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Investigation of Dead-Point Situations in the Simulation of Wet Film Evolution

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Abstract. The result of simulation of an idealized thin wet film connecting fixed points in the Euclidean plane is a length-minimizing curve. Gradually increasing the exterior pressure we are able to achieve the film configuration near to the Steiner minimal tree. This film evolution may be an interesting tool for solving the Euclidean Steiner problem, but several dead-point situations may occur for a certain location of fixed points. A continuous evolution of the film is impossible by increasing the pressure in these situations. The investigation of dead-point situations gives the ways of overcoming the difficulties of dead-point situations and continuing the film evolution by temporarily decreasing pressure.

Key words: optimization, Steiner problem, wet film, simulation.

1. Introduction

It is known that idealized soap films when subjected to some constraints possess various length or energy minimising properties. Courant and Robbins (1941) popularised the Steiner minimal tree problem and for the first time paid an attention to the fact that length-minimizing curve of a thin film may achieve a configuration close to the Steiner minimal tree.

1.1. *Euclidean Steiner Problem*

The Euclidean Steiner problem asks for the shortest network that spans a given set of n fixed points in the Euclidean plane. The addition of an arbitrary number of points (Steiner points) is allowed. A minimal network consisting of vertices and edges connecting the points must have the following properties (Gilbert and Pollak, 1966):

- all edges are straight lines;
- the network is a tree;
- the angle between any two edges meeting at a vertex is at least 120 degrees;
- all Steiner points are of degree 3, and the edges meeting at a Steiner point make angles of precisely 120 degrees with each other;
- the number of Steiner points is at most $n - 2$.

The properties are not sufficient in order that the length of the Steiner tree be minimal. The number of possible topologies grows exponentially as n increases.

1.2. *Simulation of Wet Film Evolution*

The so-called dry films used by Courant and Robbins (1941) in their physical model may result in only one of the variuos stable topologies of the Steiner tree.

A different idea is to use the mathematical model for an idealized wet film, connecting the fixed points with some liquid inside the film. The wet film subject to a constraint tries to shrink to the minimum length. The constraint can be either that the liquid has a fixed interior area, or that the liquid is under the fixed pressure.

We can find an attempt to simulate the evolution of an idealized soap film in order to solve the Steiner problem in (Jakutavičienė, 1965). The goal of this paper is to continue designing a wet film evolution simulation model introduced by the author (Šaltenis, 1999) and investigating in detail dead-point situations which are the main obstacle in the model.

Fig. 1 in (Šaltenis, 1999) illustrates five simulation stages of a wet film evolution for seven fixed points and increasing pressures (decreasing radii R).

In the beginning (Fig. 1a), the convex hull includes only six fixed points. Fig. 1b shows when a point not connected was touched by the curve. The pair of opposite arcs touched each other in Fig. 1c. Another contact of the opposite arcs is illustrated in Fig. 1d. The final phase is shown in Fig. 1e, and we can see the Steiner tree configuration when the radius of arcs is small enough.

The illustration substantiates the idea that by reducing the radius R of the length-minimizing curve, we can achieve a configuration close to the Steiner minimal tree.

The theorems in (Hass and Morgan, 1996) state that a length-minimizing curve enclosing the region of a fixed area has to be composed of:

- circular arcs of equal positive outward curvature and
- line segments multiplicity two.

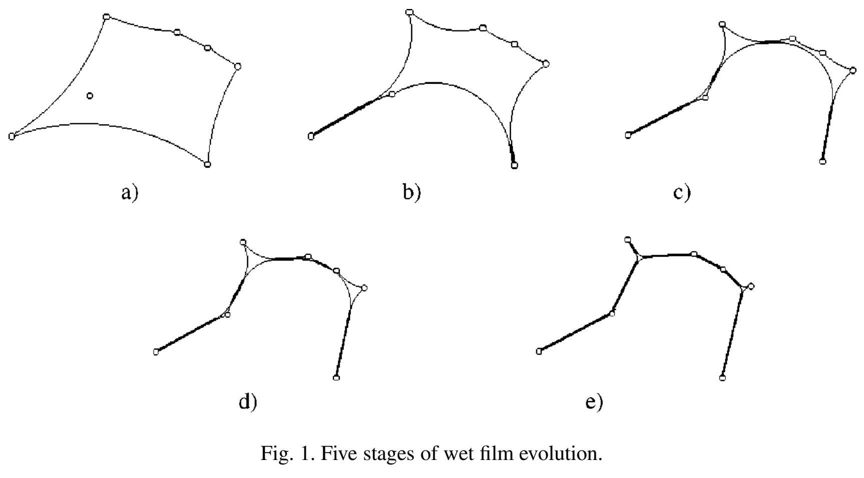


Fig. 1. Five stages of wet film evolution.

The radius R of arcs depends on the external pressure p:

$$p = 1/R.$$

2. A System of Equations

Let us present a short introduction to the author's system of equations (Šaltenis, 1999) which is used to simulate the wet film evolution.

Points that must be connected will be referred to as *fixed points* P_i $(i = 1, \ldots, n)$. Fixed points are constraints to the length-minimizing curve shrunk down around the points.

We denote the corners of the length-minimizing curve interior as C_j $(j = 1, \ldots, m)$. The corners are wet if their coordinates are the same as the coordinates of the respective fixed points (Fig. 2a); otherwise, they are dry (Fig. 2b).

Each corner C_j has its *base*. A base may be:

- a fixed point directly connected by the line segment to the corner C_j (the fixed point P_{jB} in Fig. 2b) or
- another corner directly connected by the line segment to the corner C_j (the corner C_{jB} in Fig. 2c).

The wet corner and its base have the same coordinates.

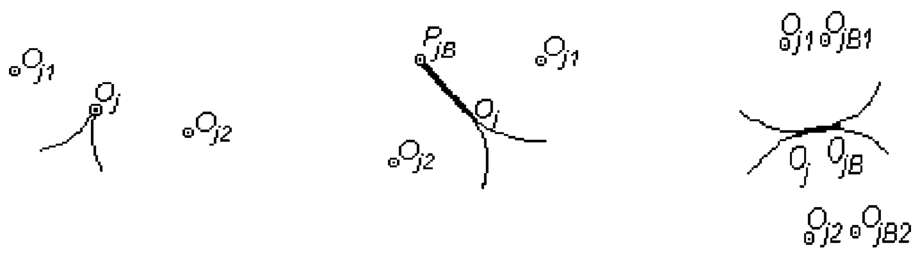
Each corner C_j has two centers of its arcs: O_{j1} and O_{j2} (Fig. 2a and 2b).

The coordinates of centers and corners satisfy the following system of nonlinear equations of three types.

1. For each wet corner C_j we have the following two equations (see Fig. 2a):

$$d(P_{jB}, O_{j1}) = R; \quad (1)$$
$$d(P_{jB}, O_{j2}) = R, \quad (2)$$

where $d(A, B)$ is an Euclidean distance between points A and B.



a) *b)* *c)*

Fig. 2. Characteristic points of the length-minimizing curve.

2. For each dry corner C_j whose base is a fixed point P_{jB} we have the following two equations (see Fig. 2b):

$$d(O_{j1}, O_{j2}) = 2R; \quad (3)$$
$$d(P_{jB}, O_{j1}) = d(P_{jB}, O_{j2}). \quad (4)$$

3. For each dry corner C_j whose base is another corner C_{jB} we have the following two equations (see Fig. 2c):

$$d(O_{j1}, O_{j2}) = 2R; \quad (5)$$
$$d(C_{jB}, O_{j1}) = d(C_{jB}, O_{j2}). \quad (6)$$

The coordinates of dry corners $C_{jB} : x(C_{jB})$ and $y(C_{jB})$ in Eq. (6) can be simply calculated because they are in the middle of two corresponding centers O_{jB1}, O_{jB2}.

The unknowns of the system are the coordinates of centers.

2.1. *Steps of Simulation*

The simulation of evolution begins with extremely low values of pressure. Only a part of the fixed points belonging to a convex hull is included. All corners are wet, therefore the system consists of equations only of type (1) and (2). The unknowns may be simply found in an explicit way at this stage of simulation.

The values of the unknowns obtained are used as initial approximations for the next step of simulation with a decreased radius R. The steepest descent method (Burden and Faires, 1993) is used for solving the system in the general case. The increments of radius ΔR from step to step are small, therefore the number of iterations is extremely small.

At each step of decreasing R we check the occurrence of next events:

- a wet corner became dry;
- a pair of opposite arcs touched each other;
- a fixed point not connected so far was touched by the length-minimizing curve.

The respective corrections must be carried out in the equation system depending on the event.

The simulation process stops when the radius R acquires low values.

3. Dead-Point Situations in the Simulation Process

The results of simulation at each step of reducing radius R are used as initial approximations for the next step of simulation. However, for a certain location of fixed points at specific stages of simulation, some dead-point situations may occur. In such a situation, the solution of equations of (1)–(6) type can not be found for the radius R less than some limit value R_c.

3.1. *Experimental Investigation of the Dead-Point Situation frequency*

The dead-point situations are rare for problems with a small number n of fixed points, but it occurs more frequently for larger n. The experimental investigation was carried out to evaluate the frequency for various numbers n. The coordinates of fixed points were obtained by independent uniform sampling in a rectangular region, and the dead-point situations were checked up during the simulation process. The experiments with random fixed points were repeated 500–1000 times for each n to evaluate the frequency f presented in Table 1.

Table 1

Frequency f of dead-point situations for the number n of fixed points

n	3	4	5	6	7	8	9
f (in %)	0	6	18	24	64	72	82

We can see that the simulation of wet film evolution for larger n is impossible without a detailed investigation of the dead-point phenomenon.

4. Evolution of the Wet Film in Dead-Point Situations

The dead-point situations makes impossible a continuous evolution of a wet film to the zero interior area by gradually reducing the radius R. After reducing the radius to some critical value R_c, we cannot find solutions of the equation system in an epsilon neighbourhood.

But really always there exists a continuous further evolution of the film at this situation. After achieving the critical radius, we must temporally pass to the stage of temporarily increasing the radius R. It does not mean the return to the same past trajectory of the film coordinates. Let us analyse some simple cases more in detail.

4.1. *Fragment of Two Wet Corners*

The simplest case is the fragment of two wet corners (see Fig. 3).

We can see that decreasing of the radius is impossible beyond the critical value $R = R_c$ in the situation where the arc angle is equal to 180^0. However, the continuous

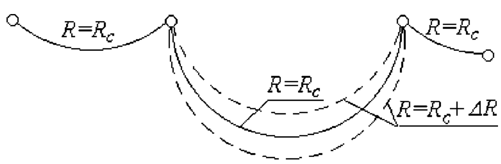


Fig. 3. The fragment of two wet corners at a dead-point situation.

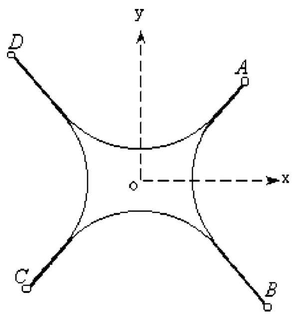


Fig. 4. Fragment of four dry corners.

evolution of the curve is possible, by temporarily increasing the radius with arc angles greater than 180^0.

4.2. *Fragment of Four Dry Corners*

The case is illustrated in Fig. 4, where the points A, B, C, and D are arbitrarily located. The points may be fixed points or some part of them may be dry corners of a wet film.

It is obvious that, if the radii of all arcs are equal, the central part of a wet film formed by the film arcs has a rectangular symmetry. Therefore, we may investigate only one quadrant of the length-minimizing curve. Positive axes of the quadrant are shown in the same Fig. 4.

In Fig. 5, we see only one quadrant of the film and the centers of two arcs: O_1 and O_2. The angle α is uniformly changing during the evolution of the film.

The straight-line fragment of wet film multiplicity two AK is perpendicular to the line connecting the centers O_1O_2 because it is tangent to the arcs.

$$d(O_1, K) = d(O_2, K) = R. \tag{7}$$

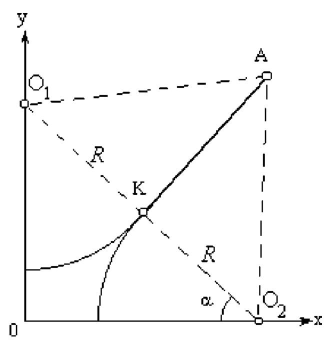


Fig. 5. One quadrant of the fragment of four dry corners.

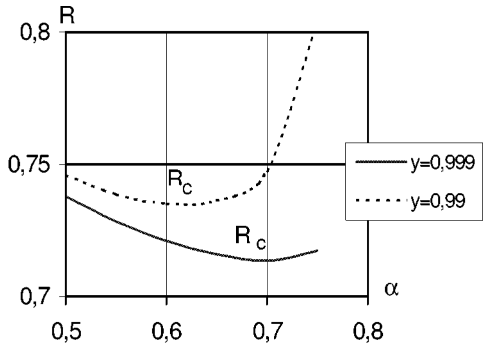


Fig. 6. Radius R of arcs versus the arc angle α.

Then, from (7) we can obtain

$$d(O_1, A) = d(O_2, A). \tag{8}$$

We shall derive the dependency of radius R on the angle α.

It is obvious (see Fig. 5) that

$$d^2(O_1, A) = x_A^2 + (2R\sin\alpha - y_A)^2, \tag{9}$$

$$d^2(O_2, A) = (x_A - 2R\cos\alpha)^2 + y_A^2, \tag{10}$$

where x_A and y_A are the coordinates of fixed point A.

After substituting (9) and (10) into (8), assuming, for simplicity, that $x_A = 1$ and $x_A > y_A$, and denoting, for simplicity, $y = y_A$, we obtain the dependency:

$$R = \frac{\cos\alpha - y\sin\alpha}{\cos 2\alpha}. \tag{11}$$

Dependency (11) shows that the radius R acquires the minimal value R_c in its evolution. The illustrations of the dependency for two values of y are presented in Fig. 6. The values of the critical angle α_c corresponding to R_c may be found from the equation:

$$\frac{dR}{d\alpha} = 0.$$

4.3. *Computational Aspects*

When the coordinates of the film come nearer to the dead-point situation, the number of iterations in solving the equation system significantly increases. After all, the system of equations has no solution. Then the changes of film coordinates may continue their evolution, but R must step by step increase. Fig. 7 illustrates the trajectory of a film angle when the interior area of the film was step by step reduced.

We can see that the trajectory continues the previous trajectory after the dead-point situation.

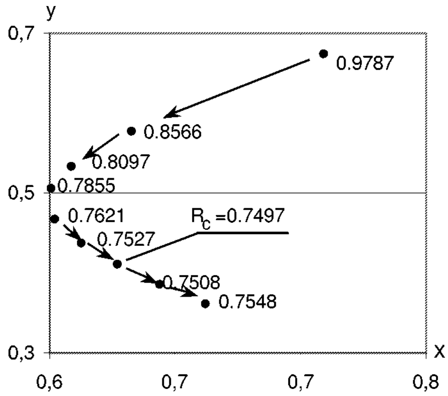


Fig. 7. The trajectory of a film angle over the dead-point.

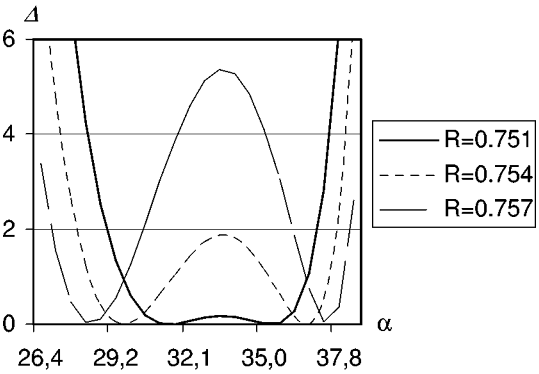


Fig. 8. Discrepancy Δ versus the arc angle along the trajectory.

The discrepancy values Δ in solving the equation system are presented in Fig. 8, when the coordinates of the film are changing along the trajectory of Fig. 7. We see two local minima for each fixed R value. One minimum (and the system solution) is for R decreasing, and the other is for R increasing evolution stages. When R is close to the critical value $R_c = 0.75$, both minima are closely located.

5. Conclusions

The wet film evolution may be an interesting tool to solve the Euclidean Steiner problem, however, some dead-point situations may occur in a certain location of fixed points. The dead-point situations are frequent for larger numbers of fixed points.

Difficulties due to the dead-point situations can be overcome by continuing the evolution process and not reducing but increasing the radius R.

The computational experiments with a larger number of fixed points and investigation of the accuracy are beyond the scope of this paper.

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Mirties taško situacijų tyrimas modeliuojant drėgnų plėvelių evoliuciją

Vydūnas ŠALTENIS

Modeliuodami idealizuotos drėgnos plėvelės, jungiančios fiksuotus taškus plokštumoje, formą, gauname minimalaus ilgio kreivę. Palaipsniui didinant išorinį slėgį, plėvelės forma palaipsniui artėja prie Euklidinio Šteinerio uždavinio sprendinio. Euklidinis Šteinerio uždavinys ieško trumpiausio ilgio tinklo, jungiančio fiksuotų taškų aibę plokštumoje. Modeliavimo eksperimentai rodo, kad kai kurioms fiksuotų taškų padėtims bei esant didesnian fiksuotų taškų skaičiui, toks evoliucijos procesas gali strigti mirties taškuose. Straipsnis skirtas tokių situacijų nagrinėjimui. Parodoma galimybė tolydžiai tęsti plėvelės evoliuciją, peržengiant mirties tašką.