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ON THE DIFFERENCE SCHEMES FOR PROBLEMS WITH NON–LOCAL BOUNDARY CONDITIONS

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Abstract. This paper is devoted to the construction and investigation of difference schemes for the solution of one-dimensional parabolic problems with non-classical boundary conditions. The stability of schemes and the convergence of a numerical solution is proved in the norms L_1 and C.

Key words: difference schemes, non-local boundary conditions, convergence in L_1 norm.

Introduction. Boundary value differential problems with non-local conditions arise in various fields of natural sciences. Bitsadze and Samarskij (1969) proposed and investigated a boundary value problem, where the value of the solution on the boundary is connected with the value of the same solution at the interior points of the domain. Numerical methods for the solution of this problem are proposed by Gordeziani (1981), Iliin and Moiseev (1986). An important class of problems deals with the solution of boundary value problems for the heat conduction equation, where one boundary condition is replaced by a non-local integral condition

$$\int_a^b p(x)\,u(x,t)\,dx = g(t), \tag{0.1}$$

where $u(x,t)$ is an unknown solution of the boundary problem. The solvability of such problems is investigated by Kaminin (1964), Jonkin (1977a, 1979), the difference schemes for the numerical solution are proposed by Jonkin (1977b), Čiegis (1984). New necessary and sufficient conditions for the existence of the solution of boundary value elliptic problems with non-local boundary conditions are given, and economical numerical methods are proposed by Sapagovas and Čiegis (1987a, 1987b). A singularly perturbed non-local problem is examined and a uniformly second order accuracy finite element scheme is proposed by Čiegis (1988).

The present paper deals with the investigation of stability and convergence in L_1 and C norm of some difference schemes for the heat conduction problem with a non-local boundary condition.

1. Equations. We start from the heat conduction problem with non-local boundary conditions

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(k(x,t)\frac{\partial u}{\partial x}\Big) - g(x,t)\,u(x,t) + f(x,t), \tag{1.1}$$

$$0 < k_0 \leqslant k(x,t) \leqslant k_1, \qquad q(x,t) \geqslant 0,$$

$$u(x,0) = u_0(x), \tag{1.2}$$

$$u(0,t) = u_1(t), \tag{1.3}$$

$$k(1,t)\frac{\partial u}{\partial x}(1,t) = k(0,t)\frac{\partial u}{\partial x}(0,t). \tag{1.4}$$

If $k(x,t) \equiv 1$, $q(x,t) \equiv 0$, then the convergence of the difference solution in C and W_2^1 norms is proved by Jonkin and Siedov (1982). A new method for the investigation of the stability of implicit difference schemes for the problem (1.1)–(1.4) is proposed by Jonkin and Furletov (1990). The convergence of a discrete solution in C norm is proved by them for the problems with the coefficients, that satisfy the auxiliary conditions $k(x) = k(1-x)$, $q(x) = q(1-x)$, $f(x) = f(1-x)$, and the mesh used in their paper is supposed to be uniform.

For the sake of simplicity at first we take the uniform mesh $\omega = \omega_h \times \omega_\tau$, too. An implicit difference scheme is constructed for the solution of problem (1.1)–(1.4)

$$\omega_\tau = \{t = t_j,\ t_j = (j-1)\tau,\ t_0 = 0,\ t_m = T\},$$

$$\omega_h = \{x = x_i,\ x_i = x_{i-1} + h,\ x_0 = 0,\ x_N = 1\},$$

$$\frac{\widehat{y} - y}{\tau} = (a\widehat{y}_{\bar{x}})_x - d_i\widehat{y}_i + \varphi_i, \tag{1.5}$$

$$y_i^0 = u_0(x_i), \tag{1.6}$$

$$y_0 = u_1(t_j), \tag{1.7}$$

$$a_N\widehat{y}_{\bar{x},N} + \frac{h}{2}(y_{t,N} + d_N\widehat{y}_N - \varphi_N) = a_1\widehat{y}_{\bar{x},1} - \frac{h}{2}(y_{t,0} + d_0\widehat{y}_0 - \varphi_0). \tag{1.8}$$

The notations and conventions here adopted are as introduced by Samarskij (1983)

$$y = y(x_i,t_j),\ \widehat{y} = y(x_i,t_{j+1}),\ y_t = (\widehat{y} - y)/\tau,$$

$$a_i = k\Big(\frac{x_i + x_{i+1}}{2}, t_{j+1}\Big),\ d_i = q(x_i,t_{j+1}),\ \varphi_i = f(x_i,t_{j+1}).$$

The realization of a one time of the difference scheme (1.5)–(1.8) is economically performed by a modified factorization algorithm (see Čiegis, 1984)

$$\widehat{y}_i = \alpha_i\widehat{y}_{j+1} + \beta_i, \quad i = \overline{0,N-1},\ \alpha_0 = 1,\ \beta_0 = U_1(t_{j+1}),$$

$$\widehat{y}_i = c_i\widehat{y}_N + d_i, \qquad c_{N-1} = \alpha_{N-1},\ d_{N-1} = \beta_{N-1},$$
$$c_i = \alpha_i c_{i+1}, \qquad d_i = \beta_i + \alpha_i d_i\ i = \overline{N-2,\,1},$$

and $\widehat{y}_N$ is determined from the boundary condition (2.8).

2. Stability and convergence. At first the stability of the difference scheme (1.4)–(1.8′) is investigated. A new term g_N is added to the boundary condition (1.8) (we have $g_N = 0$ for the exact scheme (1.5)–(1.8))

$$a_N\,\widehat{y}_{\bar{x},N} + \frac{h}{2}\,(y_{t,N} + d_N\,\widehat{y}_N - \varphi_N) = a_1\,\widehat{y}_{\bar{x},1} - \frac{h}{2}\cdot \qquad (1.8')$$
$$\cdot\,(y_{t,0} + d_0\,\widehat{y}_0 - \varphi_0) + g_N.$$

This has enabled us to prove the boundary condition stability of the difference scheme (1.5)–(1.8), too. It is sufficient to investigate the homogeneous boundary condition (1.7)

$$\widehat{y}_0 = 0. \qquad (1.7')$$

Theorem 2.1. *The difference scheme (1.5)–(1.8′) is stable in L_1 norm and the following estimation is valid for the solution of the scheme*

$$\|y(t_j)\|_1 \leqslant \|y(t_0)\|_1 + t_j \max_{i\leqslant k\leqslant j} \Big(\sum_{i=1}^{N-1} h|\varphi_i^k| + |g^k{}_N|\Big), \qquad (2.1)$$

where the notation of a discrete L_1 norm is used

$$\|y\|_1 = \sum_{i=1}^{N-1} h|y_i| + 0,5\,h|y_N|$$

Proof. We may write the scheme (1.5)–(1.8′) in the form

$$A\widehat{Y} = F,$$

where matrix A is of the form

$$A = \begin{pmatrix} c_1 & -b_2 & 0 & \dots & 0 & 0 \\ -b_2 & c_2 & -b_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_{N-1} & -b_N \\ -b_1 & 0 & 0 & \dots & -b_N & c_N \end{pmatrix}$$

$$c_i = b_i + b_{i+1} + \tau\, d_i + 1, \quad b_i = \frac{\tau}{h^2}\, a_i, \; i = \overline{0, N-1},$$
$$c_N = b_N + 0,5\,\tau\, d_N + 0,5, \quad F_i = y_i + \tau\,\varphi_i,$$
$$F_N = 0,5\, y_N + 0,5\,\tau(\varphi_0 + \varphi_N) + \frac{\tau}{h}\, g_N.$$

It is easy to prove that all the entries of the inverse matrix $A^{-1} = (g_{ij})$ are positive. This follows from the simple equality $(A^{-1})^* = (A^*)^{-1}$, where A^* is a transposed matrix. In our case A^* is a diagonally-dominant matrix, so $g_{ij} > 0$. From the equation (2.1) it follows that

$$\widehat{Y}_t = \sum_{j=1}^{N} g_{ij}\, F_j,$$
$$|\widehat{Y}_i| = \Big|\sum_{j=1}^{N} g_{ij}\, F_j\Big| \leqslant \sum_{j=1}^{N} |g_{ij}|\, |F_j| = \sum_{j=1}^{N} g_{ij} |F_j| = \widehat{W}_i,$$

where $\widehat{W}_i$ is the solution of linear problem (2.3)

$$A\,\widehat{W} = |F|. \tag{2.3}$$

From the linear system (2.3) it follows that

$$c_1\,\widehat{W}_1 = b_2\,\widehat{W}_2 + |F_1|$$
$$c_i\,\widehat{W}_i = b_i\,\widehat{W}_{i-1} + b_{i+1}\,\widehat{W}_{i+1} + |F_i|, \quad i = \overline{2, N-1}$$
$$c_N\,\widehat{W}_N = b_N\,\widehat{W}_{N-1} + b_1\,\widehat{W}_1 + |F_N|.$$

Then, summing up all the equalities, we have

$$\sum_{i=1}^{N-1} h(1+\tau\, d_i)\,\widehat{W}_i + \frac{h}{2}(1+\tau\, d_N)\,\widehat{W}_N = \sum_{i=1}^{N} |F_i|\, h, \quad (2.4)$$

and using the inequalities $|\widehat{Y}_i| \leqslant \widehat{W}_i$ from (2.4) we obtain

$$\|\widehat{Y}\|_1 \leqslant \|\widehat{W}\|_1 \leqslant \|Y\|_1 + \tau\Big(\sum_{i=1}^{N-1} h|\varphi_i| + |g_N|\Big). \quad (2.5)$$

From the inequalities (1.13), (2.5) it follows that

$$\|Y(t_j)\|_1 \leqslant \|Y(t_0)\|_1 + t_j \max_{1\leqslant k\leqslant j} \Big(\sum_{i=1}^{N-1} h|\varphi_i^k| + |g_N{}^k|\Big),$$

and this completes the proof of the theorem.

The convergence of the solution of the difference scheme (1.5)–(1.8) to an exact solution of the problem (1.1)–(1.4) follows from Theorem 2.2.

Theorem 2.2. *The solution of the difference scheme (1.5)–(1.8) converges to an exact solution of the problem (1.1)–(1.4), and the error estimation in L_1 norm is true*

$$\|y(t_j) - u(t_j)\|_1 = O(\tau + h^2).$$

Proof. We introduce a notation for the error function $z_i = y_i - u(x_i, t_j)$. It is easy to show that z_i is the solution of the difference problem

$$z_t = (a\,\widehat{z}_{\bar{x}})_x - d_i\,\widehat{z}_i + \psi_i, \quad (2.6)$$
$$z_i^0 = 0, \quad (2.7)$$
$$z_0^j = 0, \quad (2.8)$$
$$a_N\widehat{z}_{\bar{x},N} + 0,5\, h(z_{t,N} + d_N\,\widehat{z}_N) = a_1\,\widehat{z}_{\bar{x},1} + \psi_N, \quad (2.9)$$

where ψ_i is the approximation error of the difference scheme (2.4)–(2.8). A simple Taylor expansion reveals that $|\psi_i|$ can be bounded by

$$|\psi_i| \leqslant C(\tau + h^2), \quad i = \overline{1,N}.$$

Using stability inequality (2.1) for the solution of the difference scheme (2.6)–(2.9) we obtain

$$\|z\|_1 \leqslant \|z(t_0)\|_1 + t_j \max_{1 \leqslant k \leqslant j} \Big(\sum_{i=1}^{N-1} h|\psi_i^k| + |\psi_N^k| \Big) \leqslant$$
$$\leqslant t_j \cdot C(\tau + h^2) = O(\tau + h^2).$$

This completes the proof of the theorem.

Theorem 2.3. *The difference scheme (1.5)–(1.8) has a unique solution for all parameters* τ, h.

Proof. To show the existence of the solution of the difference scheme (1.5)–(1.8) it is sufficient to prove that the linear problem (1.5)–(1.8) has only a unique solution. However the solution of the difference scheme (1.5)–(1.8) with the initial data

$$\varphi(x,t) = 0, \quad u_1(t) = 0, \quad u_0(x) = 0, \quad g_N(t) = 0$$

is unique $y(x_i, t_j) \equiv 0$, as it follows from the stability inequality (2.1)

$$\|y(t_j)\|_1 \leqslant \|u_0\|_1 + t_j \max_{1 \leqslant k \leqslant j} \Big(\sum_{i=1}^{N-1} h|\varphi_i| + |g_N| \Big) = 0.$$

3. Nonuniform mesh. In this section we generalize the results of Section 2. Now we use a nonuniform mesh

$$\omega_h = \{x = x_i, \quad x_i = x_{i-1} + h_i, \quad x_0 = 1, \quad x_N = 1\},$$

and an implicit difference scheme is constructed

$$y_t = (a\,\widehat{y}_{\bar{x}}) - d_i\,\widehat{y}_i + \psi_i, \tag{3.1}$$
$$y_i^0 = u_0\,(x_i), \tag{3.2}$$
$$y_0 = u_1(t_j), \tag{3.3}$$
$$a_N\,\widehat{y}_{\bar{x},N} + \frac{h_N}{2}\,(y_{t,N} + d_N\,\widehat{y}_N - \psi) = a_1\,\widehat{y}_{\bar{x},1} - \\ - \frac{h}{2}(y_{t,0} + d_0\,\widehat{y}_0 - \psi), \tag{3.4}$$

where the notation

$$(a\,y_{\bar{x}})_{\widehat{x}} = \frac{1}{\hbar_i}\left(a_{i+1}\frac{y_{i+1} - y_i}{h_{i+1}} - a_i\frac{y_i - y_{i-1}}{h_i}\right),$$
$$\hbar_i = 0,5(h_{i+1} + h_i), \quad i = \overline{1, N-1}$$

is used.

Lemma 3.1. *The difference scheme (3.1)–(3.4) is stable in L_1 norm and the estimation (2.1) is valid, where the notation of a discrete L_1 norm on the nonuniform mesh ω_h is used*

$$||Y||_1 = \sum_{i=1}^{N-1} \hbar_i|y_i| + 0,5h_N|y_N|.$$

Proof. The proof of Lemma 3.1 is similar to that of Theorem 2.1. We again can write the difference scheme (3.1)–(3.4) in the matrix form (2.2), only the entries of the matrix are defined differently this time

$$c_i = b_i + b_{i+1} + \tau\hbar_i d_i + \hbar_i, \quad b_i = \frac{\tau}{\hbar_i}a_i,$$
$$c_N = b_N + 0,5h_N(\tau d_N + 1), \quad F_i = \hbar_i(y_i + \tau\varphi_i),$$
$$F_N = 0,5\hbar_N y_N + 0,5\tau(h_N\varphi_N + h_1\varphi_0) + \tau g_N.$$

The remaining part of the proof is analogous to the proof of Theorem 2.1.

Using the stability inequality we can prove the validity of the following estimation for the error function $z_i = y_i - u(x_i, t_j)$.

$$||z||_1 \leqslant t_j \max_{1\leqslant k\leqslant j} \Big(\sum_{i=1}^{N-1} \hbar_i |\psi_i| + |\psi_N| \Big),$$

where the approximation error ψ_i of the difference scheme (3.1)–(3.4) can be estimated by the inequalities

$$|\psi_i| = \Big| \frac{h_{i+1} - h_i}{3} u^{III}(x_i, t_j) + \frac{\hbar_i^2}{12} u^{IV}(x_i, t_j) + \frac{\tau}{2} \ddot{u}(x_i, t_j) \Big|$$
$$\leqslant O(\tau + h_i^2 + |h_{i+1} - h_i|), \quad i = \overline{1, N}.$$

Thus in general, we have a reduction of the convergence rate for the difference schemes on the nonuniform mesh

$$||z||_1 \leqslant C(\tau + h_i).$$

However, there are some important cases, when the method, used in this paper, enables us to prove the second order convergence for the non-uniform difference mesh, too. The first case is the quasiuniform difference mesh (see Čiegis, 1990), when the irregularity of the mesh is such that $h_{i+1} - h_i = O(h_i^2)$. The second case is the meshes, when only for the fixed number $M = M(h) \leqslant M_0$ of mesh points the equality $h_i = h$ is violated. In this case we have

$$\sum_{i=1}^{N-1} h_i |\psi_i| = \sum_{i=1}^{N-1}{}' h_i |\psi_i| + \sum_{l=1}^{M(h)} h_{i_l} |\psi_{i_l}|,$$

$$\sum_{l=1}^{M(h)} h_{i_l} |\psi_{i_l}| = \sum_{l=1}^{M(h)} h_{i_l} O(h_{i_l}) \leqslant M(h) O(h^2) \leqslant M_0 O(h^2),$$

$$\sum_{i=1}^{N-1}{}' h_i |\psi_i| = \sum_{i=1}^{N-1} h_i O(h^2) = O(h^2).$$

4. Stability in C norm. In this section a new method for the investigation of the difference scheme stability in C norm is applied to the differential problem (1.1)–(1.3) with another non-local boundary condition (1.4′)

$$u(1,t) = cu(a,t) + g(t). \tag{1.4'}$$

The difference scheme (1.5)–(1.8′) is constructed for the numerical solution of the problem (1.1)–(1.4′), where the approximation of non-local boundary condition (1.4′) is taken in the form

$$\widehat{y}_N = c\widehat{y}_l + g(t_{j+1}), \quad x_l = a. \tag{1.8'}$$

If $x = a$ does not coincide with any point of the difference mesh ω_h, then the difference non-local condition (1.8″) is used $(x_l < a < x_{l+1})$

$$\widehat{y}_N = c\Big(\frac{x_{l+1} - a}{h}\widehat{y}_l + \frac{a - x_l}{h}\widehat{y}_{l+1}\Big) + g(t_{j+1}). \tag{1.8''}$$

Lemma 4.1. *For* $|c| < 1$ *the solution of difference scheme (1.5)–* (1.8′) *exists, is bounded and the stability estimation is valid*

$$\max_{t_j} ||y^j||_c \leqslant \frac{1}{1 - |c|}\big(||y^0||_c + t_j \max_{t_k} ||\varphi|| + \max_{t_k}\big(|u_1(t_k)|,\ |g(t_k)|\big)\big). \tag{4.1}$$

Proof. If we introduce a new unknown function $\lambda(t_j) = y_N$, then it is possible to get a well-known stability inequality for the solution of an implicit difference scheme of the linear boundary value problem (see, Samarskij, 1983)

$$||y(t_j)||_c \leqslant ||y(t_0)|| + t_j \max_{t_k} ||\varphi^k|| + \max_{t_k}\big(|y_0^k|,\ |y_N{}^k|\big). \tag{4.2}$$

However, from the non-local boundary condition (1.8′) we have the estimation

$$|y_N| \leqslant |c||y_l| + |g| \leqslant |c| \, ||y||_c + |g|. \qquad (4.3)$$

Taking two relations (4.2), (4.3) together, it follows that

$$(1-|c|)||y(t_j)||_c \leqslant ||y(t_0)|| + t_j \max_{t_k} ||\varphi^k|| + \max_{t_k} \left(|y_0^k|, \ |g(t_k)|\right),$$

and this completes the proof of Lemma 4.1.

Using the standard approach from stability inequality (4.2) and the approximation $|\psi_i| \leqslant C(\tau + h^2)$, we can obtain the boundaries for the global error $z_i = y_i - u(x_i, t_j)$ of the solution of the difference scheme (1.5)–(1.8′)

$$||z||_c \leqslant \frac{t_j}{1-|c|} \max_{t_k} \left(||\psi^k||_c\right) \leqslant C(\tau + h^2). \qquad (4.4)$$

REMARK 4.1. Estimation (4.4) is true for a more general non-local boundary condition (4.5), too

$$u(1,t) = \sum_{i=1}^{k} c_i y(a_i, t) + g(t), \qquad (4.5)$$

$$0 < a_1 < a_2 < \ldots < a_k < 1, \qquad \sum_{i=1}^{k} |c_i| < 1.$$

A difference approximation of the boundary condition (4.5) is constructed

$$y_N = \sum_{i=1}^{k} c_i \widetilde{y}_i + g(t_j), \qquad (4.6)$$

where the notation

$$\widetilde{y}_i = \frac{x_{l_i+1} - a_i}{h} y_{l_i} + \frac{a_i - x_{l_i}}{h} y_{l_i+1},$$
$$x_{l_i} \leqslant a < x_{l_i+1}$$

is used. The remaining part of the proof is analogous to that of Lemma 4.1 and estimation (4.4).

Further we use a generalization of this method (see Sapagovas and Čiegis, 1987b,too). Now we assume that $k(x,t) = k(x)$, $g(x,t) = g(x)$ (this assumption is used only for the sake of simplicity).

Theorem 4.2. *A sufficient condition for the stability of the difference scheme (1.5)–(1.8′) is*

$$-\infty < cw(x_l) \leqslant c_0 < 1,$$

where $w(x_i)$ is the solution of the boundary value problem (4.8)–(4.9)

$$-(aw_{\bar{x}})_x + d_i w_i = 0, \qquad (4.8)$$
$$w_0 = 0, \qquad w_N = 1. \qquad (4.9)$$

Proof. We may write the solution of the difference scheme (1.5)– (1.8′) in the form

$$y = y^1(x_i, t_j) + y^2(x_i, t_j),$$

where $y^1(x_i, t_j)$ is the solution of the boundary value problem (4.10)–(4.12)

$$y_t^1 = (a\widehat{y}_{\bar{x}}^1)_x - d_i \widehat{y}^1 + \varphi, \qquad (4.10)$$
$$\widehat{y}_0^1 = u_1(t_{j+1}), \qquad \widehat{y}_N{}^1 = 0, \qquad (4.11)$$
$$y^1(x_i, t_0) = u_0(x_i). \qquad (4.12)$$

For the solution of the difference scheme (4.10)–(4.12) the stability inequality

$$||y(t_j)||_c \leqslant ||y(t_0)||_c + t_j \max_{t_k} ||\varphi^k|| + ||u_1(t_k)||_c \qquad (4.13)$$

is valid (see (4.2)). The difference function $y^2(x_i, t_j)$ is the solution of the difference scheme with the non-local boundary condition

$$y_t^2 = (a\widehat{y}_{\bar{x}}^2)_x - d_i \widehat{y}^2, \qquad (4.14)$$

$$\widehat{y}_0^2 = 0, \qquad \widehat{y}^2{}_N = c\widehat{y}_l^2 + \widetilde{g}, \qquad (4.15)$$

$$y^2(x_i, t_0) = 0, \qquad \widetilde{g} = \widehat{g} + c\widehat{y}_l^1. \qquad (4.16)$$

From the maximum principle (an implicit scheme for the parabolic problem) it follows that the function $y^2(x_i, t_j)$ may be bounded
by

$$||y^2(t_j)||_c \leqslant \frac{1}{1 - cw_l}\big(|\widetilde{g}|_c + C||y^1||_c\big).$$

This estimation is sharp for $t_j \to \infty$, when the problem (4.14)–(4.16) becomes stationary

$$-(ay_{\bar{x}}^2)_x + d_i y^2 = 0, \qquad (4.17)$$

$$y_0^2 = 0, \qquad y^2{}_N = cy_l^2 + |\widetilde{g}|_c. \qquad (4.18)$$

The solution of (4.17), (4.18) is found in the form $y_i^2 = \lambda w_i$, where w_i is the solution of the boundary value problem (4.8), (4.9).

REMARK 4.2. It follows from the maximum principle that the solution of the boundary value problem (4.8)–(4.9) is bounded $0 \leqslant w_i < 1$, so the condition (4.7) $cw_l \leqslant c_0 < 1$ is a generalization of Lemma 4.1.

REMARK 4.3. The stability of the difference scheme (1.5)– (1.8′) in the case of $-\infty < c \leqslant 0$ may be investigated straightway (see Iliin and Moiseev, 1986, too). The solution of the difference scheme, using the local basis functions, can be rewritten as

$$y(x) = \sum_{i=0}^{N} y_i w_i(x),$$

where $w_i(x)$ is a piecewise linear basis function. The non-local boundary condition (1.8′), as it follows from the mean value theorem, may be written in the form

$$y(x_N) + |c|y(a) = (|c| + 1)y(x_N + \Theta(t_j)(a - x_N)) = g(t_j),$$
$$y(x_N + \Theta(t_j)(a - x_N)) = \frac{1}{|c| + 1} g(t_j). \qquad (4.19)$$

Investigating separately the boundary value problems (1.5), (1.7), (4.19) and (1.5), (4.20), (1.8′), where

$$y_2(a) = y_i(a), \qquad (4.20)$$

we get the stability inequality

$$||y(t_j)||_c \leqslant ||y(t_0)||_c + t_j \max_{t_k} \left(|u_1(t_k)|, \ \frac{1}{|c| + 1}|g(t_k)|\right).$$

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