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Sequent Calculi for Temporal Logics of Common Knowledge and Belief

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Abstract. In this paper we consider two logics: temporal logic of common knowledge and temporal logic of common belief. These logics involve the discrete time linear temporal logic operators "next" and "until". In addition the first logic contains an indexed set of unary modal operators "agent i knows", the second one contains an indexed set of unary modal operators "agent i believes". Also the first logic contains the modality of common knowledge and the second one contains the modality of common belief. For these logics we present sequent calculi with an analytic cut rule. The soundness and completeness for these calculi are proved.

Key words: agents, temporal logic, common knowledge, common belief, sequent calculus, analytic cut.

1. Introduction

Temporal logics of knowledge and belief are becoming increasingly important in both mainstream computer science and AI. In AI, temporal logics of knowledge and belief are used as knowledge representation formalism (Catach, 1988), and may be used in the specification and verification of distributed intelligent systems (Halpern, 1987; Wooldridge, 1992) and as a subpart of logics of rational agency (Wooldridge, 1992).

In this paper we consider generalizations of the temporal logic of knowledge and the temporal logic of belief considered in (Wooldridge *et al.*, 1998). We call the considered logics CKL_n, CBL_n, respectively. These logics involve the discrete time linear temporal logic operators "next" and "until". In addition CKL_n contains an indexed set of unary modal operators "agent i knows" and CBL_n contains an indexed set of unary modal operators "agent i believes" that allow to represent the information possessed by the group of agents. These operators satisfy the analogues of the modal axioms $S5$ and $KD45$, respectively. These systems are widely accepted as logics of idealized knowledge and idealized belief. These logics contain the modality of common knowledge and the modality of common belief as well.

For these logics we present sequent calculi with an analytic cut rule. The soundness and completeness for these calculi are proved. Our work uses the ideas from (Alberucci, 2002) and (Halpern *et al.*, 2004).

We mention some related works. The temporal logic of knowledge without common knowledge operator is considered in (Wooldridge *et al.*, 1998; Dixon *et al.*, 1998). In (Wooldridge *et al.*, 1998) a tableau based decision procedure is presented for the considered logic. In (Dixon *et al.*, 1998) a resolution-based proof system is presented which is shown to be correct. The logic of common knowledge without temporal operators is considered in (Alberucci, 2002), where complete Tait-style sequent calculus with restricted cut rule for the logic is presented. The paper is organized as follows. In the next section we provide formal definitions for the logics we consider. In Section 3 we present sequent calculi and prove the soundness theorem. In Section 4 we prove the completeness of the presented sequent calculi.

2. Language and Semantics

To define the language $\mathcal{L}$ of the logics we start from a set of *primitive propositions* $P = \{p, q, \ldots\}$, the *propositional connectives* $\neg, \wedge, \vee$, the *modalities* $[1], \ldots, [n]$, the *modality* E, the *common knowledge (belief) modality* C and the *temporal modalities*: unary operator $\bigcirc$ and a binary operator U. If ϕ is a formula $[i]\phi$ says that agent i knows (believes) ϕ, a formula $E\phi$ says that every agent knows (believes) ϕ, a formula $C\phi$ says that ϕ is a common knowledge (belief) of all agents, a formula $\bigcirc\phi$ says that ϕ is true at the next time moment, a formula $\phi U \psi$ says that ϕ holds until ψ does.

In order to define semantics, we first introduce the notion of a *state*. It is assumed that the world may be in any of a set S of *states*. We generally use s to denote a state. The internal structure of states is not an issue in this work. As we interpret $\mathcal{L}$ over linear temporal structures, it is natural to introduce the notion if a *timeline*, representing the history of the system. A *timeline* l is an infinitely long, linear, discrete sequence of states, indexed by natural numbers. For convenience, we define a timeline l to be a total function $l\colon N \to S$. Let $Tlines$ be the set of all timelines. Note that timelines correspond to the *runs* of Halpern, Meyden and Vardi (Halpern *et al.*, 2004). A *point*, p, is pair (l, u), where $l \in Tlines$ is a timeline and $u \in N$ is a temporal index into l. Any point (l, u) will uniquely identify a state $l(u)$. Let the set of all points (over S) be *Points*. We then let an agent's knowledge (belief) accessibility relation R_i hold over *Points*, i.e., $R_i \subseteq$ *Points*$\times$*Points*, for all $i \in \{1, \ldots, n\}$. A *valuation* for $\mathcal{L}$ is a function that takes a point and a proposition, and says whether that proposition is true or false at that point. A *valuation*, π, is a function π: *Points* $\times P \to \{T, F\}$. We can now define models for $\mathcal{L}$.

A *model*, M, for $\mathcal{L}$, is a structure $M = (TL, R_1, \ldots, R_n, \pi)$, where:

- $TL \subseteq Tlines$ is set of timelines;
- R_i, for all $i \in \{1, \ldots, n\}$, is an agent accessibility relation over *Points*, i.e., $R_i \subseteq$*Points*$\times$*Points* and
- π: *Points* $\times P \to \{T, F\}$ is a valuation.

As usual, we define the semantics of the language via satisfaction relation "$\models$". For $\mathcal{L}$, this relation holds between pairs of the form $(M, (l, u))$, where M is a model and (l, u) is a point, and $\mathcal{L}$ formulas. $E\phi$ stands for $[1]\phi \wedge \ldots \wedge [n]\phi$.

- $(M,(l,u)) \models p$ iff $\pi(l,u)(p) = T$, where p is a primitive proposition;
- $(M,(l,u)) \models \neg\phi$ iff $(M,(l,u)) \not\models \phi$;
- $(M,(l,u)) \models \phi \vee \psi$ iff $(M,(l,u)) \models \phi$ or $(M,(l,u)) \models \psi$;
- $(M,(l,u)) \models \phi \wedge \psi$ iff $(M,(l,u)) \models \phi$ and $(M,(l,u)) \models \psi$;
- $(M,(l,u)) \models [i]\phi$ iff $\forall l' \in TL$, $\forall v \in N$, if $((l,u),\ (l',v)) \in R_i$, then $(M,\ (l',v)) \models \phi$;
- $(M,\ (l,u)) \models C\phi$ iff $(M,\ (l,u)) \models E^k\phi$ for $k = 1,\ldots$, where $E^1\phi = E\phi,\ E^{k+1}\phi = EE^k\phi$;
- $(M,\ (l,u)) \models \bigcirc\phi$ iff $(M,\ (l,u+1)) \models \phi$;
- $(M,\ (l,u)) \models \phi U \psi$ iff $\exists v \in N$ such that $v \geqslant u$ and $(M,\ (l,v)) \models \psi$ and $\forall \omega \in N$, if $u \leqslant \omega < v$ then $(M,\ (l,\omega)) \models \phi$

We use standard abbreviation: $\phi \supset \psi$ stands for $\neg\phi \vee \psi$.

An $\mathcal{L}$ formula ϕ is *satisfiable* iff there is some $(M,\ (l,u))$ such that $(M,\ (l,u)) \models \phi$, and unsatisfiable otherwise. An $\mathcal{L}$ formula ϕ is *valid* in a model M iff $(M,\ (l,u)) \models \phi$ for every point $(l,u) \in M$. If $\mathcal{C}$ is a class of models, then ϕ is valid with respect to $\mathcal{C}$ iff ϕ is valid in every model in $\mathcal{C}$. We write $\models_K \phi$ ($\models_B \phi$), if ϕ is valid with respect to the class of models of logic CKL_n (CBL_n) . An $\mathcal{L}$ model $M = (TL, R_1,\ldots,R_n,\pi)$ is a CKL_n (CBL_n) model iff for all $i \in \{1,\ldots,n\}$, R_i is an equivalence relation (Euclidean, serial and transitive relation).

It is well-known that the following axioms are valid in CKL_n models:

K: $[i]\phi \wedge [i](\phi \supset \psi) \supset [i]\psi$; T: $[i]\phi \supset \phi$; 4: $[i]\phi \supset [i][i]\phi$; 5: $\neg[i]\phi \supset [i]\neg[i]\phi$, C: $C\phi \supset (E\phi \wedge EC\phi)$. It is well-known that the axioms presented above except the axiom T and the axiom D: $[i]\phi \supset \neg[i]\neg\phi$ are valid in CBL_n models.

There is a graphical interpretation of the semantics C which is useful in the sequel. Fix a model M. A point (l',u') in M is *reachable* from a point (l,u) if there exists points $(l_0,u_0),\ldots,(l_k,u_k)$ such that $(l,u) = (l_0,u_0),\ (l',u') = (l_k,u_k)$, and for all $j = 0,\ldots,k-1$ there exists i such that $(l_j,u_j)R_i(l_{j+1},u_{j+1})$. It can be verified the following

Lemma 2.1. $(M,\ (l,u) \models_W C\phi$ *iff* $(M,\ (l',u')) \models_W \phi$ *for all points* (l',u') *reachable from* (l,u)*, where* $W \in \{K,B\}$.

3. Tait-Style Sequent Calculi

In this section we introduce a Tait-style sequent calculi KT and BT for the temporal logic of common knowledge and for the temporal logic of common belief, respectively. The calculus KT is a reformulation of the Hilbert-type calculus presented in (Halpern *et al.*, 2004).

As usual $p,q,\ldots$ stand for primitive propositions and small Greek letters for arbitrary formulas. Further, the capital Greek letters $\Gamma,\Delta,\Sigma,\ldots$ stand for finite subsets of $\mathcal{L}$ formulas which are called *sequents*. For any sequents Γ,Δ and formulas α,β the sequent $\Gamma \cup \Delta \cup \{\alpha\} \cup \{\beta\}$ is denoted by $\Gamma,\ \Delta,\ \alpha,\ \beta$. Let Γ be the sequent $\{\alpha_1,\ldots,\alpha_n\}$, we often use the following convenient abbreviations:

$\vee\Gamma = \{\alpha_1 \vee \ldots \vee \alpha_n\}$; $\neg\Gamma = \{\neg\alpha_1, \ldots, \neg\alpha_n\}$; $\neg[i]\Gamma = \{\neg[i]\alpha_1, \ldots, \neg[i]\alpha_n\}$; $[i]\Gamma = \{[i]\alpha_1, \ldots, [i]\alpha_n\}$; $\neg C\Gamma = \{\neg C\alpha_1, \ldots, \neg C\alpha_n\}$.

With the help of de Morgans laws and the law of double negation we push the negation through the propositional connectives as far as possible, i.e., if ϕ is $\alpha \wedge \beta$ then $\neg\phi$ is $\neg\alpha \vee \neg\beta$, if ϕ is $\alpha \vee \beta$, then $\neg\phi$ is $\neg\alpha \wedge \neg\beta$, if ϕ is $\neg\alpha$, then $\neg\phi$ is α.

Let us introduce the Tait-style calculus KT for the logic CKL_n. All the rules are represented as schemes.

Axiom of KT: $\Gamma,\ \alpha,\ \neg\alpha$

Basic inference rules of KCT:

$$\frac{\Gamma, \alpha, \beta}{\Gamma, \alpha \vee \beta}(\vee) \qquad \frac{\Gamma, \alpha \qquad \Gamma, \beta}{\Gamma, \alpha \wedge \beta}(\wedge),$$

$$\frac{\neg C\Lambda, \neg[i]\Gamma, [i]\Delta, \alpha}{\neg C\Lambda, \neg[i]\Gamma, [i]\Delta, [i]\alpha, \Sigma}([i]) \qquad \frac{\Gamma, \neg\alpha}{\Gamma, \neg[i]\alpha}(\neg[i]).$$

C-rules of KT:

$$\frac{\Gamma, \neg E\alpha}{\Gamma, \neg C\alpha}(\neg C1) \qquad \frac{\Gamma, \neg EC\alpha}{\Gamma, \neg C\alpha}(\neg C2),$$

$$\frac{\neg\alpha, E\alpha \wedge E\beta}{\neg\alpha, C\beta, \Sigma}(Ind_C).$$

The rules for temporal modalities:

$$\frac{\Gamma}{\bigcirc\Gamma, \Sigma}(\bigcirc) \qquad \frac{\Gamma, \bigcirc\neg\alpha}{\Gamma, \neg\bigcirc\alpha}(\neg\bigcirc),$$

$$\frac{\Gamma, \phi_2, \phi_1 \wedge \bigcirc(\phi_1 U \phi_2)}{\Gamma, \phi_1 U \phi_2}(U) \qquad \frac{\Gamma, \neg\phi_2 \qquad \Gamma, \neg\phi_1, \bigcirc\neg(\phi_1 U \phi_2)}{\Gamma, \neg(\phi_1 U \phi_2)}(\neg U),$$

$$\frac{\neg\phi\prime, \neg\psi \wedge \bigcirc\phi\prime}{\neg\phi\prime, \neg(\phi U \psi)}(Ind_U).$$

Let us introduce the sequent calculus BT for the temporal logic of common belief CBL_n. It is obtained from the calculus KT dropping the inference rule $(\neg[i])$ and replacing the basic inference rule $([i])$ of KT by the following rule of inference:

$$\frac{\neg C\Lambda, \neg\Gamma, \neg[i]\Gamma, [i]\Delta, \alpha}{\neg C\Lambda, \neg[i]\Gamma, [i]\Delta, [i]\alpha, \Sigma}([i]).$$

We did not introduce any cut rules since we want to distinguish our calculi with various additional cuts. Hence, we always mention explicitly which cut rules are admitted. Let us introduce the most general cut scheme, the *general cut rule*.

General cut:

$$\frac{\Gamma, \alpha \qquad \Gamma, \neg\alpha}{\Gamma}(G - cut).$$

In this case the designated formulas α and $\neg\alpha$ are called cut formulas of $(G - cut)$.

Let Π be a set of formulas which are closed under negation, that is we have $\neg\Pi = \Pi$. Then Π-cuts are all cuts

$$\frac{\Gamma, \alpha \qquad \Gamma, \neg\alpha}{\Gamma} (\Pi - cut)$$

such that the cut formula α belongs to Π.

Let we have a rule

$$\frac{\Gamma_1}{\Gamma} \; or \; \frac{\Gamma_1, \qquad \Gamma_2}{\Gamma}.$$

It can be verified that if $\vee\Gamma_1$ is valid or $\vee\Gamma_1$ and $\vee\Gamma_1$ are valid, then $\vee\Gamma$ is valid. So, by induction on the length of the proof it can be showed the following

Theorem 3.1 (soundness). *Let* $W \in \{K, B\}$. *If* $WT + (G - cut) \vdash \Gamma$, *then* $\models_W \vee\Gamma$.

4. Completeness

In this section we give a sketch of completeness proof of the Tait-style calculi KT and BT with the cut rule, where the cut formula is from some finite sets of formulas.

Now we define the Fisher–Ladner closure $FL(\alpha)$ of a formula α of $\mathcal{L}$. $FL(\alpha)$ is defined to be the smallest set such that: α belongs to $FL(\alpha)$; if $\neg\beta \in FL(\alpha)$, then $\beta \in FL(\alpha)$; if $\beta \vee \gamma \in FL(\alpha)$, then $\beta, \gamma \in FL(\alpha)$; if $\beta \wedge \gamma \in FL(\alpha)$, then $\beta, \gamma \in FL(\alpha)$; if $[i]\beta \in FL(\alpha)$, then $\beta \in FL(\alpha)$; if $C\beta \in FL(\alpha)$, then $E\beta, EC\beta \in FL(\alpha)$; if $\bigcirc\beta \in FL(\alpha)$, then $\beta \in FL(\alpha)$; if $\beta U \gamma \in FL(\alpha)$, then $\beta, \gamma, \bigcirc(\beta U \gamma) \in FL(\alpha)$; $FL(\alpha)$ is closed under negation.

As in (Fisher and Ladner, 1979) can be verified

PROPOSITION 4.1. For an arbitrary formula α the set $FL(\alpha)$ is finite and contains not more elements than $c|\alpha|$, where $|\alpha|$ is the length of α.

Using $FL(\alpha)$ we define sets $C_{FL_B(\alpha)}$ and $C_{FL_K(\alpha)}$ of formulas which are used as cut formulas in our proof of completeness.

Let X be a finite set of formulas. Then we write φ_X for the a finite conjunction formulas in X.

The set $FL_B(\alpha)$ is defined to be $FL(\alpha) \cup \{[i][i]\beta,\ \neg[i][i]\beta | [i]\beta \in FL(\alpha),\ 1 \leqslant i \leqslant n\} \cup \{[i]\neg[i]\beta,\ \neg[i]\neg[i]\beta | \neg[i]\beta \in FL(\alpha),\ 1 \leqslant i \leqslant n\}$. The set $FL_K(\alpha)$ is defined to be $FL(\alpha)$.

Let $W \in \{K, B\}$. The set $C'_{FL_W(\alpha)}$ is defined to be the set $\{\varphi_{M_1} \vee \ldots \vee \varphi_{M_k},\ \bigcirc(\varphi_{M_1} \vee \ldots \vee \varphi_{M_k}),\ [i](\varphi_{M_1} \vee \ldots \vee \varphi_{M_k}),\ [i]\neg(\varphi_{M_1} \vee \ldots \vee \varphi_{M_k}) | M_1, \ldots, M_k \subseteq FL_W(\alpha),\ k \geqslant 1\}$. The closure $C_{FL_W(\alpha)}$ is defined to be the set $C'_{FL_W(\alpha)} \cup \{\neg\phi | \phi \in C'_{FL_W(\alpha)}\}$.

Let $W \in \{K, B\}$, Δ be a set of formulas. A finite set of $\mathcal{L}$ formulas Γ is Δ-consistent if $WT + (\Delta - cut) \not\vdash \neg\Gamma$. We write $| \vdash_W \Gamma$ if $WT + (C_{FL_W(\alpha)} - cut) \vdash \Gamma$.

Suppose CL is a finite set of formulas with the property that for all $\phi \in CL$, either $\neg\phi \in CL$ or ϕ is of the form $\neg\phi'$ and $\phi' \in CL$. We define an *atom* of CL to be a maximal $C_{FL_W}(\alpha)$-consistent subset of CL, where $W \in \{K, B\}$.

Let $W \in \{K, B\}$. Let α be a $C_{FL_W}(\alpha)$-consistent formula. We begin the construction of the model of α by first constructing a *pre-model* $M_W(\alpha)$, which is a structure $< S_W, \rightarrow, R_1, \ldots R_n >$ consisting of a set S_W of states, a binary relation $\rightarrow$ on S_W, and for each agent i a binary relation R_i on S_W. If W is K then R_i is an equivalence relation. If W is B then R_i is serial, transitive and Euclidean relation.

Let $W \in \{K, B\}$. The set S_W consists of all atoms of $FL_W(\alpha)$. The relation $\rightarrow$ is defined so that $X \rightarrow Y$ iff $\{\phi | \bigcirc(\phi) \in X\} \subseteq Y$. For $W = K$ the relation R_i is defined so that $(X, Y) \in R_i$ iff $\{\phi | [i]\phi \in X\} = \{\phi | [i]\phi \in Y\}$. It follows that relation R_i in a pre-model $M_W(\alpha)$ is an equivalence relation. In $M_B(\alpha)$ the relation R_i is defined so that $(X, Y) \in R_i$ iff $\{\phi | [i]\phi \in X\} \subseteq Y$.

As in (Alberucci, 2002) it can be proved

Lemma 4.1. *If $X \subseteq FL_W(\alpha)$ and X is $C_{FL_W(\alpha)}$-consistent, then there exists an atom Y of $FL_W(\alpha)$ such that $X \subseteq Y$.*

As in (Sakalauskaitė, 2004) it can be proved

Lemma 4.2. $| \vdash \vee_X$*is an atom of*$_{FL_W(\alpha)} \varphi_X$.

Using the definition of R_i and the definition of $FL_B(\alpha)$ it can be verified

Lemma 4.3. *Let X, Y, Z be atoms from S_B.*

a) *for each $X \in S_B$ there exists $Y \in S_B$ such that $(X, Y) \in R_i$;*
b) *if $[i]\beta \in X$ and $(X, Y) \in R_i$, then $[i]\beta \in Y$;*
c) *if $(X, Y) \in R_i$ and $(X, Z) \in R_i$ and $[i]\beta \in Y$, then $\beta \in Z$.*

From Lemma 4.3 it follows that in a pre-model $M_B(\alpha)$ R_i is serial, transitive and Euclidean relation for each $i \in \{1, \ldots, n\}$.

Below $W \in \{K, B\}$ and s, t are states from a pre-model $M_W(\alpha)$. As in (Sakalauskaitė, 2004) we can show

Lemma 4.4. *If s, t are states such that $(s, t) \notin R_i$, then $| \vdash_W \neg\varphi_s, [i]\neg\varphi_t$.*

Let U be a set of states. We write φ_U for disjunction of the formulas φ_u for $u \in U$.

We can verify similarly as in (Sakalauskaitė, 2004).

Lemma 4.5. *Let s be a state and let U be the set of states u such that $s \rightarrow u$. Then $| \vdash_W \neg\varphi_s, \bigcirc\varphi_U$.*

Let $W \in \{K, B\}$. As in (Sakalauskaitė, 2004) it can be proved the following

Lemma 4.6. *For all formulas α, β and γ, if $| \vdash_W \neg\alpha,\ \neg\gamma$ and $| \vdash_W \neg\alpha,\ \bigcirc(\alpha \vee (\neg\beta \wedge \neg\gamma))$, then $WT + (\{\bigcirc(\alpha \vee (\neg\beta \wedge \neg\gamma))\} \cup \{C_{FL_W(\alpha)}\} - cut) \vdash \neg\alpha,\ \neg(\beta U \gamma)$.*

Define a $\rightarrow$-sequence of states to be a (finite or infinite) sequence $s_1, s_2, \ldots$ such that $s_1 \rightarrow s_2 \rightarrow \ldots$.

Let $W \in \{K, B\}$ and $s, t, s_1, \ldots, s_n$ be states in a pre-model $M_W(\alpha)$. As in (Sakalauskaitė, 2004) we can show

Lemma 4.7. a) *if $\bigcirc\phi \in FL_W(\alpha)$, then for all states t such that $s \rightarrow t$ we have $\bigcirc\phi \in s$ iff $\phi \in t$;*

b) *if $[i]\phi \in FL_W(\alpha)$, then $\neg[i]\phi \in s$ iff there is some state t such that sR_it and $\neg\phi \in t$;*

c) *if $\phi_1 U \phi_2 \in FL_W(\alpha)$, then $\phi_1 U \phi_2 \in s$ iff there exists a $\rightarrow$-sequence $s = s_0 \rightarrow s_1 \rightarrow \ldots \rightarrow s_n$, where $n \geqslant 0$ such that $\phi_2 \in s_n$ and $\phi_1 \in s_k$ for all $k < n$.*

We use Lemmas 4.5, 4.6 to prove the "if" part of the item c) of Lemma 4.7.

Lemma 4.8. *If $C\phi \in FL_W(\alpha)$, then $\neg C\phi \in s$ iff there is a state t reachable from s such that $\neg\phi \in t$.*

Proof. We prove the lemma in the case $W = B$. The proof of the lemma in the case $W = K$ is similar. We prove "only if" direction by contradiction. Let $\neg\phi \in t$ and $s = s_0 R_{i_1} s_1 \ldots s_{k-1} R_{i_k} s_k = t$. Let $C\phi \in s$. By the rule $(\neg C1)$ it follows that $| \vdash \neg C\phi, E\phi$. Thus $[1]\phi, \ldots, [n]\phi \in s$. Similarly by the rule $(\neg C2)$ we have $[1]C\phi, \ldots, [n]C\phi \in s$. By induction on k we get the following fact: if $C\phi \in s_0$, then $\phi \in s_k,\ k \geqslant 1$. Thus we get a contradiction.

We prove the converse by contradiction. Suppose that no state containing $\neg\phi$ is reachable from s by the relations R_i. Let V be the set of states reachable from s. Then $\phi \in v$ for each $v \in V$. Thus $| \vdash_B \neg\varphi_V,\ \phi$ (1). Using Lemmas 4.2, 4.4 we can show that $| \vdash_B \neg\varphi_V, [i]\varphi_V$ (2). From (1) and (2) we can show that $| \vdash_B \neg\varphi_V, [i]\varphi_V \wedge [i]\phi$. This implies $| \vdash_B \neg\varphi_V, E\varphi_V \wedge E\phi$. Then by (Ind$_C$) we get $| \vdash_B \neg\varphi_V, C\phi$ (3).

It can be verified using the rule (Ind_C) that $| \vdash_B \neg E\phi, \neg CE\phi, C\phi$. Thus, from the assumption that $| \vdash_B \neg\varphi_s, \neg C\phi$ we get $| \vdash_B \neg\varphi_s, \neg E\phi, \neg EC\phi$. Since s is a maximal $C_{FL_W}(\alpha)$-consistent subset of $FL_B(\alpha)$ it follows $| \vdash_B \neg\varphi_s, \neg E\phi$ or $| \vdash_B \neg\varphi_s, \neg EC\phi$. Thus there exists i such that $| \vdash_B \neg\varphi_s, \neg[i]\phi$ or $| \vdash_B \neg\varphi_s, \neg[i]C\phi$. By the item b) of the Lemma 4.7 there exists a state t such that sR_it and $\neg\phi \in t$ or $\neg C\phi \in t$. Since $t \in V$ this contradicts to (1) and (3).

We say that an infinite $\rightarrow$-sequence of states $(s_0, s_1, \ldots)$, is *acceptable* if for all $n \geqslant 0$, if $\phi_1 U \phi_2 \in s_n$, then there exists $m \geqslant n$ such that $\phi_2 \in s_m$ and $\phi_1 \in s_k$ for all $n \leqslant k < m$. Using part c) of Lemma 4.7 we can show

Lemma 4.9. *Every finite →-sequence of states can be extended to an infinite acceptable sequence.*

A *canonical model* for α is a tuple $(\mathcal{R}, R_1, \ldots, R_n, \pi)$, where $\mathcal{R}$ is a set of all acceptable sequences of states from the pre-model $M_W(\alpha)$; R_i is a binary relation on points in $\mathcal{R}$ such that $((r,n),(r',n')) \in R_i$ if $(r(n), r'(n')) \in R_i$, where R_i is from the pre-model $M_W(\alpha)$; $\pi(r,n)(p) = T$ iff $p \in r(n)$. The following theorem gives a sufficient condition for a formula in the Fisher–Ladner closure to hold at a point in the canonical model. Let $W \in \{K, B\}$.

Theorem 4.1. *If $\mathcal{I}$ is the canonical model for α, ϕ is in the closure $FL_W(\alpha)$, then $(\mathcal{I}, (r,n)) \models_W \phi$ if and only if $\phi \in r(n)$.*

Proof. Proof is carried on by induction on the complexity of ϕ using Lemmas 4.7, 4.8, 4.9.

COROLLARY 4.1. If $\mathcal{I} = (\mathcal{R}, R_1, \ldots, R_n, \pi)$ is a canonical model for α, (r,n) is a point of $\mathcal{I}$ such that $\alpha \in r(n)$, then $(\mathcal{I}, (r,n)) \models_W \alpha$.

Let α be $C_{FL_W(\alpha)}$-consistent formula. Let $s \in S_W$ be a state such that $\alpha \in s$. Such a state must exist as it follows from Lemma 4.1. By Lemma 4.9 there exists an acceptable sequence $r = s_0, s_1, \ldots$ with $s = s_0$. Corollary 4.1 implies that $(\mathcal{I}, (r,0)) \models_W \alpha$. This establishes the following completeness theorem of the calculi $KT + (C_{FL_K}(\alpha) - cut)$, $BT + (C_{FL_B}(\alpha) - cut)$.

Theorem 4.2 (completeness). *Let α be a valid formula of the language $\mathcal{L}$ with respect to the models of the logic CKL_n (CBL_n). Then $KT + (C_{FL_{K(\alpha)}} - cut) \vdash \alpha$ ($BT + (C_{FL_{B(\alpha)}} - cut) \vdash \alpha$).*

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Sekvenciniai skaičiavimai bendro žinojimo ir tikėjimo logikoms

Jūratė SAKALAUSKAITĖ

Nagrinėjamos bendro žinojimo ir tikėjimo laiko logikos. Šioms logikoms pateikti sekvenciniai skaičiavimai, kuriuose pjūvio formulė yra apribota. Įrodytas šių skaičiavimų nepriештaringumas ir pilnumas. Pilnumo įrodymas yra semantinis.