INFORMATICA, 2002, Vol. 13, No. 2, 177–208

Logical Formal Description of Expert Systems

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Received: October 2001

Abstract. The objective of expert systems is the use of Artificial Intelligence tools so as to solve problems within specific prefixed applications. Even when such systems are widely applied in diverse applications, as manufacturing or control systems, until now, there is an important gap in the development of a theory being applicable to a description of the involved problems in a unified way. This paper is an attempt in supplying a simple formal description of expert systems together with an application to a robot manipulator case.

Key words: artificial intelligence, expert systems, logic.

1. Introduction

Expert Systems are usually developed for specific applications (Georgeff, Firschein, 1985; Davis, 1985; Antonelli, 1983; Hinchman, Morgan, 1983) in a wide class of systems including free-dynamics systems (as, for instance, computing systems, transport-storage problem, etc.) and dynamical systems (like, for instance, physical systems or control processes). Their main characteristic is their ability to give a solution for a given problem, belonging to its competence domain, without an exhaustive interaction with the system's manager. The decision is taken based on the automatic evaluation of the process data by the so-called knowledge base. The knowledge base is a set of rules organised in a hierarchical way and derived by both the knowledge engineer and the system itself from the evaluation of the heuristic and/or analytical knowledge supplied by human experts. The main basic parts of the expert system are (see, for instance, De la Sen, Miñambres, 1987; Jackson, 1999; Veloso, Wooldridge, 1999):

- *Database:* Data set fixed from the particular environment for a given problem.
- *Knowledge base:* Set of rules and their crossed relationships which process the initial and intermediate data towards the achievement of a result.

*This work has been supported in part by the Spanish Ministry of Science and Technology through research project code DPI 2000-0244 and by the University of the Basque Country through research project code 1/UPV/EHU 00I06.I06-EB-8235/2000 and through the Ph.D. studies of Mr. Garrido.

- *Knowledge acquisition:* It consists of obtaining new rules from the initial ones, the environment and the experience of the previous system from similar problems. A parallel possibility of modifying the former rules, when necessary, must be also allowed. The knowledge acquisition implies a diagnostic analysis of the experimented situations.
- *Process monitoring to the user:* Information about the taken decisions and the followed "reasoning process" to the users.
- *Result of the evaluation procedure:* From the analysis of each problem at hand, the process gives a result to the users, which may be matched from the monitoring process. Theoretically, for a well-posed expert system and an admissible experiment, the supplied solution should be the optimal feasible one within the set of possible solutions. It turns out that such an expert system is only built after processing a set of admissible experiments within a learning context.

It may be clearly deduced from the above exposition that it is extraordinary difficult to state a mathematical formulation being valid to describe both the development process and the evaluation of performances that the admissible examples dealt with. Our main purpose in this paper is to give some preliminary theoretical solutions for the existing gaps from a logic formal viewpoint.

To organise the subsequent developments towards their applicability while taking into account the above characteristic of an expert system, the main requirements on the formulation are now stated.

1) The expert systems will be able to analyse a finite or infinite set of experiments belonging to a given family, namely, the set of possible (i.e. admissible/non-admissible) experiments.
2) Since expert systems are designed for specific applications they should be able to be organised in hierarchical structures, by applicability constraint reasons. In this context, a "master expert system" addresses each evaluation process within a coordination context with other specific expert systems of a lower hierarchy level. The number of expert systems being available for each analysis process should be finite. The hierarchy levels may be changed according to the process evolution and the knowledge stored in the database and knowledge base. Each expert system may be designed with its own database for access time saving reasons within the overall structure. The involved databases may have common parts. A finite string of states describes the system evolution for each example. The change of state is due to any system's operation.

   For the sake of a coherent formalism, it is convenient for each expert system to have a (attainable or not in finite or infinite time) final state in which the knowledge base remains invariant (old rules are not modified and no new rule is added) so that the optimal solution for any new admissible experiment is found (De la Sen, Miñambres, 1987). The final state describes the stationary system and it is not assumed to be attainable for each experiment but, at most, after a finite number of experiments. In such a case, the final state is attainable in finite time.

3) The knowledge base may be split into a number of parts, one for each level of hierarchy of each expert system, and each of them must have its own rules. The rules may be of different types, according to specific tasks. For instance: data admission, connection between another rules, rules of matching of conditions within the hierarchical structure, output decisions, etc. The rules within each level should be ordered in a priority context and this order should allow the possibility of being changed according to the analysed process evolution and the "experience" of the expert system from previously processed similar examples of the first admissibility class. When the knowledge base is invariant, each rule is invariant. This also occurs when no change occurs in the knowledge base.

4) A scalar nonnegative quality index of each level (in fact each expert system which operates in the process) is introduced in order to evaluate the degree of achievement of the requested specifications. In the case of joint analysis or different performance measurements, a vector quality index could be used instead.

   The system evolves such that, by adding knew knowledge, at least for each experiment (if repeated), the quality index diminishes. There is and asymptotic upper-bound of the index for each family of admissible experiments which indicates the stationary state of the network of expert systems and the knowledge base. The overall network of expert systems works as an "expert network" addressed by a master coordinator which depends on each experiment.

5) The solution found for an example and the evaluation of the quality index must be able to modify the knowledge basis (namely, a part of the associated rules). This is addressed by the learning functions, so as to use the experience to solve new examples. This kind of information treatment disappears in the stationary state commonly defined in practice as time tends to infinity.

6) A set of information functions which depends on the learning functions, on the state evolution and on the experiment and data is available to the rules.

The original contribution of the paper is that the expert system treats the systems under operation classified into classes with appropriate databases and knowledge bases obtained from appropriate partitions of the whole databases. In control systems, for instance, the classifications may be performed according to the plant type, its eventual discretization strategy, parametrization, control strategy and environment. In contrast to most of the work oriented to the formal study of intelligent systems, see for instance (Veloso, Wooldridge, 1999; Milne, Trave-Massuyes, 1995; Gaul, Schader, 2000; Benjamin, 2000), this paper establishes a thorough theoretical formalism applicable to dynamic systems. The main theoretical ideas rely on the partition into classes for data and knowledge bases, "admissible" and "non-admissible" experiments, and the admissible ones into equivalence classes. The classification is performed during a learning phase of the expert system when testing experiments are computed. Also, the expert system may be ruled by a "master" governing subordinated expert systems subject to priority hierarchies. The formalism is developed in an axiomatic context. The axioms have an intuitive interpretation which is briefly explained when proposed. The set of axioms is consistent for the set of given theorems in the sense that if a necessary axiom in the hypothesis of a theorem fails, then the theorem fails (i.e., Axiom A⇒Theorem A and —Axiom A ⇒—Theorem A, where—≡negation). The set of axioms is complete for the set of given theorems since no extra axiomatic hypothesis out of the given set is introduced to prove any of the given theorems. It is well known that consistency and completeness are required for an axiomatic formulation to be a priori well-posed, see, for instance (De la Sen, Almansa, 1999).

The paper is organised as follows. Section 2 states the mathematical notation in a logic formalism context to be used later. Section 3 develops a first axiomatic formalism with ideal hypothesis of existence of no interaction on system and environment and no failures that modify the knowledge performance (Wertz, 1985; Brown *et al.*, 1977; Burstall, 1968; Charniak *et al.*, 1980; Georgeff, Firschein, 1985; Davis, 1985; Antonelli, 1983; Hinchman, Morgan, 1983). Section 4 points out some extensions to results of Section 3 giving also a summary of the presented results. A supervisor, built with the given ideas, which improves the performance of a planar robot during the adaptation transient, jointly with a stability proof, are given in Section 5. Some comments about the use of the formalism are provided in Section 6, and finally conclusions end the paper.

The main simplifications and hypothesis made are:

- the modification time (access time for evaluation of data, etc.) is instantaneous although the evaluation processes last a finite non-zero time interval;
- the modification of rules (knowledge acquisition) is not discussed in detail although it is pointed out.

2. Notation and Mathematical Preliminary Axioms

2.1. *Fundamental Notation*

- $x \triangleq$ Cartesian product of sets. $\Re(\Re^+)$ is the set of real (positive real) numbers.
- Card $(S) \triangleq$ Cardinal of the set S.
- $\chi_0 \triangleq$ Cardinal of a set being infinite and numerable.
- $H_\alpha \triangleq$ Set of admissible experiments H of class $\alpha; \alpha \in A$ with $\mathrm{Card}(A) \leqslant \chi_0$ (i.e., finite or infinite numerable). In the experiments, a time argument means a fixed time instant as in H(t); two time arguments means a time interval for the operation of the experiment, like H(t', t); and time is irrelevant if the time argument is suppressed. The same conceptual framework extends to concepts like data, knowledge base and expert system.

  Time-index arguments like, for instance, H(t), $H(t', t)$, etc. denote the time instants in which data are available in the database with the joint time arguments denoting the time interval they define.

$H_\alpha(t',t)$, $\mathrm{H}(t',t)$, etc. denote, respectively, classes of experiments or sets of admissible experiments taking place at time $t' \geqslant t_0$, whose results are available in the database at time $t \geqslant t' \geqslant t_0$. The time t is in fact taken as a continuous argument with time intervals including operation or execution times as well as intermediate times located between execution times (i.e., those that imply data processing).

The cardinality may be infinity since the experiment varies with the change of any data input and different kinds of experiments may deal with it.

- $\overline{H}_\alpha$ (the complement to H_α) is the set of experiments that are not of class α; i.e., $U_\alpha = \mathrm{H}_\alpha \cup \overline{\mathrm{H}}_\alpha$.
- The universe of possible (admissible or not) experiments is $U = H \cup \overline{H}$, where $H = \underset{\alpha\in A}{\cup} H_\alpha$ is the set of admissible experiments.
- E^i_{jk} is an expert system of *hierarchical* level $i = i(t)$; order $j = j_i(t)$ (within the hierarchical level i), and state $k = k_{ij}(t)$; where i, j and k are real-valued functions of time of positive integer values, j may be a function of i and k may be a function of the i and j indices.
- If some index is deleted in $E^{(.)(.)}_{(.)(.)}$, as for instance, $E^{i\alpha}_{j}$ means that the state is irrelevant and one is referring to the expert system itself. Similarly, E^i_{jk} means that the class is irrelevant. At time t, this may be denoted by $E^{i\alpha}_j(t)$ with $k = k(t)$.
- The master expert system at time t in a hierarchical structure is denoted by $E^*(t) = E^{i\alpha}_{jk}(t)$, some positive integers α, i, j and k being values of sets A, I, J and K. The index α denotes the expert system irrespectively of its hierarchical level, order within this level and state.
- DB, KB, R and C denote the database, knowledge base, set of rules and set of connection rules, respectively.
- A datum $d \in S_d$ is a mapping $d : (t, \mathrm{H}_d, S_d, R) \to \Re$, some $\mathrm{H}_d \in H$, with the ordered string of data $S_d \triangleq \left\{E^i_{jk}; i \in I, j \in J, k \in K, \alpha \in A : d \in S_d\right\}$, I, J, K and A being subsets of Z^+, where $d \in S_d$ means in the formalism that d is processed by a set S_d of expert systems of a hierarchy i, order j, state k and numeration (an integer number being fixed independently of i, j and k). A set of rules for S_d is $R_d \triangleq \{R_k, k \in K, R_k \in S_d\}$, with R denoting rules of a knowledge base and H being a subset of Z^+; $R_k \in S_d$ means that the rule R_k is evaluated by a subset of expert system S_d which process the datum d.
- The rules R are mappings $R : (t, I_R) \to O_R$ where I_R and O_R are, respectively, the input and output sets of the rule R (i.e., data or actions like a direction of another rule). Each rule may be active or not within a given time interval but it is not considered in the subsequent formulation.

  In our formalism the rules are classified as: Those of admittance (a) or rejection (r) of arbitrary data (applicable to rules that accept or not data from the database DB only).

– Rules for connection (c) between expert systems.

– Rules for connection (c) with other rules.

– Evaluators of matching conditions of an expert system.

– Those that put out data to the database DB.

– The learning capability require the definition of:

– the rule-evaluation modification function $m(t)$ is m: $\Re^+ \to M$,

– the rule-hierarchy modification function m_h: $\Re^+ \times M \to M_h$,

– the rule-order modification function m_0: $\Re^+ \times M \times M_h \to M_0$,

where M is difficult to generically specify for arbitrary admissible experiments. For instance, in logical rule, M may be an evaluation set consisting of two elements (evaluation, non-evaluation). In an analytical rule, M may be a Cartesian product like $\{$(evaluation) $\times$ (real-valued function in $\Re^n$)$\}$, etc. M_h and M_0 are sets of positive integers denoting respectively orders between hierarchies or priorities within a hierarchy.

Each rule R: $(t, I_R) \to O_R$ is defined by six fields: $R(t, \text{hierarchy}, \text{type}, m(t), m_h(t), m_0(t))$, where:

– hierarchy is a mapping h: $DB_{\alpha_1} \times C_{\alpha_2} \times KB_{\alpha_3} \to (0,1) \times (0,1) \times (0,1)$,

– type is a mapping t_y: $DB_{\alpha_1} \times C_{\alpha_2} \times KB_{\alpha_3} \to (0,1) \times (0,1) \times (0,1)$,

and $DB_{\alpha_1} \subset DB$, $C_{\alpha_2} \subset C$, $KB_{\alpha_3} \subset KB$ for some $\alpha_1 \in A_1$, $\alpha_2 \in A_2$, $\alpha_3 \in A_3$, with A_1, A_2 and A_3 being bounded subsets of Z^+. α_i $(i = 1,2,3)$ denote, respectively, the overall set of disjoints sub-databases, sub-connection rules and sub-knowledge bases which one is dealing with. In theory, each $DB(.)$, $C(.)$ or $KB(.)$ is a component of its respective set; such that

$$DB = \underset{\alpha_1 \in A_1}{\cup}(DB_{\alpha_1}); \quad C = \underset{\alpha_2 \in A_2}{\cup}(C_{\alpha_2}); \quad KB = \underset{\alpha_3 \in A_3}{\cup}(KB_{\alpha_3});$$

- DB_{α_1}, C_{α_2} and KB_{α_3} are subsets of the database, the connection between expert systems and the knowledge basis, being accessible from a given rule.

  Each rule R_k $(k \in K)$ is characterised by its code rule $R_k =$(string of data admission S_d, string of data admission $\overline{S}_d$) $\times$ (no connection with string S_E of expert systems, connection with string S_E of expert systems)$\times$ (no connection with string S_R of rules, connection with string S_R of rules), where

  $S_d \triangleq S_d(t, R_k)$ is an ordered sequence string of data admitted by the rule R_k at time t;

  $S_E \triangleq S_E(t, R_k)$ is an ordered string of connected expert systems in which, conditions like $S_E(t, R_k) = E^i_{jk} c_1(t) E^{i'}_{jk'}$ or $E^i_{jk} c_2(t) E^{i'}_{jk'}$ mean that rule R connects E^i_{jk} with $E^{i'}_{jk'}$ at time t, if c_1 is matched (i.e., it is "active") at time t or, respectively, if c_2 is matched at time t. The general form of the string is matched at time t.

The general form of the string is

$$S_E(t, R_k) \triangleq \underset{i,j,k,l,l'}{\overset{\cup}{\cap}} E^i_{jk}\,(c_1(t), o_1(t))\, E^{i'}_{jk'}$$

($\overset{\cup}{\cap}$ means union and/or intersection of sets),

where $i, i' \in I_R$, $j, j' \in J_R$, $k, k' \in K_R$, $l \in L_R$ and $l' \in L'_R$, being I_R, J_R, K_R, L_R and L'_R subsets of Z^+, and where $c_{(.)}$ is a subset of matching conditions and $o_{(.)}$ is a set of logical or analytical evaluation rules.

In the same way, a string of rules is given by:

$$S_R(t, R) \triangleq \underset{i \in I_R, k \in K_R}{\overset{\cup}{\cap}} R\left(t, DB_{\alpha(R)} \times C_{\alpha(R)} \times KB_{\alpha(R)} \to (0,1) \times (0,1) \times (0,1)\right),$$

$$(c_i(t), o_i(t))\, R\left(t, DB_{\alpha(R_k)} \times C_{\alpha(R_k)} \times KB_{\alpha(R_k)} \to (0,1) \times (0,1) \times (0,1)\right),$$

$i \in I_R$, $k \in K_R$ (I, K being finite subsets of Z^+), with the same interpretations for $o_i(.)$ and $c_i(.)$.

The code set $(0,1) \times (0,1) \times (0,1)$ is a set of three binary ordered pairs which have the usual Boolean interpretation (no – yes).

- The database and knowledge base which are associated with an expert system $E^{i\alpha}_{jk}$ are denoted, respectively, by $DB^{i\alpha}_{jk}$ and $KB^{i\alpha}_{jk}$. The set of connection rules of $E^{i\alpha}_{jk}$ is $C^{i\alpha}_{jk}$.
- The database used by the rule R is denoted abbreviately as $DB(R) = \underset{\alpha \in A(R)}{\cup} (DB_\alpha(R))$ where $DB_\alpha(R)$ are possible sub-databases attainable (i.e., to be potentially used) by the set of rules R.
- Consider the expert system $E^{i\alpha}_{jk}(t)$ with its pair $(D_\alpha, \zeta_\alpha)(t)$; $D(t) = \left(DB^{i\alpha}_{jk}(t)_\alpha KB^{i\alpha}_{jk}(t)\right)$ and $\zeta_\alpha = (i \in I, j \in J, k \in K, \alpha \in A)$.

  The quality index at time t of the expert system $E^{i\alpha}_{jk}(t)$ for the experiment H is a scalar (or vectorial) function which takes nonnegative values (or nonnegative valued components) $Q(t, \mathrm{H}(t), E^{i\alpha}_{jk}(t)) = Q_1(t, \mathrm{H}, DB^{i\alpha}_{jk}(t), KB^{i\alpha}_{jk}(t), C^{i\alpha}_{jk}(t))$ for the current hierarchy, order and state of the expert system.

  (When details are unnecessary, the notation will be simplified as, for instance, $Q(t, \mathrm{H}(t), E^{i\alpha}_{jk}(t)) \to Q(E^{i\alpha}_{jk}(t))$).
- If possible, a similar quality index at time t may be defined for the subsets of rules R as follows:

$$Q\left(t, \mathrm{H}(t), R_k(t)\right) = Q_1\left(t, \mathrm{H}, DB_\alpha(R_k), KB_\alpha(R_k), C_\alpha(R_k)\right), \quad \forall \alpha \in A(R),\ \forall R_k \in R.$$

- The function of hierarchy, order within that hierarchy, and state of the expert system of class α, $E_{jk}^{l\alpha}(t)$ are defined by

$$\begin{cases} i = i(t) = i\left(t, Q(t, H_\alpha(t', t), E_{jk}^{i\alpha}(t))\right), & Vt' \leqslant t, \\ j = j(t) = j\left(t, i(t), Q_i(t, H_\alpha(t', t), E_{jk}^{i\alpha}(t))\right), & \forall t' \leqslant t, \ \ \forall i \in I \subset Z^+, \ \forall_\alpha \in Z^+, \\ k = k(t) = k\left(t, i(t), j(t)\right). \end{cases}$$

- The overall network of expert systems of master $E^*(t)$ at time t is $S(E^*(t)) = \left\{E_j^{i\alpha}(t), i \in I, j \in J, k \in K\right\}$. The corresponding set of rules is $R\left[S(E^*(t))\right]$.
- The modification functions for rule $R_k \in R$ are:

$$\begin{cases} m_h(t) = m_h\left(t, Q\left(t, H_\alpha(t', t), R\right), Q_\beta(t, H_\beta(t', t), E_{jk}^{i\beta}(t)\right), \\ \qquad\qquad \forall\alpha \in A(R), \quad \forall\beta \in S(E^*(t)); \\ m_0(t) = m_0\left(t, m_h(t), Q\left(t, H_\alpha(t', t), R\right), Q_\beta(t, H_\beta(t', t), E_{jk}^{i\beta}(t)\right), \\ \qquad\qquad \forall\alpha \in A(R); \quad \forall R_k \in R, \quad \forall\beta \in S(E^*(t)); \\ m(t) = m\left(t, m_h(t), m_0(t)\right). \end{cases}$$

These functions may be, in fact, new rules.

2.2. *Axiomatic Settings*

From a structural viewpoint, well-posededness of the above formulation requires, at least, the following structural *axiomatic* requirements by purely computational reasons.

The number of expert systems, i.e., the cardinality of the expert network addressed by the master expert system $E^*(t)$ at time t and the number of related rules is finite; namely

2.2.1. Card$(S(E^*(t))) < \chi_0$ (i.e., the expert network has a numerable and finite number of expert systems); $\mathrm{card}(R(S(E^*(t)))) < \chi_0$ for all $t \geqslant t_0$ (initial time)→card$(I) < \chi_0$, card$(J) < \chi_0$ for the hierarchy and order subsets of Z^+.

2.2.2. For each admissible class of experiments $H(t', t) \subset H$ at time t (i.e., supplying evaluation results at most at time t), the computer supplies results at time t with a finite number of operations, i.e., for each $E_{jk}^{i\alpha}(t), i \in I, j \in J, k \in K, \alpha \in A$ and for each experiment in the class $H(t', t)$ as subset of admissible experiments, card$(K) < \chi_0$.

2.2.3. The database has finite size for each experiment $H \in H(t', t) \subset H$, so that:

Card$(DB(\mathrm{H}(t)) < \chi_0 \Rightarrow$

$$\Rightarrow \begin{cases} \mathrm{Card}(DB^{i\alpha}_{jk}) < \chi_0; \quad \forall i \in I; \forall j \in J; \forall k \in K, \forall \alpha \in A, \\ \text{and} \begin{cases} \mathrm{Card}(KB^{i\alpha}_{jk}) < \chi_0; & \forall i \in I; \forall j \in J; \forall k \in K; \forall \alpha \in A(S(E^*(t))); \\ \mathrm{Card}(DB(R_k)) < \chi_0; & \forall R \in R\,[S(E^*(t))]; \\ \mathrm{Card}(C^{i\alpha}_{jk}) < \chi_0; & \forall i \in I, \forall j \in J, \forall k \in K, \forall \alpha \in A(S(E^*(t))). \end{cases} \end{cases}$$

2.2.4. All the strings $S_d(t, R)$, $S_E(t, R)$ and $S_R(t, R)$ have a finite number of elements.

The above requirements are introduced in order to process a finite number of data, experiments, rules, etc. which are addressed in the subsequent axiomatic formulation.

3. Axiomatic Formulation

Main Axiom. *The expert system $E^{i\alpha}_{jk}(t)$; $i \in I$, $j \in J$, $k \in K$, $\alpha \in A$ is fully defined at each $t \geqslant t_0$ by the pair $(D_\alpha, \zeta_\alpha)(t)$, where $D_\alpha(t) = (DB^{i\alpha}_{jk}(t), KB^{i\alpha}_{jk}(t), C^{i\alpha}_{jk}(t)$; $i \in I$, $j \in J$, $k \in K$, $\alpha \in A)$.*

That is, for each time instant t, all the information is contained in (D_α, ζ_α).
This motivates the following definitions.

DEFINITION 3.0.

(1) The pair $(D_\alpha(t), \zeta_\alpha(t))$ is called the *information pattern* of the α-expert system $E^{i\alpha}_{jk}(t)$ at time $t \geqslant t_0$.
(2) The triple $D_\alpha(t) = (DB^{i\alpha}_{jk}(t), KB^{i\alpha}_{jk}(t), C^{i\alpha}_{jk}(t))$ and the quadruple $\zeta_\alpha(t) = (i, j, k, \alpha) \in \zeta = (I, J, K, A)$ are called, respectively, the *data pattern* and the *configuration pattern* of $E^{i\alpha}_{jk}(t)$ at time $t \geqslant t_0$.

The information pattern and the configuration pattern (namely, the set of indexations) are the original information about data and primary rules, plus the rules derived from the knowledge base together with connecting relations, which are necessary to solve the learning problem. In this section, we introduce a set of axioms together with related results so as to join and relate the formulation of Section 2 with standard requirements for expert systems.

3.1. The admissible experiments of the same type are considered distinct when a datum or a set of data change. The set of data involved at each experiment uses a finite number of initial and intermediate data. Then,

Axiom 3.1. $\mathrm{Card}(\mathrm{H}(t', t)) < \chi_0$ *for each element* H *in the class* $H_\alpha(t', t)$; $\mathrm{card}(H_\alpha(t', t)) \leqslant \chi_0$; *and* $\mathrm{card}(\mathrm{H}(t', t)) \leqslant \chi_0$; $\mathrm{card}(\overline{H}_\alpha(t', t)) \leqslant \chi_0$ *for each element* H *in the class* $\overline{H}_\alpha(t', t) \Rightarrow \mathrm{card}(\overline{H}) \leqslant \chi_0$, *for each* $t \geqslant t_0$.

This axiom basically means that the number of data in an experiment is finite while the number of data in a class of experiments may be finite or infinite. This last cardinality is infinity when the number of (potential) distinct experiments is infinity (each one with a finite number of data). An example is when the set of admissible data takes values in an infinite set. For instance, suppose that a scalar control signal in a control problem is allowed to belong to the admissibility interval $[-1, 1]$.

Theorem 3.1. *The first part of Axiom* 3.1:
$\mathrm{Card}(\mathrm{H}(t', t)) < \chi_0$; $t \geqslant t_0$, *for each* H *in* $H(t', t)$ *is a Corollary of the structural axiomatic requirements* 2.2.1 *to* 2.2.3.

Proof. Trivial from axiomatic requirements 2.2.1 to 2.2.3 since

$$\mathrm{Card}\,(DB(\mathrm{H})) < \chi_0, \ \ \mathrm{card}\,(DB(R(\mathrm{H}))) < \chi_0, \ \ \mathrm{card}\left(DB(E_{jk}^{i\alpha}(\mathrm{H}))\right) < \chi_0,$$

where $i \in I$, $j \in J$, $k \in K$, $\alpha \in A$, and $\mathrm{card}\,(S(E^*(t))) < \chi_0$.

3.2. Each part of a system expert must have a specific task. Each expert system addressed by the unique master $E^*(t)$ (see Axiom 3.2.2 below) at time t has an objective. Besides, it is very convenient to distinguish between datum, rule and expert system as different objects (see Axiom 3.2.3 below). This eliminates ambiguities and superpositions in the involved notation. Two expert systems, addressed by the same unique master, are different. Then,

Axiom 3.2. *For each* $\mathrm{H}(t) \in H(t', t) \subset H$:

(1) $E_{jk}^{i\alpha}(t) \neq E_{j'k'}^{i'\alpha'}(t')$ *if one of the following matching conditions are fulfilled:* $i \neq i'$, $j \neq j'$, $k \neq k'$, $\alpha \neq \alpha'$, $t \neq t'$ *for each* $i, i' \in I$; $j, j' \in J$; $k, k' \in K$; $\alpha, \alpha' \in A$; $t, t' \in [t_0, \infty) \cap \Re^+$.

(2) $E^*(t) = E_{jk}^{i\alpha}(t)$ *for unique* $i \in I$, $j \in J$, $k \in K$, $\alpha \in A$ *at each* $t \in [t_0, \infty) \cap \Re^+$ *(and* $E^*(t) = \phi$ *for each* $\mathrm{H}(t) \in H(t', t) \subset \overline{H}$*).*

(3) $DB_{jk}^{i\alpha}(t) \cap KB_{jk}^{i\alpha}(t) \cap E_{jk}^{i\alpha}(t) = \phi$ *(i.e., the empty set) for all time.*

3.3. It is necessary to endow the concept of "master" with an independence of the hierarchy and the state, in such a way that it only depends on the α-index. Each master operates like such during a finite or infinite time interval. Then, one has

Axiom 3.3. $E^*(t', t) = E^*(t)$ *for* $t \geqslant t_0$, *some finite or infinite* $t' > t$ *and some* $\alpha \in A$.

3.4. The concepts of hierarchy, order and state indicate privileges between systems of different level, within a level or between sequences of operations. Also, relations of partial order may be introduced using the above concepts. These features are addressed in the result below.

Theorem 3.4.1. *The concepts of hierarchy, order and state of a family of expert systems (or network)* $S(E^*(t))$ *of master* $E^*(t)$ *at time t, formally introduce associate relations* (Rel) *of total order, namely* $E_{jk}^{i\alpha}(t)\mathrm{Rel}\,E_{j'k'}^{i'\alpha'}(t)$, *where* $E_{jk}^{i\alpha}(t)$, $E_{j'k'}^{i'\alpha'}(t) \in S(E^*(t))$; $i, i' \in I$; $j, j' \in J$; $k, k' \in K$ *and* $\alpha, \alpha' \in A$, *being these relations defined by* $\mathrm{Rel}_{\,k} :=$ $i \leqslant i'$ *or* $i \geqslant i'$; $\mathrm{Rel}_{\,0}^{\,i} := j \leqslant j'$ *or* $j \geqslant j'$ *(for each* $i \in I$*) and* $\mathrm{Rel}_{\,k}^{\,ij} := k \leqslant k'$ *or* $k \geqslant k'$ *(for each* $i \in I$, $j \in J$*).*

Proof. Any pair of experts systems $E^{i\alpha}_{jk}(t)$, $E^{i'\alpha'}_{j'k'}(t) \in S(E^*(t))$ as above may be ordered through relations Rel_h, $\mathrm{Rel}^{\,i}_0$, $\mathrm{Rel}^{\,ij}_s$ satisfying the properties reflexive, antisymmetric and transitive since by virtue of Axiom 2.2.1 and Axiom 3.3, the master exist over a time interval with $\mathrm{card}\,(S(E^*(t))) < \chi_0$, for all $t \geqslant t_0$, and those properties stand directly. Furthermore, if the expert network has infinite cardinal or no master exists, then the result fails since the calculations cannot be hierarchized or, even, performed.

Theorem 3.4.2. *The above relations may be defined as partial order relations.*

Proof. It is trivial since any subset $S'(t)$ of $S(E^*(t))$ of cardinality greater than or equal to two may be totally ordered (since it satisfies Zorn's Lemma (Blum, 1971)).

DEFINITION 3.4. The bounded subsets $A_1(t)$, $A_2(t)$, $A_3(t)$ of Z^+ such that $DB(t) = \underset{\alpha_1 \in A_1}{\cup} (DB\alpha_1(t)); C(t) = \underset{\alpha_2 \in A_2}{\cup} (C_{\alpha_2}(t)); KB(t) = \underset{\alpha_3 \in A_3}{\cup} (KB_{\alpha_3}(t))$ with the unions being formed of disjoint subsets are called, respectively, database, connection base, and knowledge base indices at time t (abbreviated by $DB(t)$-index, $C(t)$-index and $KB(t)$-index).

3.5. Each expert system $E^{i\alpha}_{jk}(t)$ of $S(E^*(t))$ at time $t \geqslant t_0$ must have a non-empty database $DB(E^{i\alpha}_{jk}(t))$ and a non-empty knowledge base $KB(E^{i\alpha}_{jk}(t))$ since, by virtue of Axioms 3.2, an expert system is not a set of data or a knowledge sub-base. The connection $C(E^{i\alpha}_{jk}(t))$ may be understood as a knowledge sub-base of the master $E^*(t)$ and any non-admissible experiment has no master expert system to process it. In practice, that means that if a set of data is detected to be not valid, it is rejected for processing. This can occur when the data from the database enter for processing or at any time when the knowledge base detects that the experiment is not admissible. These features are addressed in the subsequent result.

Theorem 3.5. *The following propositions hold for a class of experiments $H(t', t) \subset H$ and for $S(E^*(t))$.*

(1) *If $DB(E^{i\alpha}_{jk}(t)) \cap DB(E^{i'\alpha'}_{j'k'}(t)) = \varnothing$, at some time $t \geqslant t_0$ for all $i, i' \in I$; $j(i), j'(i') \in J$ and $k(i,j), k'(i',j') \in K$, then $DB(t) = \underset{\alpha \in A_1}{\cup} DB(E^{i\alpha}_{jk}(t))$ with α_1 being the $DB(t)$-index.*

(2) *A proposition similar to* (1) *may be stated for $KB(E^{i\alpha}_{jk}(t))$ and $KB(E^{i'\alpha'}_{j'k'}(t))$.*

(3) *If $C\left(E^{i\alpha}_{jk}(t)\right) = \varnothing$ at $t \geqslant t_0$, for all $i \in I$, $j \in J$, $k \in K$, then for each $\mathrm{H}(t) \in H(t', t) \subset H$, it holds that $E^*\left(t, \mathrm{H}(t)\right) = E^{i\alpha}_{jk}(t_0)$, for one and only one $i \in I$, $j \in J$, $k \in K$, $\alpha \in A$. Also, $E^*(t, \mathrm{H}(t)) = \varnothing$ for each $\mathrm{H}(t) \in H(t', t) \subset \overline{H}$.*

Proof. Propositions (1)–(2) are followed trivially from Definition 3.4, and Proposition (3) follows from Axiom 3.2(2) since if there are no connection rules, then the master is one of the expert systems on $[t_0, t]$. There is no master for a non admissible experiment and Axiom 3.2(2) is consistent for this result.

3.6. It is important to give an axiomatic differentiation between the design phase of "building an expert system" (in which examples are submitted, and the expert system takes an admissible or non-admissible solution leading to the modification of the knowledge base), and the "stationary phase" of an expert system. In this last one, any admissible example leads to an admissible solution of the expert system. The "admissibility" characterisation is outlined through the quality indices defined in Section 2. The values of the quality index diminish as the expert system building is in progress.

It must be pointed out that the action of processing a rule modifies at least the index k and perhaps i and j. The subsequent axiom is then used to obtain new formal results:

Axioms 3.6. *For a given $S(E^*(t))$ and all $t \geqslant t_0$:*

1. $Q\left(\mathrm{H}(t), E_{jk}^{i\alpha}(t)\right) = \infty$ *if* $\mathrm{H}(t) \in \overline{H}$ *or $H(t)$ is not processed by $E_{jk}^{i\alpha}(t)$, namely,* $E_{jk}^{i\alpha}(t, \mathrm{H}(t)) = E_{jk}^{i\alpha}(t', \mathrm{H}(t'))$, *all $t' \geqslant t \geqslant t_0$, all $i \in I$, $j \in J$, $k \in K$ and $\alpha \in A$.*
2. *Otherwise to conditions in* (1), $Q(\mathrm{H}(t), E_{jk}^{i\alpha}(t)) < \infty$, *all $i \in I$, $j \in J$, $k \in K$ and $\alpha \in A$.*
3. $Q(\mathrm{H}(t), E_{jk}^{i\alpha}(t)) \leqslant Q(\mathrm{H}(t'), E_{j'k'}^{i'\alpha'}(t')) < \infty$ *for all* $\mathrm{H}(t) \in H(t'', t) \subset H$, $\mathrm{H}(t') \in H(t''', t) \subset H$, *any $i, i' \in I$; $j, j' \in J$; $k, k' \in K$, $\alpha \in A$ and $t' \leqslant t$, any* $\mathrm{t}''' \leqslant \mathrm{t}''$. *Furthermore, a fixed bounded scalar (or vector of finite nonnegative components, depending on the problem dimension)* $\overline{Q}(\mathrm{H}(t))$ *exists for each* $\mathrm{H}(t) \in H(t', t) \subset H$, *which upperbounds $Q(.)$ and each $i \in I$, $j \in J$, $k \in K$, $\alpha \in A$.*

   Also, there exists $\lim\limits_{t\to\infty}\left(Q(\mathrm{H}(t), E_{jk}^{i\alpha}(t))\right) \leqslant \overline{Q}\big[\lim\limits_{t\to\infty}(\mathrm{H}(t), E_{jk}^{i\alpha}(t))\big] = Q_0(H) < \infty$.
4. *The processing of any rule $R(E_{jk}^{i\alpha}(t))$ implies that $E_{jk}^{i\alpha}(t) \neq E_{j'k'}^{i'\alpha'}(t')$, $\forall t' \geqslant t \geqslant t_0$, any $i, i' \in I$; $j, j' \in J$; $k, k' \in K$; $\alpha, \alpha' \in A$. Also, $Q(\mathrm{H}(t), E_{jk}^{i\alpha}(t)) \geqslant Q(\mathrm{H}(t'), E_{jk}^{i\alpha}(t'))$, $\forall t' \geqslant t \geqslant t_0$ being finite if $E_{jk}^{i\alpha}(t)$ processes a rule $R(E_{jk}^{i\alpha}(t))$, any $i \in I$; $j \in J$, $k \in K$ and $\alpha \in A$; and $Q(\mathrm{H}(t), E_{jk}^{i\alpha}(t)) = Q(\mathrm{H}(t'), E_{jk}^{i\alpha}(t'))$ if no-processing holds.*
5. *The quality indices fulfill the following relationships:*

$$
\begin{aligned}
Q(E^{i\alpha}(t)) \geqslant\ & \sum_{j\in J} Q\left(E_j^{(i-1)}(t)\right) \\
& \Rightarrow Q^*(t) = Q(E^*(t)) \geqslant \sum_{j\in J} Q\left(E_j^{i\alpha}(t)\right) \\
& \Rightarrow Q(E^*(t)) \geqslant \sum_{i\in I}\sum_{j\in J} Q(E_j^{i\alpha}(t)) = \sum_{\alpha\in A}\sum_{j\in J} Q(E_j^{i\alpha}(t)) \\
& \textit{and}\ \ Q(E^{i\alpha}(t)) \geqslant \sum_{k<i}\sum_{j} Q(E_j^{k\alpha}(t))
\end{aligned}
$$

for all appropriate indices i, j, k, α, any $\mathrm{H}(t) \in H(t', t) \subset H$, and all time.

Axioms 3.6 (1)–(2) mean that the quality index is infinity for a non-admissible or non-processed experiment and finite for any processed any admissible experiment. Axiom 3.6(3) means that the quality index becomes minimised (i.e., improved) as any new particular experiment not previously processed is performed along time. The second part of this axiom means that the quality indices become asymptotically stationary and finite for admissible processed experiments. Axiom 3.6(4) means that if a rule is active and the time is finite then the quality index is improved with time. This occurs according to the preceding axiom until the steady-state is reached. Finally, Axiom 3.6(5) means that the master expert system (then also called "the expert network") has for all time non less "accumulated" quality index than the overall contribution of all the expert systems. The subsequent result is directly obtained from the above axioms. It is concerned with the boundedness of the existing limits and with the fact that a limit admissible experiment as time tends to infinity is itself an admissible experiment which may be processed. The consistency of the above theorems follows since infinity quality index is assigned to non-admissible (or non-processed) experiments. A finite quality index is assigned to each expert system corresponding the highest one corresponding to the master since it contains information of the whole expert network. The existence of finite limits relies on the fact that theoretically the learning process ends in finite time.

Theorem 3.6. *The following propositions hold for all $S(E^*(t))$, all $t, t' \geqslant t_0$.*

$$
\begin{aligned}
&\text{(i)} \quad Q\left(\mathrm{H}(t'), E^*(t')\right) = \lim_{t\to\infty} Q\left(\mathrm{H}(t), E^*(t)\right) = \infty, \\
&\qquad \textit{any } \mathrm{H}(t') \in H(t'', t') \subset \overline{H}; \\
&\text{(ii)} \quad \infty > \overline{Q}\left(\lim_{t\to\infty} \left(\mathrm{H}(t), E^*(t)\right)\right) \geqslant \lim_{t\to\infty} \left(Q(\mathrm{H}(t), E^*(t))\right) \\
&\qquad \geqslant \sum_{k<\mathrm{card}(I)} \sum_{j\in J} \overline{Q}\left(\lim_{t\to\infty} \left(\mathrm{H}(t), E_j^k(t)\right)\right) \\
&\qquad \geqslant \sum_{k<\mathrm{Card}(I)} \sum_{j\in J} \lim_{t\to\infty} \left(Q\left(\mathrm{H}(t)\right), E_j^k(t)\right).
\end{aligned}
$$

Proof. Proposition (i) follows from Axiom 3.6(1) and Proposition (ii) follows from Axioms 3.6(3) and 3.6(5). Note that trivially Theorem 3.6(i) does not hold for admissible experiments, while Theorem 3.6(ii) does not hold for non-admissible or non-processed experiments, from the consistency of Axioms 6.3(1), (3) and (5).

3.7. In the mathematical preliminaries of Section 2, a set of modification functions have been introduced. This functions $m_0(.)$, $m_h(.)$ and $m_{(.)}$ remain yet unspecified. On the other hand, the subset (I, J, K) of the configuration pattern (Definition 3) is, in general, time-varying. The set A is a subset of Z^+ of maximum cardinality that of $S(E^*(t))$ at time $t \geqslant t_0$. If, for some experiment H$\in$ H, a specific expert system does not work, it may be located at the end of the string of the subset (I, J, K), namely, it has minimum hierarchical priority, last order within this priority and empty connection with its preceding elements $E_{jk}^{i\alpha}(t)$; $i \in I$, $j \in J$, $k \in K$, $\alpha \in A$. Proceeding in this way, Card(A) =Card$(S(E^*(t)))$ =constant, $\forall t \geqslant t_0$; i.e., the experts systems that do not perform at some time are assembled in the expert network with empty connections with the remaining elements of the expert network. The empty connections together with minimum priority mean in fact that such an expert system is kept out within the whole network for that particular experiment.

Also, it seems to be logical to store the information of previous experiments to the current one belonging to the same equivalence classes (stated using similarities of observed type, data and quality performances), and to use this information in the next examples of the same class in order to modify the pattern $(I, J, K)(t)$, $t \leqslant t_0$, as well as $m_0(t)$, $m_h(t)$ and $m(t)$ for the knowledge base. The D previously acquired knowledge should be stored in databases of the expert network.

According to this feature, we state a new axiomatic statement. First, we extend some of our former concepts to the overall system.

DEFINITIONS 3.7.1.

1. The set $S(E^*(t))$, $\forall t \geqslant t_0$ is called the expert net of master $E^*(t)$ at time t.

For the following definitions, review Definitions 3.0.

2. The pair $(D(t), J(t)) = \{(D_\alpha(t), J_\alpha(t))\,; \alpha \in A\}$ is called the *information pattern* of the expert set $S(E^*(t))$ at time t, where $\rightarrow$ is the pair wise-order relation of hierarchy and priority within $S(E^*(t))$. $\{J_\alpha^{\rightarrow}(t)\}$ is the set of pairs together with the order relation.
3. The *data pattern* of $S(E^*(t))$ is the set $D(t) = \{D_\alpha(t);\ \alpha \in A,\ t \geqslant t_0\}$.
4. The *configuration pattern* of $S(E^*(t))$ is the set $\zeta(t) = \{\zeta_\alpha(t);\ \alpha \in A,\ t \geqslant t_0\} = \{I, J, K, A\}(t)$.

Then, from Main Axiom 3 and Definitions 3.7.1, the following results stands.

Theorem 3.7.1. *The expert set $S(E^*(t))$ addressed by the master expert system $E^*(t)$ is fully defined for each $t \geqslant t_0$ by the information pattern as defined in Definition* 3.7.1(2).

It is now interesting to ensure axiomatically that the configuration pattern is asymptotically invariant for the classes of the admissible experiments. A logical way to proceed is to make it vary using the quality-indices of $S(E^*(t))$ through the appropriate rules of the knowledge base, that place the evaluation results in the appropriate database in order to be evaluated.

Axioms 3.7.

1. *The only way to modify the configuration pattern $\zeta(t)$ is through the evaluation of the quality indices available at time t and the configuration pattern at previous*

times, i.e., $\zeta(t) = \zeta\big(t^-, Q_s(\mathrm{H}(t), S(E^*(t))), \zeta(t^-)\big)$, $\forall \mathrm{H}(t', t) \in H$, $\forall t_0 < t$. *The sequence* $Q_s(.)$ *denotes the set* $\{Q_\alpha, \alpha \in A\}$, *i.e.,* $Q\left(\mathrm{H}(t), E^{i\alpha}_{(.)}(t)\right)$ $\forall t_0 \leqslant t' \leqslant t$.

2. *There exist classes* $H(t', t) \in H$, $\forall t_0 \leqslant t' \leqslant t$, *containing each one at least one admissible experiment* $\mathrm{H}(t)$, *such that*

$$H(t) = \{H_n(t', t) : \mathrm{H}_1(t), \mathrm{H}_2(t) \in H(t', t) \Rightarrow \zeta(t, \mathrm{H}_1(t)) = \zeta(t, \mathrm{H}_2(t))\}$$
$$\text{at time } t \geqslant t_0,$$

i.e., the information pattern of $S(E^*(t))$ *stands simultaneously for both* $\mathrm{H}_1(t)$ *and* $\mathrm{H}_2(t)$.

This motivates the following definition.

DEFINITION 3.7.2. An information pattern is invariant for two admissible experiments $\mathrm{H}_1(t), \mathrm{H}_2(t)$ processed by an expert set $S(E^*(t))$ if and only if the two experiments have such a pattern.

From Axiom 3.7(2), the following result stands directly.

Theorem 3.7.2. *The relation* β *in* $(H, S(E^*(t)))$, *for some* $t \geqslant t_0$ *defined by* $\mathrm{H}_1(t)\beta\mathrm{H}_2(t)$ *if* $\zeta(t, \mathrm{H}_1(t)) = \zeta(t, \mathrm{H}_2(t))$, *is an equivalence relation which induces de quotient set of equivalence classes* $\{[h]\} = \mathrm{H}/\beta$.

From Axiom 3.7(1) and Theorem 3.7.2, the following result is directly obtained.

Theorem 3.7.3. *If* $\mathrm{H}_1(t)\beta\mathrm{H}_2(t)$ *in* $(H, S(E^*(t)))$ *at time* $t \geqslant t_0$ *and the quality-index sequence of* $S(E^*(t))$ *is constant on* $[t, t'] \cap \Re$, *then* $\mathrm{H}_1(\tau)\beta\mathrm{H}_2(\tau)$, $\forall \tau \in (t, t'] \cap \Re^+$.

REMARKS 3.7.

1. Theorem 3.7.2 classifies the admissible experiments by the expert set $S(E^*(t))$ into categories or equivalence classes which accomplish with the configuration pattern at $t \geqslant t_0$.
2. Theorem 3.7.3 states that the configuration pattern remains constant within an interval of time if the quality-index sequence in that interval is constant. In particular the master that addresses the expert set is the same in this interval.

A previous comment at the end of Section 3.7 states that if the information pattern suffers a change in its state, it is motivated by modifications in the database originated from previous analysis of admissible experiments. Axiom 3.7.1 states that the information pattern is changed only by changes in the quality-index string of the expert network.

3.8. It has been introduced axiomatically in the formulation in such a way that each modification of a rule in a class of experiment implies a flag in the database. Thus, we have the following *dependence chain*:

$$\begin{array}{l} D \to Q \to \zeta \\ \uparrow \quad \nearrow \\ KB \end{array}$$

so that Q_α is evaluated directly from D_α. The following axiom concerns with the logic fact that a modification in the knowledge basis at some time keeps such a base modified during a nonzero length time interval.

Axiom 3.8. *Each change in $KB_{jk}^{i\alpha}(t)$, where the quadruple $(i, j, k, \alpha) \in \zeta_\alpha$; at time $t \geqslant t_0$ implies that $KB_{jk}^{i\alpha}(t) \neq KB_{jk}^{i\alpha}(t')$ for all $t' \in (t, t+\varepsilon]$, some real constant $\varepsilon > 0$.*

Now, Main Axiom of Section 3, Axiom 3.2(3), Axiom 3.8 and Definitions 3.0 and 3.7.1(2) together with Theorems 3.7.2 and 3.7.3 lead to the following result, where two new equivalence relations δ and ψ are introduced to define equivalence classes of experiments and information patterns. Those equivalence classes appear in a natural way after defining the equivalence relation β for "equivalent" experiments.

Theorem 3.8. *The following propositions hold.*

1. *The data pattern $D(.)$ fulfills the subsequent proposition: $D(t)\delta D'(t)$ in $(H, S(E^*(t)))$ at time $t \geqslant t_0$ if $\mathrm{H}(t)\beta\mathrm{H}'(t)$ at time $t \geqslant t_0$ for some equivalence relation δ.*
2. *If $\mathrm{H}(t)\overline{\beta}\mathrm{H}'(t)$ ($\overline{\beta} :=$ no relation β), then $D(t)\overline{\delta}D'(t)$ at time $t \geqslant t_0$.*
3. *Proposition 1 implies that the information pattern $(D(t), \zeta(t))$ verifies another equivalence relation ψ under the hypothesis of Theorem 3.7.3 as follows:*

   $(D(t), \zeta(t))\,\psi\,(D'(t), \zeta'(t))$ in $(H, S\,(E^*(t)))$ at time $t \geqslant t_0$ if $\mathrm{H}(t)\beta\mathrm{H}'(t)$ in $(H, E^*(t))$ at time $t \geqslant t_0$.

Proof (outline). Propositions 1 and 2 follow directly with δ defined as follows $D(t)\delta D'(t)$ in $(H, S(E^*(t)))$ if $\mathrm{H}(t)\beta\mathrm{H}'(t)$.

In Proposition (3), ψ is defined as follows: $(D(t), \zeta(t))\,\psi\,(D'(t), \zeta'(t))$ if $D(t)\delta D'(t)$ and $\mathrm{H}(t)\beta\mathrm{H}'(t)$ in $(H, S(E^*(t)))$. That is, $(D(t), \zeta(t))\,\psi\,(D'(t), \zeta'(t))$ if $\mathrm{H}(t)\beta\mathrm{H}'(t)$ in $(H, S(E^*(t)))$.

4. Some Extensions and Summary of Results

The extensions which could be made are concerned to the use of the quality indices in the modifying functions of the knowledge base rules $m(.)$, $m_h(.)$, $m_0(.)$, and the use of the asymptotic lower bounds for the quality indices so as to state the nominally optimal expert network.

Expert systems are usually designed from a practical viewpoint without the existence of a theoretical formalism, so that it is very difficult for a designer to implement the system. The difficulties are aggravated by the fact that an expert system modifies its inner structure as the number of examined examples increases. The preliminary results which have been stated in this paper about the logical formalism of expert systems are an attempt to close the gap between the practical and the theoretical design of expert systems, in order to facilitate the task of the designer when taking the necessary steps to carry out the system. Thus, an interesting objective is the introduction of a formalism which allows:

- The design of the 'usual' experimental steps.
- The capacity to introduce modifications in the data/knowledge basis, which is of interest in the learning context to build and improve an expert system.
- The distribution of complex problems into networks consisting in several simplified problems (expert networks as a set of specialised expert systems addressed by a master expert system, to which the orders for analysis are addressed).
- To give an axiomatic formulation which specifies the main hypothesis and the way to deal with it in a systematic and non-redundant fashion.

The axioms we have introduced consists mainly of the following considerations:

1. An expert network consists of a set (of at least one element) of expert systems addressed by one "master expert system".
2. There is a set of admissible experiments which are grouped into equivalence classes. For each class, the master expert system of the expert network is time-asymptotically unique (namely, under the theoretical hypothesis that no additional knowledge must be asymptotically added for learning through new rules or modification of the existing ones).
3. There is a hierarchical distribution of the expert systems within their whole expert network. For each of the above class of admissible experiments, the hierarchy is asymptotically stationary.
4. The topological connections within the hierarchy are implemented through connection rules which may be made to belong to both (or one of) the knowledge base of the "local master expert system" and to that of "local slave expert system". The concept of state of the expert system has been introduced to distinguish expert systems belonging to the same hierarchy and priority but with different performance objectives.
5. Modifications of the hierarchies and priorities between the various expert systems of the network and the rules of each knowledge base are made via modifications in the knowledge bases through evaluation of quality-indices for performed results on the current and former admissible experiments of the same class. This is assumed to be the only way to deal with the above modifications. These quality indices are also used as direct arguments in the modification functions of the rules.

The axioms and their theoretical conclusions have been introduced in the logical order following the well-known experimental steps that are usually implemented in the applications. Future work may address the improvement of the on-line updating of the modification functions of the rules according to the registered values of the quality indices when the expert system is in operation.

5. A Simple Expert Network for Improving the Adaptation Transients in Adaptive Control

5.1. *Process to Be Controlled*

A planar robot with three articulations is considered. For modeling simplification, the masses of the two first articulations are assumed concentrated at the distal end of each link while the mass center of the third element is assumed located at the center of mass of the second link with its inertia tensor assumed diagonal. If the robot parametrization is not fully known then the mechanical torque is assumed to be given by:

$$\tau = \widehat{M}(\Theta)\left(\ddot{\Theta}_d + k_v\dot{E} + k_pE\right) + \widehat{V}(\Theta, \dot{\Theta}) + \widehat{G}(\Theta) + \widehat{F}(\dot{\Theta}), \tag{1}$$

where $\widehat{M}$, $\widehat{V}$, $\widehat{G}$ and $\widehat{F}$ are the estimates of M, which is the mass matrix, V groups centrifugal and Coriolis forces, G is a gravitation force while F models the friction and Coulomb effects, Θ is the vector of relative position angles of each arm, with first and second time-derivatives $\dot{\Theta}$ (angular velocities) and $\ddot{\Theta}$ (angular accelerations) referred to the previous one. The particular expressions for M, V, G and F are time-varying and non-linear including products of angular velocities and positions and some related trigonometric-type functions $\sin(\cdot)$ and $\cos(\cdot)$ as well as similar coupling terms between the various angular variables. Note that this plant is nonlinear and time-varying. However form small variations of Θ, it may be considered as a second order linear one. Details of that parametrization are provided, for instance, in (De la Sen, Almansa, 1999). It has been assumed that four process parameters are unknown, namely, m_1, $m_2 + m_3$ (masses); I_{zz} (third component of the inertia tensor of the third link), and v_1 (first friction coefficient). K_v and K_p are proportional and derivative controller gains. Those parameters are assumed unknown and then estimated while the remaining parameters are assumed known. The principal objective of this section is to provide an adaptive controller for this robot which uses supervision techniques to improve the adaptation transients.

5.2. *The Expert System*

The expert system uses an adaptive controller at the highest hierarchical level with a least-squares estimation algorithm with time-varying free-design parameter and a sampling law with small sampling period variations at the third hierarchical level. The second level is a coordinator of the actions at level 3. The basic scheme for the first control-estimation

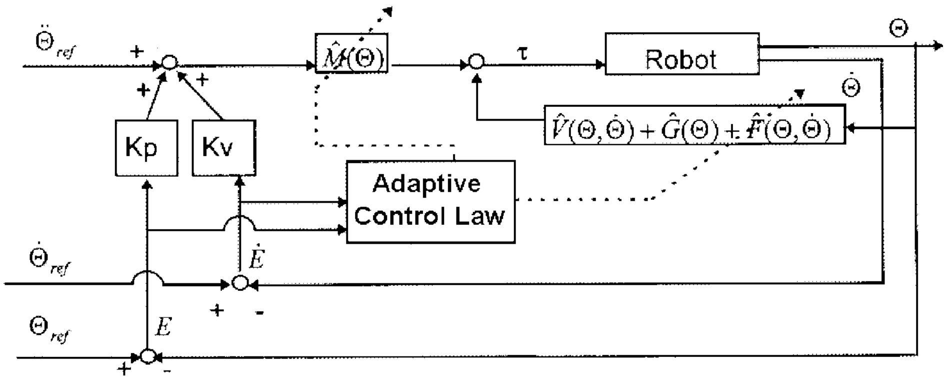

Fig. 1. Basic scheme of the first control-estimation levels.

levels is displayed in Fig. 1. The other two levels are devoted to properly supervise the basic level by taking necessary correcting actions when necessary.

The fixed parameters are taken as follows: T_0 (nominal sampling period)=0.6 msec., c (free-design gain of the estimation algorithm) $= 5 \times 10^{-3}$ and forgetting factor $\lambda = 1$ (i.e., no forgetting factor is used). The fixed part of the controller in Fig. 2 is given by the proportional and derivative gain matrices: $K_p = \text{Diag}\,(100, 100, 100)$, $K_v = \text{Diag}\,(20, 20, 20)$. The parameters of the robot are: $m_1 = 4.6$Kg; $m_2 = 2.3$Kg; $m_3 = 1$Kg; $I_{zz} = 0.1\text{Nm}^2$; $v_i = k_i = 0.5$ ($i = 1, 2, 3$). $L_i = 0.5$m ($i = 1, 2$) are the arm lengths. The initial estimates for the four unknown parameters are: $\hat{m}_1 = 9.2$; $\widehat{m_2 + m_3} = 6.6$; $\hat{I}_{zz} = 0$; $\hat{v}_1 = 0$.

Now, the whole expert network based on the main parts of the given formalism is organised as follows:

E_1^1: *Unsupervised Adaptive Controller*

$\widehat{M}$, $\widehat{V}$, $\widehat{G}$ and $\widehat{F}$ are the estimates obtained from the estimate $\widehat{p}$ of the unknown (and then estimated) parameter vector $p = (m_1, m_{2+}m_3, I_{zz}, v_1)^T$ and the known parameter vector $p' = (v_2, v_3, k_1, k_2, k_3, k_{p1}, k_{p2}, k_{p3})^T$. The corresponding regressor matrices are W and W', containing the various signals obtained from $\Theta, \dot{\Theta}$ and $\ddot{\Theta}_d$ which affect to each component of p and p'. The estimate of p is obtained from a least-square type estimation algorithm with covariance matrix F_k updating and free-design adaptation parameter c_k given at each k-th sample by:

$$\widehat{p}_{k+1} = \widehat{p}_k + \frac{F_k W_k^T E_\tau}{c_k + \|W_k F_k W_k^T\|}, \tag{2}$$

$$F_{k+1} = \frac{1}{\lambda}\left(F_k - \frac{F_k W_k^T W_k E_\tau}{c_k + \|W_k F_k W_k^T\|}\right), \tag{3}$$

with the adaptation error being $E_\tau = \widehat{M}^{-1}(\ddot{E}_d + k_v\dot{E} + k_pE)$ for all time and $\widehat{M}$ being parametrized for all time from the estimation $\widehat{p}$ of p and the known p' (detailed relations are provided in (De la Sen, Almansa, 1999)) and $F_0 = F_0^T > 0$. The basic adaptive controller implements equations (1)–(3).

E_2^3: *Updating Rules for* c_k

A loss function is defined at each sampling instant by:

$$J_k^c = \delta_{1k} \sum_{i=k-N_k}^{k} \sigma_i^k E_i^T Q E_i + \delta_{2k} \sum_{i=k+1}^{k+M_k} \sigma_i^k \widehat{E}_i^t Q \widehat{E}_i, \quad k \geqslant 0. \tag{4}$$

The c_k-parameter is on-line adjusted for each k-th sample according to the improvement of J_k^c related to J_{k-1}^c. In (6), $\delta_{1k} \in [0,1]$ and $\delta_{2k} = 1 - \delta_{1k} \in [0,1]$ are relative weights for each of the two right-hand-side terms, and $[k - N_k, k)$ is a "correction horizon" in the sense that $c_{(\cdot)}$ and then E_i, since previously occurred, cannot be modified at the current k^{th}-sample but its associate contribution to (4) is a measure of the recent registered transient performance. $[k+1, k+M_k]$ is a "prediction horizon" in the sense that the error tendency through predicted values $\widehat{E}_{(\cdot)}$ of future $E_{(\cdot)}$ contributes to J_k^c. The predictions $\widehat{E}_{(\cdot)}$ are computed through direct extrapolation of the last few measured tracking errors $E_{(\cdot)}$. The expert system becomes as follows. First calculate

$$c_k = \rho_k \left\| W_k F_k W_k^t \right\| + \bar{c} \quad (\bar{c} > 0), \tag{5}$$

so that c_k is mainly adjusted via regressor contributions if $\rho_k >> 1$, it is almost negligible if $\rho_k << 1$ and it is close to $\|W_k F_k W_k^t\|$ in (1) if $\rho_k \approx 1$ and $\bar{c}$ is small. $\bar{c} > 0$ is used to avoid $c_k = 0$, which would violate the scheme's stability constraints, and also the algorithm to fail if simultaneously $F_k = 0$. Thus, the main idea is to design c_k through (5) and a rational empirical on-line choice of ρ_k according to the evolution of the relative loss $\tilde{J}_k^c = \left|J_k^c - J_{k-1}^c\right| / \left|J_k^c\right|$ at each k sample:

Rule 1. For $\Delta\rho_k \geqslant 0$,

i) $\rho_{k+1} = \rho_k + \Delta\rho_k$ if $\rho_k < \rho_{k-1}$ and $J_{k+1} > J_k$ or if $\rho_k > \rho_{k-1}$ and $J_{k+1} < J_k$.

ii) Then calculate c_k from (7). The heuristic explanation is:

"Progress by doing identical action by increasing c_k if the relative cost is being improved when increasing c_{k-1}. Otherwise, modify the supervision strategy and decrease c_k if c_{k-1} was increased and the relative cost was worsening."

Rule 2. i) $\rho_{k+1} = \rho_k - \Delta\rho_k$ if $\rho_k < \rho_{k-1}$ and $J_{k+1} < J_k$ or if $\rho_k > \rho_{k-1}$ and $J_{k+1} > J_k$.

ii) Calculate c_k from (5).

Rule 2 is interpreted similarly: c_k is decreased if decreasing is improving the relative cost or if it was increased in the previous step and now the cost is detected to be worsening.

Rule 3. $c_k = c_{k-1}$ if $J_k = J_{k-1}$.

Since $c_k \in [c_{\min}, c_{\max}] \subset (0, \infty)$, with prefixed $c_{\min} > 0$ and $c_{\max} > c_{\min}$, then $\Delta\rho_k = l_k \Delta\rho$ with l_k integer such that $\Delta\rho = 0.05\rho_0$ and $l_k M_k \leqslant c_k \leqslant l_{k+1} M_k$, where $M_k = 0.05 J_k$. The initial $c_0 = (c_{\min} + c_{\max})/2$.

E_1^3 : *Sampling Period Updating Law*

A commonly used one is

$$T_k = \frac{C T_{k-1}}{\|E_k - E_{k-1}\|_R} \quad (C = 5 \times 10^{-4}),$$

if $T_k \in [T_{\min}, T_{\max}]$, otherwise either $T_k = T_{\min}$ or $T_k = T_{\max}$, with some $T \in [T_{\min}, T_{\max}]$ being a nominal constant running sampling period suitable in the application. The constant C is set arbitrarily so that the adaptation is typically a bang-bang rule with mutual switches between $T_{\min}$ and $T_{\max}$ at the beginning of the adaptation transient. After a set of samples, values within the admissibility interval for the sampling period are also found. $R = R^T \geqslant 0$ is a (at least) positive semidefinite matrix so that $\|X\|_R = (X^t R X)^{1/2}$ is the generalized Euclidean norm of X. Since large variations of the sampling period are not allowed in an adaptive scheme-based for systems with constant or slowly time-varying parameters, the sampling period has to be slowly varying and to converge to some $T_0 \in [T_{\min}, T_{\max}]$ in a neighbourhood of the nominal T. T_0 may be typically identical to the nominal sampling period T. Thus, we proceed as follows.

Prefix ΔT_α and ΔT_β such that $T_{\min} = T_0 - \Delta T_\alpha$ and $T_{\max} = T_0 + \Delta T_\beta$, so that $[T_0 - \Delta T_\alpha, T_0 + \Delta T_\beta] \subset [T_{\min}, T_{\max}]$.

Rule 4. For the current $\Delta T_{\alpha k}$ and $\Delta T_{\beta k}$ compute the trial current sampling period

$$\bar{T}_k = \frac{C T_{k-1}}{\|E_k - E_{k-1}\|_R} \quad \text{if } E_k \neq E_{k-1} \text{ and } \bar{T}_k = T + \Delta T_{\beta k}, \tag{6a}$$

otherwise; then choose using (6a) the sampling period as

$$\begin{aligned}
T_k &= \bar{T}_k && \text{if } \bar{T}_k \in [T - \Delta T_{\alpha k}, T + \Delta T_{\beta k}], \\
T_k &= T - \Delta T_{\alpha k} && \text{if } \bar{T}_k < T - \Delta T_{\alpha k} \\
T_k &= T + \Delta T_{\beta k} && \text{if } \bar{T}_k > T + \Delta T_{\beta k}.
\end{aligned} \tag{6b}$$

Rule 5. Decrease $\Delta T_{\alpha k}$ and $\Delta T_{\beta k}$ if $k \geqslant k_0$ (finite), so that $T_k \to T_0$ as $k \to \infty$. Otherwise, (if $k < k_0$, some finite integer $k_0 > 0$) then choose $\Delta T_{\alpha k} < \Delta T_{\alpha, k-1}$, $\Delta T_{\beta k} < \Delta T_{\beta, k-1}$ unless the transient performance is being worsened.

REMARK 5.2. The decrease in the increments $\Delta T_{\alpha k}$, $\Delta T_{\beta k}$ may be overcome with time exponentially decreasing rules, or less drastically, with functions converging to zero slower than exponentially with time. In the example discussed in this section, the choices are simplified to $\Delta T_{\alpha k} = \Delta T_{\beta k} = \Delta T_k \leqslant \Delta T$ with $T = T_0 = \frac{T_{\min}+T_{\max}}{2}$ and $\Delta T_k = m\Delta T_0 \mathrm{e}^{-k}$, $m > 0$ being small, decreases at slow exponential rate.

E_1^2 : *Supervisor of* E_1^3 *and* E_2^3

The main objectives of this supervisor are:

i) To modify the values of the weights δ_{1k}, δ_{2k} of the correction or prediction horizons according to the registered performance and to modify when necessary the sizes of the correction and prediction horizons of sizes N_k and M_k in E_2^3.

ii) To switch when necessary to another adaptive sampling law from the current one or to decide to use one of the two supervisions only to improve the system performance.

One has proceeded as follows for E_1^3 and E_2^3:

– *Choice of the weights and correction/prediction horizons.* The variations ranges for δ_{1k}, δ_{2k} have been chosen within [0, 1] with a small reduced admissible variation interval from a set of admissible experiments in H. The correction and prediction horizons have been fixed around the values $N_k \cong 3$, $M_k \cong 2$. The size of the correction horizon is large since it deals with real values of the tracking error, while in the prediction one they have to be calculated using interpolation. Since it turns out that the final effect of this supervision is the on-line adjustment of c_k, it is seen from (4) that the effects of variations in the values of N_k and M_k may lead to qualitatively similar performances than properly chosen variations in the weights δ_{1k} and δ_{2k}, respectively. As a result in the numerical example below in this section, they have been taken sample-independent as $N_k = 3$, $M_k = 2$ $(\forall k \geqslant 0)$ and $\delta_{2k} \in [0.85, 1]$, $\delta_{1k} = 1 - \delta_{2k}$ (both adjusted on-line) for all $k \geqslant 0$.

– *Choice of the sampling law.* The following set of adaptive sampling laws are obtained from those proposed in (De la Sen, Almansa, 1999) and (De la Sen, 1984), after approximating the error time-derivatives by the finite difference method with evaluations at sampling instants:

Law 1

$$\bar{T}_k = \frac{T_{\max}T_{k-1}^2}{C\left\|E_k - E_{k-1}\right\|_R^2 + T_{k-1}^2},\ a = 1,\ b = 2; \quad C = 1/3AB^2; B = 1/T_{\max} \text{ in (1).}$$

Law 2:

$$\bar{T}_k = \frac{CT_{k-1}}{\left\|E_k - E_{k-1}\right\|_R},\ a = 0,\ b = 2; \quad C = (AB)^{1/2}.$$

Law 3:

$$\bar{T}_k = T_{\max} - \frac{C\left\|E_k - E_{k-1}\right\|_R}{T_{k-1}},\ a = 1,\ b = 1; \quad C = 1/2AB^2;\ B = 1/T_{\max}.$$

Law 4:

$$\bar{T}_k = \frac{T_{\max}T_{k-1}}{C\left\|E_k - E_{k-1}\right\|_R + T_{k-1}},\ a = 0,\ b = 1; \quad C = 1/AB^2;\ B = 1/T_{\max}.$$

Table 1

Database

Robot parameters:
$m_1 = 4.6$kg, $m_2 = 2.3$kg, $m_3 = 1$kg, $I_{zz} = 0.1$kgm^2,
$v_i = k_i = 0.5$ $(i = 1, 2, 3)$, $L_1 = L_2 = 0.5$m
Nominal sampling period:
$T = T_0 = 0.6$msec, $T_{\min} = 0.5$msec, $T_{\max} = 0.7$msec
Initial conditions:
$\Theta\Big\|_t = 0 = (0, 30, -50)^T$; $\dot{\Theta}\Big\|_{t=0} = \ddot{\Theta}\Big\|_{t=0} = (0, 0, 0)^T$; $F\Big\|_{t=0} =$ Diag (10^3).
Final reference conditions:
$\Theta_d = (10, -50, -10)^T$.
Gain matrices:
$K_p =$ Diag $(1, 0, 0)$.
Initial conditions of estimates:
$\widehat{m}_1\Big\|_{t=0} = 9.2$, $\widehat{m}_{23}\Big\|_{t=0} = 6.6$, where $m_{23} = m_2 + m_3$; $\widehat{I}_{zz}\Big\|_{t=0} = 0$, $\widehat{v}_1\Big\|_{t=0} = 0$.
Saturation of the estimation algorithm for all time from "a priori" knowledge:
$0.01 \leqslant \widehat{m}_1 \leqslant 20,\ 0.01 \leqslant \widehat{m}_{23} \leqslant 20,\ 0.01 \leqslant \widehat{I}_{zz} \leqslant 1,\ 0.01 \leqslant \widehat{v}_1 \leqslant 1$.
Nominal value of c_k $(= c_0) = 5 \times 10^3$; $\lambda = \sigma = 1$(constant);
minimum value of $\bar{c}_k$ *is* $\bar{c} = \Delta\rho = 0.1$.
Correction and prediction horizon sizes:
$N \leqslant 3$, $M \leqslant 2$; $\delta_{1k} = 1 - \delta_{2k}$, $\delta_{2k} \in [0.85, 1]$.
Weighting matrix for supervision of c_k: $Q = I$.
Weighting matrix for the sampling laws: $R =$ Diag $(0, 0, 1)$.

Table 2

Percentages of performance improvement under adaptive sampling

Law 1	Law 2	Law 3	Law 4
39%	29%	16%	26%

The database used by the expert network is displayed in Table 1.

In Table 2, the percent of improvement of the time-integral of the quadratic tracking error is quantified over fifty samples for each of the four given sampling laws evaluated separately without supervision of the c_k-parameter. The percentages are computed related to the unsupervised situation of nominal constant sampling period $T_k = T = T_0 = 0.6$msecs, for all $k \geqslant 0$ and $R =$ Diag $(0, 0, 1)$.

5.3. *Closed-loop Stability*

The following result proves that the basic supervision-free system and the supervised ones are both stable.

Theorem 5.3 (Stability results). *The following two items hold:*

(i) *In the absence of supervision, the estimated parameters are bounded if their initial conditions are bounded and the initial adaptation covariance matrix is positive definite. Also, the closed-loop system is globally Lyapunov's stable so that the output, input, estimation error and tracking error are all bounded provided that the reference trajectory is bounded.*

(ii) *If only the algorithm free-parameter c_k (or, alternatively, the forgetting factor) is supervised by the given rule while respecting its positivity and boundedness (while belonging to the range (0, 1]) for all sample, then (i) holds. If the sampling period is supervised (with the free-parameter being supervised or not) during a finite time interval within its admissibility domain, then (i) still holds.*

Sketch of Proof: (i) Direct calculations with (1)–(3) yield for all sampling instants:

$$\begin{aligned} E_{\tau k} &= M_{1k}\hat{P}_k + \hat{M}_k\ddot{\Theta}_k = M_k\ddot{\Theta}_k + E'_{\tau k} - M_{1k}\tilde{P}_k, \\ E'_{\tau k} &= W_k\tilde{P}_k = M_k - M_k\ddot{\Theta}_k = M_k(K_v\ddot{\Theta}_k - K_p\ddot{\Theta}_k), \end{aligned} \tag{7}$$

where $\tilde{P}_k = P - \hat{P}_k$ is the parametrical error for the auxiliary parameter vector P. Thus,

$$E_{\tau k} = (W_k - M_{1k})P_k + M_k\ddot{\Theta}_k. \tag{8}$$

On the other hand, one gets from the estimation algorithm and the above error expression:

$$\lambda_k F_{k+1}F_k^{-1} = \left(I - \frac{F_kW_k^TW_k}{c_k + ||W_kF_kW_k^T||}\right); \quad \lambda_k F_k^{-1}\tilde{P}_k = F_{k+1}^{-1}\tilde{P}_{k+1}. \tag{9}$$

If the Lyapunov's-like sequence $V_k = \tilde{P}_k^TF_k^{-1}\tilde{P}_k$ is defined then it follows that $V_{k+1} \leqslant \lambda_kV_k \leqslant V_k \leqslant V_0$ since

$$V_{k+1} - \lambda_kV_k = -\frac{\lambda_k}{c_k + ||W_kF_kW_k^T||}\,\tilde{P}_k^TW_k^TW_k\tilde{P}_k \leqslant 0 \tag{10}$$

for the free-design parameter of the estimation algorithm $c_k \in (0, \infty)$ and the forgetting factor $\lambda_k \in (0, 1]$, all integer $k \geqslant 0$ with

$$\tilde{P}_{k+1} - \tilde{P}_k = \left(I - \frac{F_kW_k^TW_k}{c_k + ||W_kF_kW_k^T}||\right)\tilde{P}_k - \tilde{P}_k = \frac{F_kW_k^TW_k\tilde{P}_k}{c_k + ||W_kF_kW_k^T||}. \tag{11}$$

Since the sequence $\{V_k\}_0^\infty$ is nonnegative and bounded for V_0 bounded and non strictly monotonically decreasing, then it has a finite nonnegative limit so that

$$\infty > V_0 \geqslant V_k \geqslant \lambda_{\max}(F_k^{-1})||\tilde{P}_k||_E^2. \tag{12}$$

This implies that the parameter error $\tilde{P}_k$ and its associate estimate are bounded for all sample since the above maximum eigenvalue of the covariance inverse is always strictly positive. As a result, all the estimates of the direct parameters used in the calculations in (2)–(3) are bounded. If the regressor is bounded then $E_{\tau k}$ and the auxiliary one $E'_{\tau k}$ are also bounded from the initial identities of this proof and then the estimated and error torques $\hat{\tau}_k$ and $\tilde{\tau}_k$ are bounded and $W_k \tilde{P}_k$ converges asymptotically to zero. It follows that the output and the tracking error are bounded if the reference is bounded. Finally, if the regressor fulfils a standard type of asymptotic persistent excitation condition then the parametrical error converges asymptotically to zero. This proves item (i). The proof of (ii) follows in the same way since the free parameters of the basic estimation scheme always belong to their admissibility domains compatible with stability if the supervisor scheme for any of the free-parameters is in operation. Finally, assume that the sampling period is on-line updated within its admissibility domain during a finite time interval and then it is fixed to a constant value within such an interval. Thus, the overall system becomes time-invariant after a finite time which may be set as initial time for analysis and the above stability results still hold.

Note that the closed-loop stability is also ensured if the time-varying sampling period tends exponentially to any constant value within its admissibility domain. A particular situation is when such a limit is its nominal value, in practice, a good tested value for a correct operation mode in the current practical application at hand. This property may be proved by extending directly Theorem 5.3(ii) by adding to the identification and – parametrical error bounded and exponentially decaying additive terms. The key point to ensure that the closed-loop stability holds under supervision is that the free-parameter of the parameter-adaptive algorithm and the sampling period are kept within their admissible domains. Those domains are compatible with convergence of the updating algorithm and stability. Thus, a judicious supervision of the free adaptation parameters/forgetting factor and sampling period dictated, for instance, by the given updating supervisory rules maintains the global stability (see Theorem 5.3(ii)) previously guaranteed in the unsupervised scheme (see Theorem 5.3(i)) while may be able to improve very much the transient behavior in the sense that large overshoots are avoided during the adaptation transient.

5.4. *Numerical Results*

A numerical example has been performed with the data of the above expert system. The evolution of the arms positions is displayed in Figs. 2 for both the unsupervised and supervised case with c_k-supervision and constant sampling period. The improvement is apparent in the second case. Estimates of the torque at the second joint is displayed in Fig. 3, and the loss function of the time-integral of the squared tracking error and the evolution of the c_k-parameter are shown in Figs. 4.

Some related results are shown in Figs. 5 when only sampling period supervision is used.

Results for a combined supervision of c_k and T_k are shown in Figs. 6.

It is seen that the supervision improves the transient performances related to the unsupervised case, with an apparent decrease in the value of the accumulative time-integral of the tracking error during the transient adaptation.

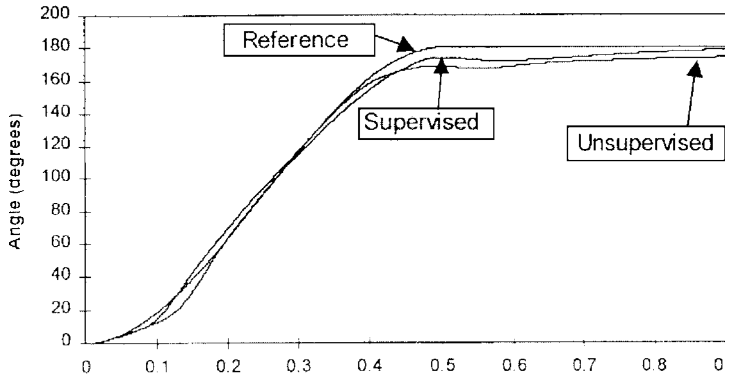

Fig. 2a. Position of the first joint.

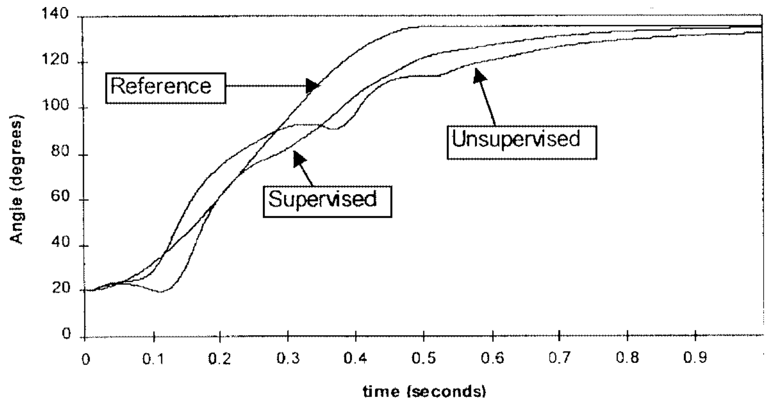

Fig. 2b. Position of the third joint.

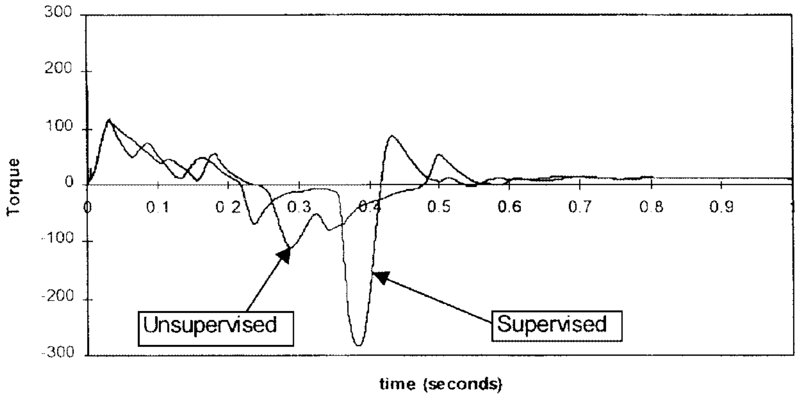

Fig. 3. Torque in the second joint with and without supervision of c.

6. Comments about the Use of the Formalism of Sections 2–3

The master expert system E_1^1 is unique and it is implemented as the basic identification-control scheme E_1^1 where the parametrizations of the identifier (Eqs. (2), (3)) and then the adaptive controller (Eq. (1)) are both readjusted each new sampling instant. Eq. (1) uses

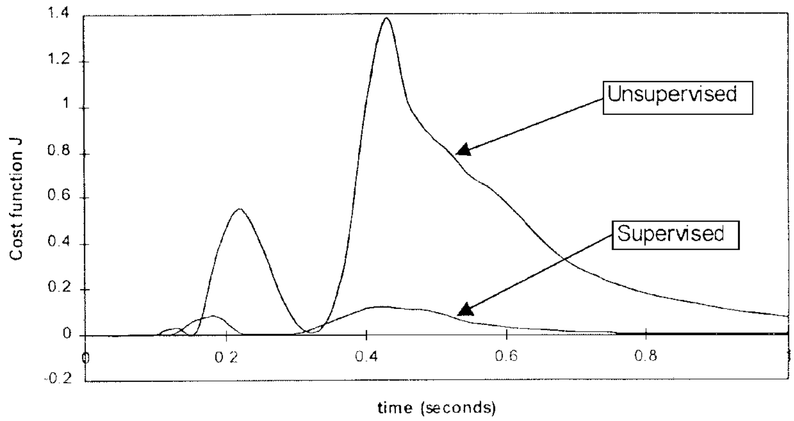

Fig. 4a. Cost function J with and without supervision.

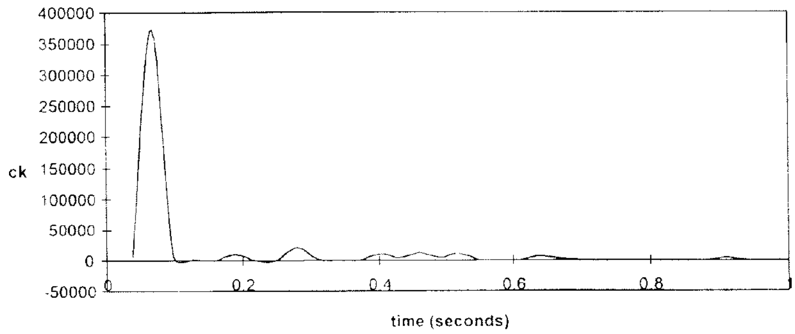

Fig. 4b. Evolution of the supervised free-design parameter c_k.

the process estimates given by (2) and (3). Note that this is the basic adaptive control philosophy where learning is based in updating values of the adaptive controller based in an on-line process of identification and tracking error measurement. As a result, the control signal is re-updated at each sampling instant. This part of the scheme plays the role of a unique master expert system (see Axioms 3.2, 3.3, and Theorem 3.4.1) which governs the slave expert systems and acts during certain time interval, i.e., the supervision of the free-design parameter (E_2^3) and sampling period (E_1^3), each having a set of rules in the knowledge base. The basic rules are Rules 1–5 described above. The processing or not processing of rules is basically organised for entire groups of rules concerned with E_2^3 and E_1^3 where any of the two parts may be switched-off from the supervision scheme during a set of samples. The generation of new rules has not been considered like that in this particular design. However, through initial experimentation, it has been decided which sampling law to use. Similarly, it has been decided which is the appropriate set of prefixed design parameters, like λ, $\bar{c}$, $\delta_{(\cdot)}$, C, T, T_0, $T_{\min}$, $T_{\max}$, etc., to use for the subsequent experiments to be processed. Note that each expert system within the network has its own database and knowledge base which is not necessary disjoint of the remaining ones (see Theorem 3.5).

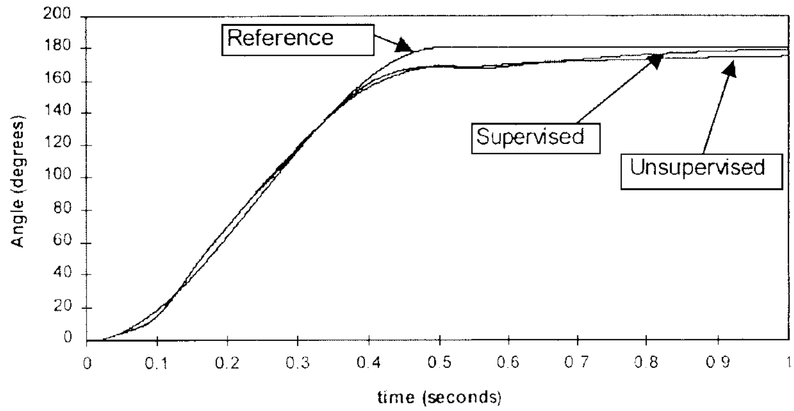

Fig. 5a. Position of the first joint with constant and with supervised sampling period.

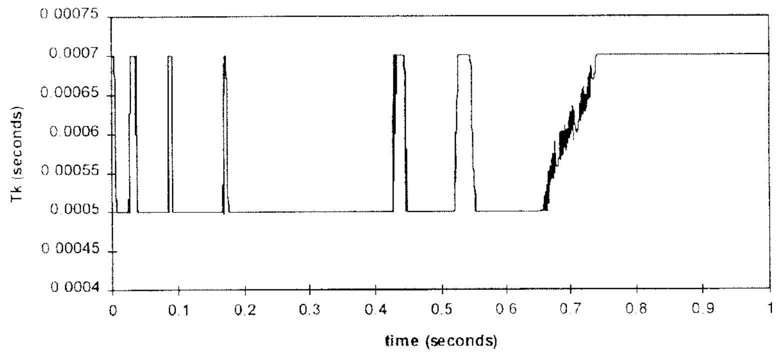

Fig. 5b. Supervision of the sampling period.

There are two main phases in the use of the formalism. The first phase is the construction of the expert system by learning through a set of experiments. The "admissible" experiments consists in processing the performance with different values of the nominal sampling period T_0 of the order of msecs, accordingly to the suitable requirements on bandwidth, stability and application requirements, and different values of the free-design parameter of the algorithm c (assumed to be constant) between its admissible variation domain (0, ∞) compatible with the closed-loop stability. Other data are obtained from the initial covariance matrix $F_0 = F_0^T > 0$ to which the transient performance is very sensitive. The various experiments are obtained for a wide variation of c in (0, ∞) but for small variation of the admissible sampling period which has to be compatible with the application and with a set of control design specifications. The reason of the small variation for the sampling period is that for controller synthesis purposes, time-invariance or quasi time-invariance processes are suitable, while the discrete plant becomes time-varying if the sampling period varies.

The sets of non-admissible experiments are possible sets which are rejected because the design requirements are not satisfied, as well as those rejected because of bad registered performances although they are initially processed. The equivalence classes are

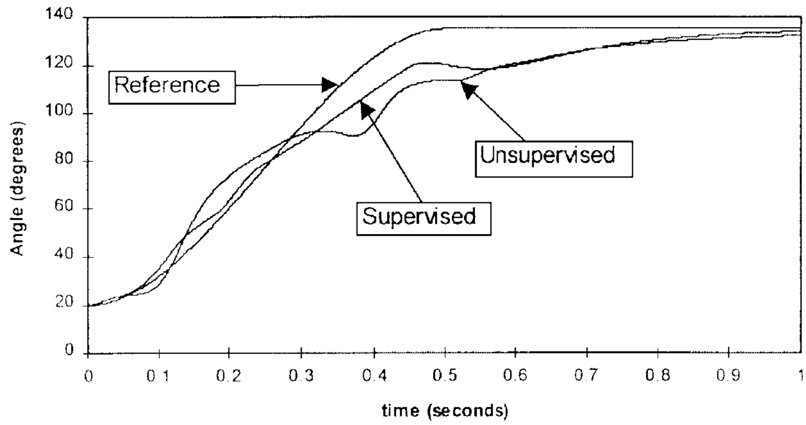

Fig. 6a. Position of the third joint with supervision of c_k and T_k and without supervision.

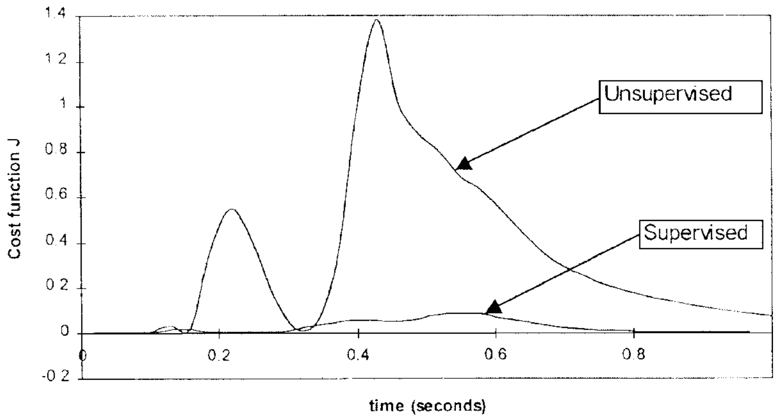

Fig. 6b. Cost function J with supervision of c_k and T_k and without supervision.

obtained form the various conditions p_0, F_0, c, T, T_0 which lead to very similar performances for the given second order plant. A classification of experiments for different types of plants is not performed in this study case since the plant is a second order one with damped oscillatory behaviour after feedback is implemented. A quality index is used in the general formalism to evaluate the transient performance. In this case, the quality index is a time-integral of the quadratic tracking error of the third arm position related to the reference signal for system E_2^3 and related to a generalized norm of the tracking error variations for system E_1^3 (see Eq. 4 and structures of sampling laws 1–3). The quality indices of the basic scheme E_1^1 and E_2^2 may also be defined with quadratic measures of the tracking errors so as to decide when to end the basic adaptation, switching between sampling laws or in-between $E_{1,2}^3$, or when to modify the weights in the loss function (Eq. 4). In this example, the quality indices of the four systems in the network (i.e., the basic adaptation scheme, the two supervisors and the coordinator) are the accumulative

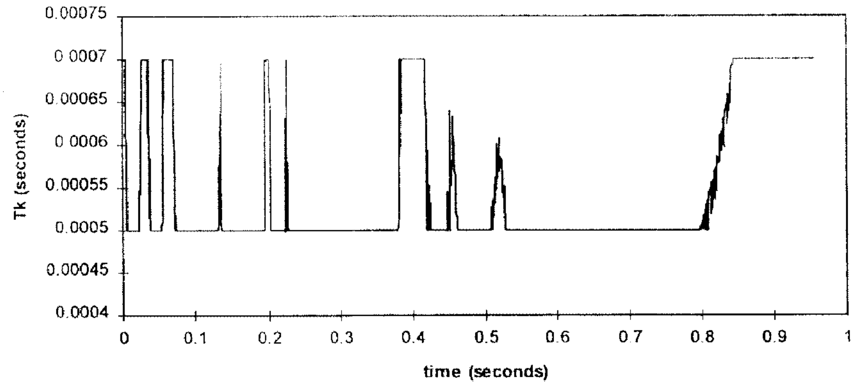

Fig. 6c. Supervision of the sampling period with combined supervision.

relative quadratic error of the third arm without weighting. The classification into examples in this particular case study has not been referred to different robot parametrizations. Such a classification has been made to design appropriate values for the magnitudes of the database (K_v, K_p, "a priori" modeling values for M, F, G, etc.) and to decide that the sampling law 1 affords the best results if it has to be chosen without alternative use of switching with another laws. Basically, this first phase is performed for the isolated master expert system E_1^1 without supervision from the other expert network components concerned with the supervisor updating process.

The second phase is an on-line hierarchized supervision procedure of the coordinator E_1^2 (hierarchical level 2) and the hierarchical level 3 with E_1^3 and E_2^3. The set of rules have been given before by using a database (Table 1) obtained from "a priori" knowledge on the controlled process and knowledge obtained from the phase 1 in which no supervision was implemented. The order of the priorities of E_1^3 and E_2^3 was decided to be kept fixed with the highest one for the sampling law after some experiments since the performance is much more sensitive to the sampling period variation than to the free-design parameter c_k. In this phase, if was also decided to use the sampling law 1 since it gave the best performances in phase 1. As explained before, it would be no difficult to incorporate the use of the combination of the various sampling laws to the scheme including the possibility of mutual switches, at the expense of a higher design complexity.

7. Conclusion

An axiomatic formalism has been provided for addressing the operation and performance of sets of expert systems grouped into expert networks. The axioms proposed are valid, in a first attempt, to distinguish between the different functions which permit to change hierarchies and priorities in the execution of admissible experiments within a learning context. The learning procedure is evaluated by stating quality indices to process the individual experiments entering the expert network. The axioms have been used to derive some mathematical results concerning the theoretical improvement of the expert network performance when learning is in progress as a result of the incorporation and processing of new admissible experiments. An application to improve the transient behavior of a robot manipulator has been also given. Work is now in progress about the axiomatic extension specifically to the self-learning aspects concerned with the rules of the knowledge base.

Acknowledgements

The authors are very grateful to the anonymous reviewers that have helped to improve the previous versions of this manuscript. They are also grateful to the Spanish Ministry of Science and Technology by its partial support of this work through Project DPI 2000-0244. They are also grateful to UPV/EHU for its support through Project 1/UPV/EHU00I06.I06-EB-8235/2000 and for its support of the Ph.D. studies of Mr. Garrido.

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Formalus ekspertinių sistemų aprašymas logikos priemonėmis

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Ekspertinės sistemos skirtos spręsti uždavinius iš anksto tiksliai apibrėžtoje dalykinėje srityje, panaudojant dirbtinio intelekto teorijos metodus. Nors ekspertinės sistemos plačiai naudojamos daugelyje dalykinių sričių, pavyzdžiui, gamybos bei valdymo sistemose, iki šiol vis dar yra didelė balta dėmė jų kūrimo teorijoje: nepasiūlyta kaip standartizuoti sprendžiamų uždavinių aprašus. Šiame straipsnyje siūloma kaip paprastu būdu sudarinėti formalius ekspertinių sistemų aprašus. Pasiūlymai gali būti taikomi ir manipuliatorių klasės robotams aprašinėti.