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Two Population Dynamics Models with Child Care

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Abstract. Two models for an age-structured nonlimited population dynamics with maternal care of offspring are presented. One of them deals with a bisexual population and includes a harmonic mean type mating of sexes and females' pregnancy. The other one describes dynamics of an asexual population. Migration is not taken into account. The existence and uniqueness theorem for the general case of vital rates is proved, the extinction and growth of the population are considered, and a class of the product (separable) solutions is obtained for these two models. The long-time behavior of the asexual population is obtained in the stationary case of vital rates.

Key words: population dynamics, random mating, age-sex-structured population, child care.

1. Introduction

In recent paper (Skakauskas, 1999) we analyzed an age-sex-structured population dynamics model without spatial dispersal and took into account coupling of sexes, females' pregnancy, and their sterility period following delivery. For the mating law we used a harmonic mean type mating function.

In the present paper we consider two nondispersing age-structured population dynamics models taking into account maternal care of offspring. According to the maternal care law offspring under maternal care die if dies his/her mother. One of the models describes dynamics of a bisexual population and generalizes the model in (Skakauskas, 1999) including, in addition, maternal care of offsping. In this model, each sex has three age grades: pre-reproductive, reproductive, and post-reproductive. Individuals of the pre-reproductive grade are divided into two classes: young (under maternal care) and juvenile. Young dies if dies his/her mother. Females of the reproductive age grade are divided into two classes (single (nonfertilized) and fertilized). We assume that pairs of sexes exist only for period of mating, which is not taken into account. Hence all males can be treated as singles in this model. It is also assumed that for each sex the commencement of each age grade and subgrade as well as the duration of the gestating period are independent of individuals or time.

The other model presented in this paper describes dynamics of an asexual population. As in the bisexual population model, all individuals have three age grades and individuals of the pre-reproductive grade are divided into young and juvenile, too. We assume that all individuals of the reproductive grade are being fertilized at the fixed ages and, for the simplicity, gestating period is not taken into account.

We prove the existence and uniqueness theorem for these two models and construct their product (separable) solutions. We also construct the long-time behavior of the asexual population in the stationary case of vital rates.

The paper is organized as follows. In Section 3 we present the models. Section 4 lists main hypotheses and results. Justification of results is given in Section 5. The proof of existence and uniqueness theorem of the general solution, a majorant for densities of single males and females, and a class of separable solutions for bisexual population model are given in 5.1, 5.2, and 5.3, respectively. In 5.4, 5.5, and 5.6 we construct a class of separable solutions for the asexual population model, prove the existence and uniqueness theorem of general solution for this model, and find its large time behavior in the case of stationary vital rates, respectively.

2. Notation

The following notation is used for the analysis of the bisexual population model:

t: time;

$\tau_1, \tau_2, \tau_3, \tau_4$: the age of male, female, embryo, and offspring under maternal care, respectively;

$\sigma_4 = (0, T_{24}]$: the age-interval of offspring under maternal care;

$u_1(t, \tau_1)$: the age-density of males aged $\tau_1 > T_{24}$ at time t;

$u_2(t, \tau_2)$: the age-density of single (nonfertilized) females aged $\tau_2 > T_{24}$ at time t;

$u_3(t, \tau_1, \tau_2, \tau_3)$: the age-density of fertilized females aged τ_2 at time t whose embryo is at age τ_3 and that were fertilized by males aged τ_1 at the mating moment;

$u_4(t, \tau_1, \tau_2, \tau_4)$: the age-density of females aged τ_2 at time t who take care of their progeny aged τ_4 (at the same time t) whose fathers were aged τ_1 at the mating moment;

$u_{4k}(t, \tau_1, \tau_2, \tau_4)$: the age-density of progeny being under maternal care and aged τ_4 at time t whose mothers are aged τ_2 (at the same time t) and whose fathers were aged τ_1 at the mating moment; $k = 1$ (resp. $k = 2$) for the male (resp. female) offspring;

$\nu_1(t, \tau_1)$ (resp. $\nu_2(t, \tau_2)$): the death rate of males aged τ_1 (resp. single females aged τ_2) at time t;

$\nu_3(t, \tau_1, \tau_2, \tau_3)$: the death rate of fertilized females aged τ_2 at time t whose embryo is at age τ_3 and who were fertilized by males aged τ_1 at the mating moment;

$\nu_4(t, \tau_1, \tau_2, \tau_4)$: the death rate of females aged τ_2 at time t who take care of their progeny aged τ_4 (at the same time t) whose fathers were aged τ_1 at the mating moment;

$\nu_{4k}(t, \tau_1, \tau_2, \tau_4)$: the death rate of progeny aged τ_4 at time t whose mothers are aged τ_2 (at the same time t) and whose fathers were aged τ_1 at the mating moment;

$h_1(t,\tau_1)$ (resp. $h_2(t,\tau_2)$): the probability that a male aged τ_1 (resp. female aged τ_2) at time t wishes to marry;

$\widetilde{p}(t,\tau_1,\tau_2)$: the fertilization rate of females from a pair formed of a female aged τ_2 and a male aged τ_1 at time t; $p(t,\tau_1,\tau_2) = h_1(t,\tau_1)h_2(t,\tau_1)\widetilde{p}(t,\tau_1,\tau_2)$;

$X_g(t,\tau_2)$: the single female gain density by the females aged τ_2 at time t whose offsprings are aged T_{24} at the same time t;

$X_l(t,\tau_2)$: the loss density of single females due to conception who are aged τ_2 at time t;

$\sigma_1 = [\tau_{11},\tau_{12}], T_{24} < \tau_{11} < \tau_{12} < \infty$: the male sexual activity age-interval;

$\sigma_3 = (0,T_{23}], 0 < T_{23} < \infty$: the female gestating period, $\overline{\sigma}_3 = [0,T_{23}]$;

$\sigma_{23}(\tau_3) = [\tau_{21}+\tau_3,\tau_{22}+\tau_3], T_{24} < \tau_{21} < \tau_{22} < \infty$;

$\sigma_{23}(0)$ and $\sigma_{23}(T_{23})$: the female fertilization and delivery age-intervals;

$\sigma_{24}(\tau_4) = [\tau_{21}+T_{23}+\tau_4,\tau_{22}+T_{23}+\tau_4]$;

$b_1(t,\tau_1,\tau_2)$ and $b_2(t,\tau_1,\tau_2)$: the average numbers of the male and female progeny produced by a female aged τ_2 at time t who was fertilized by a male aged τ_1 at the mating moment;

$u_1^0(\tau_1), u_2^0(\tau_2), u_3^0(\tau_1,\tau_2,\tau_3), u_4^0(\tau_1,\tau_2,\tau_4), u_{4k}^0(\tau_1,\tau_2,\tau_4)$: the initial distributions;

$\tau_2^1 = \tau_{21},\ \tau_2^2 = \min(\tau_{21}+\widetilde{T},\ \tau_{22}), \tau_2^3 = \max(\tau_{21}+\widetilde{T},\ \tau_{22}), \tau_2^4 = \tau_{22}+\widetilde{T},\ \widetilde{T} = T_{23}+T_{24}$;

$Q_1 = (T_{24},\infty)$, $\overline{Q}_1 = [T_{24},\infty)$, $Q_2 = (T_{24},\infty) \setminus \bigcup\limits_{s=1}^{4}\{\tau_2^s\}$, $\overline{Q}_2 = [T_{24},\infty)$;

D: a domain not necessarily bounded, $\overline{D}$: closure of D;

$S_{23} = \big\{(\tau_2,\tau_3)\colon \tau_2 \in \sigma_{23}(\tau_3), \tau_3 \in \sigma_3\big\},\ Q_3 = \sigma_1 \times S_{23}, \overline{Q}_3 = \sigma_1 \times \overline{S}_{23}$;

$S_{24} = \big\{(\tau_2,\tau_4) : \tau_2 \in \sigma_{24}(\tau_4), \tau_4 \in \sigma_4\big\},\ Q_4 = \sigma_1 \times S_{24}, \overline{Q}_4 = \sigma_1 \times \overline{S}_{24}$;

$[u_2(t,\tau_2^j)]$: a jump discontinuity of u_2 at the plane $\tau_2 = \tau_2^j$,

$C^0(\overline{D})$ (resp. $C^0(D)$): a class of bounded continuous functions in $\overline{D}$ (resp. D);

$C^1(D)$: a class of bounded continuous in D functions $f(x_1,\ldots,x_m)$ with $\partial f/\partial x_i \in C^0(D)$, $i = \overline{1,m}$,

$C^{0,1,0,\ldots,0}(D)$: a class of bounded continuous in D functions $f(x_1,\ldots,x_m)$ with $\partial f/\partial x_2 \in C^0(D)$.

For the analysis of the asexual population model we use the following notation:

t: time,

τ: the age of an individual,

$(0,T_0]$: the young age-interval,

T_1+kh, $T_1 > T_0$, $h > T_0$, $k = 0,1,2,\ldots,n$: the reproductive ages of individuals,

$u(t,\tau)$: the age-density of individuals aged τ at time t;

$v_k(t,\tau)$: the age-density of progeny aged τ at time t, who were born to mothers aged $T_1 + kh$,

$\nu_k(t,\tau)$: the death rate of progeny aged τ at time t, who were born to mothers aged $T_1 + kh$,

$\nu(t,\tau)$: the death rate of individuals aged τ at time t,

$b_k(t)$: the average number of progeny born to a mother aged $T_1 + kh$ at time t.

3. The Models

Using the balance law together with the harmonic mean type mating function and the fact that the offspring under maternal care dies if dies his/her mother, we derive the nondispersing bisexual population dynamics model which consists of the following system of integro-differential equations for $u_1, u_2, u_3, u_4, u_{41}, u_{42}$,

$$\begin{aligned}
&\partial u_1/\partial t + \partial u_1/\partial \tau_1 = -\nu_1 u_1,\ t>0, \tau_1 \in Q_1,\\
&\partial u_2/\partial t + \partial u_2/\partial \tau_2 = -\nu_2 u_2 - X_l + X_g,\ t>0, \tau_2 \in Q_2,\\
&\partial u_3/\partial t + \partial u_3/\partial \tau_2 + \partial u_3/\partial \tau_3 = -\nu_3 u_3,\ t>0, (\tau_1,\tau_2,\tau_3) \in Q_3,\\
&\partial u_4/\partial t + \partial u_4/\partial \tau_2 + \partial u_4/\partial \tau_4 = -\nu_4 u_4,\ t>0, (\tau_1,\tau_2,\tau_4) \in Q_4,\\
&\partial u_{4k}/\partial t + \partial u_{4k}/\partial \tau_2 + \partial u_{4k}/\partial \tau_4 = -(\nu_4+\nu_{4k}) u_{4k},\\
&\qquad\qquad\qquad\qquad k=1,2,\ t>0,\ (\tau_1,\tau_2,\tau_4) \in Q_4,
\end{aligned} \tag{3.1}$$

$$X_l = \begin{cases} 0, & \tau_2 \notin \sigma_{23}(0),\\ \int\limits_{\sigma_1} u_3\big|_{\tau_3=0} d\tau_1, & \tau_2 \in \sigma_{23}(0),\end{cases}$$

$$X_g = \begin{cases} 0, & \tau_2 \notin \sigma_{24}(T_{24}),\\ \int\limits_{\sigma_1} u_4\big|_{\tau_4=T_{24}} d\tau_1, & \tau_2 \in \sigma_{24}(T_{24}),\end{cases}$$

which supplemented with the conditions

$$\begin{aligned}
&u_k|_{\tau_k=T_{24}} = \int\limits_{\sigma_1} \mathrm{d}\tau_1 \int\limits_{\sigma_{24}(T_{24})} u_{4k}\big|_{\tau_4=T_{24}} \mathrm{d}\tau_2, \quad t>0,\ k=1,2,\\
&u_3\big|_{\tau_3=0} = p u_1 u_2 \Big/ \Big(\int\limits_{\sigma_1} h_1 u_1 \mathrm{d}\tau_1' + \int\limits_{\sigma_{23(0)}} h_2 u_2 \mathrm{d}\tau_2' \Big),\\
&\qquad\qquad t>0,\ (\tau_1,\tau_2) \in \sigma_1 \times \sigma_{23}(0),\\
&u_4\big|_{\tau_4=0} = u_3\big|_{\tau_3=T_{23}}, \quad t>0,\ (\tau_1,\tau_2) \in \sigma_1 \times \sigma_{24}(0),
\end{aligned}$$

$$u_{4k}\big|_{\tau_4=0} = b_k u_3\big|_{\tau_3=T_{23}}, \quad t>0,\ (\tau_1,\tau_2)\in\sigma_1\times\sigma_{24}(0),$$
$$\big[u_2\big|_{\tau_2=\tau_2^s}\big]=0, \quad t>0, \quad s=\overline{1,4},$$
$$u_k\big|_{t=0}=u_k^0 \ \text{ in } \overline{Q}_k, \quad k=1,2,3,4,$$
$$u_{4k}\big|_{t=0}=u_{4k}^0 \ \text{ in } \overline{Q}_4 \tag{3.2}$$

describes dynamics of the population. In addition to (3.2) we assume that the initial distributions $u_1^0, u_2^0, u_3^0, u_4^0, u_{4k}^0$ satisfy the following compatibility conditions

$$u_k^0\big|_{\tau_k=T_{24}} = \int\limits_{\sigma_1} d\tau_1 \int\limits_{\sigma_{24}(T_{24})} u_{4k}^0\big|_{\tau_4=T_{24}} d\tau_2, \quad k=1,2, \tag{3.3}$$
$$u_3^0\big|_{\tau_3=0} = p\big|_{t=0} u_1^0 u_2^0 \Big/ \Big(\int\limits_{\sigma_1} u_1^0 h_1\big|_{t=0} d\tau_1' + \int\limits_{\sigma_{23}(0)} u_2^0 h_2\big|_{t=0} d\tau_2'\Big),$$
$$(\tau_1,\tau_2)\in\sigma_1\times\sigma_{23}(0),$$
$$u_4^0\big|_{\tau_4=0} = u_3^0\big|_{\tau_3=T_{23}}, \quad (\tau_1,\tau_2)\in\sigma_1\times\sigma_{24}(0),$$
$$u_{4k}^0\big|_{\tau_4=0} = b_k\big|_{t=0} u_3^0\big|_{\tau_3=T_{23}}, \quad (\tau_1,\tau_2)\in\sigma_1\times\sigma_{24}(0),$$
$$\big[u_{20}(\tau_2^s)\big]=0, \quad s=\overline{1,4}.$$

As follows from the foregoing, given functions $\nu_1,\nu_2,\nu_3,\nu_4,\ \nu_{41},\nu_{42},p,h_1,h_2,b_1,b_2$, $u_1^0,u_2^0,\ u_3^0,u_4^0,\ u_{41}^0,u_{42}^0$ and the unknown ones $u_1,u_2,u_3,u_4,u_{41},u_{42}$ must be positive valued, otherwise they have no biological significance. Because of the mating law model (3.1)–(3.3) is nonlinear.

To derive the asexual population dynamics simple model we neglect the individual gestating period and assume that all individuals are being fertilized at the ages T_1+kh with $k=0,1,\dots,n<\infty$ and T_0,T_1,h do not depend on individual or time. Using the balance law, we derive the following nondispersing nonlimited asexual population dynamics model

$$\begin{cases} \partial v_k/\partial t + \partial v_k/\partial\tau = -(\nu_k(t,\tau)+\nu(t,T_1+kh+\tau))v_k(t,\tau), \\ \qquad\qquad t>0,\ \tau\in(0,T_0), \\ v_k(t,0)=b_k(t)u(t,T_1+kh), \quad t>0, \\ v_k(0,\tau)=v_{k0}(\tau), \quad \tau\in[0,T_0], \end{cases} \tag{3.4}$$

$$\begin{cases} \partial u/\partial t + \partial u/\partial\tau = -\nu(t,\tau)u(t,\tau), \quad t>0,\ \tau\in(T_0,\infty), \\ u(t,T_0)=\sum\limits_{k=0}^{n} v_k(t,T_0), \quad t>0, \\ u(0,\tau)=u_0(\tau), \quad \tau\in[T_0,\infty). \end{cases} \tag{3.5}$$

In addition to (3.4) and (3.5) we assume that initial distributions v_{k0} and u_0 satisfy the following compatibility conditions

$$\begin{cases} v_{k0}(0) = b_k(0)u_0(T_1 + kh), \\ u_0(T_0) = \sum\limits_{k=0}^{n} v_{k0}(T_0). \end{cases} \tag{3.6}$$

We observe that model (3.4)–(3.6) is linear. Given functions $\nu_k, \nu, b_k, v_{k0}, u_0$ and unknown ones $v_k,\ u$ must be positive valued.

4. Hypotheses and Results

We use the following hypotheses:

$$\begin{aligned}
&0 < h_1 \in C^0\big([0,\infty) \times \sigma_1\big), \quad 0 < h_2 \in C^0\big([0,\infty) \times \sigma_{23}(0)\big);\\
&0 < \nu_k \in C^{0,1}\big([0,\infty) \times \overline{Q}_k\big), \quad k = 1,2;\\
&0 < \nu_3 \in C^{0,0,1,1}\big([0,\infty) \times \overline{Q}_3\big);\\
&0 < \nu_4,\ \nu_{4k} \in C^{0,0,1,1}\big([0,\infty) \times \overline{Q}_4\big), \quad k = 1,2;\\
&0 < p \in C^{1,0,1}\big([0,\infty) \times \sigma_1 \times \sigma_{23}(0)\big);\\
&0 < b_k \in C^{1,0,1}\big([0,\infty) \times \sigma_1 \times \sigma_{23}(T_{23})\big), \quad k = 1,2;\\
&0 < u_k^0 \in C^1(\overline{Q}_k), \quad k = 1,2;\\
&0 < u_3^0 \in C^{0,1,1}(\overline{Q}_3);\\
&0 < u_4^0,\ u_{4k}^0 \in C^{0,1,1}(\overline{Q}_4), \quad k = 1,2;\\
&0 < \tau_{ij} = \text{const} < \infty, \quad i,j = 1,2;\\
&0 < T_{2k} = \text{const} < \infty, \quad k = 3,4.
\end{aligned} \tag{H$_1$}$$

$$\begin{aligned}
&0 < \nu_k \in C^{0,1}\big([0,\infty) \times [0,T_0]\big), \quad k = 0,1,\dots,n;\\
&0 < \nu \in C^{0,1}\big([0,\infty) \times [T_0,\infty)\big);\\
&0 < b_k \in C^1\big([0,\infty)\big), \quad k = 0,1,\dots,n;\\
&0 < v_k^0 \in C^1\big([0,T_0]\big), \quad k = 0,1,\dots,n;\\
&0 < u^0 \in C^1\big([T_0,\infty)\big).
\end{aligned} \tag{H$_2$}$$

Theorem 1. *Assume that all the hypotheses* (H$_1$) *and conditions* (3.3) *hold. Then, for any finite* $t^* > 0$*, problem* (3.1), (3.2) *has the unique strictly positive solution* $u_1, u_2, u_3, u_4, u_{41}, u_{42}$ *such that:*

$$u_1 \in C^0([0,t^*] \times \overline{Q}_1) \cap C^1\big(([0,t^*] \times \overline{Q}_1) \setminus \{t = \tau_1 - T_{24}, \tau_1, \tau_1 + T_{23}\}\big),$$

$$u_2 \in C^0([0,t^*]\times \overline{Q}_2)\cap C^1\Big(([0,t^*]\times \overline{Q}_2)\setminus \big(\bigcup_{s=1}^{4}\{\tau_2=\tau_2^s, t+\tau_2^s\}$$
$$\cup\{\tau_2 = t+T_{24},\quad t, t-T_{23}\}\big)\Big),$$
$$u_3 \in C^0([0,t^*]\times \overline{Q}_3)\cap C^{1,0,1,1}\Big(([0,t^*]\times \overline{Q}_3)\setminus \big(\bigcup_{s=1}^{3}\{t=\tau_2-\tau_2^s\}$$
$$\cup\{t=\tau_3, \tau_3+\tau_1-T_{24}, \tau_1+\tau_3, \tau_1+\tau_3+T_{23}, \tau_2, \tau_2+T_{23}, \tau_2-T_{24}\}$$
$$\cup\bigcup_{s=1}^{4}\{\tau_2=\tau_2^s\}\big)\Big),$$
$$u_4,\, u_{4k} \in C^0([0,t^*]\times \overline{Q}_4)\cap C^{1,0,1,1}\Big(([0,t^*]\times \overline{Q}_4)\setminus \big(\{t=\tau_4, \tau_4+\tau_1+T_{23},$$
$$\tau_4+\tau_1+2T_{23}, \tau_4+T_{23}, \tau_1+\tau_4+T_{23}-T_{24}, \tau_2, \tau_2+T_{23}, \tau_2-T_{24}\}$$
$$\cup\bigcup_{s=1}^{3}\{t=\tau_2-\tau_2^s\}\cup\bigcup_{s=2}^{4}\{\tau_2=\tau_2^s\}\big)\Big),\quad k=1,2,$$
$$\int_{\sigma_1} h_1u_1 d\tau_1 + \int_{\sigma_{23}(0)} h_2u_2 d\tau_2 \in C^1([0,t^*]).$$

Theorem 2. *Under the hypotheses of Theorem* 1 *and conditions* (5.17)*, for large time,* u_1 *and* u_2 *satisfying problem* (3.1)*,* (3.2) *have majorants* (5.18) *and* (5.19)*,* (5.20)*, respectively.*

Theorem 3. *Assume that:*

1. *all vital functions in problem* (3.1)*,* (3.3) *do not depend on time* t;
2. $0<\nu_k \in C^0(\overline{Q}_k),\quad k=1,2;$
3. $0<h_1\in C^0(\sigma_1),\quad 0<h_2\in C^0(\sigma_{23}(0));$
4. $0<\nu_s\in C^{0,1,0}(\overline{Q}_s),\quad s=3,4;\ \nu_{4k}\in C^{0,1,0}(\overline{Q}_4), k=1,2;$
5. $0<p\in C^{0,1}\big(\sigma_1\times\sigma_{23}(0)\big);$
6. $0<b_k\in C^{0,1}\big(\sigma_1\times\sigma_{23}(T_{23})\big),\quad k=1,2;$
7. $b_2(\tau_1,\tau_2-T_{24})\exp\Big\{-\int_0^{T_{24}}\nu_{42}(\tau_1, x+\tau_2-T_{24}, x)\,\mathrm{d}x$
$$= f b_1(\tau_1,\tau_2-T_{24})\exp\Big\{-\int_0^{T_{24}}\nu_{41}(\tau_1, x+\tau_2-T_{24}, x)\,\mathrm{d}x,\quad f=\mathrm{const}.$$

Then problem (3.1)*,* (3.2)$_{1-5}$ *has the separable solution represented by* (5.21).

Theorem 4. *Let* ν, ν_k*, and* b_k *be stationary and:* $0<\nu_k(\tau)\in C^0([0,T_0])$*,* $0<\nu(\tau)\in C^0([T_0,\infty))$*,* $0<b_k=\mathrm{const}$*. Then systems* (3.4)*,* (3.5) *without initial conditions has the separable solution of form* (5.31).

Theorem 5. *Under the hypotheses* (H$_2$) *and conditions* (3.6)*, for any finite* t^**, problem*

(3.4), (3.5) *has the unique positive solution* v_k, u *such that*

$$\begin{aligned} &v_k \in C^0\big([0,t^*]\times[0,T_0]\big)\cap C^1\Big(\big([0,t^*]\times[0,T_0]\big)\setminus\big\{t-\tau=0,\ s(T_1+kh)\\ &\qquad -T_0,\ s(T_1+kh);\ s=1,2,\dots\big\}\Big),\quad k=0,1,\dots,n,\\ &u\in C^0\big([0,t^*]\times[T_0,\infty)\big)\cap C^1\Big(\big([0,t^*]\times[T_0,\infty)\big)\setminus\big\{t-\tau=s(T_1+kh)\\ &\qquad -T_0,\ s(T_1+kh);\ s=0,1,2,\dots;\ k=0,1,\dots,n\big\}\Big). \end{aligned}$$

Theorem 6. *Let* ν, ν_k *and* b_k *be stationary and:*

$$\begin{aligned} &0<\nu_k\in C^0([0,T_0]),\quad 0<\nu\in C^0([T_0,\infty)),\quad 0<b_k=\mathrm{const},\\ &0<v_k^0\in C^1([0,T_0]),\quad 0<v^0\in C^1([T_0,\infty)). \end{aligned}$$

Then functions (5.36) *exhibit the large time behavior of the solution of problem* (3.4)*,* (3.5)*.*

5. Justification of Results

5.1. *Proof of Theorem* 1

We consider the case of multiple deliveries, i.e., $\tau_{22}-\tau_{21}>\widetilde{T}$, $\tau_2^2=\tau_{21}+\widetilde{T}$, $\tau_2^3=\tau_{22}$. The apposite case can be analyze in the similar way.

Define

$$n(t)=\int\limits_{\sigma_1} h_1u_1\mathrm{d}\tau_1+\int\limits_{\sigma_{23}(0)} h_2u_2\mathrm{d}\tau_2, \tag{5.1}$$

$$q(t,\tau_2)=\int\limits_{\sigma_1} pu_1\mathrm{d}\tau_1, \tag{5.2}$$

$$B_k(t)=u_k(t,T_{24}),\quad k=1,2. \tag{5.3}$$

We first obtain the formal integral representations for u_1,u_2,u_3,u_4,u_{4k}. Integrating Eqs. 3.1 over characteristics together with conditions (3.2), we obtain:

$$u_3=\begin{cases} u_3^0(\tau_1,\tau_2-t,\tau_3-t)\\ \quad\times\exp\Big\{-\int\limits_0^t\nu_3(x,\tau_1,x+\tau_2-t,x+\tau_3-t)\,\mathrm{d}x\Big\},\ 0\leqslant t\leqslant\tau_3,\\ (pu_1u_2/n)|_{(t-\tau_3,\tau_1,\tau_2-\tau_3)}\\ \quad\times\exp\Big\{-\int\limits_{t-\tau_3}^t\nu_3(x,\tau_1,x+\tau_2-t,x+\tau_3-t)\,\mathrm{d}x\Big\},\ 0\leqslant\tau_3\leqslant t, \end{cases} \tag{5.4}$$

$$u_4 = \begin{cases} u_4^0(\tau_1, \tau_2 - t, \tau_4 - t) \\ \quad \times \exp\Big\{ -\int\limits_0^t \nu_4(x, \tau_1, x + \tau_2 - t, x + \tau_4 - t)\,\mathrm{d}x \Big\},\ 0 \leqslant t \leqslant \tau_4, \\ u_3^0(\tau_1, \tau_2 - t, T_{23} + \tau_4 - t) \\ \quad \times \exp\Big\{ -\int\limits_0^{t-\tau_4} \nu_3(x, \tau_1, x + \tau_2 - t, x + T_{23} + \tau_4 - t)\,\mathrm{d}x \\ \quad - \int\limits_{t-\tau_4}^t \nu_4(x, \tau_1, x + \tau_2 - t, x + \tau_4 - t)\,\mathrm{d}x \Big\},\ \tau_4 \leqslant t \leqslant T_{23} + \tau_4, \\ (pu_1u_2/n)\big|_{(t-\tau_4-T_{23}, \tau_1, \tau_2-\tau_4-T_{23})} \\ \quad \times \exp\Big\{ -\int\limits_{t-\tau_4-T_{23}}^{t-\tau_4} \nu_3(x, \tau_1, x + \tau_2 - t, x + \tau_4 + T_{23} - t)\,\mathrm{d}x \\ \quad - \int\limits_{t-\tau_4}^t \nu_4(x, \tau_1, x + \tau_2 - t, x + \tau_4 - t)\,\mathrm{d}x \Big\},\ t \geqslant \tau_4 + T_{23}, \end{cases} \tag{5.5}$$

$$u_{4k} = \begin{cases} u_{4k}^0(\tau_1, \tau_2 - t, \tau_4 - t) \\ \quad \times \exp\Big\{ -\int\limits_0^t (\nu_4 + \nu_{4k})\big|_{(x, \tau_1, x+\tau_2-t, x+\tau_4-t)}\mathrm{d}x \Big\},\ 0 \leqslant t \leqslant \tau_4, \\ b_k(t - \tau_4, \tau_1, \tau_2 - \tau_4) u_3^0(\tau_1, \tau_2 - t, T_{23} + \tau_4 - t) \\ \quad \times \exp\Big\{ -\int\limits_0^{t-\tau_4} \nu_3(x, \tau_1, x + \tau_2 - t, x + T_{23} + \tau_4 - t)\,\mathrm{d}x \\ \quad - \int\limits_{t-\tau_4}^t (\nu_4 + \nu_{4k})\big|_{(x, \tau_1, x+\tau_2-t, x+\tau_4-t)}\mathrm{d}x \Big\},\ \tau_4 \leqslant t \leqslant T_{23} + \tau_4, \\ b_k(t - \tau_4, \tau_1, \tau_2 - \tau_4)(pu_1u_2/n)\big|_{(t-\tau_4-T_{23}, \tau_1, \tau_2-\tau_4-T_{23})} \\ \quad \times \exp\Big\{ -\int\limits_{t-\tau_4-T_{23}}^{t-\tau_4} \nu_3(x, \tau_1, x + \tau_2 - t, x + \tau_4 + T_{23} - t)\,\mathrm{d}x \\ \quad - \int\limits_{t-\tau_4}^t (\nu_4 + \nu_{4k})\big|_{(x, \tau_1, x+\tau_2-t, x+\tau_4-t)}\mathrm{d}x \Big\},\ t \geqslant \tau_4 + T_{23}, \end{cases} \tag{5.6}$$

$$u_1 = \begin{cases} u_1^0(\tau_1 - t) \exp\Big\{ -\int\limits_0^t \nu_1(x, x + \tau_1 - t)\,\mathrm{d}x \Big\}, \quad 0 \leqslant t \leqslant \tau_1 - T_{24}, \\ B_1(T_{24} + t - \tau_1) \exp\Big\{ -\int\limits_{T_{24}+t-\tau_1}^t \nu_1(x, x + \tau_1 - t)\,\mathrm{d}x \Big\}, \\ \quad 0 \leqslant \tau_1 - T_{24} \leqslant t, \end{cases} \tag{5.7}$$

$$u_2 = \begin{cases} u_2^0(\tau_2 - t) \exp\Big\{ -\int\limits_0^t \nu_2(x, x + \tau_2 - t)\,\mathrm{d}x \Big\}, \quad 0 \leqslant t \leqslant \tau_2 - T_{24}, \\ B_2(T_{24} + t - \tau_2) \exp\Big\{ -\int\limits_{T_{24}+t-\tau_2}^t \nu_2(x, x + \tau_2 - t)\,\mathrm{d}x \Big\}, \\ \quad 0 \leqslant \tau_2 - T_{24} \leqslant t, \end{cases} \tag{5.8}$$

for $\tau_2 \in [T_{24}, \tau_2^1]$,

$$u_2 = \begin{cases} u_{21} := u_2^0(\tau_2 - t) \exp\Big\{ -\int\limits_0^t (\nu_2 + \frac{q}{n})|_{(x,x+\tau_2-t)} \,\mathrm{d}x \Big\}, \\ \quad 0 \leqslant t \leqslant \tau_2 - \tau_2^1, \\ u_{22} := u_2(\tau_2^1 + t - \tau_2, \tau_2^1 - 0) \\ \quad \times \exp\Big\{ -\int\limits_{\tau_2^1+t-\tau_2}^t (\nu_2 + \frac{q}{n})|_{(x,x+\tau_2-t)} \,\mathrm{d}x \Big\}, \quad t > \tau_2 - \tau_2^1, \end{cases} \tag{5.9}$$

for $\tau_2 \in [\tau_2^1, \tau_2^2]$,

$$u_2 = \begin{cases} u_{23} := u_2^0(\tau_2 - t) \exp\Big\{ -\int\limits_0^t (\nu_2 + \frac{q}{n})|_{(x,x+\tau_2-t)} \,\mathrm{d}x \Big\} \\ \quad + \int\limits_0^t X_g(y, y+\tau_2-t) \exp\Big\{ -\int\limits_y^t (\nu_2 + \frac{q}{n})|_{(x,x+\tau_2-t)} \,\mathrm{d}x \Big\} \,\mathrm{d}y, \\ \quad 0 \leqslant t \leqslant \tau_2 - \tau_2^2, \\ u_{24} := u_2^0(\tau_2 - t) \exp\Big\{ -\int\limits_0^t (\nu_2 + \frac{q}{n})|_{(x,x+\tau_2-t)} \,\mathrm{d}x \Big\} \\ \quad + \int\limits_{\tau_2^2+t-\tau_2}^t X_g(y, y+\tau_2-t) \exp\Big\{ -\int\limits_y^t (\nu_2 + \frac{q}{n})|_{(x,x+\tau_2-t)} \mathrm{d}x \Big\} \,\mathrm{d}y, \\ \quad \tau_2 - \tau_2^2 \leqslant t \leqslant \tau_2 - \tau_2^1, \\ u_{25} := u_2(\tau_2^1 + t - \tau_2, \tau_2^1 - 0) \\ \quad \times \exp\Big\{ -\int\limits_{\tau_2^1+t-\tau_2}^t (\nu_2 + \frac{q}{n})|_{(x,x+\tau_2-t)} \mathrm{d}x \Big\} + \int\limits_{\tau_2^2+t-\tau_2}^t X_g(y, y+\tau_2-t) \\ \quad \times \exp\Big\{ -\int\limits_y^t (\nu_2 + \frac{q}{n})|_{(x,x+\tau_2-t)} \,\mathrm{d}x \Big\} \,\mathrm{d}y, \quad t \geqslant \tau_2 - \tau_2^1 \geqslant \widetilde{T}, \end{cases} \tag{5.10}$$

for $\tau_2 \in [\tau_2^2, \tau_2^3]$,

$$u_2 = \begin{cases} u_2^0(\tau_2 - t) \exp\Big\{ -\int\limits_0^t \nu_2(x, x+\tau_2-t) \,\mathrm{d}x \Big\} \\ \quad + \int\limits_0^t X_g(y, y+\tau_2-t) \exp\Big\{ -\int\limits_y^t \nu_2(x, x+\tau_2-t) \,\mathrm{d}x \Big\} \,\mathrm{d}y, \\ \quad 0 \leqslant t \leqslant \tau_2 - \tau_2^3, \\ u_2(\tau_2^3 + t - \tau_2, \tau_2^3 - 0) \exp\Big\{ -\int\limits_{\tau_2^3+t-\tau_2}^t \nu_2(x, x+\tau_2-t) \,\mathrm{d}x \Big\} \\ \quad + \int\limits_{\tau_2^3+t-\tau_2}^t X_g(y, y+\tau_2-t) \exp\Big\{ -\int\limits_y^t \nu_2(x, x+\tau_2-t) \,\mathrm{d}x \Big\} \mathrm{d}y, \\ \quad t \geqslant \tau_2 - \tau_2^3, \end{cases} \tag{5.11}$$

for $\tau_2 \in [\tau_2^3, \tau_2^4]$,

$$u_2 = \begin{cases} u_{20}(\tau_2 - t) \exp\Big\{ -\int\limits_0^t \nu_2(x, x + \tau_2 - t)\,\mathrm{d}x \Big\}, \quad 0 \leqslant t \leqslant \tau_2 - \tau_2^4, \\ u_2(\tau_2^4 + t - \tau_2, \tau_2^4) \exp\Big\{ -\int\limits_{\tau_2^4 + t - \tau_2}^t \nu_2(x, x + \tau_2 - t)\,\mathrm{d}x \Big\}, \\ \quad t \geqslant \tau_2 - \tau_2^4, \end{cases} \tag{5.12}$$

for $\tau_2 \in [\tau_2^4, \infty)$.

Now by (5.1), (5.9) and (5.10) we obtain the following integral equation:

$$\begin{aligned} n &= \int\limits_{\sigma_1} u_1 h_1 \,\mathrm{d}\tau_1 + \int\limits_{\tau_2^1}^{\tau_2^1 + t} u_{22} h_2 \,\mathrm{d}\tau_2 + \int\limits_{\tau_2^1 + t}^{\tau_2^2} u_{21} h_2 \,\mathrm{d}\tau_2 \\ &\quad + \int\limits_{\tau_2^2}^{\tau_2^2 + t} u_{24} h_2 \,\mathrm{d}\tau_2 + \int\limits_{\tau_2^2 + t}^{\tau_2^3} u_{23} h_2 \,\mathrm{d}\tau_2, \quad 0 \leqslant t \leqslant \widetilde{T}, \\ n &= \int\limits_{\sigma_1} u_1 h_1 \,\mathrm{d}\tau_1 + \int\limits_{\tau_2^1}^{\tau_2^2} u_{22} h_2 \,\mathrm{d}\tau_2 + \int\limits_{\tau_2^2}^{t + \tau_2^1} u_{25} h_2 \,\mathrm{d}\tau_2 \\ &\quad + \int\limits_{t + \tau_2^1}^{t + \tau_2^2} u_{24} h_2 \,\mathrm{d}\tau_2 + \int\limits_{t + \tau_2^2}^{\tau_2^3} u_{23} h_2 \,\mathrm{d}\tau_2, \quad \widetilde{T} \leqslant t \leqslant \tau_2^3 - \tau_2^2, \\ n &= \int\limits_{\sigma_1} u_1 h_1 \,\mathrm{d}\tau_1 + \int\limits_{\tau_2^1}^{\tau_2^2} u_{22} h_2 \,\mathrm{d}\tau_2 + \int\limits_{\tau_2^2}^{t + \tau_2^1} u_{25} h_2 \,\mathrm{d}\tau_2 \\ &\quad + \int\limits_{t + \tau_2^1}^{\tau_2^3} u_{24} h_2 \,\mathrm{d}\tau_2, \quad \tau_2^3 - \tau_2^2 \leqslant t \leqslant \tau_2^3 - \tau_2^1, \\ n &= \int\limits_{\sigma_1} u_1 h_1 \,\mathrm{d}\tau_1 + \int\limits_{\tau_2^1}^{\tau_2^2} u_{22} h_2 \,\mathrm{d}\tau_2 + \int\limits_{\tau_2^2}^{\tau_2^3} u_{25} h_2 \,\mathrm{d}\tau_2, \quad t \geqslant \tau_2^3 - \tau_2^1, \end{aligned} \tag{5.13}$$

if $\tau_{22} - \tau_{21} \geqslant 2\widetilde{T}$, and

$$n = \int\limits_{\sigma_1} u_1 h_1 \,\mathrm{d}\tau_1 + \int\limits_{\tau_2^1}^{t + \tau_2^1} u_{22} h_2 \,\mathrm{d}\tau_2 + \int\limits_{t + \tau_2^1}^{\tau_2^2} u_{21} h_2 \,\mathrm{d}\tau_2$$

$$+\int\limits_{\tau_2^2}^{t+\tau_2^2} u_{24}h_2\,\mathrm{d}\tau_2+\int\limits_{t+\tau_2^2}^{\tau_2^3} u_{23}h_2\,\mathrm{d}\tau_2,\quad 0\leqslant t\leqslant \tau_2^3-\tau_2^2,$$

$$n=\int\limits_{\sigma_1} u_1h_1\,\mathrm{d}\tau_1+\int\limits_{\tau_2^1}^{t+\tau_2^1} u_{22}h_2\,\mathrm{d}\tau_2+\int\limits_{t+\tau_2^1}^{\tau_2^2} u_{21}h_2\,\mathrm{d}\tau_2$$

$$+\int\limits_{\tau_2^2}^{\tau_2^3} u_{24}h_2\,\mathrm{d}\tau_2,\quad \tau_2^3-\tau_2^2\leqslant t\leqslant T_{23}, \tag{5.14}$$

$$n=\int\limits_{\sigma_1} u_1h_1\,\mathrm{d}\tau_1+\int\limits_{\tau_2^1}^{\tau_2^2} u_{22}h_2\,\mathrm{d}\tau_2+\int\limits_{\tau_2^2}^{t+\tau_2^1} u_{25}h_2\,\mathrm{d}\tau_2$$

$$+\int\limits_{t+\tau_2^1}^{\tau_2^3} u_{24}h_2\,\mathrm{d}\tau_2,\quad T_{23}\leqslant t\leqslant \tau_2^3-\tau_2^1,$$

$$n=\int\limits_{\sigma_1} u_1h_1\,\mathrm{d}\tau_1+\int\limits_{\tau_2^1}^{\tau_2^2} u_{22}h_2\,\mathrm{d}\tau_2+\int\limits_{\tau_2^2}^{\tau_2^3} u_{25}h_2\,\mathrm{d}\tau_2,\quad t\geqslant \tau_2^3-\tau_2^1,$$

if $\widetilde{T}<\tau_{22}-\tau_{21}<2\widetilde{T}$.

For the sake of brevity we write this integral equation as follows

$$n=K(n;t,X_g,B_1,B_2,q), \tag{5.15}$$

where:

$X_g(t,\tau_2)$ is defined by $(3.1)_7$ provided that $u_4|\tau_4=T_{24}$ is known,

$B_k(t)$ is defined by (5.3), $(3.2)_1$ provided that $u_4|_{\tau_4=T_{24}}$ is known,

and q is defined by (5.2), (5.7) provided that B_1 is known.

Now we examine (5.15) going along the t axis by a step of size $\widetilde{T}$.

Let $t\in(0,\widetilde{T}]$. From (5.4) we find u_3 for $t\leqslant\tau_3$ (and hence $u_3|_{\tau_3=T_{23}}$ for $t\leqslant T_{23}$). From (5.5) and (5.6) we find u_4 and u_{4k} for $0\leqslant t\leqslant\tau_4+T_{23}$ (and hence $u_4|_{\tau_4=T_{24}}$ and $u_{4k}|_{\tau_4=T_{24}}$ for $0\leqslant t\leqslant\widetilde{T}$). This allows us, by $(3.1)_7$ and (5.3), $(3.2)_1$ to construct X_g and B_k for $t\leqslant\widetilde{T}$. Then we find, by (5.7) and (5.8), u_k for $0<t\leqslant\tau_k+T_{23}$, $k=1,2$, $\tau_1\in(T_{24},\infty)$, $\tau_2\in(T_{24},\tau_{21}]$. Thus we know $u_k(t,\tau_{k1})$ for $0\leqslant t\leqslant T_{23}+\tau_{k1},\ k=1,2$. Hence q, by (5.2), and $\int_{\sigma_1}u_1d\tau_1$ are also known for $t\in[0,\tau_{11}+T_{23}]$. Thus (5.15) is an integral equation for $n(t),\ t\in[0,\widetilde{T}]$.

From (5.13) and (5.14) it is easy to see, by definition of $u_{2i}, i=\overline{1,5}$, that $K(n'',\cdot)>K(n',\cdot)$, if $n''>n'$, and $0<K(\int_{\sigma_1}u_1d\tau_1,\cdot)<K(n,\cdot)<K(\infty,\cdot)$. This allows us to solve (5.15) by the iteration method starting with $n^0=\int_{\sigma_1}u_1d\tau_1$ and to obtain

the monotonically increasing sequence $K(n^m, \cdot)$, which converges, since it is bounded. Moreover, from (5.13) and (5.14) it follows that

$$|n^{m+1} - n^m| \leqslant \kappa \int_0^t |n^m - n^{m-1}| \, \mathrm{d}x, \quad m = 1, 2, \dots, \tag{5.16}$$

where κ is a positive constant. Hence

$$|n^{m+1} - n^m| \leqslant \frac{\kappa^m t^m}{m!} c, \quad c = \sup_{t \in [0, \widetilde{T}]} \Big(K(\infty, \cdot) - \int_{\sigma_1} u_1 \, \mathrm{d}\tau_1 \Big), \quad m = 1, 2, \dots,$$

and the sequence $\{n^m\}$ converges uniformly to a strictly positive function $n \in C^0([0, \widetilde{T}])$ because of the hypotheses (H_1). Uniqueness of solution of (5.15) follows from the inequality analogous to (5.16) for two solutions n'' and n'. Knowing n, B_1, B_2, q we find, by (5.9)–(5.12), the function u_2 for $(t, \tau_2) \in [0, \widetilde{T}] \times [\tau_2^1, \infty)$.

Let $t \in (\widetilde{T}, 2\widetilde{T}]$. From (5.4) we find u_3 for $\tau_3 \leqslant t \leqslant \widetilde{T} + \tau_3$ (and hence $u_3|_{\tau_3 = T_{23}}$ for $T_{23} \leqslant t \leqslant T_{23} + \widetilde{T}$). Then, by (5.5) and (5.6), we construct u_4 and u_{4k} for $\tau_4 + T_{23} < t \leqslant \tau_4 + T_{23} + \widetilde{T}$ (and hence $u_4\big|_{\tau_4 = T_{24}}$ and $u_{4k}\big|_{\tau_4 = T_{24}}$ for $\tau_4 + T_{23} < t \leqslant \tau_4 + T_{23} + \widetilde{T}$). This allows us, by $(3.1)_7$ and (5.3), $(3.2)_1$ to construct X_g and B_k for $\widetilde{T} < t \leqslant 2\widetilde{T}$. Then by (5.7) and (5.8), we find u_k for $\tau_k + T_{23} < t \leqslant \tau_k + T_{23} + \widetilde{T}$, $k = 1, 2$, $\tau_1 \in (T_{24}, \infty)$, $\tau_2 \in (T_{24}, \tau_{21}]$. Thus, we know $u_k(t, \tau_{k1})$ for $\tau_{k1} + T_{23} \leqslant t \leqslant \tau_{k1} + T_{23} + \widetilde{T}$. Hence q, by (5.2), and $\int_{\sigma_1} u_1 d\tau_1$ are also known for $t \in [\tau_{11} + T_{23}, \tau_{11} + T_{23} + \widetilde{T}]$. Thus we again have an integral equation for $n(t)$, $t \in (\widetilde{T}, 2\widetilde{T}]$. Using the same method, as above, we construct its solution $n \in C^0([\widetilde{T}, 2\widetilde{T}])$, and then by (5.9)–(5.12), find u_2 for $(t, \tau_2) \in [\widetilde{T}, 2\widetilde{T}] \times [\tau_2^1, \infty)$.

Repeating our argument, we can construct $u_1, u_2, u_3, u_4, u_{4k}$ and (hence B_1, B_2, q) for any finite $t^* > 0$. Direct calculation together with hypotheses (H_1) show that $n \in C^1([0, t^*])$. Observe that conditions (3.3) ensure only the continuity of solution but not its differentiability at the respective planes (see Theorem 1). This completes the proof of Theorem 1.

5.2. *Proof of Theorem* 2

In this subsection, we consider the case where

$$\begin{aligned}
&\nu_2(t, \tau_2) \geqslant \nu_*(\tau_2) > 0, \quad 0 < b_k(t, \tau_1, \tau_2) \leqslant b^*(\tau_2), \\
&0 < p(t, \tau_1, \tau_2) \leqslant p^*(\tau_2), \\
&\nu_* \in C^0([T_{24}, \infty)), \quad b^* \in C^0(\sigma_{23}(T_{23})), \quad p^* \in C^0(\sigma_{23}(0)).
\end{aligned} \tag{5.17}$$

It is easy to see that u_1 and u_2 constructed in Subs. 5.1 have the following majorants $\widetilde{u}_1$ and $\widetilde{u}_2$, respectively, where

$$\widetilde{u}_1 = \begin{cases} u_{10}(\tau_1 - t)\exp\Big\{ -\int\limits_0^t \nu_1(x, x+\tau_1 - t)\,\mathrm{d}x\Big\}, \quad 0 \leqslant t \leqslant \tau_1 - T_{24}, \\ \widetilde{u}_2(T_{24} + t - \tau_1, T_{24})\exp\Big\{ -\int\limits_{T_{24}}^{\tau_1} \nu_1(x + t - \tau_1, x)\,\mathrm{d}x\Big\}, \\ \quad 0 \leqslant \tau_1 - T_{24} \leqslant t, \end{cases} \tag{5.18}$$

and $\widetilde{u}_2$ is the unique strictly positive solution of the system

$$\begin{aligned}
&\partial\widetilde{u}_2/\partial t + \partial\widetilde{u}_2/\partial\tau_2 = -\nu_*\widetilde{u}_2 \\
&\quad + \begin{cases} 0, \ \tau_2 \in (T_{24}, \infty) \setminus \sigma_{24}(T_{24}), \quad t > 0, \\ \int\limits_{\sigma_1} u_4|_{\tau_4 = T_{24}}\,\mathrm{d}\tau_1, \quad \tau_2 \in \sigma_{24}(T_{24}), \quad 0 < t \leqslant \widetilde{T}, \\ (h_*)^{-1}p^*(\tau_2 - \widetilde{T})\widetilde{u}_2(t - \widetilde{T}, \tau_2 - \widetilde{T}), \quad \tau_{24} \in \sigma_{24}(T_{24}), \quad t \geqslant \widetilde{T}, \end{cases} \\
&[\widetilde{u}_2(t, \tau_2^s)] = 0, \quad s = 2, 4, \quad t \geqslant 0, \\
&\widetilde{u}_2(0, \tau_2) = u_{20}(\tau_2), \quad \tau_2 \in [T_{24}, \infty), \\
&\widetilde{u}_2(t, T_{24}) \\
&\quad = \begin{cases} \int\limits_{\sigma_{24}(T_{24})} \mathrm{d}\tau_2 \int\limits_{\sigma_1} u_{42}|_{\tau_4 = T_{24}}\,\mathrm{d}\tau_1, \quad t \leqslant \widetilde{T}, \\ \int\limits_{\sigma_{24}(T_{24})} \mathrm{d}\tau_2 p^*(\tau_2 - \widetilde{T})(h_{1*})^{-1}b^*(\tau_2 - T_{24})\widetilde{u}_2(t - \widetilde{T}, \tau_2 - \widetilde{T}), \quad t \geqslant \widetilde{T}, \end{cases}
\end{aligned}$$

where $u_{42}|_{\tau_4 = T_{24}}$ for $0 \leqslant t \leqslant \widetilde{T}$ is defined by (5.6), and $h_{1*} = \min\, h_1 > 0$. Because of the delayed argument $t - \widetilde{T}$, function $\widetilde{u}_2$ can be constructed by using the argument similar to that used in 5.1 for construction of u_2. Moreover, for $t > \tau_2 - T_{24}$, we have

$$\begin{aligned}
&\widetilde{u}_2 = \widetilde{u}_2(T_{24} + t - \tau_2, T_{24})v(\tau_2), \\
&dv/d\tau_2 = -\nu_* v + \begin{cases} 0, \tau_2 \in (T_{24}, \infty) \setminus \sigma_{24}(T_{24}), \\ (h_{1*})^{-1}p^*(\tau_2 - \widetilde{T})v(\tau_2 - \widetilde{T}), \quad \sigma_2 \in \sigma_{24}(T_{24}), \end{cases} \\
&v(T_{24}) = 1, \quad [v(\tau_2^s)] = 0, \quad s = 2, 4.
\end{aligned} \tag{5.19}$$

Clearly, $v(\tau_2) \in C^0([T_{24}, \infty)) \cap C^1\big([T_{24}, \infty) \setminus \{\tau_2^2, \tau_2^4\}\big)$. Now we find the upper estimate for $\widetilde{u}_2(t, T_{24})$. Let $t > \tau_{22} + T_{23}$. Then

$$\begin{aligned}
\widetilde{u}_2(t, T_{24}) &= \int\limits_{\sigma_{23(0)}} (h_{1*})^{-1}p^*(x)b^*(x + T_{23})\widetilde{u}_2(t - \widetilde{T}, x)\,\mathrm{d}x \\
&= \int\limits_{\sigma_{23(0)}} (h_{1*})^{-1}p^*(x)b^*(x + T_{23})\widetilde{u}_2(t - T_{23} - x, T_{24})v(x)\,\mathrm{d}x
\end{aligned}$$

$$= \int_{t-a}^{t-b} \widetilde{u}_2(y, T_{24}) p^*(t - T_{23} - y) b^*(t - y) v(t - T_{23} - y)(h_{1*})^{-1}\, \mathrm{d}y,$$

$$b = T_{23} + \tau_{21}, a = T_{23} + \tau_{22},$$

and hence

$$\widetilde{u}_2(t, T_{24}) \leqslant \xi \int_{t-a}^{t-b} \widetilde{u}_2(y, T_{24})\, \mathrm{d}y, \quad \xi = \sup_{\tau_2} p^* b^* v h_{1*}^{-1}.$$

From this inequality we derive, by induction, the following estimate

$$\widetilde{u}_2(t, T_{24}) \leqslant \xi\eta(1 + \xi b)^j, \quad \eta = \int_0^a \widetilde{u}_2(x, T_{24})\, \mathrm{d}x$$
$$\text{for } a + jb \leqslant t \leqslant a + (j + 1)b, \quad j = 0, 1, \ldots,$$

or

$$\widetilde{u}_2(t, T_{24}) \leqslant \xi\eta(1 + \xi b)^{\frac{t-a}{b}}, \quad t \geqslant a. \tag{5.20}$$

Thus there exists the Laplace transform $f(\lambda)$ of $\widetilde{u}_2(t, T_{24})$. Letting $\rho(\tau_2) = p^*(\tau_2 - T_{23}) b^*(\tau_2) h_{1*}^{-1}$, we obtain

$$\begin{aligned}
f(\lambda) &= \int_0^\infty \exp\{-\lambda t\} \widetilde{u}_2(t, T_{24})\, \mathrm{d}t \\
&= \int_0^b \exp\{-\lambda t\} \widetilde{u}_2(t, T_{24})\, \mathrm{d}t + \int_b^\infty \exp\{-\lambda t\} \widetilde{u}_2(t, T_{24})\, \mathrm{d}t \\
&= \int_0^b \exp\{-\lambda t\} \widetilde{u}_2(t, T_{24})\, \mathrm{d}t + \int_b^\infty \exp\{-\lambda t\} \\
&\quad \times \Big(\int_b^{\min(t,a)} \rho(\tau_2) \widetilde{u}_2(t - \tau_2, T_{24}) v(\tau_2 - T_{23})\, \mathrm{d}\tau_2 \\
&\quad + \int_{\min(t,a)}^a \rho(\tau_2) \widetilde{u}_2(t - \tau_2, T_{24}) v(\tau_2 - T_{23})\, \mathrm{d}\tau_2 \Big)\, \mathrm{d}t \\
&= \int_0^b \exp\{-\lambda t\} \widetilde{u}_2(t, T_{24})\, \mathrm{d}t + \int_b^a \exp\{-\lambda t\}\, \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\times \int_t^a \rho(\tau_2)\widetilde{u}_2(t-\tau_2, T_{24}) v(\tau_2 - T_{23})\,\mathrm{d}\tau_2 \\
&+ \int_b^a \mathrm{d}\tau_2 \rho(\tau_2) v(\tau_2 - T_{23}) \int_{\tau_2}^{\infty} \exp\{-\lambda t\}\widetilde{u}_2(t-\tau_2, T_{24})\,\mathrm{d}t \\
= {}& I(\lambda) + R(\lambda) f(\lambda),
\end{aligned}
$$

where

$$
\begin{aligned}
R(\lambda) &= \int_b^a \rho(\tau_2) v(\tau_2 - T_{23}) \exp\{-\lambda\tau_2\}\,\mathrm{d}\tau_2, \\
I(\lambda) &= \int_0^b \exp\{-\lambda t\}\widetilde{u}_2(t, T_{24})\,\mathrm{d}t \\
&\quad + \int_b^a \exp\{-\lambda t\}\,\mathrm{d}t \int_t^a \rho(\tau_2)\widetilde{u}_2(t-\tau_2, T_{24}) v(\tau_2 - T_{23})\,\mathrm{d}\tau_2.
\end{aligned}
$$

Hence

$$
f(\lambda) = I(\lambda)/\{1 - R(\lambda)\}.
$$

It is well known that equation $R(\lambda) = 1$ has a unique real root λ_0 and conjugate pairs of single complex roots λ_i, $i = 1, 2, \ldots$ such that $Re\lambda_i < \lambda_0$. $I(\lambda)$ is an analytic function, thus, using the method of rectangle contour integral (Bellman and Cooke), we can evaluate the inverse Laplace transform obtaining

$$
\widetilde{u}_2(t, T_{24}) \sim \widetilde{u}_2^{as}(t, T_{24}) = \kappa \exp\{\lambda_0 t\}, \quad \kappa = I(\lambda_0)/\{-dR/d\lambda\}|_{\lambda=\lambda_0},
$$

and, by (5.19),

$$
\widetilde{u}_2(t, \tau_2) \sim \kappa v(\tau_2) \exp\{\lambda_0 (T_{24} + t - \tau_2)\}
$$

for large t. If $\lambda_0 < 0$, then solution of problem (3.1)–(3.3) decreases provided that (5.17) and hypotheses (H_1) hold.

5.3. *Proof of Theorem* 3

In this subsection we consider the case of stationary vital functions $b_1,\ b_2,\ h_1,\ h_2,\ p,\ \nu_1$, $\nu_2, \nu_3, \nu_4,\ \nu_{4k}$ and examine solution of system (3.1), $(3.2)_{1,2,3,4}$ of the form:

$$
\begin{aligned}
&u_1 = \exp\{\lambda(t-\tau_1)\}U_1^* U_1(\tau_1),\ U_1(T_{24}) = 1,\\
&u_2 = \exp\{\lambda(t-\tau_2)\}U_2^* U_2(\tau_2),\ U_2(T_{24}) = 1,\\
&u_3 = \exp\{\lambda(t-\tau_2)\}U_2^* U_3(\tau_1,\tau_2,\tau_3),\\
&u_4 = \exp\{\lambda(t-\tau_2)\}U_2^* U_4(\tau_1,\tau_2,\tau_4),\\
&u_{4k} = \exp\{\lambda(t-\tau_2)\}U_2^* U_{4k}(\tau_1,\tau_2,\tau_4),
\end{aligned}
\tag{5.21}
$$

where $\lambda,\ U_1^*,\ U_2^*$ are constants. Substituting of (5.21) into (3.1), $(3.2)_{1,2,3,4}$ leads to the following equations:

$$
U_1' = -\nu_1 U_1, \quad U_1(T_{24}) = 1, \tag{5.22}
$$

$$
\begin{aligned}
{U_2}' = -\nu_2 U_2 - U_2 &\begin{cases} 0, & \tau_2 \notin \sigma_{23}(0)\\ \int\limits_{\sigma_1} pn^{-1} U_1(\tau_1)\exp\{-\lambda\tau_1\}\,\mathrm{d}\tau_1, & \tau_2 \in \sigma_{23}(0)\end{cases}\\
&+ \begin{cases} 0, & \tau_2 \notin \sigma_{24}(T_{24})\\ \int\limits_{\sigma_1} U_4|_{\tau_4=T_{24}}\,\mathrm{d}\tau_1, & \tau_2 \in \sigma_{24}(T_{24}),\end{cases}
\end{aligned}
$$

$$
U_2(T_{24}) = 1, \quad [U_2(\tau_2^s)] = 0, \quad s = \overline{1,4}, \tag{5.23}
$$

$$
\partial U_3/\partial\tau_2 + \partial U_3/\partial\tau_3 = -\nu_3 U_3,\ U_3|_{\tau_3=0} = pn^{-1}U_1U_2\exp\{-\lambda\tau_1\}, \tag{5.24}
$$

$$
\partial U_4/\partial\tau_2 + \partial U_4/\partial\tau_4 = -\nu_4 U_4,\ U_4|_{\tau_4=0} = U_3|_{\tau_3=T_{23}}, \tag{5.25}
$$

$$
\partial U_{4k}/\partial\tau_2 + \partial U_{4k}/\partial\tau_4 = -\nu_4 U_{4k},\ U_{4k}|_{\tau_4=0} = b_k U_3|_{\tau_3=T_{23}}, \tag{5.26}
$$

$$
1 = \int\limits_{\sigma_1}\mathrm{d}\tau_1 \int\limits_{\sigma_{24}(T_{24})} U_{41}|_{\tau_4=T_{24}}\exp\{\lambda(T_{24}-\tau_2)\}\,\mathrm{d}\tau_2 U_2^*/U_1^*, \tag{5.27}
$$

$$
1 = \int\limits_{\sigma_1}\mathrm{d}\tau_1 \int\limits_{\sigma_{24}(T_{24})} U_{42}|_{\tau_4=T_{24}}\exp\{\lambda(T_{24}-\tau_2)\}\,\mathrm{d}\tau_2, \tag{5.28}
$$

where

$$
\begin{aligned}
n = &\int\limits_{\sigma_1} h_1(\tau_1)U_1(\tau_1)\exp\{-\lambda\tau_1\}\,\mathrm{d}\tau_1\\
&+ \int\limits_{\sigma_{23}(0)} h_2(\tau_2)U_2(\tau_2)\exp\{-\lambda\tau_2\}\,\mathrm{d}\tau_2 U_2^*/U_1^*,
\end{aligned}
\tag{5.29}
$$

and the prime denotes differentiation.

Equations (5.22), (5.24), (5.25), and (5.26) have the following solution

$$
\begin{aligned}
&U_1 = \exp\Big\{-\int\limits_{T_{24}}^{\tau_1} \nu_1(x)\,\mathrm{d}x\Big\},\\
&U_3 = n^{-1}p(\tau_1,\tau_2-\tau_3)\exp\{-\lambda\tau_1\}U_1(\tau_1)U_2(\tau_2-\tau_3)\\
&\quad\times\exp\Big\{-\int\limits_0^{\tau_3}\nu_3(\tau_1,x+\tau_2-\tau_3,x)\,\mathrm{d}x\Big\},\\
&U_4 = n^{-1}p(\tau_1,\tau_2-\tau_4-T_{23})\exp\{-\lambda\tau_1\}U_1(\tau_1)U_2(\tau_2-\tau_4-T_{23}) \qquad (5.30)\\
&\quad\times\exp\Big\{-\int\limits_0^{\tau_4}\nu_4(\tau_1,x+\tau_2-\tau_4,x)\,\mathrm{d}x-\int\limits_0^{T_{23}}\nu_3(\tau_1,x+\tau_2-\tau_4-T_{23},x)\,\mathrm{d}x\Big\},\\
&U_{4k} = U_4(\tau_1,\tau_2,\tau_4)b_k(\tau_1,\tau_2-\tau_4)\exp\Big\{-\int\limits_0^{\tau_4}\nu_{4k}(\tau_1,x+\tau_2-\tau_4,x)\,\mathrm{d}x\Big\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&U_4(\tau_1,\tau_2,T_{24}) = n^{-1}p(\tau_1,\tau_2-\widetilde{T})\exp\{-\lambda\tau_1\}U_1(\tau_1)U_2(\tau_2-\widetilde{T})\\
&\quad\times\exp\Big\{-\int\limits_0^{T_{24}}\nu_4(\tau_1,x+\tau_2-T_{24},x)\,\mathrm{d}x-\int\limits_0^{T_{23}}\nu_3(\tau_1,x+\tau_2-\widetilde{T},x)\,\mathrm{d}x\Big\},\\
&U_{4k}(\tau_1,\tau_2,T_{24}) = U_4(\tau_1,\tau_2,T_{24})b_k(\tau_1,\tau_2-T_{24})\\
&\quad\times\exp\Big\{-\int\limits_0^{T_{24}}\nu_{4k}(\tau_1,x+\tau_2-T_{24},x)\,\mathrm{d}x\Big\}.
\end{aligned}
$$

This together with (5.23) determines U_2 depending on τ_2,λ,n. (5.27), (5.28) and the seventh condition of Theorem 3 show that $U_2^*/U_1^* = f$. Since $\widetilde{U}_{2*}(\tau_2) \leqslant U_2 \leqslant \widetilde{U}_2^*(\tau_2)$, where $\widetilde{U}_2^*(\tau_2)$ and $\widetilde{U}_{2*}(\tau_2)$ are solutions of equations

$$
\begin{aligned}
&d\widetilde{U}_2^*/d\tau_2 = -\nu_2\widetilde{U}_2^* - \widetilde{U}_2^*\begin{cases}0, & \tau_2\notin\sigma_{23}(0),\\ p^*h_{1*}^{-1}\widetilde{U}_2^*(\tau_2-\widetilde{T}), & \tau_2\in\sigma_{24}(T_{24}),\end{cases}\\
&\widetilde{U}_2^*(T_{24}) = 1, \quad [\widetilde{U}_2^*(\tau_2^s)] = 0, \quad s=\overline{1,4},\\
&\widetilde{U}_{2*}' = -\nu_2\widetilde{U}_{2*} - \widetilde{U}_{2*}\begin{cases}0, & \tau_2\notin\sigma_{23}(0),\\ p^*h_{1*}^{-1}, & \tau_2\in\sigma_{23}(0),\end{cases}\\
&\widetilde{U}_{2*}(T_{24}) = 1, \quad [\widetilde{U}_{2*}(\tau_2^s)] = 0, \quad s=1,3,
\end{aligned}
$$

where $p^* = \max p,\ h_{1*} = \min h_1$, the right-hand side of (5.29) has the following minorant

$$\int\limits_{\sigma_1} h_1 U_1(\tau_1) \exp\{-\lambda\tau_1\}\,\mathrm{d}\tau_1 + f \int\limits_{\sigma_{23}(0)} h_2 \widetilde{U}_{2*}(\tau_2) \exp\{-\lambda\tau_2\}\,\mathrm{d}\tau_2,$$

and majorant

$$\int\limits_{\sigma_1} h_1 U_1(\tau_1) \exp\{-\lambda\tau_1\}\,\mathrm{d}\tau_1 + f \int\limits_{\sigma_{23}(0)} h_2 \widetilde{U}_2^*(\tau_2) \exp\{-\lambda\tau_2\}\,\mathrm{d}\tau_2,$$

and thus (5.29) has at least one solution $n = N(\lambda)$.

Then the right-hand side of (5.28) has the following majorant

$$f^{-1} b_2^* p^* h_{2*}^{-1} \exp\{-\lambda T_{23}\} \int\limits_{\sigma_1} \exp\{-\lambda\tau_1\} U_1(\tau_1)\,\mathrm{d}\tau_1 \to 0 \quad \text{as } \lambda \to \infty,$$

and minorant

$$\begin{aligned} &b_{2*} p_* \exp\{-\lambda T_{23}\} \exp\{-T_{24}(\nu_{4k}^* + \nu_4^*) - T_{23}\nu_3^*\} \\ &\quad / \Big(h_1^* / \int\limits_{\sigma_{23}(0)} \widetilde{U}_{2*} \exp\{-\lambda\tau_2\}\,\mathrm{d}\tau + h_2^* f / \int\limits_{\sigma_1} U_1 \exp\{-\lambda\tau_1\}\,\mathrm{d}\tau_1 \Big) \to \infty \\ &\quad \text{as} \quad \lambda \to -\infty, \end{aligned}$$

where $h_{2*} = \min h_2,\ p_* = \min p,\ h_2^* = \max h_2,\ b_2^* = \max b_2,\ b_{2*} = \min b_2$. Hence (5.28) with $n = N(\lambda)$ has at least one real root λ_0. (5.21) show that population dies if $\lambda < 0$ and increases if $\lambda > 0$.

5.4. *Proof of Theorem* 4

In this subsection we consider the case of stationary vital functions $b_k,\ \nu,\ \nu_k$ and examine solutions of system (3.4), (3.5) without initial conditions of the form:

$$\begin{aligned} v_k &= V_k(\tau) \exp\{\lambda(t-\tau)\}, \\ u &= U(\tau) \exp\{\lambda(t-\tau)\}, \end{aligned} \tag{5.31}$$

with λ a constant. Substituting (5.31) into (3.4), (3.5) leads to the following equations:

$$\begin{aligned} &U' = -\nu U,\ U(T_0) = \sum_{k=0}^{n} V_k(T_0); \\ &V_k' = -(\nu_k(\tau) + \nu(T_1 + kh + \tau)) V_k(\tau), \\ &V_k(0) = b_k U(T_1 + kh) \exp\{-\lambda(T_1 + kh)\}. \end{aligned}$$

Hence

$$U = U(T_0)\exp\Big\{-\int\limits_{T_0}^{\tau}\nu(x)\,\mathrm{d}x\Big\},$$
$$V_k = V_k(0)\exp\Big\{-\int\limits_{0}^{\tau}(\nu_k(x)+\nu(T_1+kh+x))\,\mathrm{d}x\Big\}, \tag{5.32}$$

where $U(T_0)$ is an arbitrary constant,

$$V_k(0) = U(T_0)b_k\exp\Big\{-\lambda(T_1+kh)-\int\limits_{T_0}^{T_1+kh}\nu(x)\,\mathrm{d}x\Big\},$$

and

$$1 = \sum_{k=0}^{n}\gamma_k(\lambda),$$
$$\gamma_k(\lambda) = b_k\exp\Big\{-\lambda(T_1+kh)-\int\limits_{0}^{T_0}\nu_k(x)\,\mathrm{d}x-\int\limits_{T_0}^{T_0+T_1+kh}\nu(x)\,\mathrm{d}x\Big\}. \tag{5.33}$$

It is well known that (5.33) has a unique real root λ_0 and conjugate pairs of single roots $\lambda_i,\ i=1,2,\ldots,$ such that $\mathrm{Re}\lambda_i<\lambda_0$. From (5.33) it is easy to see that $\partial\lambda_0/\partial T_0<0$ if $\nu(\tau)$ increases with τ increasing, and $\partial\lambda_0/\partial b_k>0$, $\partial\lambda_0/\partial T_1<0$, $\partial\lambda_0/\partial h<0$. This completes the proof.

5.5. *Proof of Theorem* 5

Taking into account hypotheses (H_2) and integrating (3.4), (3.5) over characteristics we obtain

$$u = \begin{cases} u_0(\tau-t)\exp\Big\{-\int\limits_0^t\nu(x,x+\tau-t)\,\mathrm{d}x\Big\}, & 0\leqslant t\leqslant\tau-T_0,\\ \sum\limits_{k=0}^{n}v_k(T_0+t-\tau,T_0)\exp\Big\{-\int\limits_{T_0+t-\tau}^{t}\nu(x,x+\tau-t)\,\mathrm{d}x\Big\}, & t\geqslant\tau-T_0,\end{cases}$$

$$v_k = \begin{cases} v_{k0}(\tau-t)\exp\Big\{-\int\limits_0^t\big(\nu_k(x,x+\tau-t)\\ \quad+\nu(x,x+T_1+kh+\tau-t)\big)\,\mathrm{d}x\Big\}, & 0\leqslant t\leqslant\tau\\ b_k(t-\tau)u(t-\tau,T_1+kh)\exp\Big\{-\int\limits_{t-\tau}^{t}\big(\nu_k(x,x+\tau-t)\\ \quad+\nu(x,x+T_1+kh+\tau-t)\big)\,\mathrm{d}x\Big\}, & t\geqslant\tau.\end{cases} \tag{5.34}$$

Hence

$$u = \begin{cases} u_0(\tau - t)\exp\Big\{-\int\limits_0^t \nu(x, x+\tau-t)\,\mathrm{d}x\Big\}, \quad 0 \leqslant t \leqslant \tau - T_0, \\ \sum\limits_{k=0}^n v_{k0}(\tau - t)\exp\Big\{-\int\limits_0^{T_0+t-\tau} \big(\nu_k(x, x+\tau-t) + \nu(x, x+T_1 \\ \quad +kh+\tau-t)\big)\,\mathrm{d}x - \int\limits_{T_0+t-\tau}^t \nu(x, x+\tau-t)\,\mathrm{d}x\Big\}, \ \tau - T_0 \leqslant t \leqslant \tau, \\ \sum\limits_{k=0}^n b_k(t-\tau)u(t-\tau, T_1+kh) \\ \quad \times \exp\Big\{-\int\limits_{t-\tau}^{T_0+t-\tau} \big(\nu_k(x, x+\tau-t) + \nu(x, x+T_1+kh+\tau-t)\big)\,\mathrm{d}x \\ \quad -\int\limits_{T_0+t-\tau}^t \nu(x, x+\tau-t)\,\mathrm{d}x\Big\}, \quad t \geqslant \tau. \end{cases} \tag{5.35}$$

Equation (5.35) includes delayed argument $t - \tau$. Therefore going along t axis by the step τ we can construct u for any finite t. Then (5.34) determines v_k. Observe that conditions (3.6) ensure the continuity of u and v_k (but not their continuous differentiability) in $[0, t^*] \times [T_0, \infty)$ and $[0, t^*] \times [0, T_0]$, respectively. The proof is complete.

5.6. *Proof of Theorem* 6

In this subsection, we analyze the case where b_k, ν, and ν_k are stationary. It is easy to see, from (5.35), that $u \leqslant A\exp\{(t\ln B)/\tau\}$, where $B = \sum\limits_{k=0}^n b_k$ and $A = \max\limits_{t\leqslant\tau} u$. Therefore there exists the Laplace transform $\widehat{u}$ of u. From (5.35) we obtain

$$\begin{aligned} u(t, T_1+sh) &= \sum_{k=0}^n u(t - T_1 - sh, T_1+kh)\kappa_k \\ &\qquad \times \exp\Big\{-\int\limits_{T_0}^{T_1+sh} \nu(x)\,\mathrm{d}x\Big\}, \quad t \geqslant T_1 + sh, \end{aligned}$$

with

$$\kappa_k = b_k\exp\Big\{-\int\limits_0^{T_0} \nu_k(x)\,\mathrm{d}x - \int\limits_{T_1+kh}^{T_0+T_1+kh} \nu(x)\,\mathrm{d}x\Big\},$$

then

$$\sum_{s=0}^n \kappa_s \int\limits_0^\infty u(t, T_1+sh)\exp\{-t\lambda\}\,\mathrm{d}t$$

$$= \sum_{k=0}^{n}\sum_{s=0}^{n}\kappa_k \int\limits_{T_1+sh}^{\infty} u(t-T_1-sh,,T_1+kh)\exp\{-t\lambda\}\,\mathrm{d}t\kappa_s$$

$$\times \exp\Big\{-\int\limits_{T_0}^{T_1+sh}\nu(x)\,\mathrm{d}x\Big\} + \sum_{s=0}^{n}\int\limits_{0}^{T_1+sh}\kappa_s u(t,T_1+sh)\exp\{-t\lambda\}\,\mathrm{d}t,$$

and finally

$$\sum_{s=0}^{n}\kappa_s\int\limits_0^{\infty}u(t,T_1+sh)\exp\{-t\lambda\}\,\mathrm{d}t$$
$$= \sum_{s=0}^{n}\int\limits_{0}^{T_1+sh}\kappa_s u(t,T_1+sh)\exp\{-t\lambda\}\,\mathrm{d}t\Big/\Big\{1-\sum_{0}^{n}\gamma_s\Big\},$$

with γ_s defined in (5.33). The numerator of the right-hand side of this equation is analytic function. Thus, using the method of rectangle contour integral [1] for large time, we evaluate the inverse Laplace transform obtaining

$$\sum_{s=0}^{n}\kappa_s u(t,T_1+sh) \sim \eta\exp\{t\lambda_0\},$$

where η is a positive constant and λ_0 is the unique real root of (5.33). Then, from (5.35) and (5.34) for large time, it follows that

$$\begin{aligned}
u(t,\tau) \sim u^{as} &= \eta\exp\Big\{(t-\tau)\lambda_0 - \int\limits_{T_0}^{\tau}\nu(x)\,\mathrm{d}x\Big\},\\
v_k(t,\tau) \sim v_k^{as} &= \eta b_k\exp\Big\{(t-\tau-T_1-kh)\lambda_0\\
&\quad - \int\limits_{T_0}^{T_1+kh+\tau}\nu(x)\,\mathrm{d}x - \int\limits_0^{\tau}\nu_k(x)\,\mathrm{d}x\Big\}.
\end{aligned} \tag{5.36}$$

(5.31), (5.32) show that functions u^{as}, v_k^{as} are the separable solution of problem (3.4), (3.5) without initial conditions.

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Du globojančios savo vaikus populiacijos dinamikos modeliai

Vladas SKAKAUSKAS

Pateikti du nelimituotos globojančios savo vaikus populiacijos amžiaus struktūros dinamikos modeliai. Vienas iš jų yra populiacijos modelis, įskaitantis harmoninio tipo kryžminimosi dėsnį ir patelių nėštumą. Kita aprašo aseksualios populiacijos dinamiką. Abiejuose modeliuose nepaisoma individų migracijos. Abiems modeliams bendruoju gyvybinių funkcijų atveju įrodyta sprendinio egzistencijos ir vienaties teorema, ištirtas populiacijos augimas ir jos išnykimas bei rasta separabelinių sprendinių klasė. Stacionariuoju gyvybinių funkcijų atveju dideliems laikams gauta aseksualios populiacijos modelio sprendinio asimptotika.