

# Sensitivity Analysis of Multivariable Systems in State Space

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**Abstract.** This paper contains measures to describe the matrix impulse response sensitivity of state space multivariable systems with respect to parameter perturbations. The parameter sensitivity is defined as an integral measure of the matrix impulse response with respect to the coefficients. A state space approach is used to find a realization of impulse response that minimizes a sensitivity measure.

**Key words:** multivariable systems, sensitivity analysis, state space.

## 1. Introduction

One possibility of converting linear systems into mathematical processes consists in the state space representations. They are characterized by the description of behavior of physical systems with constant matrices and, therefore, they establish a direct relation to the methods of linear algebra. The state space representation of a given linear system can be used in order to perform a pole-zero determination (Dooren, 1981), a stability test (Bernett and Storey, 1970), and a test concerning the passivity (Anderson and Vongpanitlerd, 1973). The solution of these problems using the methods of linear algebra is recommended from a numerical point of view as powerful software packages are available (Thiele, 1986).

The state space design of a system consists of finding a set of state space equations that realize a desired transfer function. The state space equations corresponding to the transfer function or impulse response are not unique, thus one may select among the realizations one that minimizes a suitable sensitivity measure. Also some system properties are invariant with respect to the realizations, state space realizations do affect certain properties. One important property is the sensitivity of the system with respect to the realization parameters. In applications such as digital control and filter design, it is of practical importance to have a state space realization for which the system sensitivity is minimal. One of the reasons for this is the existence of the finite word length effect to coefficient truncation and arithmetic roundoff in the implementation of a controller or filter. It is understandable that poor sensitivity may lead to the degradation in performance of an implementation (Yan and Moore, 1992).

The definition of sensitivity of a system with respect to its realization coefficients is by no means unique. One definition is in terms of the use of two different norms  $L^1$  and  $L^2$  (Thiele, 1986), termed here mixed  $L^2/L^1$  sensitivity. This mixing allows an explicit solution. The class of all mixed  $L^2/L^1$  sensitivity optimal realizations can be easily characterized. On the other hand, it seems to use the sensitivity defined via  $L^2$  norm.

Yan and Moore (1992) explored the properties of the solution of the  $L^2$ -sensitivity minimization problem and clarified some issues related to model reduction. Mullis and Roberts (1976) produced closed-form results concerning minimum roundoff noise digital filters.

Thiele (86a) derived the conditions to minimize the sensitivity measure. It means that an optimal design can be found directly. By the use of the equivalence transformation it is possible to determine realizations that are  $l_2$ -scaled under a white noise input. These realizations have the minimal norm of the output noise spectrum (Mullis and Roberts, 1976), and the minimum pole sensitivity (Barnes, 1979). As Kung has shown the minimum norm realizations of arbitrary order satisfy  $\|\mathbf{F}\|_2 < 1$ , where  $\mathbf{F}$  denotes the system matrix, can be realized to be free of overflow and limit cycles (Mills and Mullis, 1978). The numerical accuracy depends on the conditions of the chosen state space representation. Therefore, it is desirable to define appropriate measures that represent the sensitivity of the realized transfer function or impulse response with respect to parameter variations.

## 2. Statement of the Problem

Consider a state-space realization  $\{\mathbf{F}, \mathbf{G}, \mathbf{R}, \mathbf{D}\}$  of the discrete-time linear multivariable system described by Zadeh and Desoer (1963)

$$\mathbf{q}(k+1) = \mathbf{F}\mathbf{q}(k) + \mathbf{G}\mathbf{x}(k), \quad (1a)$$

$$\mathbf{y}(k) = \mathbf{R}\mathbf{q}(k) + \mathbf{D}\mathbf{x}(k), \quad (1b)$$

where  $k$  is the discrete time,  $\mathbf{q}(k)$  is the  $n$ -dimensional state vector,  $\mathbf{y}(k)$  is the  $p$ -dimensional observation vector,  $\mathbf{x}(k)$  is the  $m$ -dimensional input vector,  $\mathbf{F}$ ,  $\mathbf{G}$ ,  $\mathbf{R}$ , and  $\mathbf{D}$  are constant matrices of dimension  $n \times n$ ,  $n \times m$ ,  $p \times n$ , and  $p \times m$ , respectively.

It is assumed that the system matrix  $\mathbf{F}$  has distinct eigenvalues.

Pole sensitivity of a discrete-time linear multivariable system in the state-space depends only on the system matrix  $\mathbf{F}$ . However, the impulse response or the transfer function depends not only on poles but also on zeros (Morgan, 1966). So the impulse response sensitivity depends not only on changes in the system matrix  $\mathbf{F}$ , but also in the matrices  $\mathbf{G}$ ,  $\mathbf{R}$  and  $\mathbf{D}$ .

In practice, it is not possible to realize the coefficients of  $\{\mathbf{F}, \mathbf{G}, \mathbf{R}, \mathbf{D}\}$  exactly. Thus, for the realization of  $\{\mathbf{F}, \mathbf{G}, \mathbf{R}, \mathbf{D}\}$  we define the sensitivity of impulse response  $h_{lj}(k)$  with respect to the elements of  $\mathbf{F}$ ,  $\mathbf{G}$ ,  $\mathbf{R}$  and  $\mathbf{D}$  to be a partial derivative of impulse response with respect to these elements. The derivative of impulse response  $h_{lj}(k)$  with respect to the matrix  $\mathbf{F}$ , denoted as  $\partial h_{lj}(k)/\partial \mathbf{F}$ , is an  $n \times n$  matrix whose  $(q, r)$ th entry is  $\partial h_{lj}(k)/\partial f_{qr}$ .

In this paper, we will use a sensitivity measure for discrete-time state space systems. We will explore the  $L^2$ -sensitivity problem of multi-input, multi-output systems that realize the matrix impulse response. First we have done the equivalence transform of a state space realization. Afterwards we have derived the sensitivity of matrix impulse response to the eigenvalues. Next we have used the sensitivity of the impulse response to the eigenvalues as a tool to derive the impulse response sensitivity to the system matrices.

### 3. Eigenvalues Sensitivity

The system matrix  $\mathbf{F}$  is assumed such that the equivalence transform matrix exists (Fadeev and Fadeeva, 1963)

$$\bar{\mathbf{F}} = \mathbf{T}^+ \mathbf{F} \mathbf{V} = \text{diag}(\lambda_1, \dots, \lambda_n), \quad (2)$$

where  $\lambda_i$ ,  $i = 1, 2, \dots, n$  are eigenvalues of  $\mathbf{F}$ , the symbol "+" denotes the matrix conjugate transposition. Let  $\mathbf{v}_i$  be an eigenvector associated with the eigenvalue  $\lambda_i$ , i.e.,  $\mathbf{F}\mathbf{v}_i = \lambda_i\mathbf{v}_i$ ,  $i = 1, 2, \dots, n$ ;  $\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_n] = [v_{ij}]$  is the  $n \times n$  matrix, where  $v_{ij}$  is the  $(i, j)$ th element of  $\mathbf{V}$ . Let  $\mathbf{T} = [\mathbf{V}^{-1}]^+ = [\mathbf{t}_1, \dots, \mathbf{t}_n] = [t_{ij}]_{n \times n}$  be an  $n \times n$  matrix, where  $t_{ij}$  is the  $(i, j)$ th element of  $\mathbf{T}$ , and  $\mathbf{t}_j$  is the  $j$ th column of  $\mathbf{T}$ .

Then eigenvalue  $\lambda_i = \mathbf{t}_i^+ \mathbf{F} \mathbf{v}_i$ ,  $i = 1, 2, \dots, n$ . The perturbation of the eigenvalue

$$\begin{aligned} d\lambda_i &= dt_i^+ \mathbf{F} \mathbf{v}_i + \mathbf{t}_i^+ d\mathbf{F} \mathbf{v}_i + \mathbf{t}_i^+ \mathbf{F} d\mathbf{v}_i = \mathbf{t}_i^+ \mathbf{v}_i d\mathbf{F} + \lambda_i \mathbf{v}_i dt_i^+ + \lambda_i \mathbf{t}_i^+ d\mathbf{v}_i \\ &= \mathbf{t}_i^+ \mathbf{v}_i d\mathbf{F} + \lambda_i d(\mathbf{t}_i^+ d\mathbf{v}_i). \end{aligned} \quad (3)$$

Since  $\mathbf{t}_i^+ \mathbf{v}_i = 1$ , we have

$$d\lambda_i = \mathbf{t}_i^+ d\mathbf{F} \mathbf{v}_i = \sum_{l=1}^n \sum_{j=1}^n t_{li}^* v_{ji} df_{lj}, \quad (4)$$

where  $df_{lj}$  is the  $(l, j)$ th element of  $d\mathbf{F}$ ; "\*" is a complex conjugate symbol. On the other hand,

$$d\lambda_i = \sum_{l=1}^n \sum_{j=1}^n \frac{\partial \lambda_i}{\partial f_{lj}} df_{lj}. \quad (5)$$

From (4) and (5) we obtain

$$\frac{\partial \lambda_i}{\partial f_{lj}} = t_{li}^* v_{ji}. \quad (6)$$

Define the  $n \times n$  gradient operator as follows:

$$\nabla_{\mathbf{F}} = \left[ \frac{\partial}{\partial f_{lj}} \right]. \quad (7)$$

From (6) and (7) the  $n \times n$  eigenvalue sensitivity matrix is:

$$\nabla_{\mathbf{F}} \lambda_i = \left[ \frac{\partial \lambda_i}{\partial f_{lj}} \right] = \mathbf{t}_i^* \mathbf{v}_i^T. \quad (8)$$

From (8) it follows the sensitivity invariant condition:

$$\sum_{i=1}^n \nabla_{\mathbf{F}} \lambda_i = \sum_{i=1}^n (\mathbf{v}_i \mathbf{t}_i^+)^T = (\mathbf{V} \mathbf{T}^+)^T = \mathbf{I}. \quad (9)$$

Equation (8) is a sensitivity function. The measure of this sensitivity is defined by its Euclidean norm:

$$\|\mathbf{F}\| = \left( \sum_{l=1}^n \sum_{j=1}^n |f_{lj}|^2 \right)^{\frac{1}{2}} = \text{Tr}(\mathbf{F}^+ \mathbf{F})^{\frac{1}{2}} = \text{Tr}(\mathbf{F} \mathbf{F}^+)^{\frac{1}{2}}. \quad (10)$$

Thus,

$$\|\nabla_{\mathbf{F}} \lambda_i\|^2 = \text{Tr} \left[ (\nabla_{\mathbf{F}} \lambda_i) (\nabla_{\mathbf{F}} \lambda_i)^+ \right] = \text{Tr} (\mathbf{t}_i^* \mathbf{v}_i^T \mathbf{v}_i^* \mathbf{t}_i^T) = \|\mathbf{t}_i\|^2 \cdot \|\mathbf{v}_i\|^2, \quad (11)$$

or

$$\|\nabla_{\mathbf{F}} \lambda_i\| = \|\mathbf{t}_i\| \|\mathbf{v}_i\|. \quad (12)$$

Using the Cauchy-Schwarz inequality, a lower bound is given by

$$\|\nabla_{\mathbf{F}} \lambda_i\| = \|\mathbf{t}_i\| \|\mathbf{v}_i\| \geq \|\mathbf{t}_i^+ \mathbf{v}_i\| = 1, \quad (13)$$

and

$$\sum_{i=1}^n \|\nabla_{\mathbf{F}} \lambda_i\| \geq n. \quad (14)$$

Equality (13) is valid if  $\mathbf{v}_i$  and  $\mathbf{t}_i$  are linearly dependent. The equality in (14) is valid if all eigenvectors  $\mathbf{v}_i$  are orthogonal. If the system matrix  $\mathbf{F}$  has distinct eigenvalues, the minimum sensitivity is when  $\mathbf{F}$  is a normal matrix.

#### 4. Impulse Response Sensitivity

In a state space realization of a multivariable system, the pole sensitivity is a measure of the effects of the perturbation only on the system matrix  $\mathbf{F}$ . As transfer function or impulse response are determined by both poles and zeros, the zero sensitivity as well as the pole sensitivity has to be determined to see the effects of the perturbation of the system

parameters. Our purpose is the effects of the perturbation of the system parameters on the impulse response. Then it is desired to obtain the impulse response sensitivity instead of the pole and zero sensitivities. The aim is to define the impulse response sensitivity measure of a multivariable system with respect to the system parameters.

Consider the state space realization  $\{\mathbf{F}, \mathbf{G}, \mathbf{R}, \mathbf{D}\}$  of the system described by (1). We assume that the system matrix  $\mathbf{F}$  has distinct eigenvalues. Let  $\{\bar{\mathbf{F}}, \bar{\mathbf{G}}, \bar{\mathbf{R}}, \mathbf{D}\}$  be the equivalence transformation of  $\{\mathbf{F}, \mathbf{G}, \mathbf{R}, \mathbf{D}\}$  by means of  $\mathbf{V}$ :

$$\begin{aligned}\bar{\mathbf{F}} &= \mathbf{T}^+ \mathbf{F} \mathbf{V} = \text{diag} \{ \lambda_1, \dots, \lambda_n \}, \\ \bar{\mathbf{G}} &= \mathbf{T}^+ \mathbf{G} = [\bar{g}_{lj}] = [\bar{\mathbf{g}}_1, \dots, \bar{\mathbf{g}}_m] = [\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_n]^T, \\ \bar{\mathbf{R}} &= \mathbf{R} \mathbf{V} = [\bar{r}_{lj}] = [\bar{\mathbf{r}}_1, \dots, \bar{\mathbf{r}}_p]^T = [\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_n],\end{aligned}\quad (15)$$

$$(16)$$

where  $\bar{g}_{lj}$  and  $\bar{r}_{lj}$  are the  $(l, j)$ th element of  $\bar{\mathbf{G}}$  and  $\bar{\mathbf{R}}$ , respectively;  $\bar{\mathbf{g}}_i$  and  $\boldsymbol{\alpha}_i$  are the  $i$ th column and row of  $\bar{\mathbf{G}}$ , respectively;  $\bar{\mathbf{r}}_i$  and  $\boldsymbol{\beta}_i$  are  $i$ th row and column of  $\bar{\mathbf{R}}$ , respectively.

The matrix impulse response  $\mathbf{h}(k)$  of the system is given by

$$\mathbf{h}(k) = \begin{cases} \mathbf{D}, & \text{if } k = 0, \\ \mathbf{R} \mathbf{F}^{k-1} \mathbf{G} = \bar{\mathbf{R}} \bar{\mathbf{F}}^{k-1} \bar{\mathbf{G}}, & \text{if } k \geq 1. \end{cases} \quad (17)$$

We measure the sensitivity of the matrix impulse response  $\mathbf{h}(k)$  to the eigenvalues  $\lambda_i$  and the impulse response sensitivity to the matrices  $\mathbf{F}, \mathbf{G}, \mathbf{R}, \mathbf{D}$ .

#### 4.1. Impulse Response Sensitivity to the Perturbation of Eigenvalues

Substituting (2), (15), and (16) into (17), we obtain the  $(l, j)$ th element of the matrix impulse response as follows:

$$h_{lj}(k) = \sum_{i=1}^n \bar{r}_{li} \lambda_i^{k-1} \bar{g}_{ij}, \quad k \geq 2, \quad l = 1, 2, \dots, p, \quad j = 1, 2, \dots, m. \quad (18)$$

By differentiating the  $(l, j)$ th impulse response with respect to  $\lambda_i$  we obtain

$$\frac{\partial h_{lj}(k)}{\partial \lambda_i} = (k-1) \bar{r}_{li} \lambda_i^{k-2} \bar{g}_{ij}, \quad k \geq 2. \quad (19)$$

Let us define the vector  $\nabla_{\boldsymbol{\lambda}}$  as follows:

$$\nabla_{\boldsymbol{\lambda}} = \left[ \frac{\partial}{\partial \lambda_1}, \dots, \frac{\partial}{\partial \lambda_n} \right]^T.$$

Thus,

$$\nabla_{\boldsymbol{\lambda}} h_{lj}(k) = (k-1) [\bar{r}_{l1} \lambda_1^{k-2} \bar{g}_{1j}, \dots, \bar{r}_{ln} \lambda_n^{k-2} \bar{g}_{nj}]^T, \quad k \geq 2. \quad (20)$$

The Euclidean norm of (20)

$$\|\nabla_{\boldsymbol{\lambda}} h_{lj}(k)\| = (k-1) \left( \sum_{i=1}^n |\bar{r}_{li}|^2 |\lambda_i|^{2(k-2)} |\bar{g}_{ij}|^2 \right)^{\frac{1}{2}}, \quad k \geq 2. \quad (21)$$

The sensitivity of the  $(l, j)$ th impulse response to eigenvalues  $\lambda_i$  is an infinite summation of (21):

$$J_{\boldsymbol{\lambda}}^2(h_{lj}) = \sum_{k=2}^{\infty} \|\nabla_{\boldsymbol{\lambda}} h_{lj}(k)\|^2 = \sum_{i=1}^n |\bar{r}_{li}|^2 |\bar{g}_{ij}|^2 \sum_{k=2}^{\infty} (k-1)^2 |\lambda_i|^{2(k-2)}. \quad (22)$$

Using the expression (Korn and Korn, 1961)

$$\sum_{k=2}^{\infty} (k-1)^2 x^{k-2} = \frac{1+x}{(1-x)^3}, \quad |x| < 1, \quad (23)$$

we obtain the sensitivity of the  $(l, j)$ th impulse response to the eigenvalues as follows:

$$J_{\boldsymbol{\lambda}}(h_{lj}) = \left[ \sum_{i=1}^n |\bar{r}_{li}|^2 |\bar{g}_{ij}|^2 \frac{1+|\lambda_i|^2}{(1-|\lambda_i|^2)^3} \right]^{\frac{1}{2}}. \quad (24)$$

We define the total sensitivity of the matrix impulse response as a summation of (24), i.e.,

$$J_{\boldsymbol{\lambda}}(\mathbf{h}) = \left[ \sum_{l=1}^p \sum_{j=1}^m J_{\boldsymbol{\lambda}}^2(h_{lj}) \right]^{\frac{1}{2}}. \quad (25)$$

Substituting (24) into (25), and keeping in mind, that  $\sum_{j=1}^m |\bar{g}_{ij}|^2 = \|\boldsymbol{\alpha}_i\|^2$  and  $\sum_{l=1}^p |\bar{r}_{li}|^2 = \|\boldsymbol{\beta}_i\|^2$ , we obtain

$$\begin{aligned} J_{\boldsymbol{\lambda}}(\mathbf{h}) &= \left[ \sum_{l=1}^p \sum_{j=1}^m \sum_{i=1}^n |\bar{r}_{li}|^2 |\bar{g}_{ij}|^2 \frac{1+|\lambda_i|^2}{(1-|\lambda_i|^2)^3} \right]^{\frac{1}{2}} \\ &= \left[ \sum_{i=1}^n \|\boldsymbol{\beta}_i\|^2 \|\boldsymbol{\alpha}_i\|^2 \frac{1+|\lambda_i|^2}{(1-|\lambda_i|^2)^3} \right]^{\frac{1}{2}}, \end{aligned} \quad (26)$$

where  $\boldsymbol{\beta}_i$  is the  $i$ th column of the matrix  $\bar{\mathbf{R}}$  and  $\boldsymbol{\alpha}_i$  is the  $i$ th row of the matrix  $\bar{\mathbf{G}}$ .

In the subsequent analysis on the impulse response sensitivity to the system matrices  $\mathbf{F}$ ,  $\mathbf{G}$ ,  $\mathbf{R}$ ,  $\mathbf{D}$  we assume that every matrix is perturbed.

4.2. Impulse Response Sensitivity to the System Matrix  $\mathbf{F}$  Perturbation

We differentiate the  $(l, j)$ th element of the matrix impulse response with respect to the  $(q, r)$ th element of the system matrix  $\mathbf{F}$  as follows:

$$\frac{\partial h_{lj}(k)}{\partial f_{qr}} = \sum_{i=1}^n \frac{\partial h_{lj}(k)}{\partial \lambda_i} \cdot \frac{\partial \lambda_i}{\partial f_{qr}}, \quad k \geq 2. \quad (27)$$

Substituting (6) and (19) into (27), we obtain

$$\frac{\partial h_{lj}(k)}{\partial f_{qr}} = \sum_{i=1}^n (k-1) \bar{r}_{li} \lambda_i^{k-2} \bar{g}_{ij} t_{qi}^* v_{ri}, \quad k \geq 2. \quad (28)$$

Thus,

$$\nabla_{\mathbf{F}} h_{lj}(k) = (k-1) \mathbf{T}^* \text{diag} \{ \bar{r}_{li} \lambda_i^{k-2} \bar{g}_{ij} \} \mathbf{V}^T, \quad i = 1, 2, \dots, n. \quad (29)$$

Applying the Cauchy inequality to (29), we get

$$\begin{aligned} \|\nabla_{\mathbf{F}} h_{lj}(k)\| &= \text{Tr} \{ (\nabla_{\mathbf{F}} h_{lj}(k))^+ (\nabla_{\mathbf{F}} h_{lj}(k)) \} \\ &\geq (k-1)^2 \sum_{i=1}^n |\bar{r}_{li}|^2 |\lambda_i|^{2(k-2)} |\bar{g}_{ij}|^2. \end{aligned} \quad (30)$$

From (21) and (30), we obtain that

$$\|\nabla_{\mathbf{F}} h_{lj}(k)\| \geq \|\nabla_{\boldsymbol{\lambda}} h_{lj}(k)\|, \quad k \geq 2. \quad (31)$$

In case the system matrix  $\mathbf{F}$  is normal, the equality in (31) holds and the term on the left hand-side in (31) has the minimum. The normal matrix has a set of orthonormal eigenvectors associated with distinct eigenvalues.

The sensitivity measure of the  $(l, j)$ th impulse response and the matrix impulse response is defined as follows:

$$J_{\mathbf{F}}(h_{lj}) = \left( \sum_{k=2}^{\infty} \|\nabla_{\mathbf{F}} h_{lj}(k)\|^2 \right)^{\frac{1}{2}}, \quad (32)$$

and

$$J_{\mathbf{F}}(\mathbf{h}) = \left[ \sum_{l=1}^p \sum_{j=1}^m J_{\mathbf{F}}^2(h_{lj}) \right]^{\frac{1}{2}}. \quad (33)$$

For the normal system matrix with distinct eigenvalues, we obtain

$$J_{\mathbf{F}}(h_{lj}) = J_{\boldsymbol{\lambda}}(h_{lj}), \quad (34)$$

and

$$J_{\mathbf{F}}(\mathbf{h}) = J_{\boldsymbol{\lambda}}(\mathbf{h}). \quad (35)$$

#### 4.3. Impulse Response Sensitivity to the Matrix $\mathbf{G}$ Perturbation

According to (17) the  $(l, j)$ th element of the matrix impulse response is given by

$$h_{lj}(k) = \mathbf{r}_l \mathbf{F}^{k-1} \mathbf{g}_j, \quad k \geq 1, \quad (36)$$

where  $r_l$  is the  $l$ th row of  $\mathbf{R}$  and  $\mathbf{g}_j$  is the  $j$ th column of  $\mathbf{G}$ .

Let us define the vector  $\nabla_{\mathbf{g}_j}$  and matrix  $\nabla_{\mathbf{G}}$  as follows:

$$\nabla_{\mathbf{g}_j} = \left[ \frac{\partial}{\partial g_{1j}}, \dots, \frac{\partial}{\partial g_{mj}} \right]^T, \quad j = 1, 2, \dots, m, \quad (37)$$

$$\nabla_{\mathbf{G}} = [\nabla_{\mathbf{g}_1}, \dots, \nabla_{\mathbf{g}_m}]. \quad (38)$$

Then the gradient of  $h_{lj}(k)$  with respect to the vector  $\mathbf{g}_j$  is:

$$\nabla_{\mathbf{g}_j} h_{lj}(k) = (\mathbf{F}^T)^{k-1} \mathbf{r}_l^T = \mathbf{T}^* \text{diag} \{ \lambda_i^{k-1} \} \bar{\mathbf{r}}_l^T, \quad i = 1, 2, \dots, n, \quad (39)$$

where  $\bar{\mathbf{r}}_l$  is the  $l$ th row of the matrix  $\bar{\mathbf{R}}$ . In case the matrix  $\mathbf{T}$  is orthonormal, we obtain the Euclidean norm of (39):

$$\|\nabla_{\mathbf{g}_j} h_{lj}(k)\|^2 = \sum_{i=1}^n |\bar{r}_{li}|^2 |\lambda_i|^{2(k-1)}, \quad k \geq 1. \quad (40)$$

But  $\nabla_{\mathbf{g}_j} h_{lj}(k) = 0$  if  $i \neq j$ , then  $\|\nabla_{\mathbf{G}} h_{lj}(k)\| = \|\nabla_{\mathbf{g}_j} h_{lj}(k)\|$ .

We obtain the following sensitivity of the  $(l, j)$ th impulse response to the matrix  $\mathbf{G}$ :

$$J_{\mathbf{G}}(h_{lj}) = \left( \sum_{k=1}^{\infty} \|\nabla_{\mathbf{G}} h_{lj}(k)\|^2 \right)^{\frac{1}{2}} = \left( \sum_{i=1}^n \frac{|\bar{r}_{li}|^2}{1 - |\lambda_i|^2} \right)^{\frac{1}{2}}. \quad (41)$$

The matrix impulse response sensitivity to the matrix  $\mathbf{G}$  is the summation of  $J_{\mathbf{G}}(h_{lj})$ :

$$J_{\mathbf{G}}(\mathbf{h}) = \left( \sum_{l=1}^p \sum_{j=1}^m J_{\mathbf{G}}^2(h_{lj}) \right)^{\frac{1}{2}} = \left( m \sum_{i=1}^n \frac{\|\boldsymbol{\beta}_i\|^2}{1 - |\lambda_i|^2} \right)^{\frac{1}{2}}. \quad (42)$$

#### 4.4. Impulse Response Sensitivity to the Matrix $\mathbf{R}$ Perturbation

Define the vector  $\nabla_{\mathbf{r}_l}$  and matrix  $\nabla_{\mathbf{R}}$  as follows:

$$\nabla_{\mathbf{r}_l} = \left[ \frac{\partial}{\partial r_{l1}}, \dots, \frac{\partial}{\partial r_{lm}} \right], \quad (43)$$

$$\nabla_{\mathbf{R}} = [\nabla_{\mathbf{r}_1}, \dots, \nabla_{\mathbf{r}_p}]^T. \quad (44)$$



The gradient of  $h_{lj}(k)$  (36) with respect to the row  $\mathbf{r}_l$  of the matrix  $\mathbf{R}$  becomes

$$\nabla_{\mathbf{r}_l} h_{lj}(k) = \mathbf{g}_j^T (\mathbf{F}^T)^{k-1} = \bar{\mathbf{g}}_j^T \text{diag} \{ \lambda_i^{k-1} \} \mathbf{V}^T, \quad i = 1, 2, \dots, n, \quad (45)$$

where  $\bar{\mathbf{g}}_j$  is the  $j$ th column of the matrix  $\bar{\mathbf{G}}$ . As  $\mathbf{V}$  is orthonormal, we obtain the Euclidean norm of (45):

$$\|\nabla_{\mathbf{r}_l} h_{lj}(k)\|^2 = \sum_{i=1}^n |\lambda_i|^{2(k-1)} |\bar{g}_{ij}|^2, \quad k \geq 1, \quad (46)$$

where  $\bar{g}_{ij}$  is the  $(i, j)$ th element of the matrix  $\bar{\mathbf{G}}$ . Since  $\nabla_{\mathbf{r}_l} h_{lj}(k) = 0$ , if  $l \neq i$ , we get  $\|\nabla_{\mathbf{R}} h_{lj}(k)\| = \|\nabla_{\mathbf{r}_l} h_{lj}(k)\|$ . The sensitivity of the  $(l, j)$ th impulse response to the matrix  $\mathbf{R}$  perturbations is defined by

$$J_{\mathbf{R}}(h_{lj}) = \left( \sum_{k=1}^{\infty} \|\nabla_{\mathbf{R}} h_{lj}(k)\|^2 \right)^{\frac{1}{2}} = \left( \sum_{i=1}^n \frac{|\bar{g}_{ij}|^2}{1 - |\lambda_i|^2} \right)^{\frac{1}{2}}. \quad (47)$$

The sensitivity of the matrix impulse response to the matrix  $\mathbf{R}$  perturbations becomes

$$\begin{aligned} J_{\mathbf{R}}(\mathbf{h}) &= \left( \sum_{l=1}^p \sum_{j=1}^m J_{\mathbf{R}}^2(h_{lj}) \right)^{\frac{1}{2}} = \left( p \sum_{j=1}^m \sum_{i=1}^n \frac{|\bar{g}_{ij}|^2}{1 - |\lambda_i|^2} \right)^{\frac{1}{2}} \\ &= \left( p \sum_{i=1}^n \frac{\|\boldsymbol{\alpha}_i\|^2}{1 - |\lambda_i|^2} \right)^{\frac{1}{2}}, \end{aligned} \quad (48)$$

where  $\boldsymbol{\alpha}_i$  is the  $i$ th row of the matrix  $\bar{\mathbf{G}}$ .

#### 4.5. Impulse Response Sensitivity to the Matrix $\mathbf{D}$ Perturbation

Define the  $p \times m$  matrix  $\nabla_{\mathbf{D}}$  as follows:

$$\nabla_{\mathbf{D}} = \left[ \frac{\partial}{\partial d_{lj}} \right]. \quad (49)$$

It is clear that  $\nabla_{\mathbf{D}} h_{lj}(0) = 1$ .

The sensitivity of the  $(l, j)$ th element of the matrix impulse response to the matrix  $\mathbf{D}$  becomes

$$J_{\mathbf{D}}(h_{lj}) = \|\nabla_{\mathbf{D}} h_{lj}(0)\| = 1. \quad (50)$$

The matrix impulse response sensitivity to the matrix  $\mathbf{D}$  becomes

$$J_{\mathbf{D}}(\mathbf{h}) = \left( \sum_{l=1}^p \sum_{j=1}^m J_{\mathbf{D}}^2(h_{lj}) \right)^{\frac{1}{2}} = (pm)^{\frac{1}{2}}. \quad (51)$$

#### 4.6. Sensitivity to all System Matrices Perturbations

We define the matrix impulse response sensitivity to all system matrices as follows:

$$\begin{aligned} J_{\Sigma}(\mathbf{h}) &= [J_{\mathbf{F}}^2(\mathbf{h}) + J_{\mathbf{G}}^2(\mathbf{h}) + J_{\mathbf{R}}^2(\mathbf{h}) + J_{\mathbf{D}}^2(\mathbf{h})]^{\frac{1}{2}} \\ &= \left\{ \sum_{i=1}^n \left[ \frac{\|\beta_i\|^2 \|\alpha_i\|^2 (1 + |\lambda_i|^2)}{(1 - |\lambda_i|^2)^3} + \frac{m \|\beta_i\|^2 + p \|\alpha_i\|^2}{1 - |\lambda_i|^2} \right] + pm \right\}^{\frac{1}{2}} \end{aligned} \quad (52)$$

For poles near the unit circle,  $J_{\mathbf{F}}(\mathbf{h})$  dominates  $J_{\Sigma}(\mathbf{h})$ .

### 5. Impulse Response Sensitivity of One-Input One-Output System

We measure the impulse response sensitivity of one-input one-output system  $\{\mathbf{F}, \mathbf{g}, \mathbf{r}, d\}$  and find the condition on  $\mathbf{g}$  and  $\mathbf{r}$  for minimizing the impulse response sensitivity under the dynamic range constraint in order to control overflow. Consider that the matrix  $\mathbf{F}$  is normal.

As a special case of a multivariable system, we obtain the impulse response sensitivity measure of one-input one-output normal system  $\{\mathbf{F}, \mathbf{g}, \mathbf{r}, d\}$  that has distinct eigenvalues.

From (30) we obtain the sensitivity to the matrix  $\mathbf{F}$  perturbations:

$$\|\nabla_{\mathbf{F}} h(k)\|^2 = (k-1)^2 \sum_{i=1}^n |\gamma_i|^2 |\lambda_i|^{2(k-2)}, \quad k \geq 2,$$

and

$$J_{\mathbf{F}}(h) = \left( \sum_{k=2}^{\infty} \|\nabla_{\mathbf{F}} h(k)\|^2 \right)^{\frac{1}{2}} = \left[ \sum_{k=2}^{\infty} (k-1)^2 \sum_{i=1}^n |\gamma_i|^2 |\lambda_i|^{2(k-2)} \right]^{\frac{1}{2}}.$$

Using expression (23), we get

$$J_{\mathbf{F}}(h) = \left( \sum_{i=1}^n \frac{|\gamma_i|^2 (1 + |\lambda_i|^2)}{(1 - |\lambda_i|^2)^3} \right)^{\frac{1}{2}}, \quad (53)$$

where  $\gamma_i = \alpha_i \beta_i$ .

We obtain the impulse response sensitivity to vector  $\mathbf{g}$  perturbations from (40)

$$\|\nabla_{\mathbf{g}} h(k)\|^2 = \sum_{i=1}^n |\beta_i|^2 |\lambda_i|^{2(k-1)}, \quad k \geq 1,$$

and

$$J_{\mathbf{g}}(h) = \left( \sum_{k=1}^{\infty} \|\nabla_{\mathbf{g}} h(k)\|^2 \right)^{\frac{1}{2}} = \left( \sum_{i=1}^n \frac{|\beta_i|^2}{1 - |\lambda_i|^2} \right)^{\frac{1}{2}}. \quad (54)$$

The impulse response sensitivity to vector  $\mathbf{r}$  perturbations is obtained from (46)

$$\|\nabla_{\mathbf{r}} h(k)\|^2 = \sum_{i=1}^n |\alpha_i|^2 |\lambda_i|^{2(k-1)}, \quad k \geq 1, \quad (55)$$

and

$$J_{\mathbf{r}}(h) = \left( \sum_{k=1}^{\infty} \|\nabla_{\mathbf{r}} h(k)\|^2 \right)^{\frac{1}{2}} = \left( \sum_{i=1}^n \frac{|\alpha_i|^2}{1 - |\lambda_i|^2} \right)^{\frac{1}{2}}. \quad (56)$$

The impulse response sensitivity to  $\mathbf{F}$ ,  $\mathbf{g}$ ,  $\mathbf{r}$ ,  $d$  is obtained from (52)

$$\begin{aligned} J_{\Sigma}(h) &= [J_{\mathbf{F}}^2(h) + J_{\mathbf{g}}^2(h) + J_{\mathbf{r}}^2(h) + J_d^2(h)]^{\frac{1}{2}} \\ &= \left\{ \sum_{i=1}^n \left[ \frac{|\gamma_i|^2 (1 + |\lambda_i|^2)}{(1 - |\lambda_i|^2)^3} + \frac{|\alpha_i|^2 + |\beta_i|^2}{1 - |\lambda_i|^2} \right] + 1 \right\}^{\frac{1}{2}}. \end{aligned} \quad (57)$$

## 6. Minimization of the Impulse Response Sensitivity

We now consider the minimization of  $J_{\Sigma}(h)$  in (57).  $\gamma_i$  is determined by the scalar impulse response of  $\{\mathbf{F}, \mathbf{g}, \mathbf{r}, d\}$  and is independent of realizations. Therefore we can control  $J_{\mathbf{g}}^2(h) + J_{\mathbf{r}}^2(h)$  in order to minimize the sensitivity  $J_{\Sigma}(h)$ . In minimizing  $J_{\mathbf{g}}^2(h) + J_{\mathbf{r}}^2(h)$ , the dynamic range constraint has to be considered to control overflow. We use the dynamic range constraint described by

$$\sum_{i=1}^n \frac{|\alpha_i|^2}{1 - |\lambda_i|^2} = n\delta^2, \quad (58)$$

where  $\delta$  is a dynamic range constraint coefficient.

Thus, the minimization problem is minimizing  $J_{\mathbf{g}}^2(h) + J_{\mathbf{r}}^2(h)$  with respect to the constraint (58). This minimization is a variational problem. We form the Lagrangian  $Q$ :

$$Q = \sum_{i=1}^n \frac{|\alpha_i|^2 + |\beta_i|^2}{1 - |\lambda_i|^2} + \mu \left( \sum_{i=1}^n \frac{|\alpha_i|^2}{1 - |\lambda_i|^2} - n\delta^2 \right), \quad (59)$$

where  $\mu$  is a Lagrange multiplier and  $\beta_i = \gamma_i/\alpha_i$ .

We take the partial derivative of (59) with respect to  $|\alpha_i|^2$  and set the result to zero:

$$\frac{\partial Q}{\partial |\alpha_i|^2} = \frac{\mu + 1 - |\gamma_i|^2 |\alpha_i|^{-4}}{1 - |\lambda_i|^2} = 0. \quad (60)$$

Thus, using (60), we obtain

$$|\alpha_i|^2 = \frac{|\gamma_i|}{(1 + \mu)^{\frac{1}{2}}}. \quad (61)$$

We now evaluate  $\mu$ . By substituting (61) into (58), we obtain

$$(\mu + 1)^{\frac{1}{2}} = \frac{1}{n\delta^2} \sum_{i=1}^n \frac{|\gamma_i|}{1 - |\lambda_i|^2}. \quad (62)$$

Substituting (62) into (61) we obtain  $|\alpha_i|^2$  that minimizes  $J_{\mathbf{g}}^2(h) + J_{\mathbf{r}}^2(h)$  subject to (58):

$$|\alpha_i|^2 = \frac{n\delta^2 |\gamma_i|}{\sum_{i=1}^n \frac{\gamma_i}{1 - |\lambda_i|^2}}. \quad (63)$$

## 7. Conclusions

The scalar sensitivity measure has been presented for both the multi-input multi-output discrete-time state space realization of the matrix impulse response and the one-input one-output discrete-time state space realization of the scalar impulse response. Formulas that can be computed on a computer have been derived for the analysis of the effects of parameter changes. The impulse response sensitivity of the one-input one-output normal system that has distinct poles has been determined. We have minimized this impulse response sensitivity under a dynamic range constraint in order to control overflow. The condition that minimizes the impulse response sensitivity is the same as the condition that minimizes the roundoff noise variance in the output. The eigenvalue sensitivity is minimized if and only if the system matrix is normal.

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**Daugiamačių sistemų jautrumo analizė būsenų erdvėje**

Kazys KAZLAUSKAS

Straipsnyje nagrinėjame daugiamatės diskretinės sistemos, aprašytos būsenų lygtimis, matricinės impulsinės charakteristikos jautrumą parametrų pokyčiams. Integralinė matricinės impulsinės charakteristikos jautrumo funkcija priklauso nuo būsenų lygčių visų koeficientų pokyčių. Analizuojame vienmatės diskretinės sistemos impulsinės charakteristikos jautrumą. Suradome sąlyga, kai vienmatės sistemos impulsinė charakteristika yra mažiausiai jautri parametrų pokyčiams.