \operatorname{rot} \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}, \tag{1.1}$$
$$\operatorname{div} \vec{E} = 4\pi\rho_e, \qquad \operatorname{div} \vec{H} = 0$$

jointly with the continuity equation and the Ohm relation:

$$\frac{\partial \rho_e}{\partial t} + \operatorname{div} \vec{j} = 0, \qquad \vec{j} = \sigma \vec{E} \tag{1.2}$$

and appropriate initial and boundary conditions.

The system of equations (1.1) and (1.2) is considered in the limited region $G, t > 0$. The notations are standard: $\vec{r} = (x, y, z)$ is the radius vector; t is the time; $\vec{E}$ and $\vec{H}$ are the strengths of electric and magnetic fields, respectively; $\vec{j}$ is the density of electric field; σ is the electric conduction; ρ_e is the electric charge density; c is the light velocity. We assume that $\sigma = \sigma(r)$. The system of equations is written in dimensional units.

In some cases the description of fields may by considerably simplified (Kulikovsky and Lyubimov, 1962) by using a quasistationary approximation of the Maxwell equations. It may be done under the following conditions:

$$1/\sigma_{00}t_0 << 1\,, \tag{1.3}$$

$$\mu^2 = (x_0/ct_0)^2 << 1. \tag{1.4}$$

Here x_0 and t_0 are the characteristic scale and time of the field changes, σ_{00} is the characteristic value of conductivity,

μ is a parameter. The condition (1.3) means high electric conduction of the medium, while (1.4) – the smallness of the characteristic rate of process, compared to the light velocity. If the conditions (1.3) and (1.4) are valid, we may ignore the displacement current $\frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ in comparison with the conductivity current $\vec{j}$, and the medium may be assumed quasi-neutral, $\rho_e = 0$. In this case the electric field energy proves to be low as compared with the magnetic energy $\vec{E}^2 << \vec{H}^2$.

By analogy with Samarskii (1980, p.230) we introduce dimensionless values in (1.1) and (1.2) and use the same letters for their designation like in the dimensional case. Then the system of Maxwell equations in the quasi-stationary approximation has the form:

$$\begin{aligned} &\text{rot}\,\vec{H} = 4\pi\sigma\vec{E}, \qquad \text{rot}\,\vec{E} = -\frac{\partial \vec{H}}{\partial t}, \\ &\text{div}\,\vec{H} = 0, \qquad \vec{j} = \sigma\vec{E}. \end{aligned} \tag{1.5}$$

The condition (1.3) for applicability of the quasi-stationary approximation (1.5) is rewritten in terms of dimensionless values (including σ_{00}) in the form:

$$\mu^2/\sigma_{00} << 1. \tag{1.6}$$

The condition (1.4) remains unchanged.

The inequality (1.6) obviously means that $\sigma \neq 0$. However, for applications a typical situation is when the region under investigation contains subregions with sharply nonhomogeneous electrophysical properties (for example, conductors and dielectrics). In this case the condition (1.4) remains valid, while the condition (1.6) holds only in the conducting part. In such problems the fields in the conducting part are of main interest. It is clear that from the formal point of view the quasi-stationary approximation cannot be used in this case, and the estimates (Kulikovsky and Lyubimov, 1962) following from (1.3) and (1.4) are generally not legitimate.

Often we have to calculate the fields numerically. Then it is desirable to have a model which homogeneously describes the fields in every subregion to provide through computations. Using the complete system of Maxwell equations under the condition (1.4) leads to a necessity of solving the problem with a very small time step. In fact, we have to follow in this case the propagation of electromagnetic wave in a dielectric while basic attention is usually paid to processes in a conducting part. A model with the complete system of the Maxwell equations in the dielectric and their quasi-stationary approximation in the conductor will be nonhomogeneous. To solve such a problem numerically in the spatial multidimensional case is very difficult. Therefore, it seems attractive to use the quasi-stationary approximation of the Maxwell equations (1.5) for describing the fields in the whole region.

However, it is not clear apriory how strongly the field will be misrepresented in the conductor as compared with a complete model and whether the field in the dielectric will correspond to the full description. It should be noted that in (Kulikovsky and Lyubimov, 1962) the estimates of the solution obtained in a homogeneous conducting medium have a physical nature.

In this paper a comparison was carried out between solutions obtained for the complete system of Maxwell equations and its quasi-stationary approximation, including the case of a medium with sharply nonhomogeneous electrophysical properties, i.e., the medium consisting of a conducting and a nonconducting parts. The comparison means deriving the estimates of difference norms for the solution of complete Maxwell equations and the quasi-stationary approximation (1.5).

In the dimensionless form the first equation in the system (1.1) is

$$\text{rot}\,\vec{H} - \mu^2 \frac{\partial \vec{E}}{\partial t} = 4\pi\sigma \vec{E}. \tag{1.7}$$

The rest equations of (1.1) in the dimensionless form are obvious.

We shall further compare the solutions of the complete system and its approximation (1.5) by introducing some additions to provide uniqueness of the solution (Galanin, Poveshenko and Popov, 1988).

We consider here the initial-boundary problems of determining electromagnetic fields in conducting and mixed media. The estimates are obtained in terms of the problems input data for differences between solutions corresponding to the complete and approximate systems. These estimates may be utillized in different ways. Formally, they do not depend on how small or great the involved parameters are. We are interested in them mainly from the following point of view: do the solutions of the complete and approximate systems converge to each other in any sense as μ tends to zero. By means of these estimates the conditions have been found out for ensuring a homogeneous (in time) vicinity of the solutions in the conducting medium, their tendency to zero with growing time, etc. In particular, it is shown that in the case of a nonhomogeneous medium with sharply varying electric conduction the solutions of complete and approximate systems also converge each other in a certain sense as μ (a ratio of the characteristic process rate to the light velocity) tends to zero. The convergence occurs in the magnetic field strength in the whole medium and in the electric field strength only in the conducting part of the medium. Hence, it was proved that the quasi-stationary approximation can be applied in the mixed media too.

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2. Formulation of the problem on a comparison of solution. We consider the limited domain G. Let $\vec{E}_1$ and $\vec{H}_1$ be the solutions of the complete system of the Maxwell equations in the dimensionless form:

$$\operatorname{rot} \vec{H}_1 - \mu^2 \frac{\partial \vec{E}_1}{\partial t} = 4\pi\sigma \vec{E}_1 \qquad \vec{r} \in G_1,\ t > 0 \qquad (2.1)$$
$$\operatorname{rot} \vec{E}_1 = -\frac{\partial \vec{H}_1}{\partial t}$$

with additional conditions

$$\vec{E}_1\big|_{t=0} = \vec{E}_0(\vec{r}),\ \vec{H}_1\big|_{t=0} = \vec{H}_0(\vec{r}),\ \vec{E}_{1,\tau}\big|_{\partial G} = 0. \qquad (2.2)$$

Here and below dG is G boundary, the indices τ and n designate the components tangential and normal to dG. We denote $G = G_1 \cup G_2$, where

$$G_1 = \{\vec{r} \in G_1 : \sigma > 0\}, \qquad G_2 = \{\vec{r} \in G : \sigma = 0\}.$$

We assume that $\sigma = \sigma(r)$; G_1 and G_2 are the boundaries of G_1 and G_2, respectively; $\partial G_{12} = \partial G_1 \cap G_2$. We also assume ∂G_2 to be connected and domain G_2 to be singly connected. The subdomain G_1 is a conductor, and the subdomain G_2 is a dielectric. Further on the variants will be possible when G_1 or G_2 are empty, i.e., $G = G_2$ or $G = G_1$.

We shall assume that the initial data (2.2) satisfy the conditions

$$\begin{aligned} &\operatorname{div} \vec{H}_0 = 0 \qquad \text{in } G_1 \\ &\operatorname{div} \vec{E}_0 = 0 \qquad \text{in } G_2 \end{aligned} \qquad (2.3)$$

and, as a rule, satisfy the agreement conditions

$$\operatorname{rot} \vec{H}_0 = 4\pi\sigma \vec{E}_0 \qquad \text{in } G \qquad (2.4)$$

Note that the first condition (2.3) provide satisfaction of the relation div $\vec{H}_1 = 0$ at all times. Therefore in (2.1) this equation was omitted.

Along with (2.1) and (2.2) we consider the problem where the strengths $\vec{E}_2$ and $\vec{H}_2$ are described by the system of Maxwell equations in the quasi-stationary approximation:

$$\begin{aligned} \operatorname{rot} \vec{H}_2 &= 4\pi\sigma \vec{E}_2 \qquad \vec{r} \in G,\ t > 0 \\ \operatorname{rot} \vec{E}_2 &= -\frac{\partial \vec{H}_2}{\partial t} \end{aligned} \tag{2.5}$$

with the additional gauge equation (Galanin, Poveshenko, Popov, 1988):

$$\operatorname{div} \vec{E}_2 = 0 \qquad \text{in } G_2 \tag{2.6}$$

and with initial and boundary conditions

$$\vec{H}_2\big|_{t=0} = \vec{H}_0(\vec{r}),\ \ \vec{E}_{2,\tau}\big|_{\partial G} = 0. \tag{2.7}$$

The initial data for $\vec{E}_2$ coincide with $\vec{E}_0$ generally only in G_1 and only when the condition (2.4) are fulfilled if no additional constraints are not imposed on $\vec{E}_0$.

If the agreement condition (2.4) are fulfilled it immediately follow from (2.1) that div $\vec{E}_1 = 0$ in G_2. Hence, it appears that the solution of (2.1) passes at the limit $\mu \to 0$ into the solution of (2.5).

We shall be interested in estimating the norms of solution differences for the problems (2.1) and (2.5) through the input data. These estimates must answer also the question how the solutions of the problems converge to each other when $\mu \to 0$.

It should be noted that in (Kavashima and Shizuta, 1986) a similar question was raised about the convergence of solutions obtained in the problem on the motion of a conducting fluid, where the electromagnetic part was described by the

Maxwell equations in the complete form and in the magnetohydrodynamic approximation. However, numerous assumptions made in Kavashima and Shizuta (1986) do not allow us to use the results. Besides, the medium in (Kavashima and Shizuta, 1986) was conducting. Also we would like to mention the publication of Aleksandrov and Dmitrijev (1975), where the convergence of solutions was consider for a complete system of Maxwell equations with electric conduction tending to infinity.

The problem (2.1) and (2.2) with $\mu \to 0$ is singularly perturbed and has a small parameter at the leading (time) derivative. To do thoroughly the task of comparing the solutions of (2.1) and (2.5) one should construct an expansion of the solution to (2.1) and (2.2) with respect to μ in the manner of Su Yui-Chan (1961). Here we restrict ourselves by obtaining estimates of the differences $\vec{E} = \vec{E}_1 - \vec{E}_2$ and $\vec{H} = \vec{H}_1 - \vec{H}_2$ through the input data. The estimates will give an answer to the question about a convergence of the solutions as $\mu \to 0$. Along with this we shall consider the cases of homogeneous conducting and nonconducting mediums.

We shall assume that everywhere the solutions exist and have a desirable smoothness. Differential properties of the solutions in the problems similar to (2.1)–(2.7) are discussed by Ladyzhenskaya and Solonnikov (1960); Sakhaev (1976).

Let us write down the problem to estimate the solution difference. The fields $\vec{E}$ and $\vec{H}$ are described by the equations:

$$\begin{aligned}
&\operatorname{rot}\vec{H} - \mu^2\frac{\partial \vec{E}_1}{\partial t} = 4\pi\sigma\vec{E} \qquad \vec{r} \in G,\ t > 0 \\
&\operatorname{rot}\vec{E} = -\frac{\partial \vec{H}}{\partial t} \\
&\vec{H}|_{t=0} = 0, \qquad \vec{E}_\tau|_{\partial G} = 0.
\end{aligned} \tag{2.8}$$

The second condition (2.3), the first equation (2.1) and equation (2.6) also give

$$\operatorname{div} \vec{E} = 0 \qquad \text{in } G_2 \tag{2.9}$$

Sometimes, we shall need the problems in terms of $\vec{E}$ or $\vec{E}_1$ and $\vec{E}_2$ only. We shall write them down when necessary.

For further use we shall give here the balance equations for different energies. It is easy to see that in the problem (2.1) with complete equations the energy balance in the system may be written in the form (the designations are given in ch. 3):

$$\frac{1}{8\pi}(||\vec{H}_1||_G^2 + \mu^2||\vec{E}_1||_G^2) + ||\sqrt{\sigma}\vec{E}_1||_{\Omega_t}^2$$
$$= \frac{1}{8\pi}(||\vec{H}_0||_G^2 + \mu^2||\vec{E}_0||_G^2). \tag{2.10}$$

For the problem (2.5) we have the relation

$$\frac{1}{8\pi}||\vec{H}_2||_G^2 + ||\sqrt{\sigma}\vec{E}_2||_{\Omega_t}^2 = \frac{1}{8\pi}||\vec{H}_0||_G^2. \tag{2.11}$$

The first term in the left hand side of (2.10) and (2.11) are the electromagnetic energy in the system, second term – the power of energy release as the Joul heat, in the right hand side we have an initial energy.

For the energy difference we derive

$$\frac{1}{8\pi}(||\vec{H}_1||_G^2 + \mu^2||\vec{E}_1||_G^2 - ||\vec{H}_2||_G^2)$$
$$+ \int_0^t \int_G \sigma(\vec{E}, \vec{E}_1 + \vec{E}_2)\, dV dt = \frac{\mu^2}{8\pi}||\vec{E}_0||_G^2. \tag{2.12}$$

As was expected, good agreement between energies of the electromagnetic fields, corresponding to the complete and approximate cases, is determined by the fact that $\vec{E}_1$ and $\vec{E}_2$ are close to each other in the conducting part G_1, bounded in G_1 and the value of $\overrightarrow{E}_0$ is also bounded. The boundedness of $||\sqrt{\sigma}\overrightarrow{E}_1||_{\Omega_t}$, $||\sqrt{\sigma}\overrightarrow{E}_2||_{\Omega_t}$ ($\sim O(1)$) follows from (2.10) and (2.11).

Estimates of the solutions will be obtained by the functional-analytical method of investigating the problem (2.1), (2.5) and (2.8). We shall use the technique employed by Galanin (1990a) for restoring the function by its rotor and divergence as well as the imbedding type inequality connecting the norms, the rotor and divergence of the vector function, namely,

$$||\overrightarrow{U}||^2 \leqslant c\left(||\text{rot}\ \overrightarrow{U}||^2 + ||\text{div}\ \overrightarrow{U}||^2\right). \qquad (2.13)$$

This inequality is valid for the vector function $\overrightarrow{U}$ with zero tangential or normal components at the boundary, where the rotor and the divergence are quadratically summable. In (2.13) c is a positive constant depending on the domain and its boundary only. Everywhere below c will designate constants, while the light velocity will be used only through μ. As for (2.13), the reader is referred to Galanin (1990a) and its references.

The calculations omitted here may be found in Galanin (1990b).

3. The case of the conducting medium. We consider the case $G = G_1$, $\sigma \geqslant \sigma_0 > 0$, and introduce the designation $\Omega_t = G \times [0, t]$, $\Omega_{i,t} = G_i \times [0, t]$, $i = 1, 2$. All the norms encountered below have the sense of integral norms in $L^2(G), L^2(\Omega_t)$ (as well as $L^2(G_i)$, $L^2(\Omega_{i,t})$, $i = 1, 2$). For simplicity we shall omit the symbol L^2 and will use instead the sign of the set in which the norm is calculated.

1. Let us consider the problem (2.8). We multiply equations (2.8) by $\overrightarrow{E}$ and $\overrightarrow{H}$, respectively, and make summation.

After the integration with respect to Ω_t we obtain, with involvement of initial and boundary conditions, the equation

$$\frac{1}{8\pi}||\vec{H}||^2_G + ||\sqrt{\sigma}\vec{E}||^2_{\Omega_t} + \frac{1}{4\pi}\mu^2 \int_0^t \int_G \left(\frac{\partial \vec{E}_1}{\partial t}, \vec{E}\right) dtdV = 0. \tag{3.1}$$

We transfer the third term in (3.1) to the right hand side, calculate the upper bound by the Canchy-Bunyakovsky inequality and the ε-inequality. As a result we obtain

$$\frac{1}{8\pi}||\vec{H}||^2_G + \frac{1}{2}||\sqrt{\sigma}\vec{E}||^2_{\Omega_t} \leqslant \frac{\mu^4}{32\pi^2}\left\|\frac{1}{\sqrt{\sigma}}\frac{\partial \vec{E}_1}{\partial t}\right\|^2_{\Omega_t}. \tag{3.2}$$

To obtain final estimates in terms of input data we must estimate the right hand side of (3.2)

2. We write down the problem for $\vec{E}_1$by assuming that the initial data are consistent with (2.4). Then

$$\begin{aligned} &\mu^2\frac{\partial^2 \vec{E}_1}{\partial t^2} + 4\pi\sigma\frac{\partial \vec{E}_1}{\partial t} + \operatorname{rot}\operatorname{rot}\vec{E}_1 = 0 \ \vec{r} \in G, \ t > 0 \\ &\vec{E}_1|_{t=0} = \vec{E}_0(\vec{r}), \ \left.\frac{\partial \vec{E}_1}{\partial t}\right|_{t=0} = 0, \quad \vec{E}_{1,\tau}|_{\partial G} = 0. \end{aligned} \tag{3.3}$$

We multiply (3.3) by $\frac{\partial \vec{E}_1}{\partial t}$ and integrate the result. We obtain (with involvement of all input data)

$$\frac{1}{8\pi}\mu^2\left\|\frac{\partial \vec{E}_1}{\partial t}\right\|^2_G + \left\|\sqrt{\sigma}\frac{\partial \vec{E}_1}{\partial t}\right\|^2_{\Omega_t} + \frac{1}{8\pi}||\operatorname{rot}\vec{E}_1||^2_G = \frac{1}{8\pi}||\operatorname{rot}\vec{E}_0||^2_G. \tag{3.4}$$

Proceeding from this and using the boundedness of below we obtain the estimate (see Galanin, 1990b):

$$\left\|\frac{1}{\sqrt{\sigma}}\frac{\partial \vec{E}_1}{\partial t}\right\|^2_{\Omega_t} \leqslant \frac{1}{8\pi\sigma_0^2}\Big(1-\exp\Big(-\frac{8\pi\sigma_0}{\mu^2}t\Big)\Big)\|\mathrm{rot}\,\vec{E}_0\|^2_G.$$

Finally we have the inequality

$$\frac{1}{8\pi}\|\vec{H}\|^2_G + \frac{1}{2}\|\sqrt{\sigma}\vec{E}\|^2_{\Omega_t} \leqslant \frac{\mu^4}{256\pi^3\sigma_0^2}\Big(1-\exp\Big(-\frac{8\pi\sigma_0}{\mu^2}t\Big)\Big)\|\mathrm{rot}\,\vec{E}_0\|^2_G. \quad (3.5)$$

Unlike (3.2) we have now in the right hand side a completely determined expression. The estimate (3.5) shows that as μ tends to zero $\|\vec{H}\|_G \to 0$ for all times as well as $\|\vec{E}\|_{\Omega_t} \to 0$ too.

3. The inequality (3.5) testifies to the absence of boundary layer of $\vec{H}$ near $t = 0$. It is possible that when the agreement conditions (2.4) are satisfied the boundary layer of $\vec{E}$ near $t = 0$ is absent too. Let us study this situation.

We differentiate (2.8) with respect to t and make the same calculations as in p.1, but only for the time variables. As a result, we obtain an analog of (3.2):

$$\frac{1}{8\pi}\left\|\frac{\partial \vec{H}}{\partial t}\right\|^2_G + \frac{1}{2}\left\|\sqrt{\sigma}\frac{\partial \vec{E}}{\partial t}\right\|^2_{\Omega_t} \leqslant \frac{\mu^4}{32\pi^2}\left\|\frac{1}{\sqrt{\sigma}}\frac{\partial^2 \vec{E}_1}{\partial t^2}\right\|^2_{\Omega_t}. \quad (3.6)$$

Then we differentiate (3.3) with respect to t. Instead of (3.4) we obtain the identity:

$$\frac{\mu^2}{8\pi}\left\|\frac{\partial^2 \vec{E}_1}{\partial t^2}\right\|^2_G + \left\|\sqrt{\sigma}\frac{\partial^2 \vec{E}_1}{\partial t^2}\right\|^2_{\Omega_t} + \frac{1}{8\pi}\left\|\mathrm{rot}\,\frac{\partial \vec{E}_1}{\partial t}\right\|^2_G = \frac{1}{8\pi\mu^2}\|\mathrm{rot}\,\mathrm{rot}\,\vec{E}_0\|^2_G. \quad (3.7)$$

By analogy with p.1 we have

$$\Big\|\frac{1}{\sqrt{\sigma}}\frac{\partial^2\overrightarrow{E}_1}{\partial t^2}\Big\|^2_{\Omega_t} \leqslant \frac{1}{8\pi\mu^2\sigma_0}\Big(1-\exp\Big(-\frac{8\pi\sigma_0}{\mu^2}t\Big)\Big)\|\text{rot}\,\text{rot}\,\overrightarrow{E}_0\|^2_G. \quad (3.8)$$

The result related to (3.6)forms the inequality

$$\frac{1}{8\pi}\|\text{rot}\,\overrightarrow{E}\|^2_G+\frac{1}{2}\Big\|\sqrt{\sigma}\frac{\partial\overrightarrow{E}}{\partial t}\Big\|^2_{\Omega_t} \leqslant \frac{\mu^2}{256\pi^3\sigma_0^2}\Big(1-\exp\Big(-\frac{8\pi\sigma_0}{\mu^2}t\Big)\Big)\|\text{rot}\,\text{rot}\,\overrightarrow{E}_0\|^2_G. \quad (3.9)$$

This estimate testified to the fact that $\|\text{rot}\,\overrightarrow{E}\|_G$ uniformly (in t) tends to zero and $\|\frac{\partial\overrightarrow{E}}{\partial t}\|_{\Omega_t}$ also tends to zero as $\mu\to 0$.

4. From the inequalities (3.5) and (3.9) we obtain the following results.

Proceeding from the above estimates we have

$$\|\sqrt{\sigma}\overrightarrow{E}\|^2_G = 2\int_0^t\int_G\Big(\frac{\partial E}{\partial t},\overrightarrow{E}\Big)\sigma dV dt \leqslant 2\Big\|\sqrt{\sigma}\frac{\partial\overrightarrow{E}}{\partial t}\Big\|_{\Omega_t}\|\sqrt{\sigma}\overrightarrow{E}\|_{\Omega_t}.$$

In this manner, by combining (3.5) and (3.9) we obtain

$$\|\sqrt{\sigma}\overrightarrow{E}\|^2_G \leqslant \frac{\mu^3}{64\pi^3\sigma_0^2}\Big(1-\exp\Big(-\frac{8\pi\sigma_0}{\mu^2}t\Big)\Big) \times \|\text{rot}\,\overrightarrow{E}_0\|_G\|\text{rot}\,\text{rot}\,\overrightarrow{E}_0\|_G. \quad (3.10)$$

This expression is a uniform (in t) estimate of the integral norm of the solution difference. It testifies to the absence of the boundary layer near $t = 0$.

Note that, for example in the case $\sigma = \sigma_0 = \text{const} > 0$ we can immediately obtain from (2.1)–(2.7) that $\operatorname{div} \vec{E} = 0$. In this manner we may estimate $\| \vec{E} \|_G^2$ also from the inequalities (3.9) and (2.13). However, we can obtain only $\| \vec{E} \|_G^2 \simeq O(\mu^2)$. The estimate (3.10) is stronger.

We consider now the case when electric conduction depends on $\vec{r}$. Let us have the condition

$$\frac{1}{\sigma}|\operatorname{grad} \sigma| \leqslant \nu < +\infty \qquad (3.11)$$

which is satisfied in this case. Then it is not difficult to obtain an estimate for $\|\operatorname{div} \vec{E}\|_G^2$. As a result of the calculations given in (Galanin, 1990b) we obtain that $\|\operatorname{div} \vec{E}\|_G^2 \to 0$ when $\mu \to 0$, like $O(\mu^2)$ tends to zero, i.e., in the same manner like $\|\operatorname{rot} \vec{E}\|_G^2$ in (3.9). Finally, according to (Plotnitsky, 1976) we have that $\|\operatorname{grad} \vec{E}\|_G^2 = \sum\limits_{i=1}^{3} \|\operatorname{grad} \vec{E}_i\|_G^2 = O(\mu^2)$. Hence, all the derivatives of the solution also tend to zero when $\mu \to 0$.

5. We consider the estimates $\| \vec{E}_1 \|_{\Omega_t}$ and $\left\| \frac{\partial \vec{E}_1}{\partial t} \right\|_{\Omega_t}$ following from (2.10) and (3.4). They testify to the t-uniform boundedness of norms in Ω_t, including the case when $t \to +\infty$. Hence, for $t \to +\infty$ we have $\| \vec{E}_1 \|_G \to 0$. The proof is similar to that given by Ladyzhenskaya and Solonnikov (1960 p.172).

In the same manner it follows from the estimates (3.5) and (3.9) that $\| \vec{E} \|_G$ tends to zero as $t \to +\infty$. And hence $\| \vec{E}_2 \|_G \to 0$ too.

6. Let the agreement conditions for initial data (2.4) are not fulfilled. What is changed in this case?

Obviously, the identity (3.1) and the inequality (3.2) remain valid. The summand $\frac{1}{8\pi\mu^2}\|4\pi\sigma\vec{E}_0 - \text{rot}\,\vec{H}_0\|_G^2$ will be added to the right hand side of (3.4). As a result, instead of the estimate $O(\mu^4)$ in (3.5) we obtain only $O(\mu^2)$ at disagreement of initial data in the analog (3.5). Nevertheless, in this case we also obtain the t-uniform estimates of $\|\vec{H}\|_G$ and $\|\sqrt{\sigma}\vec{E}\|_{\Omega_t}$. If we repeat our calculations further on, we shall obtain an inequality of the form (3.6) with the additional term $\frac{1}{8\pi}\left\|\text{rot}\left(\vec{E}_0 - \frac{1}{4\pi\sigma}\text{rot}\,\vec{H}_0\right)\right\|_G^2$ in the right hand side. Then instead of (2.7) we have the identity with the right hand side

$$\frac{1}{8\pi\mu^2}\|\text{rot}\,\text{rot}\,\vec{E}_0 + \frac{4\pi\sigma}{\mu^2}(\text{rot}\,\vec{H}_0 - 4\pi\sigma\vec{E}_0)\|_G^2$$
$$+\frac{1}{8\pi\mu^4}\|\text{rot}\,(\text{rot}\,\vec{H}_0 - 4\pi\sigma\vec{E}_0)\|_G^2.$$

Hence, instead of the estimate (3.8) we obtain only the estimate $O(\mu^{-6})$. In the right hand side of the analog (3.9) we shall have $O(\mu^{-2})$. The estimate (3.10) cannot be obtained. In the right hand side of the inequality we shall have only $O(1)$.

7. Proceeding from (2.10) and (2.11), the boundedness of $\|\sqrt{\sigma}\vec{E}_1\|_{\Omega_t}$ and $\|\sqrt{\sigma}\vec{E}_2\|_{\Omega_t}$ following from (2.10) and (2.11), the boundedness of $\|\vec{E}_0\|_G$, the estimate (3.5) we obtain that the energies of electromagnetic field described by complete and approximate equations will differ by $O(\mu^2)$ in the case of agreement in initial data and by $O(\mu)$ in the case of their disagreement.

4. The case of a nonconducting medium. It is obvious that at $G = G_2$, i.e., for the dielectric type medium, the quasi-stationary approximation (2.5)–(2.7) has nothing to do with solution of the complete equations. Let us illustrate this situation by means of estimates.

The solution of the problem (2.5)–(2.7) may easily by obtained. From the first equation of (2.6) it follows that rot $\vec{H}_2 = 0$, from the second equation rot rot $\vec{E}_2 = 0$. The latter in combination with (2.6) and (2.7) gives

$$\vec{E}_2 = 0, \quad \vec{H}_2 = \vec{H}_0 \tag{4.1}$$

For the solution difference, instead of (2.8) we have

$$\begin{gathered}
\text{rot}\,\vec{H} - \mu^2\frac{\partial \vec{E}}{\partial t} = 0,\ \text{rot}\,\vec{E} = -\frac{\partial H}{\partial t},\ \vec{r} \in G,\ t > 0 \\
\text{div}\,\vec{E} = 0 \\
\vec{H}|_{t=0} = 0,\ \vec{E}|_{t=0} = \vec{E}_0,\ \vec{E}_\tau|_{\partial G} = 0
\end{gathered} \tag{4.2}$$

From this conditions we obtain the identity

$$\frac{1}{2}\|\vec{H}\|_G^2 + \frac{1}{2}\mu^2\|\vec{E}\|_G^2 = \frac{1}{2}\mu^2\|\vec{E}_0\|_G^2. \tag{4.3}$$

After differentiation of (4.2) with respect to t we shall get an analog of (4.3) for the derivatives, i.e.

$$\frac{1}{2}\left\|\frac{\partial \vec{H}}{\partial t}\right\|_G^2 + \frac{1}{2}\mu^2\left\|\frac{\partial \vec{E}}{\partial t}\right\|_G^2 = \frac{1}{2}\|\text{rot}\,\vec{E}_0\|_G^2. \tag{4.4}$$

We shall further use the imbedding inequality (2.13) valid for the vector field with zero tangential components (in this case $\vec{E}$ or a zero normal component (in this case, $\vec{H}$; see about the normal component $\vec{H}$ (Duvault and Lions, 1980 p.330) on the surface. Thus, we have

$$\begin{aligned}
\|\vec{H}\|_G^2 &\leqslant c(\|\text{rot}\,\vec{H}\|_G^2 + \|\text{div}\,\vec{H}|_G^2) \\
&= c\|\text{rot}\,\vec{H}\|_G^2 = c\mu^4\left\|\frac{\partial \vec{E}}{\partial t}\right\|_G^2
\end{aligned}$$

$$\left\|\frac{\partial \vec{H}}{\partial t}\right\|_G^2 = \|\text{rot}\,\vec{E}\|_G^2 = \|\text{rot}\,\vec{E}\|_G^2 + \|\text{div}\,\vec{E}\|_G^2 \geqslant \frac{1}{c}\|\vec{E}\|_G^2.$$

Hence, from (4.3) and (4.4) we obtain the estimation from above and below.

$$c\|\mathrm{rot}\,\vec{E}_0\|_G^2 \geqslant c\mu^2\left\|\frac{\partial\vec{E}}{\partial t}\right\|_G^2 + \|\vec{E}\|_G^2 \geqslant \|\vec{E}_0\|_G^2. \qquad (4.5)$$

The inequality (4.5) testifies to the fact that the strength $\vec{E}$ is not stabilized and does not tend to zero when $\mu \to 0$. It means that sometimes the $\|\vec{E}\|_G$ vanishes or is very small, but the solution can be taken out of this state because its change rate differs from zero. in other words, the solution behaves as an oscillatory function without damping.

Nevertheless, as it follows from (4.3) when $\mu \to 0$ we have $\|\vec{H}\|_G \to 0$ in contrast to $\vec{E}$. We have also convergence of the field energies as it follows from (2.12).

The estimate obtained completely corresponds to a physical picture of the phenomenon described by (2.1)–(2.4) at $G = G_2$. In this case there is excitation of electromagnetic waves in the domain G. Due to absence of damping, the wave energy is maintained, its distribution occurring between the electric and magnetic fields. It is obvious that such a field has nothing to do with the solution (4.1).

4. Medium with sharply nonhomogeneous electrophysical properties. Let us consider a general case of the medium consisting of two subdomain (the conductor G_1 and the dielectric G_2) with distinctly different electrophysical properties. The problem of comparison if the solutions has been formulated in ch. 2. We should only add the condition $0 < t < \tilde{t}_0 < +\infty$, i.e., consider the problems given in ch. 2 over a limited time interval.

We shall assume that everywhere in G_1 the condition (3.11) is fulfilled and $0 < \sigma_0 < \sigma < \sigma_0^* < +\infty$. The condition (2.4) for the initial data agreement is also satisfied.

1. It is obvious that in the case under consideration the identity (3.1) remains valid. We divide the integral term into two part: in G_1 and G_2. The integral over G_1 is estimated in the same manner like in ch. 3, while the integral over G_2 is transformed by using the relation $\vec{E}_1 = \vec{E} + \vec{E}_2$. As a result, we have

$$\frac{1}{8\pi}||\vec{H}||_G^2 + \frac{1}{2}||\sqrt{\sigma}\vec{E}||_{\Omega_{1,t}}^2 + \frac{\mu^2}{8\pi}||\vec{E}||_{G_2}^2$$
$$\leqslant \frac{\mu^4}{32\pi^2}\left|\left|\frac{1}{\sqrt{\sigma}}\frac{\partial \vec{E}_1}{\partial t}\right|\right|_{\Omega_{1,t}}^2 + \frac{\mu^2}{8\pi}||\vec{E}_0$$
$$- \vec{E}_0^*||_{G_2}^2 - \frac{\mu^2}{4\pi}\int\limits_0^t\int\limits_{G_2}\left(\frac{\partial \vec{E}_2}{\partial t}, \vec{E}\right)dV dt. \qquad (5.1)$$

Here and below $\vec{E}_0^* = \vec{E}_2\big|_{t=0}$. As it follows from (2.4)–(2.7), $\vec{E}_0^* = \vec{E}_0$ in G_1, while in G_2 it is the solution of the problem

$$\begin{aligned} \operatorname{rot}\operatorname{rot}\vec{E}_0^* &= 0 \qquad \text{in } G_2 \\ \operatorname{div}\vec{E}_0^* &= 0 \qquad \text{in } G_2 \\ \vec{E}_{0,\tau}^*\Big|_{\partial G_2} &= \vec{E}_{0,\tau}\Big|_{\partial G_2} \end{aligned} \qquad (5.2)$$

By estimating the difference $\vec{E}_0 - \vec{E}_0^*$ through the inequality (2.13) we obtain (c is the constant from (2.13)):

$$||\vec{E}_0 - \vec{E}_0^*||_{G_2}^2 \leqslant c||\operatorname{rot}\vec{E}_0||_{G_2}^2. \qquad (5.3)$$

It is obvious that the identity (3.4) is valid in this situation too. From it we get the inequality

$$\left|\left|\frac{1}{\sqrt{\sigma}}\frac{\partial \vec{E}_1}{\partial t}\right|\right|_{\Omega_{1,t}}^2 \leqslant$$

$$\leqslant \frac{1}{8\pi\sigma_0^2}\left(1-\exp\left(-\frac{8\pi\sigma_0}{\mu^2}t\right)\right)||\mathrm{rot}\,\vec{E}_0||_G^2. \qquad (5.4)$$

It differs from the similar one is ch. 3 only by its domain where the norm is calculated in the left hand side. In the analogous manner the inequality (3.8) is transformed.

Let us make the inequality (5.1) stronger by omitting its two first terms in the left hand side and integrate the result in t from 0 to t. We obtain

$$\frac{\mu^2}{8\pi}||\vec{E}||_{\Omega_{2,t}}^2 \leqslant \frac{\mu^4}{256\pi^3\sigma_0^2}\left[t-\frac{\mu^2}{8\pi\sigma_0}\left(1-\exp\left(-\frac{8\pi\sigma_0}{\mu^2}t\right)\right)\right]$$
$$\times\cdot||\mathrm{rot}\,\vec{E}_0||_G^2+\frac{\mu^2}{8\pi}c||\mathrm{rot}\,\vec{E}_0||_{G_2}^2\cdot t$$
$$+\frac{\mu^2}{4\pi}\int_0^t\int_{G_2}(t-\tau)\left|\left(\frac{\partial\vec{E}_2}{\partial t},\vec{E}\right)\right|dVdt. \qquad (5.5)$$

In the last summand we make one integration in time by changing the integration order. Then the integral term in the right hand side of (5.5) is estimated by the Cauchy–Bunyakovsky inequality and the ε-inequality. We isolate $||\vec{E}||_{\Omega_{2,t}}^2$ and bound this value from above. The obtained inequality may be minimized in ε, however, we shall restrict ourselves by a cruder, not so bulky, estimate. It has the form

$$||\vec{E}||_{\Omega_{2,t}}^2 \leqslant \frac{\mu^2}{16\pi^2\sigma_0^2}\left[t-\frac{\mu^2}{8\pi\sigma_0}\left(1-\exp-\left(\frac{8\pi\sigma_0}{\mu^2}t\right)\right)\right]$$
$$\times\,||\mathrm{rot}\,\vec{E}_0||_G^2+2c||\mathrm{rot}\,\vec{E}_0||_{G_2}^2\cdot t+4\left\|(t-\tau)\left(\frac{\partial\vec{E}_2}{\partial\tau}\right)\right\|_{\Omega_{2,t}}^2 \qquad (5.6)$$

as a result, from (5.1) we have

$$\frac{1}{8\pi}||\vec{H}||_G^2+\frac{1}{2}||\sqrt{\sigma}\vec{E}||_{\Omega_{1,t}}^2+\frac{\mu^2}{8\pi}||\vec{E}||_{G_2}^2 \leqslant$$

$$\leqslant \frac{\mu^4}{256\pi^3\sigma_0^2}\Big(1-\exp\Big(-\frac{8\pi\sigma_0}{\mu^2}t\Big)\Big)||\mathrm{rot}\,\vec{E}_0||_G^2+$$

$$+\frac{\mu^2}{8\pi}c||\mathrm{rot}\,\vec{E}_0||_{G_2}^2+\frac{\mu^2}{4\pi}\Big|\Big|\frac{\partial \vec{E}_2}{\partial t}\Big|\Big|_{\Omega_{2,t}}||\vec{E}||_{\Omega_{2,t}}. \qquad (5.7)$$

Here, in last term the quantity $||\vec{E}||_{\Omega_{2,t}}$ is estimated by means of (5.6).

Thus, we managed to obtain the inequality where each term is the known function of μ. The inequality (5.7) contains the norm $\frac{\partial \vec{E}_2}{\partial t}$. However, in the problem (2.4)–(2.7) for $\vec{E}_2$ and $\vec{H}_2$ the value μ takes no part. Hence, under the boundedness of all norms and for the limited time interval (as was assumed earlier) the right hand side of (5.7) is $O(\mu^2)$. It means that for $\mu \to 0$ we have $||\vec{H}||_G \to 0$ and $||\vec{E}||_{\Omega_{1,t}} \to 0$, i.e., the strength $\vec{H}$ uniformly in time tends to zero throughout the entire domain G, while $\vec{E}$ tends to zero only in its integral norm depending on the considered time t and only in the domain G_1.

2. From the formal point of view, the inequality (5.7) solves the posed problem. However, in the right hand side of it we have the quantity $\Big|\Big|\frac{\partial \vec{E}_2}{\partial t}\Big|\Big|_{\Omega_{2,t}}$, which is not expressed in terms of input data. This norm was estimated in Galanin (1990b) by using the inequality and the imbedding theorem (Sobolev, 1988; Plotnitsky, 1976; Galanin, 1990a). Here we shall restrict ourselves by giving the results. It is shown in Galanin (1990b) that there is constant c, depending only on geometries of the domains G, G_1 and G_2 on the parameters σ_0, σ_0^* and ν, such that the inequality

$$\Big|\Big|\frac{\partial \vec{E}_2}{\partial t}\Big|\Big|_{\Omega_t}^2 \leqslant c\Big|\Big|\frac{1}{\sqrt{\sigma}}\mathrm{rot}\,\mathrm{rot}\,\vec{E}_0\Big|\Big|_{G_1}^2. \qquad (5.8)$$

is satisfied. If necessary the constant c can fully be determined through the parameters σ_0, σ_0^* and ν and the constants in the imbedding type inequalities. Along with (5.6) and (5.7) the estimate (5.8) gives the solution of the posed problem, i.e., we obtain the estimate of the solution (2.8) and (2.9) in terms of input data.

3. In the case of a conducting media (ch. 3) the initial data agreement condition (2.4) enabled us to obtain the t-uniform estimate of $\overrightarrow{E}$ proximity. Consider a nonhomogeneous medium.

By analogy with ch. 3 we obtain the identity of the form (3.1) for the time derivatives:

$$\frac{1}{8\pi}\left\|\frac{\partial \overrightarrow{H}}{\partial t}\right\|_G^2 + \left\|\sqrt{\sigma}\frac{\partial \overrightarrow{E}}{\partial t}\right\|_{\Omega_{1,t}}^2 + \frac{\mu^2}{4\pi}\int_0^t\int_G \left(\frac{\partial^2 \overrightarrow{E}_1}{\partial t^2}, \frac{\partial \overrightarrow{E}}{\partial t}\right) dV dt$$

$$= \frac{1}{8\pi}||\mathrm{rot}(\overrightarrow{E}_0 - \overrightarrow{E}_0^*)||_{G_2}^2. \tag{5.9}$$

We shall transform this identity in the same manner like(5.1) and use the estimate (3.8) corresponding to (5.4). Now we estimate the integral term (5.9) as above. After some obvious calculations we obtain

$$\left\|\frac{\partial \overrightarrow{E}_2}{\partial t}\right\|_{\Omega_{2,t}}^2 = O(\mu^{-2})$$

and after substituting this relation into (5.9) and estimating all the terms we derive the inequality of the form (5.7) with O(1) as the right hand side for the case (5.9).

By combining this result with (5.7) according to the algorithm of deriving the estimate (3.10) we get $||\overrightarrow{E}||_{G_1}^2 = O(\mu)$. Thus we proved that when $\mu \to 0$ the strength $\overrightarrow{E} \to 0$ in the sense of $||\overrightarrow{E}||_{G_1}$.

All the missing calculations may be easily performed by the algorithm used above. It is clear that to obtain the estimate for $\left|\left|\frac{\partial^2 \vec{E}_2}{\partial t^2}\right|\right|^2_{\Omega_{2,t}}$ we shall need smoother initial data.

4. The estimate (5.7) shows that in mixed media the electromagnetic field energy difference for the complete and approximate equations behaves at least like $O(\mu)$ when $\mu \to 0$.

6. Conclusions. The paper deals with proving the applicability of the Maxwell equations in the quasi-stationary approximation to media containing nonconducting subdomains. The main result consist in proving that the magnetic field strength in the entire domain and the electric field strength in the conducting subdomain, described by approximate equations, converge to the respective strength described by complete equations if the μ-ratio of the characteristic rate of the process to the light velocity-tends to zero. By the convergence we mean that the integral norm of the solutions difference, calculated in spatial variables, tends to zero for all considered time instants if the conditions of agreement for initial data are fulfilled. This result was obtained for the medium containing subdomains with sharply nonhomogeneous electric conduction in the problem on a limited time interval. The problem for a conducting medium was studied without this requirement. Different variants of agreed and disagreed initial data were analyzed. The estimates of solution differences were obtained in terms of input data, which testified to the presence or absence of boundary layers. The solution estimates are given in the case of homogeneous nonconducting medium, which testified that the solutions were not close to each other. Comparison of the solutions obtained for the complete and approximate Maxwell equations was made in the initial-boundary value problem with zero tangential components of the electric field strength at the boundary of the domain.

Note that from the formal point of view the convergence of the solutions to each other as $\mu \to 0$ occurs independently of the electric conduction in the conducting subdomain. It is important only that σ_0 (the lower bound of electric conductivity in the conductor) must be nonzero. When discussing the question about the solutions convergence under variation of one parameter we assumed that there are problems like (2.1) and (2.2), which differ between themselves in only one parameter. The rest parameters must be fixed. However, even comparing the conditions (1.3) and (1.6) evidences that normalization of the dimensional electric conductivity is performed by using the parameter μ. Only considering different μ may give fixed σ. The parameter μ participates in normalization of $\vec{E}$. But if we study singular perturbed problems like (2.1) we always assume that the involved data values are about unity.

Here, in ch. 5 we restricted ourselves by determining how the solutions difference depended on one parameter μ. We obtained in ch. 3 the estimates for the solutions difference, which explicitly contained the problem parameters. Hence, those estimates allow us to judge how close are the solutions depending on any parameter involved in the problem. Note, for example, that the condition (1.6) for applicability of the quasi-stationary approximation is very natural. The parameters μ and σ_0 enter into the right hand side of the condition (3.5) only in the combination (1.6). Hence, to provide the closeness of magnetic field strength at all times it is necessary to fulfill the condition (1.6). However, for the closeness of electric field strength in the norm (3.5) depending on time (or more so uniform in time (3.10)) it is necessary that the ratio μ^4/σ_0^3 and μ^3/σ_0^3 must be respectively small.

Finally, it should be noted that a different point of view is also possible as to the relationship between (2.1) and (2.5). Suppose that for some reason it is not convenient to solve the second problem although we are interested exactly in its solution. Then, as the obtained estimates show, solving the problem (2.1) we approach to the solution of (2.5). In this case the parameter μ plays a formal role and acts as artificial viscosity in hydrodynamics or a small parameter in the quasi-inversion method or like something else.

REFERENCES

Aleksandrov, A.P., and V.I. Dmitrijev (1975). On the Cauchy problem for the Maxwell equations in an anisotropic conducting medium. In *Computational Methods and Programming,* **24**, Moscow University, Moscow. pp. 23–37.

Duvault, G., and J.-L. Lions (1980). *Les Inequations en Mecanique et en Physique.* Nauka, Moscow. pp. 384 (in Russian).

Galanin, M.P. (1990a). On one inequality of the imbedding type for vector functions. *Preprint, Keldysh Inst. Appl. Math., USSR Ac. Sci.,* **72**, pp. 16.

Galanin, M.P. (1990b). Estimating the closeness between the solutions of the Maxwell equations system and its quasi-stationary approximation. *Preprint, Keldysh Inst. Appl. Math., USSR Ac. Sci.,* **129**, pp. 27.

Galanin, M.P., Yu.A. Poveshenko, and Yu.P. Popov (1988). On formulating initial boundary value problems for determining quasi-stationary electromagnetic fields in media consisting of conductors and dielectrics. *Preprint, Keldysh Inst. Appl. Math., USSR Ac. Sci.,* **59**, pp. 21.

Kawashima, S., and Y. Shizuta (1986). Magnetohydrodynamic approximation of the complete equations for an electromagnetic fluid. II, *Proc. Japan Acad.,* **62**(5), ser. A, 181–184.

Kulikovsky, A.G., and G.A. Lyubimov (1962). *Magnetic Hydrodynamics.* Fizmatgiz, Moscow. pp. 248 (in Russian).

Ladyzhenskaya, A., and V.A. Solonnikov (1960). Solving some nonstationary problems of magnetohydrodynamics for a viscous incompressible fluid. *Proc. Steklov Math. Inst., USSR Acad. Sci.,* **59**, 115–173.

Plotnitsky, T.A. (1976). On some properties of the vector analysis operators in the spaces of S.L. Sobolev. In Yu.A. Mitropolsky (Ed.), *Boundary Value Problems of Conducting Media Electrodynamics,* Inst. of Math., Ukr.SSR Acad. Sci., Kiev. 149–165.

Sakhaev Sh. (1976). Solvalibility of some nonstationary boundary value electrodynamics problems in the space $W^{2,1}_{pxt}(Q_T)$, $p>1$ and $C^{2+\alpha,1+\alpha/2}_{xt}(Q_T)$. In *Theorems of Imbeedding and Applications, Proceedings of the All-Union Simposium (Alma-Ata, 1973),* Nauka of Kaz.SSR, Alma-Ata. 140–145.

Samarskii, A.A., and Yu.P. Popov (1980). *Difference Methods of Solving Gas Dinamics Problems.* Nauka, Moscow. pp. 352 (in Russian).

Sobolev, S.L. (1988). *Some Applications of Functional Analysis in Mathematical Physics.* Nauka, Moscow. pp. 336 (in Russian).

Su Yui-Chan (1961). Asymptotic of solutions of some degenerating quasi-linear hyperbolic second-order equations. *DAN SSSR,* **138**(1), 63–66.

Tamm, I.E. (1966). *Fundamentals of the Electricity Theory.* Nauka, Moscow. pp. 624 (in Russian).

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