INFORMATICA, 1991, Vol.2, No.3, 331-351

OPTIMIZATION OF POLIMODAL LOCALLY MONOTONE PSEUDOBOOLEAN FUNCTIONS

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Abstract. The characteristics of the polymodal locally strictly monotone pseudoboolean functions and ones having constancy sets are investigated in this paper; the searchal algorithms for their optimization are proposed; analytical investigation of the proposed algorithms effectiveness is carried out. The paper is a continuation of the authors' researches which were begun before.

Key words: global pseudoboolean optimization, monotone functions, searchal regular algorithms, effectiveness of optimization.

Introduction. The classical problem of pseudoboolean optimization (see Saaty, 1970) is solved:

$$f(X) \longrightarrow \min_{X \in B_n},$$

where $f : B_n \longrightarrow \mathbb{R}^1$, $B_n = \{0,1\}^n$ is boolean hypercube.

As far as the optimized function is concerned it is assumed that it is given implicitly (as an output of some technical system) or algorithmically, i.e., the function has no evident analytical form.

In introduction we shall list the necessary definitions and formulate some statements on the punctiform sets of the boolean variables space and the unimodal pseudoboolean functions characteristics proved by Antamoshkin, Saraev and Semionkin (1990).

DEFINITION 0.1. We shall call points $X^1, X^2 \in B_n$ k-neighboring if they differ only in the values of k coordinates ($k = \overline{1,n}$). 1-neighboring points will be called simply neighbouring.

DEFINITION 0.2. The set $O_k(X)$ ($k = \overline{1,n}$) of points that are k-neighboring to the point $X \in B_n$ will be called the k-th level of point $X (O_0(X) = X)$. The point $X \in B_n$ is introduced as k-neighbouring to the set $A \subset B_n$ if $A \cap O_k(X) \neq \varnothing \wedge \forall l = \overline{0, k-1} : A \cap O_l(X) = \varnothing$. The set $O_k(A) \subset B_n$ of all points of B_n which are k-neighboring to the set A will be called the k-th level of set A, $O_0(A) = A$.

REMARK 0.1. It is obvious that for any $k = \overline{1,n}$: card $O_k(X) = C_n^k$. Here (and in the sequel) C_n^k is the number of combinations from n on k.

The function $f : B_n \longrightarrow \mathbb{R}^1$ will be called a pseudoboolean function.

DEFINITION 0.3. A point $X^* \in B_n$ for which $f(X^*) < f(X)$ $\forall X \in O_1(X^*)$ will be called a local minimum of the pseudoboolean function f.

DEFINITION 0.4. A pseudoboolean function which has only one local minimum on B_n will be called unimodal.

DEFINITION 0.5. A unimodal pseudoboolean function f will be called strictly monotone on B_n if

$$f(X^{k-1}) < f(X^k) \; \forall X^{k-1} \in O_{k-1}(X^*) \wedge \forall X^k \in O_k(X^*),$$
$$k = \overline{1,n}.$$

DEFINITION 0.6. The set of points $W(X^0, X^l) = \{X^1, X^2, \dots, X^i, \dots, X^l\} \subset B_n$ will be called the curve between the points X^1 and X^l if for all $i = \overline{1, l}$, the point X^i is neighbouring for the point X^{i-1}.

DEFINITION 0.7. The set $A \subset B_n$ is called connected set if for any X^0, $X^l \in A$ exists a curve $W(X^0, X^l) \subset A$.

DEFINITION 0.8. The connected set of points $\sqcap_C \subset B_n$, card $\sqcap_C \geqslant 2$ such that $f(X) = C$ (C = const) for any $X \in \sqcap_C$ is called constancy set of the function f on B_n.

DEFINITION 0.9. A unimodal function f will be called monotone on B_n if $f(X^{k-1}) \leqslant f(X^k)$ $\forall X^{k-1} \in O_{k-1}(X^*) \wedge \forall X^k \in O_k(X^*)$, $k = \overline{1, n}$.

DEFINITION 0.10. We shall call the first points of the set $\sqcap_C$ the points of the set $\{\overline{X}_j^I\} = O_I(X^*) \cap \sqcap_C$ where $\sqcap_C$ is a constancy set of a unimodal pseudoboolean function f if

$$O_I(X^*) \cap \sqcap_C \neq \varnothing \wedge \forall k = \overline{1, I-1} : O_k(X^*) \cap \sqcap_C = \varnothing.$$

DEFINITION 0.11. We shall call the last points of set $\sqcap_C$ the points of the set $\{\underline{X}_j^L\} = O_L(X^*) \cap \sqcap_C$ where $\sqcap_C$ is a constancy set of a unimodal pseudoboolean function f if

$$O_L(X^*) \cap \sqcap_C \neq \varnothing \wedge \ \forall k = \overline{L+1, n} : O_k(X^*) \cap \sqcap_C = \varnothing.$$

Lemma 0.1. *If $\sqcap_C$ is a constancy set of a unimodal monotone on B_n function f then for any $X_j^t \in O_t(X^*)$ $(I < t < L,\ j = 1, \ldots, C_n^t) : X_j^t \in \sqcap_C$.*

COROLLARY 0.1. For any $\sqcap_C \subset B_n$ of a unimodal function f

$$\sqcap_C = \{\overline{X}_j^I\} \cup \Big(\bigcup_{t=I+1}^{L-1} O_t(X^*) \Big) \cup \{\underline{X}_j^L\}$$

Algorithm 1 for unimodal strictly monotone pseudoboolean functions optimization and **Algorithm 2** for the optimization of the unimodal monotone having constancy sets pseudoboolean functions were suggested by Antamoshkin, Saraev and Semionkin (1990).

Algorithm 1 for locating of the local minimum point from any initial point requires calculating of the function values in the initial point and all neighbouring to it points.

Algorithm 2 ensures going out of the constancy set (if the initial point was found in it) along the optimal trajectory to a point of strict monotoneness of the function then **Algorithm 1** is used.

As the estimate of the algorithms effectiveness we mean the number of the function computates which are required for locating an extremum of the function from any initial point. Then it is clear that for **Algorithm 1** such estimate is $(n+1)$. The following statements were proved for **Algorithm 2** by Antamoshkin, Saraev and Semionkin (1990).

Theorem 0.1. *Locating of the minimum point X^* of a unimodal function f monotone on B_n for which the condition*

$$f(X^n) \neq f(X_j^{n-1})\ \forall X_j^{n-1} \in O_{n-1}(X^*),\ \ X^n \in O_n(X^*),$$

is true, from the initial point $X^0 \in O_k(X^*) \subset \sqcap_C$ such that $O_1(X^O) \subset \sqcap_C$ by **Algorithm 2** *requires T_1 computations of f*

$$T_1 = \sum_{i=0}^{M} C_n^i + S + 1,$$

$$M = \min\{L - k,\ k - I\}$$

(I and J are the level issues of the first and the last points of the set $\sqcap_C$),

$$S = \begin{cases} I - 1, & \text{if } M = k - I, \\ n - L, & \text{if } M = L - k. \end{cases}$$

COROLLARY 0.2.

$$\max T_1 = \sum_{i=0}^{\alpha} C_n^i + S + 3,$$

where

$$\alpha = \begin{cases} (L-1)/2, & \text{if } (L-1) \text{ is even,} \\ \text{the integer part of the number} \\ (L-1)/2, & \text{if } (L-1) \text{ is odd.} \end{cases}$$

$$\overline{T}_1 = \max_{L,I} \max_{k} T_1 = \sum_{i=0}^{\beta} C_n^i + 2,$$

where

$$\beta = \begin{cases} (n-2)/2, & \text{if } (n-2) \text{ is even,} \\ \text{the integer part of the number} \\ (n-2)/2, & \text{if } (n-2) \text{ is odd.} \end{cases}$$

1. The polymodal locally monotone function and it's characteristics

DEFINITION 1.1. A pseudoboolean function having on B_n more than one local minimum will be called polymodal one.

DEFINITION 1.2. The point $Y^* \in B_n$ for which

$$f(Y^*) > f(Y) \; \forall Y \in O_1(Y^*)$$

we shall call a local maximum of the pseudoboolean function f.

We denote $Y_f = \{Y_1^*, Y_2^*, \dots, Y_p^*\} \subset B_n$ the set of all local maxima of function f.

DEFINITION 1.3. The curve $W_-^l(X^0, X^l) \subset B_n(W_+^l(X^0, X^l) \subset B_n)$ will be called the curve of decrease (increase) of function f on B_n if $f(X^{i+1}) < f(X^i) \; \forall i = \overline{1,l}$ $(f(X^{i+1}) > f(X^i) \; \forall i = \overline{1,l})$.

DEFINITION 1.4. The set $\overline{G}_l$ of all points $X \in B_n$ for which the curve of decrease $W_-^k(X, X_l^*)$, $k \geqslant 0$, exists will be called the zone of attraction of the local minimum X_l^*.

DEFINITION 1.5. If

$$O_{r_l}(X_l^*) \cap \overline{G}_l \neq \varnothing \wedge O_k(X_l^*) \cap \overline{G}_l = \varnothing \quad \forall k = \overline{r_l + 1, n}$$

then the number r_l will called the radius of attraction zone $\overline{G}_l$.

We denote $G_l = \overline{G}_l \backslash Y_f^*$.

DEFINITION 1.6. The polymodal pseudoboolean function f having Q local minima on B_n will be called the locally strictly monotone on B_n one if $G_i \cap G_j = \varnothing \; \forall i \neq j$ and for any point $X \in B_n \backslash (\{X_1^*, X_2^*, \ldots, X_Q^*\} \cup Y_f^*)$ is a certain $j = \overline{1, Q}$ such that $X \in G_j \cap O_k(X_j^*)$ and

$$f(X^{k-1}) < f(X) < f(X^{k+1}) \; \forall X^i \in O_i(X_j^*), \; i = k-1, k+1.$$

Lemma 1.1. *$O_1(G_l) \cap G_t = \varnothing$ for $l \neq t$.*

Proof. Let $X \in G_l$, $Y \in G_t$, $X \in O_1(Y) \cap O_k(X_l^*)$. It is evident that $Y \in O_{k-1}(X_l^*) \cup O_{k+1}(X_l^*)$ therefore $f(Y) < f(X)$ or $f(X) > f(Y)$.

If $f(Y) > f(X)$ then we may construct the curve of decrease $W_-^{s+1}(Y, X_l^*) = W_-^1(Y, X) \cup W_-^s(X, X_l^*)$ as $W_-^s(X, X_l^*)$ exists always due to the fact that $X \in G_l$. Hence $Y \in G_l$ by Definition 1.4. But $G_l \cap G_t = \varnothing$ by Definition 1.6. So $f(Y) < f(X)$. But in this case it is possible to construct the curve of decrease $W_-^{m+1}(X, X_t^*) = W_-^1(X, Y) \cup W_-^m(Y, X_t^*)$ as $W_-^m(Y, X_t^*)$ exists always due to the fact that $Y \in G_t$. Hence $X \in G_t$, i.e., $G_l \cap G_t \neq \varnothing$. We have contradiction in the given case too, this fact proves the lemma.

DEFINITION 1.7. The points of set G_l, $l = \overline{1, Q}$ will be called the interior points of the zone of attraction $\overline{G}_l$ of the local minimum X_l^*.

DEFINITION 1.8. The set H_l of points $X \in \overline{G}_l$, $l = \overline{1,Q}$, such that $O_1(X) \cap G_t \neq \varnothing$, $t \neq l$, we shall call the boundary of attraction zone $\overline{G}_l$ and the points of set H_l we shall call the boundary points.

REMARK 1.1. According to Definition 1.6 and Definition 1.8 from Lemma 1.1 it follows that only the points of local maxima may be the boundary points for the locally monotone function.

Lemma 1.2. *If* $X \in O_k(X_l^*) \cap G_l$ *then* $\forall Y \in O_k(X_l^*)$: $Y \in G_l$.

Proof. Let $Y \in O_2(X) \cap O_k(X_l^*)$, $Y \notin G_l$. Then three cases are possible:

1) $Y = X_s^*$, $s \neq l$;
2) $Y = Y_s^*$ – the point of local maximum;
3) $Y \in G_t$, $t \neq l$.

In first case for point $X^1 \in O_1(X) \cap O_1(Y) \cap O_{k-1}(X_l^*)$ the relation $f(X^1) > f(Y)$ and $f(X^1) < f(X)$ are correct. Hence the curve of decrease $W_-^2(X,Y) = \{X, X^1, Y\}$ exists, i.e., $X \in G_l \cap G_s$, $l \neq s$, but it contradicts Definition 1.6.

For the second case we shall consider the point $X^1 \in O_1(X) \cap O_1(Y) \cap O_{k+1}(X_l^*)$. As $f(Y) > f(X^1)$ then X^1 is not the point of local maximum, and as $f(X^1) > f(X)$ then X^1 is not the point of local minimum too. As $f(X) > f(X^1)$ and $Y \in O_k(X_l^*)$, $X^1 \in O_{k+1}(X_l^*)$ then $X^1 \notin G_l$. Finally if $X^1 \in G_t$, $t \neq l$, then $X^1 \in G_t \cap O_1(G_l)$, that contradicts to Lemma 1.1. Thus for X^1 any possibility is excepted. It means that Y may not be the point of local maximum.

In third case we shall reconsider the point $X^1 \in O_1(Y) \cap O_1(X) \cap O_{k-1}(X_l^*)$. First of all $f(X^1) < f(X)$ as $X \in \in O_k(X_l^*) \subset G_l$. If $F(X^1) > f(Y)$ then the curve of decrease $W_-^{s+1}(X^1, X_t^*) = W_-^1(X^1, Y) \cup W_-^s(Y, X_t^*)$ exists, i.e., $X^1 \in G_t$. As $X^1 \in O_1(X)$, $X \in G_l$, then $G_l \cap O_1(G_t) \neq \varnothing$ but it contradicts to Lemma 1.1. It means that $f(X^1) < f(Y)$. It is evident that X^1 may not be the maximum point. $X^1 \notin G_t$ and $X^1 \notin G_l$ as in this case $O_1(G_l) \cap O_1(G_t) \neq \varnothing$, and $X^1 \notin G_s$, $s \neq l \wedge s \neq t$, as then $O_1(G_l) \cap G_s \neq \varnothing$ and $G_s \cap O_1(G_t) \neq \varnothing$.

And finally X^1 may not be the minimum point as then X and Y will appertain to the attraction zone of it that contradicts Lemma 1.1 besides. Thus we have contradiction in this case too.

Uniting 1–3 according to Definition 1.6 we shall obtain $Y \in G_l$.

Let now $Y \in O_{2m}(X)$. We shall construct the curve $W^{2m}(X,Y) \subset O_k(X_l^*) \cup O_{k-1}(X_l^*)$. If $Y \notin G_l$ then $X^{2m-2} \notin G_l$ too and so on. As $X \in G_l$ then $X^2 \in G_l$ and so on. It is clear that there are the points $X^{i-1} \in W^{2m}(X,Y)$ and $X^{i+1} \in W^{2m}(X,Y)$ such that $X^{i-1} \in G_l$ and $X^{i+1} \notin G_l$ moreover $X^{i+1} \in O_2(X^{i-1}) \cap O_k(X_l^*)$. As it was shown before it was not possible. The lemma is proved.

COROLLARY 1.1. If $X \in O_k(X_l^*) \cap H_l$ then $\forall Y \in O_k(X_l^*)$: $Y \in H_l$.

Proof. If $Y \in G_l$ then Lemma 1.2 $X \notin H_l$. In addition as $X \in H_l$ then $\exists X^1 \in G_l \cap O_{k-1}(X_l^*)$. And so $O_{k-1}(X_l^*) \in G_l$. If $Y \in G_s$, $s \neq l$, then $\exists X^1 \in O_{k-1}(X_l^*) \cap O_1(Y)$ and hence $O_1(G_s) \cap G_l \neq \varnothing$, that contradicts Lemma 1.1. Y may not be the local minimum point by similar reasons. The corollary is proved.

Lemma 1.3. *If* $O_k(X_l^*) \subset G_l$ *then* $O_{k-1}(X_l^*) \subset G_l$.

Proof. Let $X \in O_{k-1}(X_l^*)$ and $X \notin G_l$. Then the point X may not be a local maximum as $f(Y) > f(X)\ \forall Y \in O_k(X_l^*)$. $X = X_s^*$ $(s \neq l)$ is not possible too as in this case $\forall Y \in O_1(X) \cap O_k(X_l^*) : Y \in G_s \cap G_t$ is correct that contradicts Definition 1.6. As $X \in O_1(Y)$ and $Y \in O_k(X_l^*) \cap G_l$ then for case $X \in G_s$, $s \neq l$, we have $X \in O_1(G_s) \cap G_l$ that contradicts Lemma 1.1. The lemma is proved.

Lemma 1.4. *If the radius of attraction zone of local minimum X_j^* for the polymodal locally strictly monotone function f is equal r_j then*

$$\overline{G}_j = \bigcup_{i=0}^{r_j} O_i(X_j^*)$$

and also $H_j = O_{r_j}(X_j^*)$.

Proof. If in accord with Definition 1.5 $O_{r_j}(X_j^*) \cap \overline{G}_j \neq \varnothing$ then there is the point $X \in O_{r_j}(X_j^*) \cap \overline{G}_j$. By Lemma 1.2 and Corollary 1.1 $O_{r_j}(X_j^*) \subset \overline{G}_j$.

Let $\exists X \in O_{r_j}(X_j^*) \cap G_j$. Consider the point $Y \in O_{r_j+1}(X_j^*)$. $Y \notin G_l, l \neq j$, as it contradicts Lemma 1.1. Y is not the minimum point by similar reasons. It means that Y is the maximum point. But then $Y \in H_j$ as $O_1(Y) \cap G_j \ni X$ that contradicts the definition of r_j. Thus $\forall X \in O_{r_j}(X_j^*) : X \in H_j$, i.e., X is a maximum point. As the maximum points may not be 1-neighbouring points (Definition 1.2) then all the points from $O_{r_j-1}(X_j^*)$ belong to G_j. By Lemma 1.3 all the points from $O_{r_j-2}(X_j^*)$ belong to G_j too. Evidently that it is true for all the levels of the point X_j^* having the number less than r_j. The lemma is proved.

COROLLARY 1.2. Under conditions of Lemma 1.4 it is correct: if $Y \in B_n$ is a local maximum point then $\forall X \in O_1(Y) : X \in G_l$ for certain $l = \overline{1, Q}$.

COROLLARY 1.3. Under conditions of Lemma 1.4 it is correct: if $X_1^* \in O_k(X_2^*)$ then $\overline{G}_1 \cap \overline{G}_2 \neq \varnothing$ if and only if $r_1 + r_2 = k$.

If we put " $\leqslant$ " instead of the strict inequality in Definition 1.6 then wi'll have the definition of the polymodal locally monotone pseudoboolean function. It is not difficult to see that similarly as in the unimodal case, such definition assumes the possibility of the constancy sets presence (Definition 0.8) inside the attraction zones of local minima.

REMARK 1.2. It is not difficult to show that if $\sqcap_C$ is the constancy set of a polymodal locally monotone pseudoboolean function, G_l is the attraction zone of the local minimum X_l^*, $l = \overline{1, Q}$, and $\sqcap_C \subset G_l$ then $\sqcap_C$ has form indicated in Corollary 0.1.

The proof of this fact may be obtained easily by repeating of the Lemma 0.1 arguments supposing the change of B_n on G_l and X^* on X_l^*.

In this connection we may assert that all the characteristics of constancy sets for the unimodal case remain true for polymodal function too. This statement is easy to explain considering the fact that a polymodal locally monotone pseudoboolean function is unimodal monotone one in the attraction zone of it's any local minimum.

2. The optimization algorithm for the locally strictly monotone functions. Now we consider the polymodal locally strictly monotone on B_n pseudoboolean function f having $Q > 1$ local minima.

As it was noted above, the function f was the unimodal strictly monotone one inside the attraction zone of any local minimum. It means that from arbitrary intrinsic point of any zone of attraction the local minimum may be located by Algorithm 1 after $(n+1)$-th computations of the function. Note that here it is essential to distinguish the intrinsic and boundary points.

For maximal utilization of whole information obtained on previous step it is best to locate the local minimum from any point which is 1-neighbouring one to a boundary point as first of all this point is intrinsic one for the attraction zone of some local minimum by all means and secondly in this case for locating of the local minimum it will be done with two function computations less.

Lemma 1.4 permits to organize the local minima examination so that the reiterations and random walk are excluded. Moreover with the help of Lemma 1.4 it is possible to work out the stop criterion, i.e., a rule for determination of instant when all the local minima are located. Let us discuss it more explicitly.

After locating of the first local minimum it is necessary to locate an initial point for the search of a next local minimum. Let X_1^* is the located local minimum and $\overline{G}_1$ is the attraction zone of it. It is necessary to know the radius r_1 of the attraction zone $\overline{G}_1$ for to locate such initial point X^0. The locating r_1 is possible by examination of with step by step moving off X_1^*. We may consider r_1 determined after diminishing of the function value in the next point in its turn. Evidently that the discribed procedure for locating r_1 is not necessary if the first initial point is the local maximum point. Further if as the new initial point X^0 to choose the point in which the function value has diminished (when determining r_1) then it will be 1-neighboring point to the boundary H_2 of the attraction zone $\overline{G}_2$ of the new local minimum X_2^* which may be located by **Algorithm 1.**

If $X^0 \in O_k(X_2^*)$ then $r_2 = k + 1$, i.e., there are not computations of the function for locating of r_2.

If the local minima $X_1^*, X_2^*, \ldots, X_m^*$ and their attraction zone are known then the new initial point for locating X_{m+1}^* from the condition

$$X^0 \in O_1\Big(\bigcup_{j=1}^{m} \overline{G}_j\Big)$$

is chosen. After locating of all minima, i.e., when the condition

$$B_n \backslash \Big(\bigcup_{j=1}^{m} \overline{G}_j\Big) = \varnothing$$

is fulfilled, the search is stopped.

Taking into account the statements mentioned above, we proposed the following algorithm for optimization of the polymodal locally strictly monotone pseudoboolean functions.

Algorithm 3.

1. Choose the point $X^0 \in B_n$ arbitrarily. Suppose $t = 1$.
2. Determine $X^j \in O_1(X^0)$, $j = \overline{1,n}$.
3. Computate the values $f(X^0)$ and $f(X^j)$, $j = \overline{1,n}$, if they are still unknown.
4. If there is $X^j \in O_1(X^0)$ such that $f(X^j) > f(X^0)$ then pass to 6.
5. Suppose $X^0 = X^j$ for arbitrary $X^j \in O_1(X^0)$ and pass to 2.
6. Determine X_t^* by the rule:

$$x_{t,j}^* = \begin{cases} x_j^0, & \text{if } f(X^j) > f(X^0), \\ 1 - x_j^0, & \text{if } f(X^j) > f(X^0), \ j = \overline{1,n}. \end{cases}$$

7. Determine for the attraction zone the radius r_t.
8. If $\bigcup_{j=1}^{t} \overline{G}_j = B_n$ then pass to 11.
9. From the condition:

$$X^0 \in O_1\Big(\bigcup_{j=1}^{t} \overline{G}_j\Big)$$

   choose a new initial point X^0.
10. Suppose $t = t + 1$ and pass to 2.
11. Determine a global minimum X^{**} from the condition:

$$f(X^{**}) = \min_{j=\overline{1,t}} f(X_j^*).$$

It is not difficult to see that **Algorithm 3** is a generalization of Algorithm 1 for the case of polymodal locally strictly monotone function. This is necessary to understand in the sense that an unimodal strictly monotone function may be optimized by **Algorithm 3** too. In this case the expenditures for establishment of the function unimodality (strictly speaking for establishment of the fact that the attraction zone radius equals n) are added to the expenditures for locating of the minimum. In the "worst" case it will be done $2n$ computations of the function (for $X^0 = X^*$).

3. The optimization algorithm for the polymodal locally monotone functions having constancy sets. Here we shall take into account the connected constancy sets only as the unconnected ones do not influence the function optimization.

As it has been discussed before the polymodal locally monotone pseudoboolean function f was the unimodal one on compact subsets of the local minima attraction zone, i.e., for optimization of it might be used **Algorithm 2.** Taking into account that outside the constancy sets f is the strictly locally monotone function, i.e., it may be optimized by **Algorithm 3,** it is sufficient to define on optimization strategy on the points of $O_1(\sqcap_C)$, where $\sqcap_C$ is a constancy set, depending on the $\sqcap_C$ situation with respect to the local minima attraction zones boundaries.

Let $\sqcap_C \subset G_l$. If in addition

$$O_1(\sqcap_C) \cap O_{r_l}(X_j^*) = \varnothing$$

then we do with the unimodal monotone onto G_l function f and from the point $X^0 \in G_l$ it may be optimized by **Algorithm 2.**

If ever $\exists X \in O_1(\sqcap_C) \cap O_{r_l}(X_l^*)$ or that the same $\exists X \in O_1(\sqcap_C) \cap H_l$ then the point X is the local maximum point and so $\forall X^1 \in O_1(X) : f(X^1) < f(X)$. If in according to the step 5 of Algorithm 3 the arbitrary point $X^1 \in O_1(X) \cap$

G_l will be chosen then we shall have: $\exists k = \overline{1,n}$, such that $f(X_j^1) = C$, $j = \overline{1,k}$, and $\forall j = \overline{k+1,n} : f(X_j^1) > f(X^1)$ where $X_j^1 \in O_1(X^1)$, $j = \overline{1,n}$. From here it is clear that $X^1 \in O_1(X) \cap O_{r_j - 1}(X_l^*)$ and X_l^* is determined uniquely. If ever we choose $X^1 \in O_1(X) \cap G_s$, $s \neq l$, then either we have the considered above situation, but for another constancy set $\sqcap_{C_1} \subset G_s$ (by the way it is possible that $C_1 = C$ and X_s^* will be determined uniquely) or the constancy set is absent at all and we have the case of strictly monotone function which may be optimized by **Algorithm 3.**

Note that the troubles connected with the choice of an initial point (according to **Algorithm 3**) and the going out of constancy set (according to **Algorithm 2**) will influence the optimization during the first stage only, i.e., up to the moment when the first local minimum is chosen (the step 9 of **Algorithm 3**) as 1-neighbouring one to a boundary point and next in turn the local minimum is determined uniquely always from it (see above).

Everything said above permit to propose the following algorithm for the optimization of the polymodal locally monotone pseudoboolean function having constancy sets.

Algorithm 4.

1. Choose the point $X^0 \in B_n$ arbitrarily. Suppose $t = 1$.
2. Determine $X^j \in O_1(X^0)$, $j = \overline{1,n}$.
3. Computate the values $f(X^0)$ and $f(X^j)$, $j = \overline{1,n}$, if they are still unknown.
4. If $f(X^j) < f(X^0)$ $\forall j = \overline{1,n}$ then suppose $X^0 = X^j$ for arbitrary $j = \overline{1,n}$, and pass to 2.
5. Using the corresponding means of **Algorithm 2** for given X^0 locate the local minimum point X_t^*.
6. Determine the radius r_t of the attraction zone of local minimum X_t^*.
7. If $B_n = \bigcup_{j=1}^{t} \overline{G}_j$ then pass to 10.

8. Choose the new initial point X^0 from the condition:

$$X^0 \in O_1\Big(\bigcup_{j=1}^{t} \overline{G}_j\Big).$$

9. Suppose $t = t + 1$ and pass to 2.
10. Determine a global minimum X^{**} from the condition:

$$f(X^{**}) = \min_{j=\overline{1,t}} f(X_j^*).$$

The **Algorithm 4** is a generalization of both **Algorithm 2** and **Algorithm 3** for the case of the polymodal locally monotone pseudoboolean function having constancy sets.

In the case when the optimized function is an unimodal monotone function the working of **Algorithm 4** coinsiders with the working of **Algorithm 2** but after the minimum determination **Algorithm 4** does some additional computations of function (from 0 to $(n-1)$) before establishing of unimodality.

When optimizing the polymodal locally strictly monotone function the working of **Algorithm 4** coincides with the working of **Algorithm 3** completely.

4. The global optimization algorithms effectiveness. State the value of rate of convergence of **Algorithm 3.**

Theorem 4.1. *For locating of the global minimum X^{**} of the polymodal strictly monotone on B_n pseudoboolean function f such that $r_j > 1$ $\forall j = \overline{1,Q}$ from the initial point $X^0 \in O_k(X_1^*)$ by* **Algorithm 3** *it is necessary to do R_1 computations of f.*

$$R_1 = Q(n+1) + S, \quad S = \begin{cases} r_1 - k - 2, & \text{if } k < r_1, \\ n - 3, & \text{if } k = r_1. \end{cases}$$

Here Q is the function f local minima general number, r_j, $j = \overline{1,Q}$, are the attraction zones radii, X_1^* is the local minimum which will be located the first.

Proof. The number of the function f computations for locating X_1^* depends on the choice of X^0. Let $X^0 \in O_k(X_1^*)$. If $k < r_1$ then when locating X_1^* according to **Algorithm 3** $(n+1)$ computations of f will be done.

For determining r_1 there will done $r_1 - (k+1)$ more computations of the function f after that the point $X \in O_{r_2-1}(X_2^*)$ will be found. If $k = r_1$ then this fact will be established after $(n+1)$ computations (X^0 is the local maximum point). After the arbitrary point $X \in O_1(X^0)$ choice is done it is necessary to do $(n-1)$ more the function f computations for locating X_1^* (the values of f in the points X and $X^0 \in O_1(X)$ have been known already). Evidently that r_1 is known, and there is no necessity to determine it. Thus for locating X_1^* it is necessary to do $(n + 1 + r_1 - (k+1))$ computations of the function if $k < r_1$ and $((n+1)+(n-1))$ computations of one if $k = r_1$.

For locating X_2^* from the point $X \in O_{r_2-1}(X_2^*)$ when $k < r_1$ it is necessary to do n computations of the function f as in one point from $O_1(X) \cap O_{r_1}(X_1^*)$ the value f is calculated when determining r_1. When $k = r_1$ the point X_2^* may be located after $(n-1)$ computations of the function if the point $Y \in O_1(X^0) \cap O_{r_1+1}(X_1^*)$ is chosen as the initial point. The fact is that card $\{O_1(X) \cap O_1(Y)\} = 2$, i.e., the function values have been known already for two points.

In that way for locating of X_1^* and X_2^* $(n + 1 + r_1 - k - 1 + n)$ computations will have been done when $k < r_1$ and $(n + 1 + 2(n-1))$ computations when $k = r_1$.

Let us assume that $X_1^*, X_2^*, \dots X_m^*$, $m \geqslant 2$, have been known already. Then in accordance with **Algorithm 3** X_{m+1}^* is determined from the point $X^0 \in O_1(\bigcup_{j=1}^{m} G_j)$, i.e.,

$X^0 \in O_{r_{m+1}-1}(X^*_{m+1})$. For locating X^*_{m+1} from the point X^0 it is sufficient to evaluate the function in all points of $O_1(X^0)$, i.e., to do $(n+1)$ computations of the function f. For the reason that $r_j > 1\ \forall j = \overline{1,Q}$, the function values are unknown in all points of $O_1(X^0)$. These discourses remain valid for all $m = \overline{2,Q}$. I.e., when X_1^* and X_2^* are known for locating the over $(Q-2)$ local minima it is necessary to do $(Q-2)(n+1)$ computations of the function f.

Therefore for locating of all local minima including and global one it is necessary to do $((Q-2)(n+1)+(n+1)+r_1-k-1+n) = (Q(n+1)+r_1-k-2)$ computations of the function f in the case when $k < r_1$ and $((Q-2)(n+1)+(n+1)+2(n-1)) = (Q(n+1)+n-3)$ computations for the case when $r_1 = k$.

The theorem is proved.

REMARK 4.1. If the function f has the local minima with unit radius of attraction zones then the estimate of the theorem is reduced at the expense of the fact that when locating a next local minimum the function computations, done in the intersection points of the corresponding attraction zones when locating the previous local minima, are taken into consideration.

COROLLARY 4.1. $\max\limits_{Q} R_1 = 2^n$.

Proof. The maximal values of R_1 is reached in the case when $r_j = 1\ \forall j = \overline{1,Q}$, i.e., when $Q = 2^{n-1}$. In this case all points of the set B_n are either the local minima points or the 1-neughbouring points to them.

Theorem 4.2. *The locating of the minimum point X^* for the unimodal strictly monotone pseudoboolean function f from an initial point $X^0 \in O_k(X^*)$, $k < n$, requires R_2 function evaluations, where*

$$R_2 = 2n - k.$$

Proof. For locating of a minimum point **Algorithm 3** makes $R_2^1 = n + 1$ computations of f (the step 3). After that minimum attraction zone radius is determined (the step 7). As in the points of $O_{k+1}(X^*)$ the value of f is known the next point is taken from $O_{k+2}(X^*)$ then from $O_{k+3}(X^*)$ and so on. The last point is chosen from $O_n(X^*)$. There will be $R_2^2 = n - (k+1) = n - k - 1$ such point all in all. Summing up R_2^1 and R_2^2 we have R_2.

COROLLARY 4.2.

$$\max_{0 \leqslant k \leqslant n} R_2 = 2n \qquad \min_{0 \leqslant k \leqslant n} R_2 = n + 1.$$

Review the **Algorithm 4** effectiveness.

When optimizing the polymodal locally strictly monotone function the **Algorithm 4** work consider completely with **Algorithm 3** work and so in this case the effectiveness estimates will coincide completely with the **Algorithm 3** estimates given in the present paragraph.

The following theorem gives an effectiveness estimate for **Algorithm 4** in general case.

Theorem 4.3. *Let f is a polymodal locally monotone pseudoboolean function having constancy sets. $\overline{G}_1$ is the local minimum X_1^* attraction zone, r_1 is the radius of it, $\sqcap_C \subset G_1$ is the constancy set of the function f, I and J are the level numbers for the first and the last points of the set $\sqcap_C$ in $\overline{G}_1$ with respect to X_1^* (see the definitions 0.10 and 0.11).*

Then the function f optimization from an initial point $X^0 \in O_k(X_1^*)$, $I < k < J$, requires the function values computation not more than in R_3 points of the space B_n.

$$R_3 = Q(n+1) + \sum_{i=0}^{M+1} C_n^i - C_k^{M+1} + n + r_1 - 2M - k,$$

where Q is the function f local minima number,

$$M = \min\{k - I, J - k\}.$$

The Theorem 4.3 proof leans on Theorem 0.1, the **Algorithm 4** discription and the fact that it is not necessary to get over more than one constancy set.

COROLLARY 4.3.

$$\max_{I<k<J} R_3 = Q(n+1) + \sum_{i=0}^{\alpha+1} C_n^i - C_{I+\alpha-1}^{\alpha+1} + n + r_1 + 3\alpha,$$

where

$$\alpha = \begin{cases} \frac{J-I}{2} & \text{for even } (J-I), \\ \text{the nearest integer of the number} \\ \frac{J-I}{2} & \text{for odd } (J-I). \end{cases}$$

COROLLARY 4.4.

$$R_3^* = \max_{1 \leqslant I \leqslant J \leqslant n-1} \max_{I<k<J} R_3 = \sum_{i=0}^{S+1} C_n^i + 4n - S,$$

where

$$S = \begin{cases} \frac{n-2}{2} & \text{for even } n, \\ \text{the nearest integer of the number} \\ \frac{n-2}{2} & \text{for odd } n. \end{cases}$$

The Corollaries 4.3 and 4.4 proofs lean on Corollary 0.2 and Theorem 4.3.

It is seen well from Corollary 4.4 that even in the worst case **Algorithm 4** makes examination of a little more than half of all points of the space B_n, i.e., it has effectiveness which is almost twice higher than the total examination effectiveness. The advantage of Algorithm 4 over the total examination increases when dimension increases.

A few more words about the **Algorithm 4** work. When there are constancy sets having large cardinalities, the probability of event, that the point X^0 lies inside the constancy set, increases, it is the reason of the large expenditures on optimization. In this connection it is possible to make an interesting conclusion. The **Algorithm 3** convergence rate will make an increase when the local minima and maxima points number increase and hence the possible constancy sets cardinality will decrease. However when $Q = 2^{n-1}$ the constancy sets are absent at all but the expenditures on optimization are the highest. Hence there is certain Q for which **Algorithm 4** has maximal effectiveness (on the average). Unfortunately it is not possible to state this number because of the impossibility to take into account all intercommunications among the local minima number, the constancy sets number, situation of constancy sets inside of the local minima attraction zones and others.

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Received July 1991

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