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STACKELBERG STRATEGIES FOR SINGULAR DYNAMIC GAMES

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Abstract. This paper is concerned with the derivation of open-loop Stackelberg (OLS) solutions of a class of continuous-time two-player nonzero-sum differential games characterized by quadratic cost functionals and linear singular systems. By applying the calculus of variations, necessary conditions are derived under which the open-loop Stackelberg solution of the leader exists. Under the transformation by which the matrix E has diagonal form, we derive a matrix Riccati differential equation from the necessary conditions. An example is given to illustrate the results of the paper.

Key words: Stackelberg strategy, singular systems, dynamic games.

I. Introduction. A great deal of attention has been paid to methods of design and analysis of Stackelberg strategies in nonzero-sum dynamic games, e.g., Chen and Cruz (1972), Simaan and Cruz (1973), Cruz (1978), Basar and Olsder (1982), Ho et.al.(1982).

In the recent years, there has been a growing interest in the system-theoretic problems of singular systems due to the extensive applications of singular systems in large scale systems, control theory, and other areas (Luenberger, 1977; Cobb, 1984; Bender and Laub, 1987; Liu and Zhang, 1989 etc.). Because the state-variable description does not always exit in modelling large scale systems (Luenberger, 1977), the dynamic game theory based on that has not satisfied the needs of the practical applications. Consequently, it is important to extend the dynamic game theory to singular systems. The purpose of this paper is to introduce the Stackelberg solution concept to singular systems.

In section II, Stackelberg game problems characterized by quadratic cost functions and linear time-invariant continuous singular systems are considered. In section III, by using the calculus of variations, necessary conditions for the existance of open-loop Stackelberg strategies are given. In section IV, under the transformation by which the matrix E has diagonal form, the matrix Riccati differential equation is derived. An example is given to illustrate the results of the paper in section V.

II. Problem formulation. Consider a Stackelberg game for a linear singular system

$$E\dot{x}(t) = Ax(t) + B^1u^1(t) + B^2u^2(t), \qquad Ex(0) = Ex_0 \quad (2.1)$$

with associated cost functional for each decision maker P_i (where P_1 denotes the follower; P_2, the leader)

$$J_i(u^1, u^2) = 1/2 \int_0^T \left[x(t)'Q^i x(t) + \sum_{j=1}^{2} u^j(t)'R^{ij}u^j(t)\right] dt + 1/2x(T)'E'Q^i(T)Ex(T), \qquad i = 1,2, \quad (2.2)$$

where E is a square matrix with $\operatorname{rank}(E) = n_1 < n$, and $\det[sE - A] \not\equiv 0$, $x(t)$ is an n-dimensional descriptor vector, $u^j(t)$ is an r_j – vector control function of P_j, $Q^i \geq 0$, $Q^i(T) \geq 0$, $R^{ij} \geq 0$, $i, j = 1, 2$.

Within the framework of the linear-quadratic singular game problem (2.1) and (2.2), the open-loop Stackelberg (OLS) solution concept can be introduced as follows:

DEFINITION. For the singular differential game posed above, a strategy $\tilde{u}^2$ constitutes an open-loop Stackelberg (OLS) strategy for the leader if

$$\sup_{u^1 \in R(\tilde{u}^2)} J_2(u^1, \tilde{u}^2) \leq \sup_{u^1 \in R(u^2)} J_2(u^1, u^2) \tag{2.3a}$$

for any u^2, where $R(u^2)$ is the rational reaction set of the follower which is defined by

$$R(u^2) = \{\tilde{u}^1 : \ J_1(\tilde{u}^1, u^2) \leq J_1(u^1, u^2), \text{ for any } u^1\}. \tag{2.3b}$$

For simplicity in the later notation, let us assume that the matrices E, A, and B^j take the form:

$$\{ E \mid A \mid B^j \} = = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \middle| \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \middle| \begin{pmatrix} B_1^j \\ B_2^j \end{pmatrix} \right\} \tag{2.4a}$$

Both Q^j and $Q^j(T)$ have the corresponding form

$$\{ Q^j \mid Q^j(T) \} = = \left\{ \begin{pmatrix} Q_{11}^j & Q_{12}^j \\ (Q_{12}^j)' & Q_{22}^j \end{pmatrix} \middle| \begin{pmatrix} Q_{11}^j(T) & Q_{12}^j(T) \\ (Q_{12}^j(T))' & Q_{22}^j(T) \end{pmatrix} \right\} \tag{2.4b}$$

REMARKS 1.

(**a**). In fact, for any $n \times n$ matrix E with $\text{rank}(E) = n_1 < n$, there exist $n \times n$ nonsingular matrices U amd V and $n_1 \times n_1$

unit matrix I such that $UEV = \text{diag}\{I, 0\}$. So when U and V are applied to the singular system (2.1), we have

$$UEVV^{-1}\dot{x}(t) = UAVV^{-1}x(t) + UB^1u^1(t) + UB^2u^2(t)$$

If we define $V^{-1}x(t) = [x_1(t)', x_2(t)']'$ and the following matrices

$$UAV = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad UB^j = \begin{pmatrix} B_1^j \\ B_2^j \end{pmatrix},$$

then the system equation of notion (2.1) becomes

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1^1u^1(t) + B_1^2u^2(t)$$
$$x_1(0) = x_{10} \qquad (2.5a)$$

$$O = A_{21}x_1(t) + A_{22}x_2(t) + B_2^1u^1(t) + B_2^2u^2(t) \qquad (2.5b)$$

in which the matrices E, A, and B^j have taken the form (2.4a), where $x_{10} = (1\ 0)V^{-1}x_0$.

(**b**). For the singularly perturbed system

$$\dot{x}_1(t) = A_{11}^*x_1(t) + A_{12}^*x_2(t) + B_1^{1*}u^1(t) + B_1^{2*}u^2(t),$$
$$x_1(0) = x_{10}$$
$$\mu\dot{x}_2(t) = A_{21}^*x_1(t) + A_{22}^*x_2(t) + B_2^{1*}u^1(t) + B_2^{2*}u^2(t),$$
$$x_2(0) = x_{20}$$

the following slow subsystem can be obtained by letting $\mu = 0$ (Khalil and Medanic, 1980).

$$\dot{x}_1(t) = A_{11}^*x_1(t) + A_{12}^*x_2(t) + B_1^{1*}u^1(t) + B_1^{2*}u^2(t)$$
$$x_1(0) = x_{10}$$
$$0 = A_{21}^*x_1(t) + A_{22}^*x_2(t) + B_2^{1*}u^1(t) + B_2^{2*}u^2(t)$$
$$x_2(0) = x_{20}$$

which is the same as (2.5) in the form. As far as the authors know, Stackelberg strategies for singularly perturbed linear-quadratic problems have been dealt with in the continuous-time systems (Khalil and Medanic, 1980). But, all of the previuos results about the singularly perturbed Stackelberg problems were obtained with the assumption that A_{22} is non-singular. But the method of this paper can be used to solve the slow Stackelberg problems of singularly perturbed continuous-time games without such assumption.

With the possibility of impulses in $x(t)$ (and $u^j(t)$), we immediately face the question of how to interpret the cost integral (2.2). We do this in the following assumption.

ASSUMPTION 1. The integral (2.2) is assumed to be defined in the same way as in Bender and Laub (1987); that is as a distributional integral. This type of integral has the property that

$$\int_0^T \| \delta(t)v \|_2 \, \mathrm{dt} < \infty \quad \text{but} \quad \int_0^T \| \delta(t)v \|_2^2 \, \mathrm{dt} = \infty$$

where $\delta(t)v$ is the impulse function along v defined by

$$< \delta(t)v, f(t) >= f(0)$$

Thus an impulse function is integrable but its square is not.

III.Necessary conditions for the existance of OLS solution. In order to derive the open-loop Stackelberg solution of the leader, we must first determine the rational reaction of the followerP_1 to control u^2 which is declared by the leader P_2. Since the underlying information pattern is open-loop, the optimization problem faced by the follower P_1 is reduced to the following optimal control problem.

$$\min_{u^1} J_1(u^1, u^2) \tag{3.1}$$

subject to (2.1) for each fixed u^2.

In fact, the optimization problem (3.1) is an optimal control problem of singular systems and was solved in Jonckheere (1988) and Bender & Laub (1987). By extending the analysis of Jonckheere (1988), one can obtain the following conclusion.

Lemma. *Assume that*
(1) (2.1) is controllable at infinity by the follower;
(2) $Q^1_{22} > 0,\ R^{11} > 0$.
Then, the rational reaction of the follower to u^2 *exists, is unique and continuous, and is defined by the unique solution to the following two-point boundary value problem.*

$$E\dot{x}(t) = Ax(t) + B^1u^1(t) + B^2u^2(t),\qquad Ex(0) = Ex_0 \tag{3.2a}$$

$$E'\dot{p}^1(t) = -Q^1x(t) - A'p^1(t),\qquad E'p^1(T) = E'Q^1(T)Ex(T) \tag{3.2b}$$

$$0 = R^{11}u^1(t) + B^{1'}p^1(t) \tag{3.2c}$$

Proof (Sketch). If $u^2(t) = 0$, then the problem (3.1) is the same one as in Bander and Laub (1987), and Lemma has been proven in Jonckheere (1988). But, it doesn't influence the second-order variation of the cost functional whether $u^2(t)$ is zero or not. Therefore, by extending Jonckheere's proof, it follows that sufficient and necessary condition for the existence of optimal control $u^1(t)$ is (3.2) under the assumption of Lemma.

In addition, by extending the analysis of Bander and Laub (1987), it is easy to prove that for any given continuous $u^2(t)$ there exists a unique continuous solution to (3.2). Q.E.D.

REMARK 2. (2.1) is controllable at infinity by the follower if all its impulsive modes can be excited from zero initial conditions by an input $u^1(t)$ containing no impulse. (2.1) is controllable at infinity by the follower iff there exists a feedback K such that $(sE - A - B^1K)^{-1}$ has no dynamic modes at infinity. Further, iff (2.1) is controllable at infinity by the follower, (A_{22}, B_2^1) has full row rank.

Now, to obtain the Stackelberg strategy of the leader, we have to minimize $J_2(u^1, u^2)$, in view of the unique rational reaction of the follower to be determined from (3.2). Therefore, the problem faced by the leader is the following optimization problem.

$$\min_{u^2} J_2(u^1, u^2) \tag{3.3}$$

subject to (3.2).

In fact, the optimization problem (3.3) also is an optimal control problem of singular systems. But, from Lemma, it follows that for any continuous function $u^2(t)$ there exists a unique continuous solution to the constraint (3.2). Thus, the standard variational calculus, for example, Sage (1968, 53–66), can be used to solve the problem (3.3). To solve this optimization problem, let us append the constraints (3.2) to $J_2(u^1, u^2)$ by using Lagrange multipliers $p^2(t), n^1(t), m^1(t)$ and $n^1(T)$.

$$\begin{aligned} J_2(u^1, u^2) =&(1/2)x(T)'E'Q^2(T)Ex(T) \\ &- n^1(T)'[E'Q^1(T)Ex(T) - E'p^1(T)] \\ &+ \int_0^T \{(1/2)[x(t)'Q^2x(t) + u^1(t)'R^{21}u^1(t) \\ &+ u^2(t)'R^{22}u^2(t)] \\ &+ p^2(t)'[Ax(t) + B^1u^1(t) + B^2u^2(t) - E\dot{x}(t)] \\ &+ n^1(t)'[-Q^1x(t) - A'p^1(t) - E'\dot{p}^1(t)] \\ &+ m^1(t)'[R^{11}u^1(t) + B^{1'}p^1(t)]\}\mathrm{d}t \end{aligned} \tag{3.4}$$

By using the standard variational techniques, the necessary conditions that characterize u^2 being the optimal solution of optimization (3.3) take the form

$$E\dot{x}(t) = Ax(t) + B^1u^1(t) + B^2u^2(t),$$
$$Ex(0) = Ex_0 \tag{3.5a}$$
$$E'\dot{p}^1(t) = -Q^1x(t) - A'p^1(t),$$
$$E'p^1(T) = E'Q^1(T)Ex(T) \tag{3.5b}$$
$$0 = R^{11}u^1(t) + B^{1'}p^1(t) \tag{3.5c}$$
$$E'\dot{p}^2(t) = -Q^2x(t) - A'p^2(t) + Q^1n^1(t)$$
$$E'p^2(T) = E'Q^2(T)Ex(T) - E'Q^1(T)En^1(T) \tag{3.5d}$$
$$E\dot{n}^1(t) = An^1(t) - B^1m^1(t), \qquad En^1(0) = 0 \tag{3.5e}$$
$$0 = R^{21}u^1(t) + B^{1'}p^2(t) + R^{11}m^1(t) \tag{3.5f}$$
$$0 = R^2u^2(t) + B^{2'}p^2(t) \tag{3.5g}$$

IV. Characterization of optimal solution. If the following vectors are defined

$$\bar{x}(t)' = (x_1(t)', -n_1^1(t)') \tag{4.1a}$$
$$\bar{p}(t)' = (p_1^2(t)', p_1^1(t)') \tag{4.1b}$$
$$\bar{u}(t)' = (p_2^2(t)', -n_2^1(t)', m^1(t)', p_2^1(t)',$$
$$x_2(t)', u^1(t)', u^2(t)') \tag{4.1c}$$

then, under the assumption (2.3), the two-point boundary value problem (3.5) can be rewritten in the compact form as follows:

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t),$$
$$\bar{x}(0)' = (x_{10}', 0) \tag{4.2a}$$
$$\dot{\bar{p}}(t) = -\bar{Q}(t)\bar{x}(t) - \bar{A}'\bar{p}(t) - \bar{S}\bar{u}(t),$$
$$\bar{p}(T) = \bar{Q}(T)\bar{x}(T) \tag{4.2b}$$
$$0 = \bar{S}'\bar{x}(t) + \bar{B}'\bar{p}(t) + \bar{R}\bar{u}(t) \tag{4.2c}$$

where:

$$\bar{A} = \mathrm{diag}\{A_{11}, A_{11}\} \tag{4.3a}$$

$$\bar{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & A_{12} & B_1^1 & B_1^2 \\ 0 & A_{12} & B_1^1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.3b}$$

$$\bar{S} = \begin{pmatrix} A'_{21} & Q_{12}^1 & 0 & 0 & Q_{12}^2 & 0 & 0 \\ 0 & 0 & 0 & A'_{21} & Q_{12}^1 & 0 & 0 \end{pmatrix} \tag{4.3c}$$

$$\bar{Q}(T) = \begin{pmatrix} Q_{11}^2(T) & Q_{11}^1(T) \\ Q_{11}^1(T) & 0 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} Q_{11}^2 & Q_{11}^1 \\ Q_{11}^1 & 0 \end{pmatrix} \tag{4.3d}$$

$$\bar{R} = \begin{pmatrix} 0 & \bar{R}^{11} & \bar{B}_2^2 \\ \bar{R}^{11\prime} & \bar{Q}_{22}^2 & 0 \\ \bar{B}_2^{2\prime} & 0 & R^{22} \end{pmatrix} \tag{4.3e}$$

with

$$\bar{R}^{11} = \begin{pmatrix} 0 & A_{22} & B_2^1 \\ A'_{22} & Q_{22}^1 & 0 \\ B_2^{1\prime} & 0 & R^{11} \end{pmatrix}, \qquad \bar{Q}_{22}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Q_{22}^2 & 0 \\ 0 & 0 & R^{21} \end{pmatrix},$$

$$\bar{B}_2^2 = \begin{pmatrix} B_2^2 \\ 0 \\ 0 \end{pmatrix} \tag{4.3f}$$

The system (4.2) is a singular system in its own right. Moreover, the matrix of this system already has the form (2.4a). In order to solve the two-point boundary problem (4.2), it is necessary for $\bar{R}$ to be invertible. Towards this end, we shall state some sufficient conditions for the invertbility of $\bar{R}$ as follows:

ASSUMPTION 2.
(a). (2.1) is controllable at infinity by the follower;
(b). $Q_{22}^1 > 0$, $R^{11} > 0$;
(c). $Q_{22}^2 \geq 0$, $R^{21} \geq 0$, $R^{22} > 0$.

REMARKS 3.

(**1**) From Lemma 12 of Bender and Laub (1987), it follows that Assumption 2 – (a) and (b), are one possible set of sufficient conditions for $\bar{R}^{11}$ to be nonsingular.

(**2**) If A is nonsingular, then the relation below is true.

$$Z = \begin{pmatrix} 0 & A & C \\ A' & B & 0 \\ C' & 0 & D \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & C'(A')^{-1} & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -C'(A')^{-1}BA^{-1} & 0 & I \end{pmatrix}$$
$$\times \begin{pmatrix} 0 & A & C \\ A' & B & 0 \\ 0 & 0 & D + C'(A')^{-1}BA^{-1}C \end{pmatrix}$$

Thus, we can get that Z is nonsingular if A is invertable, $B \geq 0$ and $D > 0$. According to Assumption 2, it is easy to prove that R is nonsingular.

If the Assumption 2 is satisfied, then $\bar{u}(t)$ can be uniquely determined from (4.2c) as:

$$\bar{u}(t) = -\bar{R}^{-1}[\bar{S}'\bar{x}(t) + \bar{B}'\bar{p}(t)]. \tag{4.4}$$

Substituting it into (4.2a-b) leads to

$$\begin{pmatrix} \dot{\bar{x}}(t) \\ \dot{\bar{p}}(t) \end{pmatrix} = \begin{pmatrix} \bar{A} - \bar{B}\bar{R}^{-1}\bar{S}' & -\bar{B}\bar{R}^{-1}\bar{B}' \\ -(\bar{Q} - \bar{S}\bar{R}^{-1}\bar{S}') & -(\bar{A} - \bar{B}\bar{R}^{-1}\bar{S}')' \end{pmatrix} \times \begin{pmatrix} \bar{x}(t) \\ \bar{p}(t) \end{pmatrix} \tag{4.5a}$$

with the boundary condition

$$x(0)' = [x'_{10}, 0], \qquad \bar{p}(T) = \bar{Q}(T)\bar{x}(T). \tag{4.5b}$$

When introducing the linear transformation

$$\bar{p}(t) = P(t)\bar{x}(t) \tag{4.6}$$

then from (4.5) and eliminating $\bar{x}(t)$, we can obtain that $P(t)$ satisfies the following matrix Riccati differential equation.

$$\begin{aligned} -\dot{P}(t) =& (\bar{Q} - \bar{S}\bar{R}^{-1}\bar{S}') + P(t)(\bar{A} - \bar{B}\bar{R}^{-1}\bar{S}') \\ &+ (\bar{A} - \bar{B}\bar{R}^{-1}\bar{S}')'P(t) - P(t)\bar{B}\bar{R}^{-1}\bar{B}'P(t) \end{aligned}$$

$$P(T) = \bar{Q}(T) \tag{4.7}$$

According to the definitions of $\bar{Q}$, $\bar{Q}(T)$ and $\bar{R}$, it is easy to prove that the matrix Riccati differential equation (4.7) is time- invariant and its solution is symmetric. Therefore, the eigenvector method, for example, Vaughan (1969), can be used to solve (4.7).

From (4.4) if the solution of (4.7) exists and is unique, then the optimal solution to (3.5) exists, is unique, and can be determined by

$$\bar{u}(t) = \bar{R}^{-1}[\bar{S}' + \bar{B}'P(t)]\bar{x}(t) \tag{4.8a}$$

where $\bar{x}(t)$ satisfies

$$\begin{aligned} \dot{\bar{x}}(t) =& \{\bar{A} - \bar{B}\bar{R}^{-1}[\bar{S}' + \bar{B}'P(t)]\}\bar{x}(t), \\ & \bar{x}(0)' = [x'_{10}, 0] \end{aligned} \tag{4.8b}$$

Now we shall conclude the discussion above with the following theorem.

Theorem. *Consider the linear-quadratic singular Stackelberg game problem (2.1) and (2.2) together with Assumption 1–2. Then, if the solution to the matrix Riccati differential equation (4.7) exists and is unique, then there exists a unique* *solution to the necessary conditions (3.5), and this solution is given by (4.8).*

Proof. The proof has been given prior to the statment of the Theorem.

V. Illustrative example. Consider the following Stackelberg game problem

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = $$
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u^1(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u^2(t)$$

$$J_j = \int_0^3 \{1/2x(t)'x(t) + 1/2[u^j(t)]^2\}\mathrm{dt}$$
$$+ 1/2[x_1(3)]^2, \qquad j = 1, 2,$$

where $x_1(0) = x_{10}$ and $x(t)' = [x_1(t), x_2(t)]$.

In this example, the necessary conditions for existance of OLS solution are as follows:

$$\begin{pmatrix} \dot{x}_1(t) \\ -\dot{n}_1^1(t) \\ \dot{p}_1^2(t) \\ \dot{p}_1^1(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & -1 & 0 \\ 2 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ -n_1^1(t) \\ p_1^2(t) \\ p_1^1(t) \end{pmatrix}$$

$$\begin{pmatrix} x_1(0) \\ -n_1^1(0) \\ p_1^2(T) \\ p_1^1(T) \end{pmatrix} = \begin{pmatrix} x_{10} \\ 0 \\ x_1(T) - n_1^1(T) \\ x_1(T) \end{pmatrix}$$

By using eigenvector method, for example, Abou-Kandil and Bertrand (1985), this two-point boundary value problem has a unique solution, where the optimal controls $u^1(t)$ and $u^2(t)$ are

$$u^1(s) =(-3.303 + 10.75e^{-s} - .2341e^{-4.606s} - 2.269e^{-3.606s}) \\ /(1 + 35.51e^{-s} + .0704e^{-4.606s} - 7.495e^{-3.606s})$$

$$u^1(s) =(2.303 - 46.26e^{-s} + .1620e^{-4.606s} + 9.765e^{-3.606s}) \\ /(1 + 35.51e^{-s} + .0704e^{-4.606s} - 7.495e^{-3.606s})$$

with $s = 3 - t$. It should be noted that $u^1(t)$ and $u^2(t)$ are only candidates for OLS strategies which satisfy the necessary conditions.

VI. Conclusion. This paper discussed OLS strategy for Stackelberg games characterized by linear continuous-time singular systems and quadratic cost functionals. By using the calculus of variations , necessary conditions for the existance of OLS strategy were given. With the Riccati transformation, the matrix Riccati differential equation has been derived from the necessary conditions of the existance of OLS strategy. The results of the paper can be straightforward extended to multi-level Stackelberg problems. In addition, when E is nonsingular, the singular system will become the state-space system. So the results of the paper include the corresponding results of all previous papers.

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