A Note about Total Stability of a Class of Hybrid Systems

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Abstract. Robust stability results for nominally linear hybrid systems are obtained from total stability theorems for purely continuous-time and discrete-time systems. The class of hybrid systems dealt with consists of, in general, coupled continuous-time and digital systems subject to state perturbations whose nominal (i.e., unperturbed) parts are linear and time-varying, in general. The obtained sufficient conditions on robust stability are dependent on the values of the parameters defining the over-bounding functions of the uncertainties and the weakness of the coupling between the analog and digital sub-states provided that the corresponding uncoupled nominal subsystems are both exponentially stable.

Key words: dynamic hybrid systems, stability, total stability.

1. Introduction

Stability of both continuous-time and discrete-time singularly perturbed systems has received much attention in the last years (Kafri and Abed, 1996; Kolev, 1993; Shieh *et al.*, 1986; Oppenheimer and Michel, 1988). Also, stability analysis of discrete-time singularly perturbed systems with calculations of parameter bounds has been reported in (Kolev, 1993; Shieh *et al.*, 1986). An assumption used in previous work to carry out the stability analysis of singularly perturbed systems is relaxed in (Kafri and Abed, 1996) where an upper-bound on the singular perturbation parameters is included to derive such an analysis. On the other hand, the so-called hybrid models are a very important tool for analysis in the modern computers and control technologies since they describe usual situations where continuous-time and either discrete-time and /or digital systems are coupled (De la Sen, 1996; Kabamba and Hara, 1993). A usual example, very common in practice, is the case when a digital controller operates over a continuous-time plant to stabilize it or to improve its performance. The systems described in (De la Sen, 1996; Kabamba and Hara, 1993) are more general since the controlled plant can also possess an hybrid nature since all the continuous-time and digital state- variables can be mutually coupled and to possess internal delays (De la Sen and Alastruey, 2004; De la Sen, 2004; De la Sen and

Luo, 2004). Other studies and development concerned with hybrid systems in the contexts of presence of delays in their dynamics, fundamental properties of dynamic systems and manufacturing systems have been recently performed in (De la Sen, 2006; Bargelis *et al.*, 2004; Chaib *et al.*, 2005; Marchenko and Poddubnaya, 2005). In this brief paper, stability results are obtained for a wide class of such systems whose nominal (i.e., unperturbed) parts are linear and, in general, time-varying while the state perturbations are allowed to be, in general, non-linear, time-varying and of a dynamic nature. The results about robust stability are obtained by firstly deriving sufficient stability conditions related to total stability for an extended discrete system which describes the overall state trajectory at sampling instants via the discretization of the continuous-time sub-state. Subsequently, a result about total stability of the continuous-time sub-state is carried out to ensure the system' s stability during the inter-sample intervals. Some links with the results given in (Kafri and Abed, 1996) about singularly perturbed systems are also given for a special hybrid system within the given class.

Notation. $\lambda_{\max}(M)$ and $\det(M)$ denote, respectively the maximum eigenvalue and determinant of the square matrix $M = (M^{(ij)})$. The symbol $\otimes$ denotes the direct Kronecker product of matrices. Particular norms for functions, sequences or matrices are denoted by the appropriate subscript. In the expressions being valid for any norms, those subscripts are omitted.

2. Problem Statement

A) *Plant Description* Σ

Consider the following, in general, time-varying hybrid plant

System Σ:

$$\dot{x}_c(t) = A_c(t)x_c(t) + A_{cd}(t)x_d[k] + \delta_c(t), \tag{1}$$

$$x_d[k+1] = A_d[k]x_d[k] + A_{dc}[k]x_c[k] + \delta_d[k], \tag{2}$$

$$\delta_c(t) = f_{cc}(t, x_c(t)) + f_{cd}(t, x_d[k]) + g_{cc}(t, x_c(t)) + g_{cd}(t, x_d[k]), \tag{3}$$

$$\delta_d[k] = f_{dc}(k, x_c[k]) + f_{dd}(k, x_d[k]) + g_{dc}(k, x_c[k]) + g_{dd}(k, x_d[k]) \tag{4}$$

for all time $t \in [kT, (k+1)T)$ and discrete time integer index $k \geqslant 0$ for sampling period T where $x_c(t)$ and $x_d[k]$ are, respectively, the n_c continuous-time (or analog) substate and n_d discrete-time (or digital) substate. The continuous-time and discrete-time variables are denoted by (t) and $[k]$, respectively. The discretized analog substate at sampling instants is denoted as a digital signal; i.e., $x_c(kT) = x_c[k]$. The matrix functions $A_c(t)$, $A_{cd}(t)$, $A_d[k]$ and $A_{dc}[k]$ are of dimensions being compatible with the corresponding vectors in (1)–(2). $\delta_c(t)$ and $\delta_d[k]$ are disturbances being, in general, nonlinear and time-varying subject to the following set of constraints on the functions $f_{(.)}$ and $g_{(.)}$.

Constraints C:

C1) $A_d[k]$, $A_{dc}[k]$, $f_{dc}(k, x_c[k])$, $f_{dd}(k, x_d[k])$, $g_{dc}(k, x_c(t))$, $g_{dd}(k, x_d[k])$ matrix and vector sequences of k of bounded entries. The entries of $A_c(t)$, $A_{cd}(t)$, $f_{cc}(t, x_c(t))$,

$f_{cd}(t, x_d[k])$, $g_{cc}(t, x_c(t))$ and $g_{cd}(t, x_d[k])$ are locally integrable functions of t for each fixed x in the ball $\text{Max}\,(\|x_c\| \leqslant r, \|x_d\| \leqslant r)$ and all integer $k \geqslant 0$ and all $t \geqslant 0$.

C2) $f_{cc}(t, 0) = f_{dc}(t, 0) = 0 \in \mathbf{R}^{n_c}, \quad f_{dc}(k, 0) = f_{dd}(k, 0) = 0 \in \mathbf{R}^{n_d},$ (5)

C3) $\|h_{cc}(t, x_{c1}) - h_{cc}(t, x_{c2})\| \leqslant \beta_{cc}^h \|x_{c1} - x_{c2}\|,$

$\qquad \|h_{cd}(t, x_{c1}) - h_{cd}(t, x_{c2})\| \leqslant \beta_{cd}^h \|x_{c1} - x_{c2}\|,$ (6)

$\qquad \|h_{dc}(k, x_{d1}) - h_{dc}(k, x_{d2})\| \leqslant \beta_{dc}^h \|x_{d1} - x_{d2}\|,$

$\qquad \|h_{dd}(k, x_{d1}) - h_{dd}(k, x_{d2})\| \leqslant \beta_{dd}^h \|x_{d1} - x_{d2}\|,$ (7)

C4) $\|g_{cc}(t, x_{c1})\| \leqslant \beta_{cc}^g r, \quad \|g_{cd}(t, x_{d1})\| \leqslant \beta_{cd}^g r,$ (8)

$\qquad \|g_{dc}(t, x_{c1})\| \leqslant \beta_{dc}^g r, \quad \|g_{dd}(t, x_{d1})\| \leqslant \beta_{dd}^g r,$ (9)

for all $\|x_{ci}\| \leqslant r$, $\|x_{di}\| \leqslant r$ and all integer $k \geqslant 0$ and all $t \geqslant 0$, with h being any of the vector real functions or sequences of (3)–(4) and $\beta_{(\cdot)}^h$ ($h = f$ or $h = g$) being known nonnegative real constants. *The problem dealt with in this brief is the investigation of the robust stability of Σ (i.e., that of (1)–(2) with dynamic state disturbances (3)–(4)) subject to the set of constraints C. For this purpose, the state-trajectory of Σ at sampling instants is calculated in the following subsection.*

B) *Extended Discrete System Σ_d*

Direct calculation of the solution of Σ at sampling instants (i.e., $t = kT$, k being any nonnegative integer) yields the following discrete extended system:

$$\Sigma_d: \ x[k+1] = A[k]x[k] + \delta[k], \quad \text{all integer} \ k \geqslant 0 \qquad (10)$$

with $x[k] = (x_c^T[k], x_d^T[k])^T$ subject to $x[0] = (x_c^T[0], x_d^T[0])^T$, with $x_c[0] = x_c(0)$, and

$$A[k] = \begin{bmatrix} \Phi_c[k] & \Gamma_c[k] \\ A_{dc}[k] & A_d[k] \end{bmatrix}, \quad \Gamma_c[k] = \int_0^T \Phi_c((k+1)T - \tau) A_{cd}(kT + \tau)\, \mathrm{d}\tau, \quad (11)$$

$$\delta[k] = \left(\delta_c^{'T}(t), \delta_d^T[k] \right) = \left(\int_0^T \delta_c^T(kT + \tau)\Phi_c^T((k+1)T - \tau)\, \mathrm{d}\tau, \delta_d^T[k] \right)^T, \quad (12)$$

$\Phi_c(t)$ and $\Gamma_c(t)$ being defined at sampling instants as $\Phi_c[k] = \Phi_c(kT) = \Psi_c((k+1)T, kT)$ and $\Gamma_c[k] = \Gamma_c(kT)$ are the k-th intersample state transition and control matrices of the continuous subsystem, respectively, (i.e., $\dot{\Psi}_c(t, 0) = A_c(t)\Psi_c(t, 0)$; $\Psi_c(0, 0) = I_{n_c}$ for all $t \in [kT, (k+1)T)$ and all integer $k \geqslant 0$.

3. Main Results

The robust stability of Σ subject to the constraints C under the knowledge of the constants $\beta_{(\cdot)}^{(\cdot)}$ is now investigated. The results on robust stability are useful for both local and global stability in the sense that stability is ensured for initial conditions of (1)–(4) being constrained to the balls $\|x_c(0)\| \leqslant r$, $\|x_d[0]\| \leqslant r$ where the radius r is arbitrary but compatible with the validity of the constraints C on Σ.

A) *Exponential Stability of the Nominal Extended System* Σ_d^*

The nominal Σ is defined by zeroing $\delta_c(t)$ and $\delta_d[k]$ in (1)–(2). This results into the nominal version Σ_d^* of Σ_d in (10)–(12) satisfying $x^*[k+1] = A[k]x^*[k]$ with $x^*[0] = (x_c^T[0], x_d^T[0])^T$. The following assumption is given:

ASSUMPTION 1. The nominal uncoupled continuous-time and digital subsystems $\dot{x}_c^*(t) = A_c(t)x_c^*(t)$ and $x_d^*[k+1] = A_d[k]x_d^*[k]$ are both exponentially stable, i.e., thee exist norm-dependent real constants $K_c \geqslant 1$ and $K_d \geqslant 1$ such that $\|\Psi_c(t_2, t_1)\| \leqslant K_c e^{-a_c(t_2-t_1)}$ and $\|\Psi_d(k_2, k_1)\| \leqslant K_d a_d^{k_2-k_1}$ for some real constants $a_c > 0$ and $a_d \in [0, 1)$ where $\Psi_c(., .)$ and $\Psi_d[., .]$ are the state-transition matrices of the uncoupled continuous-time and digital subsystems in Σ (i.e., $\dot{\Psi}_c(t, 0) = A_c(t)\Psi_c(t, 0)$; $\Psi_c(0, 0) = I_{n_c}$ with $\Psi_c(k_2 T, k_1 T) = \prod_{j=k_1}^{j=k_2-1} \Phi[j]$ between two sampling instants and $\Psi_d[k_2, k_1] = \prod_{j=k_1}^{j=k_2-1} A_d[j]$ with $\Psi_d[0, 0] = I_{n_d}$ for all $t \geqslant 0$, any real $t_2 \geqslant t_1 \geqslant 0$ and any integers $k_2 \geqslant k_1 \geqslant 0$).

The following stability result holds for the nominal extended system (i.e., $\delta \equiv 0$ in (10)).

PROPOSITION 1. Define

$$
\rho_k = \mathrm{Max}\left(\max_{1 \leqslant i \leqslant n_c} \left(\sum_{j=1}^{n_c} \int_0^T \left| \Psi_c^{(ij)}((k+1)T, kT+\tau) A_{cd}(\tau)\, \mathrm{d}\tau \right| \right), \right.
$$
$$
\left. \max_{1 \leqslant i \leqslant n_d} \left(\sum_{j=1}^{n_d} |A_{dc}^{(ij)}[k]| \right) \right). \tag{13}
$$

Thus, the nominal extended discrete system is exponentially stable if Assumption 1 holds and $\rho_k < 1 - \mathrm{Max}(e^{-a_c T}, a_d)$ for all integer $k \geqslant 0$.

Proof. Decompose $A[k] = A_0[k] + \widetilde{A}[k]$ in (11) with $A_0[k] = \mathrm{Block\,Diag}(\Phi_c[k], A_d[k])$, $\Phi_c[k] = \Psi_c((k+1)T, kT)$ and $A_d[k] = \Psi_d[(k+1)T, k]$ being the one sampling period kth transition matrices. Thus, $x^*[k+1] = A[k]x^*[k]$ is exponentially stable if there exist real constants $\overline{K} \geqslant 1$ (being norm-dependent) and $\overline{a} \in [0, 1)$ such that its state transition matrix $\Psi[k_2, k_1] = \prod_{j=k_1}^{k_2-1} A[j]$ satisfies $\|\Psi[k_2, k_1]\| \leqslant \overline{K}\, \overline{a}^{k_2-k_1} \overline{K} = 1$ for the l_2-matrix norm given by the maximum modulus within the whole set of eigenvalues. Also, $\rho_k = \|\widetilde{A}[k]\|_2$, from the definition of ρ_k and $\widetilde{A}[k] = A[k] - A_0[k]$, lies in the union $\cup_{i=1}^{n_c} R_i$ of the discs $R_i = \{z: |z| \leqslant \sum_{j=1}^{n_c+n_d} |\widetilde{A}^{(ij)}[k]|\}$ from Gerschgorin's circle theorem (Kincaid and Cheney, 1991). Therefore, $\|A[k]\|_2 \leqslant \mathrm{Max}(e^{-a_c T}, a_d) + \rho_k \leqslant \overline{a} < 1$ for all integer $k \geqslant 0$ if Assumption 1 and (13) hold. Thus, the nominal extended system is exponentially stable and the result has been proved.

B) *Stability of the Discrete Disturbed Extended System* Σ_d

The following result gives sufficient conditions for stability of the extended discrete system (10) within a closed ball of the extended state $x[.]$.

PROPOSITION 2. Assume that Proposition 1 holds under Assumption 1 (i.e., the nominal extended system (10) is exponentially stable) under the stronger condition $\|A[k]\|_2 \leqslant \overline{a} < 1 - \frac{\overline{K_d}}{\overline{K}} < 1$ where the real constants $\overline{K}$ and $\overline{a}$ are related to the state transition matrix of Σ_d^e Eq. 10 and defined in the proof of Proposition 1, and

$$\overline{K}_d = K_c a_c^{-1}\beta_c + K_d\beta_d, \tag{14}$$
$$\beta_h = \beta_{hc}^f + \beta_{hd}^f + \beta_{hc}^g + \beta_{hd}^g \tag{15}$$

for $h = c, d$; and $\overline{K}_d < 1$. Thus, the state vector is uniformly bounded according to

$$\|x[k]\| \leqslant \left(\overline{a}^k + \overline{K}\,\overline{K}_d\frac{1 - \overline{a}^k}{1 - \overline{a}}\right)r \leqslant r \tag{16}$$

for all integer $k \geqslant 0$ provided that $\mathrm{Max}(\|x_c[0]\|, \|x_d[0]\|) < \frac{r}{2\overline{K}} \leqslant \frac{r}{2}$.

Proof. First, note from direct calculus from (6)–(9) that the disturbance signal $\delta[k]$ in (10) satisfies

$$\|\delta[k]\| \leqslant \|\delta_c'[k]\| + \|\delta_d[k]\| \leqslant \overline{K}_d r \tag{17}$$

provided that $\mathrm{Max}(\|x_c[k]\|, \|x_d[k]\|) < r/2$ for all integer $k \geqslant 0$ since $\overline{K} > 1$ and $\overline{a} < 1 - \frac{\overline{K}_d}{\overline{K}}$ imply $\overline{a} + \overline{K}_d < 1$. Consider the set of sequences $\{y[k], k \geqslant 0\}$ equipped with the ℓ_∞ norm for sequences $\|y\|_\infty = \mathrm{Max}_{0 \leqslant k \leqslant \infty}(\|y[k]\|)$. Thus, the operator $\mathbf{T_d}$ defined by $(\mathbf{T_d}\,y)[k] = A[k]y[k] + \delta[k]$ is a contraction on the closed subset $\mathbf{R_d}$ of bounded n_d-vector sequences $\{y[k], k \geqslant 0: \|y\|_\infty \leqslant r\}$. By the contraction mapping theorem (Hale, 1980; Kincaid and Cheney, 1991) there is a unique solution $y[k+1] = (\mathbf{T_d}\,y)[k]$ (fixed point) with sequences in $\mathbf{R_d}$, and

$$\|x[k]\| = \left\|\Psi[k, 0]x[0] + \sum_{i=0}^{k-1}\Psi[k, i+1]\delta[i]\right\| \leqslant \overline{K}\,\overline{a}^k\|x[k]\| + \overline{K}\,\overline{K}_d\,r\sum_{i=0}^{k-1}\overline{a}^i \tag{18}$$

which leads directly to (16) since $\overline{a} < 1 - \frac{\overline{K}_d}{\overline{K}}\overline{K} < 1$ implies

$$\sum_{i=0}^{k-1}\overline{a}^i = \frac{1 - \overline{a}^k}{1 - \overline{a}}; \quad \overline{a}^k + \overline{K}\,\overline{K}_d\frac{1 - \overline{a}^k}{1 - \overline{a}} \leqslant 1 + \overline{a}^k - \overline{a} \leqslant 1. \tag{19}$$

C) *Stability of the Continuous-Time Substate Inbetween Sampling Instants*

Now, the solution to (1) subject to (2)–(3) is analyzed by taking into account that $\|x[k]\| \leqslant r$ provided that Proposition 2 holds. A total stability argument is used as main tool for the proof of stability of the continuous-time subsystem.

PROPOSITION 3. Assume that Proposition 2 holds, $\mathrm{Sup}_{0 \leqslant t < \infty}(\|A_{cd}(t)\|) \leqslant a_{cd}$, $\|x_c(0)\| \leqslant \frac{r}{2K_c}$ and $\frac{K_c\overline{K}_c}{a_c} < 1$, where

$$\overline{K}_c = K_c^f + K_c^g; \quad K_c^f = \beta_{cc}^f + \beta_{cd}^f; \quad K_c^g = a_{cd} + \beta_{cc}^f + \beta_{cd}^g. \tag{20}$$

Thus, there is a unique solution $x_c(t)$ to (1) such that for all $t \geqslant 0$:

$$\|x_c(t)\| \leqslant K_c \mathrm{e}^{-(a_c - K_c K_c^f)t}\|x_0\| + \frac{K_c K_c^g}{a_c - K_c K_c^f} r(1 - \mathrm{e}^{-(a_c - K_c K_c^f)t}) \leqslant r. \quad (21)$$

Proof. One gets directly from (1),

$$x_c(t) = \Psi_c(t,0)x_c(0) + \int_0^t \Psi_c(t-\tau)\delta_c^0(\tau)\,\mathrm{d}\tau \quad (22)$$

with $x_c(0) = x_c[0]$ and $\delta_c^0(t) = A_{cd}(t)x_d[k] + \delta_c(t)$. Under the set of constraints C, $\|\delta_c^0(t)\| \leqslant \overline{K}_c r$ for all $t \geqslant 0$ subject to (20). Using similar arguments as in the proof of Proposition 2, consider the Banach space $\mathbf{B_c} = C[0,\infty)$ of continuous, bounded n_c-vector sequences defined on $[0,\infty)$ and equipped with the L_∞-norm $\|y\|_\infty = \mathrm{Sup}_{0 \leqslant t \leqslant \infty}(|y(t)|)$. The operator $\mathbf{T_c}$ is defined via

$$(\mathbf{T_c}\,y)(t) = \Psi_c(t,0)x_0 + \int_0^t \Psi_c(t,\tau)\delta_c^0(\tau)\,\mathrm{d}\tau \quad (23)$$

is a contraction of the closed subset $\mathbf{R_c} = \{y \in \varepsilon\mathbf{B_c}\colon \|y\|_\infty \leqslant r\}$ of $\mathbf{B_c}$, because for $\|y_i\|_\infty \leqslant r$ $(i = 1, 2)$, one gets from (20) and (23)

$$|(\mathbf{T_c}y)(t)| \leqslant K_c\Big\{\mathrm{e}^{-a_c t}\|x_c(0)\| + \frac{\overline{K}_c}{a_c}(1 - \mathrm{e}^{-a_c t})r\Big\} \leqslant r$$
$$\Rightarrow \|\mathbf{T_c}y_1 - \mathbf{T_c}y_2\| \leqslant \overline{K}_c a_c^{-1}\|y_1 - y_2\| \leqslant \|y_1 - y_2\| \quad (24)$$

for $\|x_c(0)\| = \|x_c[0]\| \leqslant \frac{r}{2K_c} \leqslant \frac{r}{2\overline{K}} \leqslant r$ since $\|x_d[k]\| \leqslant \frac{r}{2\overline{K}} \leqslant r$ for all $k \geqslant 0$ from Proposition 2. By the contraction mapping theorem, (Hale, 1980; Kincaid and Cheney, 1991), there exists a unique solution of (23) in $\mathbf{R_c}$, the fixed point of $\mathbf{T_c}$. Thus, one gets from (23) that

$$\|x(t)\| \leqslant K_c\Big\{\mathrm{e}^{-a_c t}\|x_0\| + K_c^f \int_0^t \mathrm{e}^{-a_c(t-\tau)}\|x_c(\tau)\|\,\mathrm{d}\tau + K_c^g r \int_0^t \mathrm{e}^{-a_c(t-\tau)}\,\mathrm{d}\tau\Big\} \quad (25)$$

which leads to (21) from Bellman–Gronwall Lemma (Hale, 1980).

REMARK 1 (combined interpretation of Propositions 1–3). Assumption 1 and Propositions 1–3 yield the following robust stability conditions for the system Σ by using l_2 vector and matrix norms, i.e., $K_c = K_d = 1$, provided that $\|x_c[0]\| \leqslant \frac{r}{2}$ and $\|x_d[0]\| \leqslant \frac{r}{2}$:

$$\rho^* + \rho + \beta_c a_c^{-1} + \beta_d < 1, \quad \overline{K}_c a_c^{-1} < 1, \quad (26)$$
$$\rho^* = \mathrm{Max}_{0 \leqslant k \leqslant \infty}(\mathrm{e}^{-a_c T}, a_d), \quad \rho = \mathrm{Max}_{0 \leqslant k \leqslant \infty}(\rho_k) \quad (27)$$

with β_c and β_d being real constants defined in (15) related to the set of constraints C, ρ_k and $\overline{K}_c$ defined in (12) and (20). In particular: (a) $\rho^* < 1$ guarantees the exponential

stability of the uncoupled nominal continuous-time and digital subsystems (i.e., $\delta_c \equiv 0$, $\delta_d \equiv 0$). (b) $\rho^* + \rho < 1$ guarantees that the exponential stability is not destroyed in the nominal extended system Σ_d^* by the existence of linear couplings between the continuous-time and digital substates. (c) The first inequality in (26) guarantees that the state disturbances in Σ are sufficiently small in terms of the real constants defining their overbounding functions while satisfying C so that *the extended discrete system Σ_d maintains the stability of its nominal description Σ_d^*. If, furthermore, the second constraint of* (26) *holds then the signal boundedness is kept in-between sampling instants according to* (21) *and the overall hybrid system Σ is robustly stable.*

D) *Links With Singular Perturbation Theory*

In some particular descriptions within the class Σ, the perturbation theory can be combined with the above analysis. Assume, for instance that the linear dynamics of Σ is subject to variations defined by a small parameter ε, A_{dc} and A_{dd} are time-invariant and $A_c(t) = \varepsilon A_c$ for all $t \geqslant 0$ and $A_{cd}(t) = \rho(\varepsilon) \mathrm{e}^{\varepsilon A_c t}$ with $\rho(\varepsilon) \leqslant \overline{\rho} < \infty$ for all $\varepsilon \in [0, \varepsilon^*)$. Thus, a direct series expansion around εT of the state transition matrix of the continuous subsystem yields

$$
\begin{aligned}
\Psi_c((k+1)T, kT) &= \mathrm{e}^{\varepsilon A_c T} \\
&= I_{n_c} + \varepsilon A_c T + \Delta(\varepsilon, T, A_c) \int_{kT}^{(k+1)T} \Psi_c((k+1)T) A_{cd}(\tau) \, \mathrm{d}\tau \\
&= \left(I_{n_c} + \varepsilon A_c T + \Delta(\varepsilon, A_c, T) \right) \int_{kT}^{(k+1)T} \mathrm{e}^{-\varepsilon A_c \tau} A_{cd}(\tau) \, \mathrm{d}\tau \\
&= \left(\varepsilon A_c + \rho(\varepsilon) \left[I_{n_c} + \Delta(\varepsilon, T, A_c) \right] \right) T.
\end{aligned}
$$

Note that $\| \Delta(\varepsilon, T, A_c) \|_2 \leqslant 1 + \varepsilon A_c T + \varepsilon T |\lambda_{\max}(A_c)| = \delta(\varepsilon) \leqslant \overline{\delta} < \infty$ for all $\varepsilon \in [0, \varepsilon^*)$ and $\rho(\varepsilon) T [I_{n_c} + \Delta(\varepsilon, T, A_c)] \leqslant \overline{\rho} T (1 + \overline{\delta}) < \infty$ for all $\varepsilon \in [0, \varepsilon^*)$. Thus,

$$
\begin{aligned}
A(\varepsilon) &= A^*(\varepsilon) + \Delta\overline{A}(\varepsilon), \\
A^*(\varepsilon) &= \begin{bmatrix} I_{n_c} + \varepsilon A_c T & \varepsilon A_c T \\ A_{dc} & A_d \end{bmatrix}, \\
\Delta\overline{A}(\varepsilon) &= \begin{bmatrix} \Delta(\varepsilon, T, A_c) & \rho(\varepsilon) T (I_{n_c} + \Delta(\varepsilon, T, A_c)) \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

is time-invariant in (10). Thus, the discrete system Σ_d of (10) satisfies equivalently,

$$
z[k+1] = \widehat{A}^*(\varepsilon) z[k] + \left(\delta[k] + \Delta\overline{A}(\varepsilon) z[k] \right) \tag{28}
$$

by defining (see (Kafri and Abed, 1996))

$$
\widehat{A}^*(\varepsilon) = \hat{I} + \widehat{A}_{21} + \widehat{A}_{22} + \varepsilon(\widehat{A}_{11} + \widehat{A}_{12}) \quad \text{for } \varepsilon \in [0, \varepsilon^*) \tag{29}
$$

through the extended $n = n_c + n_d$-matrices

$$
\widehat{A}_{11} = \begin{bmatrix} TA_c & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{A}_{12} = \begin{bmatrix} 0 & TA_c \\ 0 & 0 \end{bmatrix}, \quad \hat{I} = \begin{bmatrix} I_{n_c} & 0 \\ 0 & 0 \end{bmatrix}, \tag{30}
$$

$$
\widehat{A}_{21} = \begin{bmatrix} 0 & 0 \\ A_{dc} & 0 \end{bmatrix}, \quad \widehat{A}_{22} = \begin{bmatrix} 0 & 0 \\ 0 & A_d \end{bmatrix}. \tag{31}
$$

Note that Schur' s stability of $\widehat{A}^*(\varepsilon)$ is equivalent to exponential stability of the unforced time-invariant system Σ_d^*: $z^*[k + 1] = \widehat{A}^*(\varepsilon)z^*[k]$ since $\widehat{A}^*(\varepsilon)$ has its eigenvalues in $|z| < 1$ for all $\varepsilon \in [0, \varepsilon^*)$. Thus, the subsequent result follows directly from Proposition 2 by using a previous result in (Kafri and Abed, 1996):

PROPOSITION 4. Define $\nu(\varepsilon) := \nu(\widehat{A}(\varepsilon)) = \det(\widehat{A}(\varepsilon) \otimes \widehat{A}(\varepsilon) - \hat{I} \otimes \hat{I})$ where $\otimes$ denotes the direct Kronecker product of matrices which is a matrix of dimension $(n+m)^2 \times (n+m)^2$. Thus, the following items hold:

   (i) If $\nu(\varepsilon)$ has no positive zeros, then either $\widehat{A}(\varepsilon)$ is Schur stable for all $\varepsilon > 0$ or it is not Schur stable for any $\varepsilon > 0$.
  (ii) If $\nu(\varepsilon)$ has positive zeros, let $\overline{\varepsilon}$ be the smallest such zero. If $\widehat{A}(\varepsilon_1)$ is Schur stable for any $\varepsilon_1 \in (0, \overline{\varepsilon})$ then $\varepsilon^* = \overline{\varepsilon}$. Otherwise, $\widehat{A}(\varepsilon)$ is not Schur stable for all sufficiently small and positive values of ε.
 (iii) The extended discrete system $\Sigma_d(\varepsilon)$ is stable for all $\varepsilon \in [0, \varepsilon^*)$ satisfying $\text{Max}(\|x_c[0], x_d[0]\|) \leqslant \frac{r}{2\overline{K}}$ and $\|A^*(\varepsilon)\|_2 + \overline{K}_d + \text{Max}_{0 \leqslant \varepsilon \leqslant \varepsilon^*}(\|\Delta A(\varepsilon)\|_2) < 1$ with $\overline{K}_d$ defined in (14).

4. Simulated Example

The following third-order system, whose state-space description lie within the class of hybrid plants (1), is considered:

$$
\ddot{y}(t) + a_1\dot{y}(t) + a_2 y(t) + a_3 y[k] + 4y[k-1]
$$
$$
= b_0 u(t) + b_1\dot{u}(t) + b_2 u[k] + 3u[k-1] + 0.3(z[k] + \delta(t)),
$$
$$
z[k+1] = 0.2z[k] + 1.1u[k] + 1.3y[k], \quad \dot{\delta}(t) = -7\delta(t) + 8.5u(t)
$$

for all $t \in [kT, (k+1)T)$ and any integer $k \geqslant 0$. The signal $u(t)$ is a stabilizing output-feedback control signal generated from an hybrid controller as follows:

$$
u(t) = \frac{G_1(D, q)}{L(D, q)}u(t) + \frac{G_2(D, q)}{L(D, q)}y(t),
$$
$$
G_1(D, q) = D^2q^2 - q^2D + D^2q + 1.25q^2 - Dq + 0.25q
$$
$$
\qquad\qquad - 1.44187D^2 + 0.206426D - 2.54251,
$$
$$
G_2(D, q) = 1.12792(D^2q^2 - 0.269774q^2D + 1.10629),
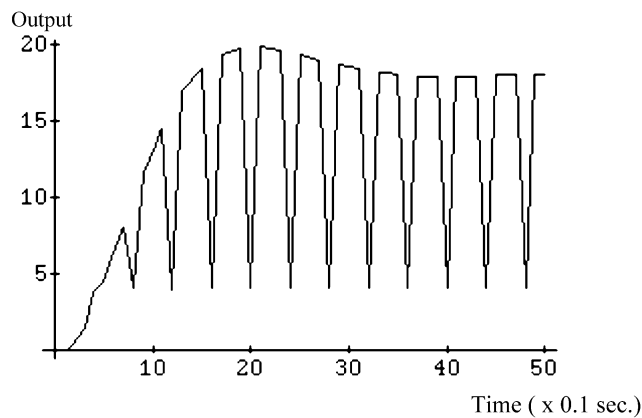$$
$$
L(D, q) = (D - 0.5)^2(q + 0.5)^2,
$$

Fig. 1. Output versus time plot.

where q is the discrete one-step advance operator and D is the time-derivative operator. After substituting the control law in the plant description, the resulting closed-loop system is of the general form given while driven only by the disturbance $\delta(t)$. The signal $\delta(t) = \delta_c(t)$ is a perturbation which satisfies the general assumptions – constraints C of the theory of total stability. There are six parameters to be estimated by the estimation schemes are $a_1 = -1$, $a_2 = 2$, $a_3 = 3$, $b_0 = 1$, $b_1 = b_2 = 2$ and $b_3 = 3$. The sampling period is $T = 0.4$. Finally, the reference model is a third-order highly damped one of discrete regulation. The plant output is shown in Fig. 1.

Note that both the extended discrete system and the continuous one are stable since the output is bounded for all time. Remark 1, which is a combined interpretation of Propositions 1–3, holds with all the signals in the loop being uniformly bounded for all time, i.e., "at" and "in-between" sampling instants. If the disturbance $\delta(t)$ is zeroed then the closed-loop system is globally asymptotically stable.

5. Concluding Remarks

The robust stability of a class of linear and time-varying hybrid systems has been investigated. The obtained sufficient conditions on robust stability are obtained by ensuring the total stability of the overall discretized system (whose state includes the digital sub-state plus the sampled continuous-time sub-state) at sampling instants and that of the continuous-time sub-state in-between sampling instants. The obtained sufficient conditions on total stability require the exponential stability of the uncoupled nominal continuous-time and digital subsystems, the sufficient weakness of the couplings between the linear parts of both sub-states and the sufficient smallness of the over-bounding functions of the state perturbations. Some links with previous work have been established for a special case of hybrid description through a combination of the obtained results with Schur's-type stability criteria for singularly perturbed systems.

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Apie totalų vienos hibridinių sistemų klasės stabilumą

Manuel De la SEN

Taikant totalinio stabilumo teoremas, sukurtas tolydžiojo bei diskrečiojo laiko sistemoms, gauti patvariojo stabilumo rezultatai nominaliai tiesinėms hibridinėms sistemoms. Išnagrinėta hibridinių sistemų klasė, susidedanti iš tolydžiojo laiko ir skaitmeninių sistemų junginio su perturbacijomis, kurių nominalios (t.y. neperturbuotos) dalys, bendru atveju, esti tiesinės bei besikeičiančios laiko atžvilgiu. Gautos pakankamos patvariojo stabilumo sąlygos priklauso nuo parametrų verčių, apibrėžiančių neapibrėžtumų funkcijas bei analoginės ir skaitmeninės posistemių junginio taikymo pagrįstumą, numatant iš anksto, kad atitinkamos nesujungtos nominalios posistemės esti eksponentiškai stabilios.