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On the Coexistence of Different Religions

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Abstract. A general model for pair formation in age, sex, and sociologically structured interacting human communities is presented. More precisely, the religion factor is taken into account. The model describes dynamics of interacting religions which tolerate both uniconfessional pairs and those with different religions. Two particular models are analyzed. One of them describes the uniconfessional pairs dynamics and allows the religion change only for the sake of marriage. The other one demonstrates the evolution of communities forbidding any confession change. In the case of constant vital rates solutions of these two models are constructed and the longtime behavior of the total numbers of single adults and pairs of each community is demonstrated.

Key words: population dynamics, random mating, age-sex-structured population, demography.

1. Introduction

In recent years there has been a considerable interest in the dynamics of two-sex populations in which pair formation is the major step to reproduction. Such models are of great importance for genetics (see, e.g., Svirezhev and Passekov (1990) and references therein), demography, and epidemiology, in particular for modelling sexually transmitted diseases (see, e.g., references in Hadeler (1993), Prüss and Schappacher (1994)). Both random mating (without formation of permanent male-female couples (see, e.g., Skakauskas (1998b)) and monogamous marriage models (Frederickson, 1971; Hoppensteadt, 1975; Staroverov, 1977; Hadeler, 1993, and references there) are usually used. In human demography one has considered the number of monogamous marriages as an important quantitative feature of the social structure. In fact, monogamy is the customary mating system of most present-day societies. Nevertheless, Murdock (1957) reports that at least seventy percent of all societies exhibit some degree of polygamy. The most general sex-age-structured population deterministic model, which takes into account marriages, has been proposed by Hoppensteadt (1975) and Staroverov (1977), and consists of a system of three integro-differential equations for the density $x(t, a)$ of single (unmarried) females at age a, density $y(t, b)$ of single males at age b, and density $p(t, a, b, c)$ of pairs which are formed of females at age a, males at age b, and which have existed for a time c. Hadeler (1993) simplified this model by introducing a maturation period τ into the mating law. This simplified model reads (Skakauskas, 1998a)

$$\partial x/\partial t + \partial x/\partial a$$
$$= -\mu_x\, x + \begin{cases} 0, \quad a < \tau, \quad t > 0, \\ -\int\limits_{\tau}^{\infty} p|_{c=0}\,\mathrm{d}a + \int\limits_{0}^{a-\tau} \mathrm{d}c \int\limits_{\tau+c}^{\infty} (\tilde{\mu}_y + \sigma)\, p\,\mathrm{d}b, \quad a > \tau,\, t > 0, \end{cases}$$

$$\partial y/\partial t + \partial y/\partial b$$
$$= -\mu_y\, y + \begin{cases} 0, \quad b < \tau, \quad t > 0, \\ -\int\limits_{\tau}^{\infty} p|_{c=0}\,\mathrm{d}b + \int\limits_{0}^{b-\tau} \mathrm{d}c \int\limits_{\tau+c}^{\infty} (\tilde{\mu}_x + \sigma)\, p\,\mathrm{d}a, \quad b > \tau,\, t > 0, \end{cases} \tag{1.1}$$

$$\partial p/\partial t + \partial p/\partial a + \partial p/\partial b + \partial p/\partial c = -(\tilde{\mu}_x + \tilde{\mu}_y + \sigma)\, p,$$
$$t > 0,\ \ a, b > \tau,\ \ c \in (0,\, \min(\mathrm{a} - \tau, \mathrm{b} - \tau)],$$
$$x(t, 0) = \int\limits_{0}^{\infty} \mathrm{d}c \int\limits_{\tau+c}^{\infty} \mathrm{d}a \int\limits_{\tau+c}^{\infty} \beta_x\, p\,\mathrm{d}b, \quad [x(t, \tau)] = 0,\ \ t > 0,$$
$$y(t, 0) = \int\limits_{0}^{\infty} \mathrm{d}c \int\limits_{\tau+c}^{\infty} \mathrm{d}a \int\limits_{\tau+c}^{\infty} \beta_y\, p\,\mathrm{d}b, \quad [y(t, \tau)] = 0,\ \ t > 0,$$
$$p|_{c=0} = f(t, a, b; x, y), \quad t > 0,\ \ (a, b) \in [\tau, \infty) \times [\tau, \infty),$$
$$x|_{t=0} = x^0,\ a \geqslant 0; \quad y|_{t=0} = y^0,\ b \geqslant 0,$$
$$p|_{t=0} = p^0, \quad a \geqslant \tau,\ \ b \geqslant \tau,\ \ c \in [0,\, \min(\mathrm{a} - \tau, \mathrm{b} - \tau)].$$

Here τ is a maturation period, t time, $[x(t,\tau)]$ and $[y(t,\tau)]$ are jumps of x and y at the lines $a = \tau$ and $b = \tau$, respectively, x^0, y^0, p^0 denote the initial age distributions of single females, single males, and pairs, respectively, β_j the birth rates of males $(j = y)$ or females $(j = x)$, μ_y resp. μ_x the death rates of single males resp. single females, σ means the divorce rate of pairs, $\tilde{\mu}_y$ resp. $\tilde{\mu}_x$ the death rates of married males resp. females, and f the mating function. The harmonic mean type mating law

$$f(x, y) = 2mxy \left(\int\limits_{\tau}^{\infty} h_x\, x\,\mathrm{d}a + \int\limits_{\tau}^{\infty} h_y\, y\,\mathrm{d}b \right)^{-1} \tag{1.2}$$

is usually used. The nonnegative functions μ_x, μ_y, m, h_x, h_y, $\tilde{\mu}_x$, $\tilde{\mu}_y$, σ, β_x, β_y, x^0, y^0, p^0 and maturation age τ are assumed to be prescribed, in particular $h_x = h_y = 1$ and $\tau = \mathrm{const}$.

In ecology an individual can be characterized by age and sex. In genetics it can be done by age, sex and a genotype parameter, and by age, sex, and a disease parameter in epidemiology. We extend the list of essential parameters characterizing an individual by adding a sociological factor and consider the sociologically-structured population dynamics without spatial dispersal. More precisely, we take into account the confession (religion) factor of the interacting monogamous people communities and generalize the Hoppensteadt–Staroverov–Hadeler deterministic model (1.1), (1.2) permitting pairs divorce. We use the harmonic mean type function as the mating law. We first describe the general model of the interacting religions, which tolerate both uniconfessional pairs and those with different religions. Then we consider a model of religions tolerating only the uniconfessional pairs (the confession change is allowed only for the sake of marriage, and, for offspring, parents may choose a faith, not necessarily their own), and a model of religions not tolerating (forbidding) any confession change (the offspring may belong to a religion, not necessarily to either maternal or paternal one).

The plan for this paper is as follows. Section 3 presents the general model for interacting religion communities. Section 4 analyzes the uniconfessional pair model. This section is divided into three subsections. In 4.1 we give the symmetric uniconfessional communities model. In 4.2 we examine the case of constant vital rates for the model in 4.1. In 4.3 the large time behavior of the total numbers of single adults and pairs of each community is demonstrated. In Section 5 we present and analyze a model of the forbidden change of religions. The structure of this section is analogous to that of Section 4. The discussion in Section 6 concludes this paper.

2. Notations

$u_i^1(t,\tau_1)d\tau_1$ – the expectation at time t of the number of single (unmarried) males of age $\xi \in [\tau_1, \tau_1 + d\tau_1]$ and of the ith religion; n – the number of religions;

$u_i^2(t,\tau_2)d\tau_2$ – the expectation at time t of the number of single (unmarried) females of age $\xi \in [\tau_2, \tau_2 + d\tau_2]$ and of the ith religion;

$u_{ij}^3(t,\tau_1,\tau_2,\tau_3)d\tau_1 d\tau_2 d\tau_3$ – the expectation at time t of the number of pairs which are formed of males of age $\xi \in [\tau_1, \tau_1 + d\tau_1]$ and of the ith religion, females of age $\xi \in [\tau_2, \tau_2 + d\tau_2]$ and of the jth religion, and which have existed for a time $\xi \in [\tau_3, \tau_3 + d\tau_3]$; $(\tau_1,\tau_2,\tau_3; i, j)$ – the characteristic of this pair; $u_i^3 = u_{ii}^3$;

$\nu_i^1(t,\tau_1)dt$ – the probability that a single male of age τ_1 and of the ith religion will die in the time interval $[t, t+dt]$;

$\nu_i^2(t,\tau_2)dt$ – the probability that a single female of age τ_2 and of the ith religion will die in the time interval $[t, t+dt]$;

$\nu_{ij}^1(t,\tau_1,\tau_2,\tau_3)dt$ – the probability that a married male of a pair with the characteristic $(\tau_1,\tau_2,\tau_3; i, j)$ will die in the time interval $[t, t+dt]$; $\nu_i^{13} = \nu_{ii}^1$;

$\nu_{ij}^2(t,\tau_1,\tau_2,\tau_3)dt$ – the probability that a married female of a pair with the characteristic $(\tau_1,\tau_2,\tau_3; i, j)$ will die in the time interval $[t, t+dt]$; $\nu_i^{23} = \nu_{ii}^2$;

$\sigma_{ij}(t,\tau_1,\tau_2,\tau_3)dt$ – the probability that a pair with the characteristic $(\tau_1,\tau_2,\tau_3; i, j)$ will divorce in the time interval $[t, t+dt]$; $\sigma_i = \sigma_{ii}$;

$b^1_{ij}(t,\tau_1,\tau_2,\tau_3)dt$ – the expectation of the number of males produced in the time interval $[t,t+dt]$ by a pair with the characteristic $(\tau_1,\tau_2,\tau_3;i,j)$; $b^1_i = b^1_{ii}$;

$b^2_{ij}(t,\tau_1,\tau_2,\tau_3)dt$ – the expectation of the number of females produced in the time interval $[t,t+dt]$ by a pair with the characteristic $(\tau_1,\tau_2,\tau_3;i,j)$; $b^2_i = b^2_{ii}$;

$L^1_i(t,\tau_1)d\tau_1 dt$ – the expectation of the number of males of age $\xi \in [\tau_1,\tau_1+d\tau_1]$ and of the ith religion which will marry in the time interval $[t,t+dt]$;

$L^2_i(t,\tau_2)d\tau_2 dt$ – the expectation of the number of females of age $\xi \in [\tau_2,\tau_2+d\tau_2]$ and of the ith religion which will marry in the time interval $[t,t+dt]$;

$S^1_i(t,\tau_1)d\tau_1 dt$ – the expectation of the number of males of age $\xi \in [\tau_1,\tau_1+d\tau_1]$ and of the ith religion which will become single in the time interval $[t,t+dt]$ due to divorce of pairs and because of death of married females;

$S^2_i(t,\tau_1)d\tau_2 dt$ – the expectation of the number of females of age $\xi \in [\tau_2,\tau_2+d\tau_2]$ and of the ith religion which will become single in the time interval $[t,t+dt]$ due to divorce of pairs and because of death of married males;

$f_{sk}(t,\tau_1,\tau_2)d\tau_1 d\tau_2 dt$ – the expectation of the number of new pairs (marriages) formed in time interval $[t,t+dt]$ of males of age $\xi \in [\tau_1,\tau_1+d\tau_1]$ and of the sth religion and females of age $\xi \in [\tau_2,\tau_2+d\tau_2]$ and of the kth religion;

$h^1_i(t,\tau_1)$ – the probability that at time t a male of age τ_1 and of the ith religion wishes to marry;

$h^2_i(t,\tau_2)$ – the probability that at time t a female of age τ_2 and of the ith religion wishes to marry;

$\tilde{m}_{sk}(t,\tau_1,\tau_2)dt$ – the probability that a male of age τ_1 and of the sth religion and a female of age τ_2 and of the kth religion will marry in the time interval $[t,t+dt]$ provided that the pair characteristic $(\tau_1,\tau_2,\tau_3;s,k)$ is given; $m_{sk}(t,\tau_1,\tau_2) = \frac{1}{2}h^1_s\, h^2_k\, \tilde{m}_{sk}$;

$\omega^{ij}_{sk}(t,\tau_1,\tau_2)$ – the probability that at time t, for the sake of marriage, a male of age τ_1 and of the sth religion and a female of age τ_2 and of the kth religion will choose the ith and the jth religions, respectively; $\omega^i_{sk} = \omega^{ii}_{sk}$;

$\Omega^i_{sk}(t,\tau_1,\tau_2,\tau_3)$ – the probability that a pair of the characteristic $(\tau_1,\tau_2,\tau_3;s,k)$ will choose the ith religion for its offspring produced at time t; $\Omega^i_s = \Omega^i_{ss}$;

τ – the maturation period;

$[u^k(t,\tau_k)]$ – the jump discontinuity of u^k_i at the line $\tau_k = \tau,\ k=1,2$;

$u^{10}_i(\tau_1),\ u^{20}_i(\tau_2),\ u^{30}_{ij}(\tau_1,\tau_2,\tau_3),\ u^{30}_i(\tau_1,\tau_2,\tau_3)$ – the initial age distributions;

$Q^1 = Q^2 = Q = (0,\tau)\cup(\tau,\infty)$;

$Q^3 = (\tau,\infty)\times(\tau,\infty)\times(0,\,\min(\tau_1-\tau,\tau_2-\tau)]$,

$\overline{Q}^3 = [\tau,\infty)\times[\tau,\infty)\times[0,\,\min(\tau_1-\tau,\tau_2-\tau)]$;

D – a domain in Euclidean space E^m of dimension m, not necessarily bounded;

$\overline{D}$ – closure of D;

$C^0(\overline{D})$ – a class of bounded continuous functions in $\overline{D}$;

$C^0(D)$ – a class of bounded functions belonging to $C^0(\overline{D}')$ for any compact subdomain $\overline{D}' \subset D$;

$C^1(D)$ – a class of functions $f(x)$, $x = (x_1, \cdots, x_m)$ such that $\partial f/\partial x_i \in C^0(D)$, $i = \overline{1, m}$;

$L^1(D)$ – the Banach space of integrable on D functions $f(x)$ with norm $||f||_{L^1} = \int\limits_D |f| \mathrm{d}x$.

3. The General Model

In this section we describe the general model of religion-structured communities which tolerate both uniconfessional pairs and those with different religions. Using the balance relations, we derive the following system for nonmigrating interacting communities:

$$
\begin{aligned}
&\partial u_i^1/\partial t + \partial u_i^1/\partial \tau_1 = -\nu_i^1\, u_i^1 - L_i^1 + S_i^1, \quad t > 0,\ \ \tau_1 \in Q^1,\\
&\partial u_i^2/\partial t + \partial u_i^2/\partial \tau_2 = -\nu_i^2\, u_i^2 - L_i^2 + S_i^2, \quad t > 0,\ \ \tau_2 \in Q^2,\\
&\partial u_{ij}^3/\partial t + \sum_{k=1}^{3} \partial u_{ij}^3/\partial \tau_k = -(\nu_{ij}^1 + \nu_{ij}^2 + \sigma_{ij})\, u_{ij}^3,\\
&\qquad\qquad\qquad\qquad t > 0,\ \ (\tau_1, \tau_2, \tau_3) \in Q^3,
\end{aligned}
\tag{3.1}
$$

$$
L_i^1 = \begin{cases} 0, & \tau_1 < \tau, \\ \sum\limits_{s=1}^{n} \int\limits_{\tau}^{\infty} f_{is}\, \mathrm{d}\tau_2, & \tau_1 > \tau, \end{cases} \qquad L_i^2 = \begin{cases} 0, & \tau_2 < \tau, \\ \sum\limits_{s=1}^{n} \int\limits_{\tau}^{\infty} f_{si}\, \mathrm{d}\tau_1, & \tau_2 > \tau, \end{cases}
$$

$$
S_i^1 = \begin{cases} 0, & \tau_1 < \tau, \\ \int\limits_{0}^{\tau_1 - \tau} \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^{\infty} \sum\limits_{s=1}^{n} (\nu_{is}^2 + \sigma_{is}) u_{is}^3\, \mathrm{d}\tau_2, & \tau_1 > \tau, \end{cases}
$$

$$
S_i^2 = \begin{cases} 0, & \tau_2 < \tau, \\ \int\limits_{0}^{\tau_2 - \tau} \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^{\infty} \sum\limits_{s=1}^{n} (\nu_{si}^1 + \sigma_{si}) u_{si}^3\, \mathrm{d}\tau_1, & \tau_2 > \tau \end{cases}
$$

subject to the conditions

$$
u_i^1|_{\tau_1=0} = \int\limits_0^{\infty} \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^{\infty} \mathrm{d}\tau_1 \int\limits_{\tau_3+\tau}^{\infty} \sum_{s,j=1}^{n} u_{sj}^3\, b_{sj}^1\, \Omega_{sj}^i\, \mathrm{d}\tau_2, \quad t > 0,
$$

$$
\begin{aligned}
&u_i^2|_{\tau_2=0} = \int\limits_0^\infty \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int\limits_{\tau_3+\tau}^\infty \sum_{s,j=1}^n u_{sj}^3\, b_{sj}^2\, \Omega_{sj}^i \,\mathrm{d}\tau_2, \quad t>0, \\
&u_{ij}^3|_{\tau_3=0} = \sum_{s,k=1}^n f_{sk}\omega_{sk}^{ij}, \quad t>0,\ \ \tau_1,\tau_2 \in [\tau,\infty), \\
&u_i^1|_{t=0} = u_i^{10}, \quad \tau_1 \in [0,\infty), \\
&u_i^2|_{t=0} = u_i^{20}, \quad \tau_2 \in [0,\infty), \\
&u_{ij}^3|_{t=0} = u_{ij}^{30}, \quad (\tau_1,\tau_2,\tau_3) \in \overline{Q}^3, \\
&\big[u_i^1|_{\tau_1=\tau}\big] = [u_i^2|_{\tau_2=\tau}] = 0, \quad t>0,
\end{aligned}
\tag{3.2}
$$

where

$$
\begin{aligned}
&\sum_{i=1}^n \Omega_{sk}^i = 1 \ \text{ for all } \ s,k, \ \text{ and } \ t \geqslant 0,\ (\tau_1,\tau_2,\tau_3) \in \overline{Q}^3, \\
&\sum_{i,j=1}^n \omega_{sk}^{ij} = 1 \ \text{ for all } \ s,k, \ \text{ and } \ t \geqslant 0,\ \tau_1,\ \tau_2 \in [\tau,\infty).
\end{aligned}
$$

In addition to (3.2), we assume that u_i^{10}, u_i^{20}, u_{ij}^{30} satisfy the following compatibility conditions:

$$
\begin{aligned}
&u_i^{10}(0) = \int\limits_0^\infty \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int\limits_{\tau_3+\tau}^\infty \sum_{s,j=1}^n u_{sj}^{30}\,(b_{sj}^1\,\Omega_{sj}^i)|_{t=0}\,\mathrm{d}\tau_2, \\
&u^{20}(0) = \int\limits_0^\infty \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int\limits_{\tau_3+\tau}^\infty \sum_{s,j=1}^n u_{sj}^{30}\,(b_{sj}^2\,\Omega_{sj}^i)|_{t=0}\,\mathrm{d}\tau_2, \\
&u_{ij}^{30}|_{\tau_3=0} = \sum_{s,k=1}^n \big(f_{sk}\omega_{sk}^{ij}\big)|_{t=0}, \quad \tau_1,\tau_2 \in [\tau,\infty), \\
&[u_i^{10}(\tau)] = [u_i^{20}(\tau)] = 0.
\end{aligned}
\tag{3.3}
$$

As follows from the foregoing, the maturation time τ and the given functions ν_i^1, ν_i^2, ν_{ij}^1, ν_{ij}^2, σ_{ij}, b_{ij}^1, b_{ij}^2, f_{ij}, h_i^1, h_i^2, ω_{sk}^{ij}, Ω_{sk}^i, u_i^{10}, u_i^{20}, u_{ij}^{30} as well as the unknown ones u_i^1, u_i^2, u_{ij}^3 must be positive-valued, otherwise they have no biological significance. In what follows we use the harmonic mean type function

$$
f_{ij} = 2m_{ij}u_i^1u_j^2\bigg(\sum_{l=1}^2\sum_{s=1}^n \int\limits_\tau^\infty u_s^l\,h_s^l\,\mathrm{d}\tau_l\bigg)^{-1},
\tag{3.4}
$$

with given positive functions m_{ij} and h_s^l.

Usually, for the sake of marriage, individuals of different religions do not change their faith or choose the partner's one, and, for offspring, parents of a pair with different confessions choose one of the pair religions. Hence, we have the following most probable rules:

$$
\begin{aligned}
&\omega_{ss}^s = 1,\ \omega_{ss}^i = 0 \quad \text{if } i \neq s;\\
&\omega_{sk}^s > 0,\ \omega_{sk}^k > 0 \quad \text{if } s \neq k;\ \ \omega_{sk}^i = 0 \quad \text{if } s \neq k \text{ and } i \neq s, k;\\
&\omega_{kk}^{ij} = 0 \quad \text{if } i, j \neq k;\ \ \omega_{ik}^{ik} > 0,\ \ i \neq k;\\
&\omega_{sk}^{ij} = 0 \quad \text{if } s \neq k \text{ and } i \neq s,\ \ j \neq k,\ \ i \neq j;\\
&\Omega_s^s = 1,\ \Omega_s^i = 0 \quad \text{if } i \neq s;\ \ \Omega_{sk}^i = 0 \quad \text{if } s \neq k \text{ and } i \neq s, k;\\
&\Omega_{sk}^s > 0,\ \Omega_{sk}^k > 0 \quad \text{if } s \neq k.
\end{aligned}
\tag{3.5}
$$

By using (3.4) and (3.5), we can simplify conditions (3.2). Indeed, since

$$
\begin{aligned}
\sum_{s,k=1}^{n} m_{sk}\, u_s^1\, u_k^2 \omega_{sk}^i &= \sum_k \Big(m_{ik} u_i^1 u_k^2 \omega_{ik}^i + \sum_{s\neq i} m_{sk} u_s^1 u_k^2 \omega_{sk}^i \Big)\\
&= \sum_k m_{ik} u_i^1 u_k^2 \omega_{ik}^i + \sum_{s \neq i} \Big(m_{si} u_s^1 u_i^2 \omega_{si}^i + \sum_{k \neq i} m_{sk} u_s^1 u_k^2 \omega_{sk}^i \Big)\\
&= \sum_{s \neq i} \big(m_{is} u_i^1 u_s^2 \omega_{is}^i + m_{si} u_s^1 u_i^2 \omega_{si}^i \big) + m_{ii} u_i^1 u_i^2 \omega_{ii}^i + \sum_{s,k \neq i} m_{sk} u_s^1 u_k^2 \omega_{sk}^i,
\end{aligned}
$$

$$
\sum_{i=1}^{n} \Omega_{sk}^i = \Omega_{sk}^s + \Omega_{sk}^k + \sum_{i \neq s,k} \Omega_{sk}^i,
$$

$$
\begin{aligned}
\sum_{s,k=1}^{n} b_{sk}^\alpha u_{sk}^3 \Omega_{sk}^i &= \sum_{k=1}^{n} \Big(b_{ik}^\alpha u_{ik}^3 \Omega_{ik}^i + \sum_{s \neq i} b_{sk}^\alpha u_{sk}^3 \Omega_{sk}^i \Big)\\
&= \sum_{k=1}^{n} b_{ik}^\alpha u_{ik}^3 \Omega_{ik}^i + \sum_{s \neq i} \Big(b_{si}^\alpha u_{si}^3 \Omega_{si}^i + \sum_{k \neq i} b_{sk}^\alpha u_{sk}^3 \Omega_{sk}^i \Big)\\
&= b_{ii}^\alpha u_{ii}^3 \Omega_{ii}^i + \sum_{k \neq i} \big(b_{ik}^\alpha u_{ik}^3 \Omega_{ik}^i + b_{ki}^\alpha u_{ki}^3 \Omega_{ki}^i \big) + \sum_{k,s \neq i} b_{sk}^\alpha u_{sk}^3 \Omega_{sk}^i, \quad \alpha = 1, 2,
\end{aligned}
$$

and by (3.5), it follows that

$$
\Omega_{sk}^s + \Omega_{sk}^k = 1,
$$

$$
\sum_{s,k=1}^{n} m_{sk}\, u_s^1\, u_k^2\, \omega_{sk}^i = \sum_{s \neq i} (m_{is}\, u_i^1\, u_s^2\, \omega_{is}^i + m_{si}\, u_s^1\, u_i^2\, \omega_{si}^i) + m_{ii}\, u_i^1\, u_i^2, \tag{3.6}
$$

$$
\sum_{s,k=1}^{n} m_{sk}\, u_s^1\, u_k^2\, \omega_{sk}^{ij} = m_{ij}\, u_i^1\, u_j^2, \quad i \neq j, \tag{3.7}
$$

$$
\sum_{s,k=1}^{n} b_{sk}^\alpha\, u_{sk}^3\, \Omega_{sk}^i = \sum_{k \neq i} (b_{ik}^\alpha\, u_{ik}^3\, \Omega_{ik}^i + b_{ki}^\alpha\, u_{ki}^3\, \Omega_{ki}^i) + b_{ii}^\alpha u_{ii}^3, \quad \alpha = 1,\, 2. \tag{3.8}
$$

4. The Uniconfessional Pair Model

In this section we consider the special case of (3.1)–(3.4) where all the religions forbid the pair with different confessions of individuals, tolerate the change of religions only for the sake of marriage, and let parents to choose a religion not necessarily their own for their offspring. Then $u^3_{ij} = 0$, $\omega^{ij}_{sk} = 0$, $\nu^1_{ij} = \nu^2_{ij} = 0$, $\sigma_{ij} = 0$, $\Omega^s_{ij} = 0$ if $i \neq j$, and from (3.1)–(3.4) we obtain the following system

$$
\begin{aligned}
&\partial u^1_i/\partial t + \partial u^1_i/\partial \tau_1 = -\nu^1_i\, u^1_i - L^1_i + S^1_i, \quad t > 0,\ \ \tau_1 \in Q^1,\\
&\partial u^2_i/\partial t + \partial u^2_i/\partial \tau_2 = -\nu^2_i\, u^2_i - L^2_i + S^2_i, \quad t > 0, \quad \tau_2 \in Q^2,\\
&\partial u^3_i/\partial t + \sum_{k=1}^{3} \partial u^3_i/\partial \tau_k = -(\nu^{13}_i + \nu^{23}_i + \sigma_i)\, u^3_i, \quad t > 0,\ \ (\tau_1,\tau_2,\tau_3) \in Q^3,\\
&L^1_i = \begin{cases} 0, \quad \tau_1 < \tau,\\ 2u^1_i \sum\limits_{s=1}^{n} \int\limits_{\tau}^{\infty} m_{is}\, u^2_s\, \mathrm{d}\tau_2 \,\big(\sum\limits_{k=1}^{2}\sum\limits_{s=1}^{n}\int\limits_{\tau}^{\infty} u^k_s\, h^k_s\, \mathrm{d}\tau_k\big)^{-1}, \quad \tau_1 > \tau, \end{cases}\\
&L^2_i = \begin{cases} 0, \quad \tau_2 < \tau,\\ 2u^2_i \sum\limits_{s=1}^{n} \int\limits_{\tau}^{\infty} m_{si}\, u^1_s\, \mathrm{d}\tau_1 \,\big(\sum\limits_{k=1}^{2}\sum\limits_{s=1}^{n}\int\limits_{\tau}^{\infty} u^k_s\, h^k_s\, \mathrm{d}\tau_k\big)^{-1}, \quad \tau_2 > \tau, \end{cases}\\
&S^1_i = \begin{cases} 0, \quad \tau_1 < \tau,\\ \int\limits_{0}^{\tau_1-\tau} \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^{\infty} (\nu^{23}_i + \sigma_i) u^3_i\, \mathrm{d}\tau_2, \quad \tau_1 > \tau, \end{cases}\\
&S^2_i = \begin{cases} 0, \quad \tau_2 < \tau,\\ \int\limits_{0}^{\tau_2-\tau} \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^{\infty} (\nu^{13}_i + \sigma_i) u^3_i\, \mathrm{d}\tau_1, \quad \tau_2 > \tau \end{cases}
\end{aligned}
\tag{4.1}
$$

subject to the conditions

$$
\begin{aligned}
&u^1_i|_{\tau_1=0} = \int_0^\infty \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int_{\tau_3+\tau}^\infty \sum_{j=1}^{n} b^1_j\, u^3_j\, \Omega^i_j \mathrm{d}\tau_2, \quad t > 0,\\
&u^2_i|_{\tau_2=0} = \int_0^\infty \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int_{\tau_3+\tau}^\infty \sum_{j=1}^{n} b^2_j\, u^3_j\, \Omega^i_j \mathrm{d}\tau_2, \quad t > 0,\\
&u^3_i|_{\tau_3=0} = 2 \sum_{s,k=1}^{n} m_{sk}\, u^1_s\, u^2_k\, \omega^i_{sk} \left(\sum_{k=1}^{2}\sum_{s=1}^{n} \int_\tau^\infty u^k_s\, h^k_s\, \mathrm{d}\tau_k \right)^{-1},\\
&\qquad\qquad\qquad t > 0, \quad \tau_1, \tau_2 \in [\tau, \infty),\\
&u^1_i|_{t=0} = u^{10}_i, \quad \tau_1 \in [0,\infty),\\
&u^2_i|_{t=0} = u^{20}_i, \quad \tau_2 \in [0,\infty),\\
&u^3_i|_{t=0} = u^{30}_i,\ (\tau_1,\tau_2,\tau_3) \in \overline{Q}^3,\\
&[u^1_i|_{\tau_1=\tau}] = [u^2_i|_{\tau_2=\tau}] = 0, \quad t > 0,
\end{aligned}
\tag{4.2}
$$

where

$\sum_{i=1}^{n} \Omega_j^i = 1$ for all j and $t \geqslant 0$, $(\tau_1, \tau_2, \tau_3) \in \overline{Q}^3$,

$\sum_{i=1}^{n} \omega_{sk}^i = 1$ for all s, k, and $t \geqslant 0$, $\tau_1,\ \tau_2 \in [\tau, \infty)$.

In addition to (4.2), we assume that u_i^{10}, u_i^{20}, u_i^{30} satisfy the following compatibility conditions:

$$
\begin{aligned}
&u_i^{10}(0) = \int_0^\infty \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int_{\tau_3+\tau}^\infty \sum_{j=1}^n \big(b_j^1\, \Omega_j^i\big)|_{t=0}\, u_i^{30} \mathrm{d}\tau_2,\\
&u^{20}(0) = \int_0^\infty \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int_{\tau_3+\tau}^\infty \sum_{j=1}^n \big(b_j^2\, \Omega_j^i\big)|_{t=0} u_i^{30} \mathrm{d}\tau_2, \qquad (4.3)\\
&u_i^{30}|_{\tau_3=0} = 2 \sum_{s,k=1}^n \quad u_s^{10}\, u_k^{20}\, \big(m_{sk}\, \omega_{sk}^i\big)|_{t=0} \left(\sum_{k=1}^2 \sum_{s=1}^n \int_\tau^\infty u_s^{k0}\, h_s^k|_{t=0}\, \mathrm{d}\tau_k\right)^{-1},\\
&\qquad\qquad \tau_1,\ \tau_2 \geqslant \tau,\\
&[u_i^{10}(\tau)] = [u_i^{20}(\tau)] = 0.
\end{aligned}
$$

4.1. *The Symmetric Communities Model*

In this subsection we consider the case where uniconfessional communities are symmetric in the sense that the following functions are symmetric in ages $\tau_1,\ \tau_2$ and do not depend on sexes, i.e.:

$\nu_i^k(t,\xi) = \nu_i(t,\xi),\ h_i^k(t,\xi) = h_i(t,\xi),$

$\nu_i^{k3}(t,\tau_1,\tau_2,\tau_3) = \nu_i^3(t,\tau_1,\tau_2,\tau_3) = \nu_i^3(t,\tau_2,\tau_1,\tau_3),$

$u_i^{k0}(\xi) = u_i^0(\xi)$ for $k = 1, 2$,

$u_i^{30}(\tau_1,\tau_2,\tau_3) = u_i^{30}(\tau_2,\tau_1,\tau_3),\ \sigma_i(t,\tau_1,\tau_2,\tau_3) = \sigma_i(t,\tau_2,\tau_1,\tau_3),$

while

$m_{ij}(t,\tau_1,\tau_2,\tau_3) = m_{ji}(t,\tau_2,\tau_1,\tau_3),\ \omega_{ks}^i(t,\tau_1,\tau_2) = \omega_{sk}^i(t,\tau_2,\tau_1),$

$b_i^k(t,\tau_1,\tau_2,\tau_3) = b_i(t,\tau_1,\tau_2,\tau_3)$ and Ω_j^i are unnecessarily symmetric in $\tau_1,\ \tau_2$.

Then

$u_i^k(t,\xi) = u_i(t,\xi),\ L_i^k(t,\xi) = L_i(t,\xi),\ S_i^k(t,\xi) = S_i(t,\xi)$ for $k = 1, 2$,

$u_i^3(t,\tau_1,\tau_2,\tau_3) = u_i^3(t,\tau_2,\tau_1,\tau_3),$

and model (4.1)–(4.3) can be written as follows

$$
\begin{aligned}
&\partial u_i/\partial t + \partial u_i/\partial \tau_1 = -\nu_i\, u_i - L_i + S_i, \quad t > 0,\ \ \tau_1 \in Q,\\
&\partial u_i^3/\partial t + \sum_{k=1}^3 \partial u_i^3/\partial \tau_k = -(2\nu_i^3 + \sigma_i)\, u_i^3, \quad t > 0,\ \ (\tau_1,\tau_2,\tau_3) \in Q^3,
\end{aligned}
$$

$$L_i = \begin{cases} 0, & \tau_1 < \tau, \\ u_i \sum\limits_{s=1}^{n} \int\limits_{\tau}^{\infty} m_{is}\, u_s \,\mathrm{d}\tau_2 \,\big(\sum\limits_{s=1}^{n} \int\limits_{\tau}^{\infty} u_s\, h_s \,\mathrm{d}\xi\big)^{-1}, & \tau_1 > \tau, \end{cases} \tag{4.1.1}$$

$$S_i = \begin{cases} 0, & \tau_1 < \tau, \\ \int\limits_{0}^{\tau_1-\tau} \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^{\infty} (\nu_i^3 + \sigma_i) u_i^3 \,\mathrm{d}\tau_2, & \tau_1 > \tau, \end{cases}$$

$$u_i|_{\tau_1=0} = \int_0^\infty \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int_{\tau_3+\tau}^\infty \sum_{j=1}^n b_j\, u_j^3 \Omega_j^i \,\mathrm{d}\tau_2, \quad t > 0,$$

$$u_i^3|_{\tau_3=0} = \sum_{s,k=1}^n m_{sk}\, u_s(t,\tau_1)\, u_k(t,\tau_2)\, \omega_{sk}^i \left(\sum_{s=1}^n \int_\tau^\infty u_s\, h_s \,\mathrm{d}\xi\right)^{-1},$$

$$t > 0, \quad \tau_1, \tau_2 \geqslant \tau,$$

$$u_i^3|_{t=0} = u_i^{30}, \quad (\tau_1, \tau_2, \tau_3) \in \overline{Q}^3,$$

$$u_i|_{t=0} = u_i^0 \text{ for } \tau_1 > 0, \quad [u_i(t,\tau)] = 0 \text{ for } t > 0,$$

where

$$\sum_{i=1}^n \Omega_j^i = 1 \text{ for all } j \text{ and } t \geqslant 0, \ (\tau_1,\tau_2,\tau_3) \in \overline{Q}^3,$$

$$\sum_{i=1}^n \omega_{sk}^i = 1, \text{ for all } s,k, \text{ and } t \geqslant 0, \ \tau_1, \ \tau_2 \in [\tau, \infty).$$

Of course, we must add the following compatibility conditions:

$$u_i(0) = \int_0^\infty \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int_{\tau_3+\tau}^\infty \sum_{j=1}^n (b_j \Omega_j^i)|_{t=0}\, u_i^{30} \,\mathrm{d}\tau_2, \quad [u_i^0(\tau)] = 0,$$

$$u_i^{30}|_{\tau_3=0} = \sum_{s,k=1}^n u_s^0 u_k^0 \left(m_{sk}\omega_{sk}^i\right)|_{t=0} \left(\sum_{s=1}^n \int_\tau^\infty u_s^0 h_s|_{t=0} \,\mathrm{d}\xi\right)^{-1}, \quad \tau_1, \tau_2 \geqslant \tau.$$

4.2. *The Case of Constant Vital Rates*

In this subsection we analyze the symmetric communities model (4.1.1) in the case where $\nu_i = \nu_i^3 = \nu$, $\sigma_i = \sigma$, $m_{sk} = m$ are some positive constants and $h_i = 1$, $\omega_{is}^i = \omega_{si}^i = 1/2$ if $s \neq i$, $\omega_{ii}^i = 1$, $\omega_{sk}^i = 0$ if $s, k \neq i$. Despite of the logical requirement $\Omega_j^i = 0$ if $i \neq j$ and $\Omega_i^i = 1$ for the uniconfessional pair, we consider a more general case and let Ω_j^i be arbitrary function satisfying the condition $\sum\limits_{i=1}^n \Omega_j^i = 1$. Then from (4.1.1) we obtain the following system

$$\partial u_i/\partial t + \partial u_i/\partial \tau_1 = -\nu\, u_i - L_i + S_i, \quad t > 0, \ \tau_1 \in Q, \tag{4.2.1}$$

$$\partial u_i^3/\partial t + \sum_{k=1}^3 \partial u_i^3/\partial \tau_k = -(2\nu + \sigma)\, u_i^3, \quad t > 0, \ (\tau_1,\tau_2,\tau_3) \in Q^3, \tag{4.2.2}$$

$$L_i = \begin{cases} 0, & \tau_1 < \tau, \\ u_i\, m, & \tau_1 > \tau, \end{cases} \qquad S_i = \begin{cases} 0, & \tau_1 < \tau, \\ (\nu+\sigma) \int\limits_0^{\tau_1-\tau} \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^{\infty} u_i^3\, \mathrm{d}\tau_2, & \tau_1 > \tau, \end{cases}$$

$$u_i|_{\tau_1=0} = \int\limits_0^{\infty} \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^{\infty} \mathrm{d}\tau_1 \int\limits_{\tau_3+\tau}^{\infty} \sum_{j=1}^{n} b_j\, u_j^3 \Omega_j^i\, \mathrm{d}\tau_2, \quad [u_i(t,\tau)] = 0,\ t > 0, \tag{4.2.3}$$

$$u_i^3|_{\tau_3=0} = \frac{m}{2} \sum_{s=1}^{n} \big(u_i(t,\tau_1)\, u_s(t,\tau_2) + u_s(t,\tau_1)\, u_i(t,\tau_2)\big)$$
$$\times \left(\sum_{s=1}^{n} \int\limits_{\tau}^{\infty} u_s\, \mathrm{d}\xi \right)^{-1}, \quad t > 0, \quad \tau_1, \tau_2 \geqslant \tau, \tag{4.2.4}$$

$$u_i|_{t=0} = u_i^0, \quad \tau_1 \in [0,\infty),$$
$$u_i^3|_{t=0} = u_i^{30}, \quad (\tau_1,\tau_2,\tau_3) \in \overline{Q}^3 \tag{4.2.5}$$

subject to the compatibility conditions

$$u_i^0(0) = \int\limits_0^{\infty} \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^{\infty} \mathrm{d}\tau_1 \int\limits_{\tau_3+\tau}^{\infty} \sum_{j=1}^{n} (b_j \Omega_j^i)|_{t=0}\, u_j^{30}\, \mathrm{d}\tau_2, \quad [u_i^0(\tau)] = 0, \tag{4.2.6}$$

$$u_i^{30}|_{\tau_3=0} = \frac{m}{2} \sum_{s=1}^{n} \big(u_i^0(\tau_1)\, u_s^0(\tau_2) + u_s^0(\tau_1)\, u_i^0(\tau_2)\big) \left(\sum_{s=1}^{n} \int\limits_{\tau}^{\infty} u_s^0\, \mathrm{d}\xi \right)^{-1},$$
$$\tau_1, \tau_2 \geqslant \tau.$$

Our purpose is to solve problem (4.2.1)–(4.2.6). We do it by two steps. We first construct the formal solution and then justify it.

Now we realize the first step. Integrating (4.2.2), (4.2.4), $(4.2.5)_2$ and (4.2.1), (4.2.3), $(4.2.5)_1$ over characteristics, we find

$$u_i^3 = \begin{cases} u_i^{30}(\tau_1 - t, \tau_2 - t, \tau_3 - t) \exp\{-(2\nu+\sigma)t\}, & 0 \leqslant t \leqslant \tau_3, \\ u_i^3(t - \tau_3, \tau_1 - \tau_3, \tau_2 - \tau_3, 0) \exp\{-(2\nu+\sigma)\tau_3\}, & 0 \leqslant \tau_3 \leqslant t, \end{cases} \tag{4.2.7}$$

and

$$u_i = \begin{cases} u_i^0(\tau_1 - t) \exp\{-\nu t\}, & 0 \leqslant t \leqslant \tau_1 \leqslant \tau, \\ u_i(t - \tau_1, 0) \exp\{-\nu \tau_1\}, & 0 \leqslant \tau_1 \leqslant \min(\mathrm{t}, \tau). \end{cases} \tag{4.2.8}$$

Let

$$z_i(t,\tau_1) = \int\limits_0^{\tau_1-\tau} \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^{\infty} u_i^3\, \mathrm{d}\tau_2, \quad z_i(t,\tau) = 0,$$

$$z_i^0(\tau_1) = z_i(0,\tau_1) = \int\limits_0^{\tau_1-\tau} \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^{\infty} u_i^{30}\, \mathrm{d}\tau_2, \tag{4.2.9}$$

for $\tau_1 \geqslant \tau$, and *assume* $u_i^3 \to 0$ as $\tau_2 \to \infty$.

Then from (4.2.1) with $\tau_1 > \tau$, (4.2.2), and (4.2.4) it follows that

$$\begin{aligned}
&\partial u_i/\partial t + \partial u_i/\partial \tau_1 = -(\nu + m)u_i + (\nu + \sigma)z_i,\\
&\partial z_i/\partial t + \partial z_i/\partial \tau_1 = -(2\nu + \sigma)\, z_i + \frac{m}{2} u_i\\
&\qquad\qquad + \frac{m}{2}\sum_{s=1}^{n} u_s(t,\tau_1)\frac{\int_\tau^\infty u_i(t,\tau_2)\,\mathrm{d}\tau_2}{\sum\limits_{s=1}^{n}\int_\tau^\infty u_s(t,\xi)\,\mathrm{d}\xi},
\end{aligned} \tag{4.2.10}$$

Further, we define

$$\begin{aligned}
&U(t,\tau_1) = \sum_{i=1}^{n} u_i(t,\tau_1), \quad Z(t,\tau_1) = \sum_{i=1}^{n} z_i(t,\tau_1),\\
&Z^0(\tau_1) = \sum_{i=1}^{n} z_i^0(\tau_1), \quad U^0(\tau_1) = \sum_{i=1}^{n} u_i^0(\tau_1), \quad U(t,0) = \sum_{i=1}^{n} u_i(t,0),
\end{aligned} \tag{4.2.11}$$

and from (4.2.9), (4.2.8) find

$$\begin{aligned}
&Z(t,\tau) = 0,\\
&U(t,\tau) = U^*(t) = \begin{cases} U^0(\tau - t)\exp\{-\nu t\}, & 0 \leqslant t \leqslant \tau,\\ U(t-\tau,0)\exp\{-\nu\tau\}, & t > \tau. \end{cases}
\end{aligned} \tag{4.2.12}$$

Now we use (4.2.10)–(4.2.12) to obtain the system

$$\begin{aligned}
&\partial U/\partial t + \partial U/\partial \tau_1 = -(\nu + m)U + (\nu + \sigma)Z,\\
&U(0,\tau_1) = U^0, \quad U(t,\tau) = U^*,\\
&\partial Z/\partial t + \partial Z/\partial \tau_1 = -(2\nu + \sigma)\, Z + m\, U,\\
&Z(0,\tau_1) = Z^0, \quad Z(t,\tau) = 0,
\end{aligned} \tag{4.2.13}$$

then the equation

$$\begin{aligned}
&\partial(U+Z)/\partial t + \partial(U+Z)/\partial \tau_1 = -\nu(U+Z),\\
&(U+Z)|_{t=0} = U^0 + Z^0, \quad (U+Z)|_{\tau_1=\tau} = U^*,
\end{aligned}$$

having the following solution

$$\begin{aligned}
&U + Z = g(t,\tau_1) =\\
&= \begin{cases} \big(U^0(\tau_1 - t) + Z^0(\tau_1 - t)\big)\exp\{-\nu\, t\}, & 0 \leqslant t \leqslant \tau_1 - \tau,\\ U^0(\tau_1 - t)\exp\{-\nu\, t\}, & 0 \leqslant \tau_1 - \tau \leqslant t \leqslant \tau_1,\\ U(t-\tau_1,0)\exp\{-\nu\,\tau_1\}, & t \geqslant \tau_1. \end{cases}
\end{aligned} \tag{4.2.14}$$

This allows us to rewrite system (4.2.13) as follows

$$\begin{aligned}
&\partial U/\partial t + \partial U/\partial \tau_1 = -(2\nu + m + \sigma)U + (\nu + \sigma)g,\\
&U|_{t=0} = U^0,\ U|_{\tau_1=\tau} = U^*,\\
&\partial Z/\partial t + \partial Z/\partial \tau_1 = -(2\nu + m + \sigma)\, Z + m\, g,\\
&Z|_{t=0} = Z^0, \quad Z|_{\tau_1=\tau} = 0.
\end{aligned} \tag{4.2.15}$$

Hence, by (4.2.12) and (4.2.14), we have

$$U = \begin{cases} \gamma_{u1}(t)\, U^0(\tau_1 - t) + \gamma_{u2}(t)\, Z^0(\tau_1 - t), & 0 \leqslant t \leqslant \tau_1 - \tau,\\ \gamma_{u3}(t,\tau_1)\, U^0(\tau_1 - t), & 0 \leqslant \tau_1 - \tau \leqslant t \leqslant \tau_1,\\ \gamma_{u4}(\tau_1)\, U(t - \tau_1, 0), & t \geqslant \tau_1, \end{cases} \tag{4.2.16}$$

$$Z = \begin{cases} \gamma_{z1}(t)\, U^0(\tau_1 - t) + \gamma_{z2}(t)\, Z^0(\tau_1 - t), & 0 \leqslant t \leqslant \tau_1 - \tau,\\ \gamma_{z3}(t,\tau_1)\, U^0(\tau_1 - t), & 0 \leqslant \tau_1 - \tau \leqslant t \leqslant \tau_1,\\ \gamma_{z4}(\tau_1)\, U(t - \tau_1, 0), & t \geqslant \tau_1, \end{cases} \tag{4.2.17}$$

where

$$\begin{aligned}
\gamma_{u1}(t) &= \big\{\nu + \sigma + m \exp\{-(\nu + m + \sigma)t\}\big\} \exp\{-t\nu\}/(\nu + \sigma + m),\\
\gamma_{u2}(t) &= \frac{\nu + \sigma}{\nu + \sigma + m}\big\{1 - \exp\{-(\nu + m + \sigma)t\}\big\} \exp\{-t\nu\},\\
\gamma_{u3}(t,\tau_1) &= \big\{\nu + \sigma + m \exp\{-(\nu + m + \sigma)(\tau_1 - \tau)\}\big\}\\
&\qquad \times \exp\{-t\nu\}/(\nu + \sigma + m),\\
\gamma_{u4}(\tau_1) &= \big\{\nu + \sigma + m \exp\{-(\nu + m + \sigma)(\tau_1 - \tau)\}\big\}\\
&\qquad \times \exp\{-\nu\tau_1\}/(\nu + \sigma + m),\\
\gamma_{z1}(t) &= \big\{m + (\nu + \sigma) \exp\{-(\nu + m + \sigma)t\}\big\} \exp\{-t\nu\}/(\nu + \sigma + m),\\
\gamma_{z2}(t) &= \frac{m}{\nu + \sigma + m}\big\{1 - \exp\{-(\nu + m + \sigma)t\}\big\} \exp\{-t\nu\},\\
\gamma_{z3}(t,\tau_1) &= \frac{m}{\nu + \sigma + m}\big\{1 - \exp\{-(\nu + m + \sigma)(\tau_1 - \tau)\}\big\} \exp\{-t\nu\},\\
\gamma_{z4}(\tau_1) &= \frac{m}{\nu + m + \sigma}\big\{1 - \exp\{-(\nu + m + \sigma)(\tau_1 - \tau)\}\big\} \exp\{-\nu\tau_1\}.
\end{aligned}$$

Observe that, even though U^0, Z^0, and $U(t,0)$ possess continuous derivatives, functions $U(t,\tau_1)$, $Z(t,\tau_1)$, and $g(t,\tau_1)$ are not differentiable in the direction crossing the lines $t = \tau_1 - \tau$ and $t = \tau_1$. Similarly, in general case functions (4.2.7) and (4.2.8) are not differentiable at the lines $t = \tau_3$, $t = \tau_1 - \tau$, $t = \tau_1$, $t = \tau_2 - t$, $t = \tau_2$ and $t = \tau_1$, respectively.

Assume u_i^0, $z_i^0 \to 0$ as $\tau_2 \to \infty$. Then from Eqs. (4.2.16) and (4.2.17) it follows that so do U, Z for $t \in [0, \tau_1]$.

Now define

$$\varphi_i(t) = \int_\tau^\infty u_i \,\mathrm{d}\tau_1, \quad \psi_i(t) = \int_\tau^\infty z_i \,\mathrm{d}\tau_1,$$
$$\varphi_i^0 = \varphi_i(0) = \int_\tau^\infty u_i^0 \,\mathrm{d}\tau_1, \quad \psi_i^0 = \psi_i(0) = \int_\tau^\infty z_i^0 \,\mathrm{d}\tau_1, \tag{4.2.18}$$

and *assume* $u_i,\, z_i \to 0$ as $\tau_1 \to \infty$. From Eqs. (4.2.10) we get the system

$$\begin{aligned} d\varphi_i/dt &= -(\nu + m)\varphi_i + (\nu + \sigma)\psi_i + u_i(t,\tau),\\ d\psi_i/dt &= -(2\nu + \sigma)\psi_i + m\varphi_i, \end{aligned} \tag{4.2.19}$$

then the equation

$$d(\varphi_i + \psi_i)/dt = -\nu(\varphi_i + \psi_i) + u_i(t,\tau),$$

having the solution

$$\varphi_i + \psi_i = g_i(t),$$
$$g_i(t) = (\varphi_i^0 + \psi_i^0)\ \exp\{-\nu\, t\} + \int_0^t \exp\{-\nu\,(t-\xi)\}\, u_i(\xi,\tau)\,\mathrm{d}\xi,$$

which allows us to rewrite system (4.2.19) as follows

$$\begin{aligned} d\varphi_i/dt &= -(2\nu + m + \sigma)\varphi_i + (\nu + \sigma)g_i + u_i(t,\tau), \quad \varphi(0) = \varphi_i^0,\\ d\psi_i/dt &= -(2\nu + m + \sigma)\psi_i + m g_i, \quad \psi_i^0 = \psi_i(0). \end{aligned}$$

Hence

$$\begin{aligned} \varphi_i = {}& \varphi_i^0\ \exp\{-(2\nu + m + \sigma)\, t\}\\ &+ \int_0^t \exp\{-(2\nu + m + \sigma)\,(t-\xi)\}\big\{u_i(\xi,\tau) + (\nu+\sigma)\, g_i(\xi)\big\}\,\mathrm{d}\xi, \end{aligned}$$

$$\begin{aligned} \psi_i = {}& \psi_i^0\ \exp\{-(2\nu + m + \sigma)\, t\}\\ &+ \int_0^t \exp\{-(2\nu + m + \sigma)\,(t-\xi)\}\, m\, g_i(\xi)\,\mathrm{d}\xi, \end{aligned} \tag{4.2.20}$$

where, by definition and (4.2.8) with $\tau_1 = \tau$,

$$g_i(t) = \left\{ \varphi_i^0 + \psi_i^0 + \int\limits_{\tau_t}^{\tau} u_i^0(\eta)\, d\eta \right\} \exp\{-\nu\, t\}, \quad 0 \leqslant t \leqslant \tau,$$

$$g_i(t) = \tilde{g}_i(t) + \exp\{-\nu\, t\} \int\limits_0^{t-\tau} \exp\{\nu\, \eta\}\, u_i(\eta, 0)\, d\eta, \quad t \geqslant \tau$$

with

$$\tilde{g}_i(t) = \left\{ \varphi_i^0 + \psi_i^0 + \int\limits_0^{\tau} u_i^0(\eta)\, d\eta \right\} \exp\{-\nu\, t\}.$$

This and (4.2.20) with $(4.2.8)_1$ show that, for $t \leqslant \tau$, φ_i and ψ_i do not depend on $u_i(t, 0)$ and are known, while, for $t > \tau$, φ_i and ψ_i can be written in the form

$$\varphi_i = \tilde{\varphi}_i(t) + \frac{\nu + \sigma}{\nu + m + \sigma} \exp\{-\nu t\} \int\limits_0^{t-\tau} \exp\{\nu \eta\} u_i(\eta, 0)\, d\eta$$

$$+ \frac{m}{\nu + m + \sigma} \int\limits_0^{t-\tau} \exp\{-(2\nu + m + \sigma)(t - \tau - \eta) - \nu \tau\} u_i(\eta, 0)\, d\eta, \qquad (4.2.21)$$

$$\psi_i = \tilde{\psi}_i(t) + \frac{m}{\nu + m + \sigma}$$

$$\times \int\limits_0^{t-\tau} \exp\{-\nu(t - \eta)\} \big\{1 - \exp\{-(\nu + m + \sigma)(t - \tau - \eta)\}\big\} u_i(\eta, 0) d\eta,$$

where

$$\begin{aligned} \tilde{\varphi}_i(t) \;=\; & \varphi_i^0 \exp\{-(2\nu + m + \sigma)\, t\} \\ & + \int\limits_0^{\tau} \exp\{-(2\nu + m + \sigma)\, (t - \xi)\} \\ & \qquad \times \big\{ u_i^0(\tau - \xi) \exp\{-\nu\, \xi\} + (\nu + \sigma) g_i(\xi) \big\}\, \mathrm{d}\xi \\ & + (\nu + \sigma) \int\limits_{\tau}^{t} \exp\{-(2\nu + m + \sigma)\, (t - \xi)\}\, \tilde{g}_i(\xi)\, \mathrm{d}\xi, \end{aligned}$$

$$\tilde{\psi}_i(t) \;=\; \psi_i^0 \exp\{-(2\nu + m + \sigma)\, t\}$$

$$+m\int_0^\tau \exp\{-(2\nu+m+\sigma)\,(t-\xi)\}\,g_i(\xi)\,\mathrm{d}\xi$$
$$+m\int_\tau^t \exp\{-(2\nu+m+\sigma)\,(t-\xi)\}\,\tilde{g}_i(\xi)\,\mathrm{d}\xi.$$

Now define

$$\alpha=\big(2(\nu+\sigma)\big)^{-1}\big\{\nu+\sigma-m+\big((\nu+\sigma)^2+m^2\big)^{1/2}\big\},\quad \alpha\in(1/2,1),$$
$$q_1=\nu+(\sigma+\nu)(1-\alpha),$$
$$\kappa_i(t)=(m/2)\varphi_i(t)/\sum_{s=1}^n\varphi_i(t),$$

and consider system (4.2.10) from which and by (4.2.8), (4.2.9) we get the equation

$$\partial(\alpha\,u_i+z_i)/\partial t+\partial(\alpha\,u_i+z_i)/\partial\tau_1=-q_1\,(\alpha\,u_i+z_i)+U\,\kappa_i$$

subject to the following conditions

$$(\alpha\,u_i+z_i)|_{t=0}=\alpha\,u_i^0+z_i^0,$$
$$(\alpha\,u_i+z_i)|_{\tau_1=\tau}=\alpha\begin{cases}u_i^0(\tau-t)\exp\{-\nu t\}, & 0\leqslant t\leqslant\tau,\\ u_i(t-\tau,0)\exp\{-\nu\tau\}, & t>\tau.\end{cases}$$

Hence

$$\alpha\,u_i+z_i=G_i(t,\tau_1), \tag{4.2.22}$$

with

$$G_i=\begin{cases}\beta_{i1}(t)\,u_i^0(\tau_1-t)+\beta_{i2}(t)\,z_i^0(\tau_1-t)+\beta_{i3}(t)U^0(\tau_1-t)\\ \qquad+\beta_{i4}(t)\,Z^0(\tau_1-t),\quad 0\leqslant t\leqslant\tau_1-\tau,\\ \beta_{i5}(t,\tau_1)\,u_i^0(\tau_1-t)+\beta_{i6}(t,\tau_1)\,U^0(\tau_1-t),\\ \qquad\qquad 0\leqslant\tau_1-\tau\leqslant t\leqslant\tau_1,\\ \beta_{i7}(\tau_1)\,u_i(t-\tau_1,0)+\beta_{i8}(t,\tau_1)\,U(t-\tau_1,0),\quad t\geqslant\tau_1,\end{cases} \tag{4.2.23}$$

where

$$\beta_{i1}(t)=\alpha\exp\{-q_1t\},\quad \beta_{i2}(t)=\exp\{-q_1t\},$$
$$\beta_{i3}(t)=\int_0^t\exp\{-q_1(t-y)\}\kappa_i(y)\gamma_{u1}(y)dy,$$

$$\beta_{i4}(t) = \int_0^t \exp\{-q_1(t-y)\}\kappa_i(y)\gamma_{u2}(y)dy,$$
$$\beta_{i5}(t,\tau_1) = \alpha\exp\{-\nu t - (q_1-\nu)(\tau_1-\tau)\},$$
$$\beta_{i6}(t,\tau_1) = \int_{t+\tau-\tau_1}^t \exp\{-q_1(t-y)\}\kappa_i(y)\gamma_{u3}(y,y+\tau_1-t)dy,$$
$$\beta_{i7}(\tau_1) = \alpha\exp\{-\nu\tau - q_1(\tau_1-\tau)\},$$
$$\beta_{i8}(t,\tau_1) = \int_{t+\tau-\tau_1}^t \exp\{-q_1(t-y)\}\kappa_i(y)\gamma_{u4}(y+\tau_1-t)dy.$$

Combining $(4.2.10)_1$ and (4.2.22), we derive the equation

$$\partial u_i/\partial t + \partial u_i/\partial\tau_1 = -q\,u_i + (\nu+\sigma)G_i, \quad q = \nu + m + \alpha(\nu+\sigma),$$

which supplemented with the conditions

$$u_i|_{t=0} = u_i^0, \quad u_i|_{\tau_1=\tau} = \begin{cases} u_i^0(\tau-t)\exp\{-\nu t\}, & 0 \leqslant t \leqslant \tau, \\ u_i(t-\tau,0)\exp\{-\nu\tau\}, & t > \tau, \end{cases}$$

has the following solution

$$u_i = \begin{cases} \rho_{i1}(t)\,u_i^0(\tau_1-t) + \rho_{i2}(t)\,z_i^0(\tau_1-t) + \rho_{i3}(t)\,U^0(\tau_1-t) \\ \quad +\rho_{i4}(t)\,Z^0(\tau_1-t), \quad 0 \leqslant t \leqslant \tau_1-\tau, \\ \rho_{i5}(t,\tau_1)\,u_i^0(\tau_1-t) + \rho_{i6}(t,\tau_1)\,U^0(\tau_1-t), \\ \qquad 0 \leqslant \tau_1-\tau \leqslant t \leqslant \tau_1, \\ \rho_{i7}(\tau_1)\,u_i(t-\tau_1,0) + \rho_{i8}(t,\tau_1)\,U(t-\tau_1,0), \quad t \geqslant \tau_1, \end{cases} \tag{4.2.24}$$

where

$$\rho_{i1}(t) = \exp\{-qt\} + (\nu+\sigma)\int_0^t \exp\{-q(t-y)\}\beta_{i1}(y)dy,$$
$$\rho_{is}(t) = (\nu+\sigma)\int_0^t \exp\{-q(t-y)\}\beta_{is}(y)dy, \quad s = 2,3,4,$$
$$\rho_{i5}(t,\tau_1) = \exp\{-qt - (q-\nu)(\tau_1-\tau)\}$$
$$\qquad + (\nu+\sigma)\int_{t+\tau-\tau_1}^t \exp\{-q(t-y)\}\beta_{i5}(y,y+\tau_1-t)dy,$$
$$\rho_{i6}(t,\tau_1) = (\nu+\sigma)\int_{t+\tau-\tau_1}^t \exp\{-q(t-y)\}\beta_{i6}(y,y+\tau_1-t)dy,$$

$$\rho_{i7}(\tau_1) = \exp\{-\nu\tau - q(\tau_1 - \tau)\} + (\nu + \sigma) \int\limits_{\tau}^{\tau_1} \exp\{-q(\tau_1 - y)\}\beta_{i7}(y)dy,$$

$$\rho_{i8}(t, \tau_1) = (\nu + \sigma) \int\limits_{t+\tau-\tau_1}^{t} \exp\{-q(t - y)\}\beta_{i8}(y, y + \tau_1 - t)dy.$$

Now we can consider condition (4.2.3). Using (4.2.7), we obtain

$$\begin{aligned} u_i(t,0) \;&=\; \int\limits_0^t \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^{\infty} \mathrm{d}\tau_1 \int\limits_{\tau_3+\tau}^{\infty} \sum_{j=1}^{n} b_j\, u_j^3 \Omega_j^i\, \mathrm{d}\tau_2 \\ &\quad + \int\limits_t^{\infty} \mathrm{d}\tau_3 \int\limits_{\tau_3+\tau}^{\infty} \mathrm{d}\tau_1 \int\limits_{\tau_3+\tau}^{\infty} \sum_{j=1}^{n} b_j\, u_j^3 \Omega_j^i\, \mathrm{d}\tau_2 \\ &= \int\limits_0^t \exp\{-(2\nu+\sigma)(t-\rho)\}\, H_i(t,\rho)\, d\rho + f_i(t), \end{aligned} \tag{4.2.25}$$

where

$$\begin{aligned} f_i(t) &= \exp\{-(2\nu+\sigma)\, t\} \\ &\quad \times \int\limits_0^{\infty} d\rho_1 \int\limits_{\rho_1+\tau}^{\infty} \mathrm{d}\xi \int\limits_{\rho_1+\tau}^{\infty} \sum_{j=1}^{n} u_j^{30}(\xi,\eta,\rho_1)\,(b_j\Omega_j^i)|_{(t,\xi+t,\eta+t,\rho_1+t)}\, d\eta, \end{aligned}$$

$$H_i(t,\rho) = \int\limits_{\tau}^{\infty} \mathrm{d}\tau_1 \int\limits_{\tau}^{\infty} \sum_{j=1}^{n} (b_j\Omega_j^i)|_{(t,t+\tau_1-\rho,t+\tau_2-\rho,t-\rho)} u_j^3(\rho,\tau_1,\tau_2,0)\mathrm{d}\tau_2,$$

$0 \leqslant \rho \leqslant t$, with $u_i^3(t,\tau_1,\tau_2,0)$ defined by (4.2.4) and (4.2.24) or,

$$H_i(t,\rho) = \sum_{s,k=1}^{2} \int\limits_{I_s} \mathrm{d}\tau_1 \int\limits_{I_k} \sum_{j=1}^{n} (b_j\Omega_j^i)|_{(t,t+\tau_1-\rho,t+\tau_2-\rho,t-\rho)} u_j^3(\rho,\tau_1,\tau_2,0)\mathrm{d}\tau_2,$$

$0 \leqslant \rho \leqslant \min(\mathrm{t},\tau)$, with $I_1 = [\tau, \tau+\rho]$, $I_2 = [\tau+\rho, \infty)$, and

$$H_i(t,\rho) = \sum_{s,k=1}^{3} \int\limits_{I_s} \mathrm{d}\tau_1 \int\limits_{I_k} \sum_{j=1}^{n} (b_j\Omega_j^i)|_{(t,t+\tau_1-\rho,t+\tau_2-\rho,t-\rho)} u_j^3(\rho,\tau_1,\tau_2,0)\mathrm{d}\tau_2,$$

$t \geqslant \rho > \tau$, with $I_1 = [\tau, \rho]$, $I_2 = [\rho, \rho+\tau]$, $I_3 = [\rho+\tau, \infty)$.

Now we examine Eqs. (4.2.25), (4.2.4), (4.2.24), (4.2.23), (4.2.21), (4.2.20), (4.2.16) going along the t axis by the step τ.

Let $t \in [0, \tau]$. Since, by $(4.2.16)_{1,2}$ and (4.2.20), $U(t, \tau_1)$ and $\varphi_i(t)$, $\psi_i(t)$ (and, hence, $\kappa_i(t)$) are known, $G_i(t, \tau_1)$ is also known by (4.2.23). Then, using $(4.2.24)_{1,2}$, we find $u_i(t, \tau_1)$, after then, by (4.2.4), we construct $u_i^3|_{\tau_3=0}$, and, by (4.2.25), obtain $u_i(t, 0)$. Finally, from $(4.2.11)_5$ we find $U(t, 0)$.

Let $t \in (\tau, 2\tau]$. From Eq. $(4.2.16)_3$ we obtain $U(t, \tau_1)$, then, by (4.2.21), we find φ_i and ψ_i. Now, by virtue of (4.2.23), we can construct $G_i(t, \tau_1)$, then, using (4.2.24), obtain $u_i(t, \tau_1)$, after then, by (4.2.4), construct $u_i^3|_{\tau_3=0}$, and, by (4.2.25), we get $u_i(t, 0)$. Finally, by $(4.2.11)_5$, we find $U(t, 0)$.

Repeating this argument we can construct the formal solution of problem (4.2.1)–(4.2.6) for any finite t.

It remains to justify the formal analysis. Assume that:

(H.4.1) $b_i,\ \Omega_j^i \in C^1\big([0,\infty) \times \overline{Q}^3\big)$ are positive,

(H.4.2) $u_i^0 \in C^1([0,\infty)) \cap L^1((0,\infty))$ is positive,

(H.4.3) $u_i^{30} \in C^1(\overline{Q}^3) \cap L^1(Q^3)$ is positive,

(H.4.4) $\int\limits_\tau^\infty \Big|\dfrac{\partial}{\partial \tau_s} u_i^{30}(\tau_1, \tau_2, \tau_3)\Big| \mathrm{d}\tau_2,\ \ s = 1, 3$ converges uniformly with respect to all $\tau_1,\ \tau_3$,

(H.4.5) $u_i^0,\ u_i^{30}$ satisfy compatibility conditions (4.2.6).

Then, for any finite $T > 0$, Eqs. (4.2.7), (4.2.8), (4.2.16), (4.2.20)– (4.2.25) show that:

$$
\begin{aligned}
& u_i(t,\tau_1) \in C^0\big([0,T]\times[0,\infty)\big) \cap C^1\big([0,T]\times\big([0,\infty)\setminus\{\tau,t,t+\tau\}\big)\big) \ \text{ and} \\
& u_i(t,\cdot),\ \partial u_i(t,\cdot)/\partial t \in L^1(0,\infty), && (4.2.26) \\
& z_i(t,\tau_1) \in C^0\big([0,T]\times[\tau,\infty)\big) \cap C^1([0,T]\times\big([\tau,\infty)\setminus\{t,t+\tau\}\big)\big) \ \text{ and} \\
& z_i(t,\cdot),\ \partial z_i(t,\cdot)/\partial t \in L^1(0,\infty), && (4.2.27) \\
& u_i^3(t,\tau_1,\tau_2,\tau_3) \in C^0\big([0,T]\times\overline{Q}^3\big) \cap C^1\big([0,T]\times\big([\tau,\infty)\setminus\{t,t+\tau\}\big) \\
& \quad \times\big([\tau,\infty)\setminus\{t,t+\tau\}\big)\times\big((0,\ \min(\tau_1-\tau,\tau_2-\tau)]\setminus\{\mathrm{t}\}\big)\big) \ \text{ and} \\
& u_i^3(t,\cdot),\ \partial u_i^3(t,\cdot)/\partial t \in L^1(Q^3), && (4.2.28) \\
& \varphi_i,\ \psi_i \in C^0([0,T])) \cap C^1((0,T)). && (4.2.29)
\end{aligned}
$$

All partial derivatives of u_i and u_i^3 have finite jumps discontinuity at the planes $\tau_1 = t,\ t+\tau$ and $\tau_1 = t,\ t+\tau,\ \tau_2 = t, t+\tau,\ \tau_3 = t$, respectively, while u_i and u_i^3 are continuous because of the hypothesis (H.4.5). Hypotheses (H.4.2) and (H.4.3) ensure the extinction of u_i, z_i, and u_i^3 at infinity for any finite t.

Now we have to justify derivation of Eq. $(4.2.10)_2$. Using (H.4.2), we have

$$
-u_j^0(\tau) + \int\limits_{t+\tau}^{\infty} \frac{\partial}{\partial t} u_j^0(\tau_2 - t)\mathrm{d}\tau_2 = -u_j^0(\tau) - \int\limits_{t+\tau}^{\infty} \frac{\partial}{\partial \tau_2} u_j^0(\tau_2 - t)\mathrm{d}\tau_2 = 0.
$$

On the other hand $\frac{\partial}{\partial t}\int\limits_{t+\tau}^{\infty} u_j^0(\tau_2-t)\mathrm{d}\tau_2 = \frac{\partial}{\partial t}\int\limits_{\tau}^{\infty} u_j^0(x)\mathrm{d}x = 0$. Hence

$$\frac{\partial}{\partial t}\int\limits_{t+\tau}^{\infty} u_j^0(\tau_2-t)\mathrm{d}\tau_2 = -u_j^0(\tau) + \int\limits_{t+\tau}^{\infty}\frac{\partial}{\partial t}u_j^0(\tau_2-t)\mathrm{d}\tau_2. \tag{4.2.30}$$

Then, by virtue of (4.2.26),

$$\begin{aligned}
&\frac{\partial}{\partial t}\int\limits_{\tau_3+\tau}^{\infty} u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2 = \frac{\partial}{\partial t}\int\limits_{t+\tau}^{\infty} u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2\\
&\quad+\begin{cases}\frac{\partial}{\partial t}\int\limits_{\tau_3+\tau}^{t+\tau} u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2, & t<\tau,\\ \frac{\partial}{\partial t}\Big\{\int\limits_{\tau_3+\tau}^{t} u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2 + \int\limits_{t}^{t+\tau} u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2\Big\}, & \tau\leqslant t,\end{cases}\\
&= u_j(t-\tau_3,t+\tau-\tau_3) + \frac{\partial}{\partial t}\int\limits_{t+\tau}^{\infty} u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2\\
&\quad+\int\limits_{\tau_3+\tau}^{t+\tau}\frac{\partial}{\partial t}u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2.
\end{aligned} \tag{4.2.31}$$

Letting $u_j(t,\tau_2) = \sum\limits_{s=1}^{4}\rho_{js}\tilde{u}_{js}^0(\tau_2-t)$ with $\tilde{u}_{j1}^0(\tau_2) = u_j^0(\tau_1)$, $\tilde{u}_{j2}^0(\tau_2) = z_j^0(\tau_2)$, $\tilde{u}_{j3}^0(\tau_2) = U^0(\tau_2)$, $\tilde{u}_{j4}^0(\tau_2) = Z^0(\tau_2)$ and using (4.2.24), (4.2.30), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\int\limits_{t+\tau}^{\infty} u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2\\
&= \sum_{s=1}^{4}\Big\{\int\limits_{t+\tau}^{\infty}\tilde{u}_{js}^0(\tau_2-t)\mathrm{d}\tau_2\frac{\partial}{\partial t}\rho_{js} + \rho_{js}\frac{\partial}{\partial t}\int\limits_{t+\tau}^{\infty}\tilde{u}_{js}^0(\tau_2-t)\mathrm{d}\tau_2\Big\}\\
&= \sum_{s=1}^{4}\Big\{\int\limits_{t+\tau}^{\infty}\tilde{u}_{js}^0(\tau_2-t)\mathrm{d}\tau_2\frac{\partial}{\partial t}\rho_{js} + \rho_{js}\Big\{-\tilde{u}_{js}^0(\tau) + \int\limits_{\tau+t}^{\infty}\frac{\partial}{\partial t}\tilde{u}_{js}^0(\tau_2-t)\mathrm{d}\tau_2\Big\}\Big\}\\
&= \int\limits_{t+\tau}^{\infty}\frac{\partial}{\partial t}u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2 - u_j(t-\tau_3,t+\tau-\tau_3).
\end{aligned}$$

This together with (4.2.31) yields

$$\frac{\partial}{\partial t}\int\limits_{\tau_3+\tau}^{\infty} u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2 = \int\limits_{\tau_3+\tau}^{\infty}\frac{\partial}{\partial t}u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2,$$

and, similarly,

$$\begin{aligned}\frac{\partial}{\partial \tau_3}&\int\limits_{\tau_3+\tau}^{\infty} u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2\\ &= -u_j(t-\tau_3,\tau)+\int\limits_{\tau_3+\tau}^{\infty}\frac{\partial}{\partial \tau_3}u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2\\ &= \int\limits_{\tau_3+\tau}^{\infty}\frac{\partial}{\partial \tau_2}u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2+\int\limits_{\tau_3+\tau}^{\infty}\frac{\partial}{\partial \tau_3}u_j(t-\tau_3,\tau_2-\tau_3)\mathrm{d}\tau_2.\end{aligned}$$

Then for $\tau_3 < t$, by $(4.2.7)_2$, it follows that

$$\begin{aligned}\frac{\partial}{\partial t}&\int\limits_{\tau_3+\tau}^{\infty} u_i^3\mathrm{d}\tau_2+\frac{\partial}{\partial \tau_1}\int\limits_{\tau_3+\tau}^{\infty} u_i^3\mathrm{d}\tau_2+\frac{\partial}{\partial \tau_3}\int\limits_{\tau_3+\tau}^{\infty} u_i^3\mathrm{d}\tau_2\\ &= \int\limits_{\tau_3+\tau}^{\infty}\{\frac{\partial}{\partial t}u_i^3+\frac{\partial}{\partial \tau_1}u_{ij}^3+\frac{\partial}{\partial \tau_2}u_i^3+\frac{\partial}{\partial \tau_3}u_i^3\}\mathrm{d}\tau_2. \qquad (4.2.32)\end{aligned}$$

Similarly, by using hypotheses (H.4.3), (H.4.4), we can obtain the same equality for $t < \tau_3$. It is easy to see that each term in the left hand side of Eq. (4.2.32), being a function of variable (t,τ_1,τ_3), belong to $C^0\big((0,T]\times\big((\tau,\infty)\setminus\{t,t+\tau\}\big)\times\big((0,\,\min(\tau_1-\tau,\tau_2-\tau)]\setminus\{\mathrm{t}\}\big)\big)$, and has a bounded jump discontinuity at the lines $\tau_1 = t,\, t+\tau, \quad \tau_3 = t$. We also observe that

$$\begin{aligned}z_i(t,\tau_1) &= \int\limits_0^t \Big\{\frac{m}{2}u_i(t-\tau_3,\tau_1-\tau_3)+U(t-\tau_3,\tau_1-\tau_3)\kappa_i(t-\tau_3)\Big\}\\ &\qquad\times\exp\{-(2\nu+\sigma)\tau_3\}\,\mathrm{d}\tau_3\\ &\quad+\int\limits_t^{\tau_1-\tau}\mathrm{d}\tau_3\int\limits_{\tau_3+\tau}^{\infty}u_i^{30}(\tau_1-t,\tau_2-t,\tau_3-t)\mathrm{d}\tau_2\\ &\qquad\times\exp\{-(2\nu+\sigma)t\},\quad t\leqslant\tau_1-\tau,\\ z_i(t,\tau_1) &= \int\limits_0^{\tau_1-\tau}\Big\{\frac{m}{2}\,u_i(t-\tau_3,\tau_1-\tau_3)+U(t-\tau_3,\tau_1-\tau_3)\kappa_i(t-\tau_3)\Big\}\\ &\qquad\times\exp\{-(2\nu+\sigma)\tau_3\}\mathrm{d}\tau_3,\quad t>\tau_1-\tau.\end{aligned}$$

Let $0 < t \leqslant \tau_1 - \tau$. Then

$$\begin{aligned}
&\frac{\partial}{\partial t} z_i + \frac{\partial}{\partial \tau_1} z_i - m u_i/2 - U\kappa_i \\
&= \frac{\partial}{\partial t}\Big(\int_0^t \mathrm{d}\tau_3 \int_{\tau_3+\tau}^{\infty} u_i^3 \mathrm{d}\tau_2 + \int_t^{\tau_1-\tau} \mathrm{d}\tau_3 \int_{\tau_3+\tau}^{\infty} u_i^3 \mathrm{d}\tau_2\Big) \\
&\quad + \frac{\partial}{\partial \tau_1}\Big(\int_0^t \mathrm{d}\tau_3 \int_{\tau_3+\tau}^{\infty} u_i^3 \mathrm{d}\tau_2 + \int_t^{\tau_1-\tau} \mathrm{d}\tau_3 \int_{\tau_3+\tau}^{\infty} u_i^3 \mathrm{d}\tau_2\Big) - m u_i/2 - U\kappa_i \\
&= \int_0^{\tau_1-\tau} \Big(\frac{\partial}{\partial t} \int_{\tau_3+\tau}^{\infty} u_i^3 \mathrm{d}\tau_2 + \frac{\partial}{\partial \tau_1} \int_{\tau_3+\tau}^{\infty} u_i^3 \mathrm{d}\tau_2\Big)\mathrm{d}\tau_3 \\
&\quad + \Big(\int_{\tau_3+\tau}^{\infty} u_i^3 \mathrm{d}\tau_2\Big)\Big|_{\tau_3=\tau_1-\tau} - \Big(\int_{\tau_3+\tau}^{\infty} u_i^3 \mathrm{d}\tau_2\Big)\Big|_{\tau_3=0} \\
&= \int_0^{\tau_1-\tau} \Big(\frac{\partial}{\partial t} \int_{\tau_3+\tau}^{\infty} u_i^3 \mathrm{d}\tau_2 + \frac{\partial}{\partial \tau_1} \int_{\tau_3+\tau}^{\infty} u_i^3 \mathrm{d}\tau_2 + \frac{\partial}{\partial \tau_3} \int_{\tau_3+\tau}^{\infty} u_i^3 \mathrm{d}\tau_2\Big)\mathrm{d}\tau_3.
\end{aligned}$$

This together with (4.2.32) show that

$$\begin{aligned}
&\frac{\partial}{\partial t} z_i + \frac{\partial}{\partial \tau_1} z_i - m u_i/2 - U\kappa_i \\
&= \int_0^{\tau_1-\tau} \mathrm{d}\tau_3 \int_{\tau_3+\tau}^{\infty} \big(\frac{\partial}{\partial t} u_i^3 + \frac{\partial}{\partial \tau_1} u_{ij}^3 + \frac{\partial}{\partial \tau_2} u_i^3 + \frac{\partial}{\partial \tau_3} u_i^3\big)\mathrm{d}\tau_2.
\end{aligned}$$

Similarly, we can prove this formula for $\max(\tau, \mathrm{t}) < \tau_1 < \mathrm{t} + \tau$ and $\tau < \tau_1 < t$. This ends the proof of the derivation of Eqs. $(4.2.10)_2$. As a result we have proved

Theorem. *Assume hypotheses* (H.4.1)–(H.4.5) *hold and let* m, ν, *and* σ *be some positive constants. Then, for any* $T > 0$, *problem* (4.2.1)–(4.2.5) *has a unique positive solution satisfying* (4.2.26)–(4.2.29).

4.3. *The Longtime Behavior of* φ_i *and* ψ_i

In this subsection we obtain the upper estimate of φ_i and ψ_i determined by (4.2.18)–(4.2.20) and (4.2.25) with bounded b, and in the case of constant b and $\Omega_i^i = 1$, $\Omega_j^i = 0$ if $j \neq i$ demonstrate the asymptotic behavior of φ_i and ψ_i as time goes to infinity.

We first consider a more general case of vital parameters. Assume $h_s = 1$ and let $\nu_i^3 = \nu_i$, m_{sk}, ω_{sk}^i, b_i, σ_i, Ω_j^i be some positive constants. Then from $(4.1.1)_{1,2,5,6}$, analogously to (4.2.19), we derive the formal system

$$\begin{aligned}
&\psi_i' = -(2\nu_i+\sigma_i)\psi_i + \sum_{s,k=1}^{n} m_{sk}\omega_{sk}^i\varphi_s\varphi_k \Big/ \sum_{s=1}^{n}\varphi_s,\\
&\varphi_i' = u_i(t,\tau) - \varphi_i\left(\nu_i + \sum_{s=1}^{n} m_{is}\varphi_s \Big/ \sum_{s=1}^{n}\varphi_s\right) + (\nu_i+\sigma_i)\psi_i,\\
&u_i(t,\tau) = \begin{cases} u_i^0(\tau-t)\exp\{-\nu_i t\}, & 0\leqslant t\leqslant \tau,\\ \sum\limits_{j=1}^{n} b_j\Omega_j^i\psi_j(t-\tau)\exp\{-\nu_i\tau\}, & \tau\leqslant t,\end{cases}
\end{aligned} \tag{4.3.1}$$

where the prime indicates differentiation. Assume φ_i^0, $\psi_i^0 > 0$ and let $0 \leqslant u_i^0(\tau_1) \in C([0,\tau])$. Then conditions of the existence and uniqueness theorem are satisfied for this system. Hence, Eqs. (4.3.1) have a local solution which, as it is easy to see, is positive. If we replace the nonlinear terms in $(4.3.1)_1$ and $(4.3.1)_2$ by 0 and $\varphi_i \sum\limits_s m_{is}$, respectivly, we obtain a linear minorant of system (4.3.1). Analogously, replacing these terms by $\sum\limits_{s,k} m_{sk}\omega_{sk}^i\varphi_k$ and 0, respectively, we obtain a linear majorant of (4.3.1). Hence, for any finite time t, system (4.3.1) has a positive solution.

Now we consider two special cases of system (4.3.1). Assume $m_{ij} = m$, then

$$\begin{aligned}
m^{-1}\sum_{s,k} m_{sk}\omega_{sk}^i\varphi_s\varphi_k &= \sum_{s,k}\omega_{sk}^i\varphi_s\varphi_k\\
&= \omega_{ii}^i\varphi_i^2 + 2\sum_{k\neq i}\omega_{ik}^i\varphi_i\varphi_k + \sum_{s,k\neq i}\omega_{s,k}^i\varphi_s\varphi_k.
\end{aligned}$$

Hence, in the case where $\omega_{sk}^i = 0$ if $s,k\neq i$, $2\omega_{ik}^i = 2\omega_{ki}^i = 1$ if $k\neq i$, and $\omega_{ii}^i = 1$, we have $\sum\limits_{s,k}\omega_{sk}^i\varphi_s\varphi_k = \varphi_i\sum\limits_k\varphi_k$. If, in addition, $\Omega_j^i = 0$ when $i\neq j$, and $\Omega_i^i = 1$, then Eqs. (4.3.1) become

$$\begin{aligned}
&\psi_i' = -(2\nu_i+\sigma_i)\psi_i + m\varphi_i,\\
&\varphi_i' = u_i(t,\tau) - (\nu_i+m)\varphi_i + (\nu_i+\sigma_i)\psi_i,\\
&u_i(t,\tau) = \begin{cases} u_i^0(\tau-t)\exp\{-\nu_i t\}, & 0\leqslant t\leqslant\tau,\\ b_i\psi_i(t-\tau)\exp\{-\nu_i\tau\}, & \tau\leqslant t.\end{cases}
\end{aligned} \tag{4.3.2}$$

In the other case when $b_i = b$, $\nu_i = \nu$, $\sigma_i = \sigma$, $m_{ij} = m$ and ω_{sk}^i, Ω_j^i are arbitrary constants from (4.3.1), for $\varphi = \sum\limits_i\varphi_i$, $\psi = \sum\limits_i\psi_i$, we derive the system

$$\begin{aligned}
&\psi' = -(2\nu+\sigma)\psi + m\varphi,\\
&\varphi' = \sum_i u_i(t,\tau) - (\nu+m)\varphi + (\nu+\sigma)\psi,\\
&\sum_i u_i(t,\tau) = \begin{cases} U^0(\tau)\exp\{-t\nu\},\ U^0(\tau) = \sum\limits_i u_i^0(\tau), & t\leqslant\tau,\\ \psi(t-\tau)b\exp\{-\tau\nu\}, & t\geqslant\tau.\end{cases}
\end{aligned} \tag{4.3.3}$$

Since the solution of system (4.3.1) is positive, $\varphi_i \leqslant \varphi$ and $\psi_i \leqslant \psi$.

Eqs. (4.3.3) are a particular case of (4.3.2). Thus, it remains to investigate system (4.3.2). It is linear and can be easily solved for $t \in (k\tau, (k+1)\tau],\ k = 0, 1, \dots$ Now we obtain the upper estimate for its solution. From (4.3.2) we find

$$(\varphi_i + \psi_i)' = u_i(t, \tau) - \nu_i(\varphi_i + \psi_i),\ (\varphi_i + \psi_i)|_{t=0} = \varphi_i^0 + \psi_i^0.$$

Hence

$$\varphi_i + \psi_i = (\varphi_i^0 + \psi_i^0)\exp\{-t\nu_i\} + \int\limits_0^t u_i(\xi, \tau)\exp\{-\nu_i(t - \xi)\}\mathrm{d}\xi$$

if $u_i^0 \in C([0, \tau])$, and, by $(4.3.2)_3$,

$$\varphi_i + \psi_i = \left(\varphi_i^0 + \psi_i^0 + \int\limits_{\tau - t}^{\tau} u_i^0(x)\mathrm{d}x\right)\exp\{-t\nu_i\}, \quad t \in [0, \tau],$$

$$\varphi_i + \psi_i = \left(\varphi_i^0 + \psi_i^0 + \int\limits_0^{\tau} u_i^0(x)\mathrm{d}x + b_i\int\limits_0^{t-\tau}\psi_i(x)\exp\{x\nu_i\}\mathrm{d}x\right) \times \exp\{-t\nu_i\}, \quad t \geqslant \tau. \tag{4.3.4}$$

Obviously, $\varphi_i,\ \psi_i$ are nonnegative. Hence $\varphi_i,\ \psi_i \leqslant \varphi_i + \psi_i$, and from (4.3.4) we derive the inequalities

$$\psi_i \leqslant c_i\exp\{-t\nu_i\}, \quad c_i = \varphi_i^0 + \psi_i^0 + \int\limits_0^{\tau} u_i^0(x)\mathrm{d}x, \quad t \in [0, \tau],$$

$$\psi_i \leqslant \exp\{-t\nu_i\}\left\{c_i + b_i\int\limits_0^{\tau}\psi_i(x)\exp\{x\nu_i\}\mathrm{d}x\right\} \leqslant \exp\{-t\nu_i\}\,c_i(1 + b_i\tau), \quad t \in (\tau, 2\tau].$$

Then, by induction, the estimate

$$\varphi_i,\ \psi_i \leqslant c_i\exp\{-t\nu_i\}(1 + b_i\tau)^k, \quad t \in (k\tau, (k+1)\tau], \quad k = 0, 1, \dots .$$

immediately follows, or more roughly

$$\varphi_i,\ \psi_i \leqslant \tilde{q}_i(t),$$
$$\tilde{q}_i = c_i\exp\{-t(\nu_i - \tau^{-1}\ln(1 + \tau b_i))\} \leqslant c_i\exp\{-t(\nu_i - b_i)\}, \tag{4.3.5}$$

and

$$\tilde{q}_i \to \begin{cases} \exp\{-t(\nu_i - b_i)\} & \text{if } \tau \to 0, \\ \exp\{-t\nu_i\} & \text{if } \tau \to \infty. \end{cases}$$

Now we turn our attention to the asymptotic behavior of φ_i and ψ_i satisfying Eqs. (4.3.2). Because of the estimate (4.3.5) there exists the Laplace transforms $\widehat{\varphi}_i(\lambda)$ and $\widehat{\psi}_i(\lambda)$ of φ_i and ψ_i, respectively. Applying this transform to Eqs. (4.3.2), we obtain

$$\widehat{\varphi}_i = \Phi_i(\lambda)/\delta_i(\lambda),\ \widehat{\psi}_i = \Psi_i(\lambda)/\delta_i(\lambda),$$

where

$$\delta_i(\lambda) = f_i(\lambda) - mb_i \exp\{-\tau(\nu_i + \lambda)\},$$
$$f_i(\lambda) = \lambda^2 + \lambda(3\nu_i + m + \sigma_i) + \nu_i(2\nu_i + \sigma_i + m),$$
$$\Phi_i(\lambda) = \tilde{\varphi}_i^0(\lambda)(\lambda + 2\nu_i + \sigma_i) + \psi_i^0(\nu_i + \sigma_i + b_i \exp\{-\tau(\lambda + \nu_i)\}),$$
$$\tilde{\varphi}_i^0(\lambda) = \varphi_i^0 + \int_0^\tau u_i^0(\tau - t) \exp\{-(\nu_i + \lambda)t\} dt,$$
$$\Psi_i(\lambda) = \psi_i^0(\lambda + \nu_i + m) + m\varphi_i^0(\lambda).$$

A graphical analysis shows that δ_i has a unique real root $\lambda_{i0} > -\nu_i$ such that $\mathrm{sign}\lambda_{i0} = \mathrm{sign}\{\mathrm{mb_i} \exp\{-\tau\nu_\mathrm{i}\} - \mathrm{f_i}(0)\}$, and all complex roots occur in conjugate pairs.

We want now to find conditions under which the real part of all complex roots λ_{ik}, $k = 1, 2, \ldots$ is smaller then λ_{i0}. To do this, we make the change of variable $\lambda = \lambda_{i0} + \tilde{\lambda}$, $\tilde{\lambda} = \xi + \sqrt{-1}\eta$, obtaining

$$\tilde{\delta}_i(\tilde{\lambda}) = \tilde{\lambda}^2 + \tilde{\lambda}\tilde{a}_i + \tilde{b}_i(1 - \exp\{-\tau\tilde{\lambda}\}), \tag{4.3.6}$$

with $\tilde{a}_i = 2\lambda_{i0} + 3\nu_i + m + \sigma_i$, $\tilde{b}_i = mb_i \exp\{-\tau(\nu_i + \lambda_{i0})\}$. Setting $\xi = 0$, from (4.3.6) we get the curve

$$\tilde{b}_i = \eta^2/(1 - \cos\tau\eta), \quad \tilde{a}_i = -\eta \sin\tau\eta/(1 - \cos\tau\eta),$$
$$\tau\eta \in I_k = [\pi(1 + 2k), 2\pi(1 + k)],$$

$k = 0, 1, \ldots$, on which $\Re\lambda_{ik} = \xi_{ik} = 0\,\forall k$. Moreover, $d\tilde{b}_i/d\eta$, $d\tilde{a}_i/d\eta > 0$ for $\tau\eta \in \tilde{I}_k$. Hence, this curve increases with η increasing and, as it is easy to see, becomes unbounded as $\tau\eta \to 2\pi(1 + k)$.

Differentiating $\tilde{\delta}_i(\lambda) = 0$ w.r.t. $\tilde{b}_i$, we obtain

$$d\tilde{\lambda}/d\tilde{b}_i|_{\xi_{ik}=0} = \big(\exp\{-\tau\eta\sqrt{-1}\} - 1\big)/\big(2\eta\sqrt{-1} + \tilde{a}_i + \tau\tilde{b}_i \exp\{-\tau\eta\sqrt{-1}\}\big).$$

Hence

$$\begin{aligned} &\mathrm{sign}(\mathrm{d}\xi_\mathrm{ik}/\mathrm{d}\tilde{\mathrm{b}}_\mathrm{i})|_{\xi_\mathrm{ik}=0} \\ &\quad = \mathrm{sign}\{(\cos\tau\eta - 1)(\tilde{\mathrm{a}}_\mathrm{i} + \tau\tilde{\mathrm{b}}_\mathrm{i}\cos\tau\eta) - \sin\tau\eta(2\eta - \tau\tilde{\mathrm{b}}_\mathrm{i}\sin\tau\eta)\} \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{sign}\{\tilde{\mathrm{b}}_{\mathrm{i}}(-\sin\tau\eta/\eta + \tau\cos\tau\eta)(\cos\tau\eta - 1) - \sin\tau\eta(2\eta - \tau\tilde{\mathrm{b}}_{\mathrm{i}}\sin\tau\eta)\} \\
&= \operatorname{sign}\{\tilde{\mathrm{b}}_{\mathrm{i}}[\tau\sin^2\tau\eta + (\cos\tau\eta - 1)(-\sin\tau\eta/\eta + \tau\cos\tau\eta)] - 2\eta\sin\tau\eta\} \\
&= \operatorname{sign}\{\tilde{\mathrm{b}}_{\mathrm{i}}[\tau - \tau\cos\tau\eta - (\sin\tau\eta/\eta)(\cos\tau\eta - 1)] - 2\eta\sin\tau\eta\} \\
&= \operatorname{sign}\{\tilde{\mathrm{b}}_{\mathrm{i}}(1 - \cos\tau\eta)(\tau + \sin\tau\eta/\eta) - 2\eta\sin\tau\eta\} = 1 \ \text{ for } \ \tau\eta \in \tilde{\mathrm{I}}_{\mathrm{k}}.
\end{aligned}
$$

Thus, we get a sequence of domains $D_k = \{(\tilde{a}_i, \tilde{b}_i) : \tilde{a}_i \in (0,\infty), \tilde{b}_i \in (0, \tilde{b}_{ik}(\tilde{a}_i))\}$, $k = 0, 1, \dots$ in which $\Re\lambda_{ik} < 0$. Hence $\xi_{ik} < 0$ in $D = \bigcap_{k=0}^{\infty} D_k$.

But

$$
-\tilde{a}_i/\eta = \cot(\tau\eta/2), \quad \tilde{b}_i = (\eta^2 + \tilde{a}_i^2)/2. \tag{4.3.7}
$$

A graphical analysis of $(4.3.7)_1$ shows, that, for a fixed $\tau\tilde{a}_i$, there exists a unique root $\tau\eta_k = u_k(\tau\tilde{a}_i) \in I_k$. This together with $(4.3.7)_2$ show that $D_k \subset D_{k+1}\,\forall k$. Hence $D = D_0$.

Since $\Phi_i(\lambda)$ and $\Psi_i(\lambda)$ are analytic functions we can estimate the inverse Laplace transform by using the method of rectangle contour integral, obtaining

$$
\begin{aligned}
&\varphi_i \sim \varphi_i^{as} \overset{\text{def}}{=} \Big(\Phi_i(\lambda_{i0})/\delta_i'(\lambda_{i0})\Big)\exp\{t\lambda_{i0}\}, \\
&\psi_i \sim \varphi_i^{as} \overset{\text{def}}{=} \Big(\Psi_i(\lambda_{i0})/\delta_i'(\lambda_{i0})\Big)\exp\{t\lambda_{i0}\},
\end{aligned} \tag{4.3.8}
$$

where the prime denotes differentiation. Clearly, the asymptotic values of φ_i and ψ_i are positive. It is easy to check that the pair φ_i^{as}, ψ_i^{as} is the explicit solution of Eq. (4.3.2) for $t \in (0,\infty)$ provided that $u_i^0(\tau_1) = \big(b_i\Psi_i(\lambda_{i0})/\delta_i'(\lambda_{i0})\big)\exp\{-\tau_1(\lambda_{i0} + \nu_i)\}$, $\tau_1 \in [0,\tau]$ and $m\Phi_i(\lambda_{i0}) = (\lambda_{i0} + 2\nu_i + \sigma_i)\Psi_i(\lambda_{i0})$.

Observe that if ν_i, b_i, and σ_i do not depend on i, then so do δ_i and λ_{i0}, but not Ψ_i and Φ_i. Assume $b =$ const, $\Omega_{\mathrm{i}}^{\mathrm{i}} = 1$, and $\Omega_j^i = 0$ if $j \neq i$. Then, by (4.2.3), $u_i(t,0) = b\psi_i(t)$. Hence, by (4.2.8),

$$
u_i(t,\tau) = \begin{cases} u_i^0(\tau - t)\exp\{-t\nu\},\ 0 \leqslant t \leqslant \tau, \\ b\psi_i(t-\tau)\exp\{-\nu\tau\}, \end{cases} \tag{4.3.9}
$$

and formulas (4.3.8), in particular case, exhibit asymptotics for φ_i and ψ_i satisfying Eqs. (4.2.19) and (4.3.9).

5. The Model of the Forbidden Change of Religions

In this section we present a model of religions not tolerating any confession change, but letting parents to choose a religion not necessarily one of their ones for their offspring. We put $\omega_{sk}^{sk} = 1$ and $\omega_{sk}^{ij} = 0$ if $i \neq s$, $j \neq k$ into (3.1)–(3.4) to obtain the following system for u_i^1, u_2^2, u_{ij}^3 :

$$
\partial u_i^1/\partial t + \partial u_i^1/\partial\tau_1 = -\nu_i^1\, u_i^1 - L_i^1 + S_i^1, \quad t > 0,\ \tau_1 \in Q^1,
$$

$$\partial u_i^2/\partial t + \partial u_i^2/\partial \tau_2 = -\nu_i^2\, u_i^2 - L_i^2 + S_i^2, \quad t > 0,\ \tau_2 \in Q^2, \tag{5.1}$$

$$\partial u_{ij}^3/\partial t + \sum_{k=1}^{3} \partial u_{ij}^3/\partial \tau_k = -(\nu_{ij}^1 + \nu_{ij}^2 + \sigma_{ij})\, u_{ij}^3, \quad t > 0,\ (\tau_1, \tau_2, \tau_3) \in Q^3,$$

$$L_i^1 = \begin{cases} 0, \quad \tau_1 < \tau, \\ 2u_i^1 \sum\limits_{s=1}^{n} \int\limits_{\tau}^{\infty} m_{is}\, u_s^2 \,\mathrm{d}\tau_2 \,\big(\sum\limits_{k=1}^{2} \sum\limits_{s=1}^{n} \int\limits_{\tau}^{\infty} u_s^k\, h_s^k \,\mathrm{d}\tau_k\big)^{-1}, \quad \tau_1 > \tau, \end{cases}$$

$$L_i^2 = \begin{cases} 0, \quad \tau_2 < \tau, \\ 2u_i^2 \sum\limits_{s=1}^{n} \int\limits_{\tau}^{\infty} m_{si}\, u_s^1 \,\mathrm{d}\tau_1 \,\big(\sum\limits_{k=1}^{2} \sum\limits_{s=1}^{n} \int\limits_{\tau}^{\infty} u_s^k\, h_s^k \,\mathrm{d}\tau_k\big)^{-1}, \quad \tau_1 > \tau, \end{cases}$$

$$S_i^1 = \begin{cases} 0, \quad \tau_1 < \tau, \\ \int\limits_{0}^{\tau_1 - \tau} \mathrm{d}\tau_3 \int\limits_{\tau_3 + \tau}^{\infty} \sum\limits_{s=1}^{n} (\nu_{is}^2 + \sigma_{is}) u_{is}^3 \,\mathrm{d}\tau_2, \quad \tau_1 > \tau, \end{cases}$$

$$S_i^2 = \begin{cases} 0, \quad \tau_2 < \tau, \\ \int\limits_{0}^{\tau_2 - \tau} \mathrm{d}\tau_3 \int\limits_{\tau_3 + \tau}^{\infty} \sum\limits_{s=1}^{n} (\nu_{si}^1 + \sigma_{si}) u_{si}^3 \,\mathrm{d}\tau_1, \quad \tau_2 > \tau \end{cases}$$

subject to the following conditions

$$u_i^1|_{\tau_1=0} = \int_0^\infty \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int_{\tau_3+\tau}^\infty \sum_{s,j=1}^{n} u_{sj}^3\, b_{sj}^1\, \Omega_{sj}^i \,\mathrm{d}\tau_2, \quad t > 0,$$

$$u_i^2|_{\tau_2=0} = \int_0^\infty \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int_{\tau_3+\tau}^\infty \sum_{s,j=1}^{n} u_{sj}^3\, b_{sj}^2\, \Omega_{sj}^i \,\mathrm{d}\tau_2, \quad t > 0, \tag{5.2}$$

$$u_{ij}^3|_{\tau_3=0} = 2m_{ij}\, u_i^1\, u_j^2 \,\Big(\sum_{k=1}^{2} \sum_{s=1}^{n} \int_\tau^\infty u_s^k\, h_s^k \,\mathrm{d}\tau_k\Big)^{-1}, \quad t > 0,\ \tau_1, \tau_2 \in [\tau, \infty),$$

$$u_i^1|_{t=0} = u_i^{10}, \quad \tau_1 \in [0, \infty),$$

$$u_i^2|_{t=0} = u_i^{20}, \quad \tau_2 \in [0, \infty),$$

$$u_{ij}^3|_{t=0} = u_{ij}^{30}, \quad (\tau_1, \tau_2, \tau_3) \in \overline{Q}^3,$$

$$[u_i^1|_{\tau_1=\tau}] = [u_i^2|_{\tau_2=\tau}] = 0, \quad t > 0,$$

where

$$\sum_{i=1}^{n} \Omega_{sk}^i = 1 \ \text{ for all } \ s, k, \ \text{ and } \ t \geqslant 0,\ (\tau_1, \tau_2, \tau_3) \in \overline{Q}^3.$$

In addition to (5.2) we assume that u_i^{10}, u_i^{20}, u_{ij}^{30} satisfy the following compatibility conditions:

$$u_i^{10}(0) = \int_0^\infty \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int_{\tau_3+\tau}^\infty \sum_{s,j=1}^{n} u_{sj}^{30}\, (b_{sj}^1\, \Omega_{sj}^i)|_{t=0} \,\mathrm{d}\tau_2,$$

$$u^{20}(0) = \int_0^\infty \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int_{\tau_3+\tau}^\infty \sum_{s,j=1}^n u_{sj}^{30}\,(b_{sj}^2\,\Omega_{sj}^i)|_{t=0}\,\mathrm{d}\tau_2, \tag{5.3}$$

$$u_{ij}^{30}|_{\tau_3=0} = 2m_{ij}|_{t=0}\,u_i^{10}\,u_j^{20}\left(\sum_{k=1}^2\sum_{s=1}^n\int_\tau^\infty u_s^{k0}\,h_s^k|_{t=0}\,\mathrm{d}\tau_k\right)^{-1}, \quad \tau_1,\,\tau_2 \geqslant \tau,$$

$$[u_i^{10}(\tau)] = [u_i^{20}(\tau)] = 0.$$

5.1. *The Symmetric Communities Model*

In this subsection we examine the case where communities are symmetric in the sense that the following demographic functions are symmetric in ages $\tau_1,\ \tau_2$ and religions, and do not depend on sexes, i.e.:

$$\begin{aligned}
&\nu_i^k(t,\xi) = \nu_i(t,\xi), \quad h_i^k(t,\xi) = h_i(t,\xi),\\
&\nu_{ij}^k(t,\tau_1,\tau_2,\tau_3) = \nu_{ij}(t,\tau_1,\tau_2,\tau_3) = \nu_{ji}(t,\tau_2,\tau_1,\tau_3),\\
&\sigma_{ij}(t,\tau_1,\tau_2,\tau_3) = \sigma_{ji}(t,\tau_2,\tau_1,\tau_3), \quad m_{ij}(t,\tau_1,\tau_2) = m_{ji}(t,\tau_2,\tau_1),\\
&u_i^{k0}(\xi) = u_i^0(\xi), \quad u_{ij}^{30}(\tau_1,\tau_2,\tau_3) = u_{ji}^{30}(\tau_2,\tau_1,\tau_3),
\end{aligned}$$

while $b_{ij}^k(t,\tau_1,\tau_2,\tau_3) = b_{ij}(t,\tau_1,\tau_2,\tau_3)$ and Ω_{sj}^i are unnecessary symmetric in $\tau_1,\ \tau_2$. Then $u_i^k(t,\xi) = u_i(t,\xi)$, $u_{ij}^3(t,\tau_1,\tau_2,\tau_3) = u_{ji}^3(t,\tau_2,\tau_1,\tau_3)$, and from (5.1)–(5.3) we obtain the system

$$\begin{aligned}
&\partial u_i/\partial t + \partial u_i/\partial\tau_1 = -\nu_i\,u_i - L_i + S_i, \quad t>0,\ \tau_1\in Q,\\
&\partial u_{ij}^3/\partial t + \sum_{k=1}^3 \partial u_{ij}^3/\partial\tau_k = -(2\nu_{ij}+\sigma_{ij})\,u_{ij}^3, \quad t>0,\ (\tau_1,\tau_2,\tau_3)\in Q^3, \qquad (5.1.1)\\
&L_i = \begin{cases} 0, & \tau_1<\tau,\\ u_i \sum_{s=1}^n \int_\tau^\infty m_{is}\,u_s\,\mathrm{d}\tau_2\,\big(\sum_{s=1}^n\int_\tau^\infty u_s\,h_s\,\mathrm{d}\xi\big)^{-1}, & \tau_1>\tau,\end{cases}\\
&S_i = \begin{cases} 0, & \tau_1<\tau,\\ \int_0^{\tau_1-\tau}\mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \sum_{s=1}^n(\nu_{is}+\sigma_{is})u_{is}^3\,\mathrm{d}\tau_2, & \tau_1>\tau,\end{cases}
\end{aligned}$$

with the conditions

$$\begin{aligned}
&u_i|_{\tau_1=0} = \int_0^\infty \mathrm{d}\tau_3\int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1\int_{\tau_3+\tau}^\infty \sum_{s,j=1}^n u_{sj}^3\,b_{sj}\,\Omega_{sj}^i\,\mathrm{d}\tau_2, \quad t>0,\\
&u_{ij}^3|_{\tau_3=0} = m_{ij}u_i(t,\tau_1)u_j(t,\tau_2)\left(\sum_{s=1}^n\int_\tau^\infty u_s h_s\,\mathrm{d}\xi\right)^{-1},\ t>0,\ \tau_1,\tau_2 >\in[\tau,\infty),\\
&u_i|_{t=0} = u_i^0, \quad \tau_1\in[0,\infty), \qquad (5.1.2)
\end{aligned}$$

$$
\begin{aligned}
&u_{ij}^3|_{t=0} = u_{ij}^{30}, \quad (\tau_1, \tau_2, \tau_3) \in \overline{Q}^3, \\
&[u_i|_{\tau_1=\tau}] = 0, \quad t > 0,
\end{aligned}
$$

where

$$
\sum_{i=1}^{n} \Omega_{sk}^i = 1 \text{ for all } s, k, \text{ and } t \geqslant 0, \ (\tau_1, \tau_2, \tau_3) \in \overline{Q}^3.
$$

In addition to (5.1.2) we assume that u_i^0, u_{ij}^{30} satisfy the following compatibility conditions:

$$
\begin{aligned}
&u_i^0(0) = \int_0^\infty \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int_{\tau_3+\tau}^\infty \sum_{s,j=1}^n u_{sj}^{30} \big(b_{sj}\, \Omega_{sj}^i\big)|_{t=0}\, \mathrm{d}\tau_2, \qquad (5.1.3) \\
&u_{ij}^{30}|_{\tau_3=0} = m_{ij}|_{t=0}\, u_i^0(\tau_1)\, u_j^0(\tau_2) \bigg(\sum_{s=1}^n \int_\tau^\infty u_s^0\, h_s|_{t=0}\, \mathrm{d}\xi \bigg)^{-1}, \quad \tau_1, \, \tau_2 \geqslant \tau, \\
&[u_i^0(\tau)] = 0.
\end{aligned}
$$

5.2. *The Case of Constant Vital Rates*

In this subsection we solve system (5.1.1)–(5.1.3) with $h_s = 1$, arbitrary functions b_{ij}, Ω_{sj}^i, and $\nu_i = \nu_{ij} = \nu$, $\sigma_{ij} = \sigma$ being some positive constants. Then from (5.1.1)–(5.1.3) we derive the system:

$$
\begin{aligned}
&\partial u_i/\partial t + \partial u_i/\partial \tau_1 = -\nu\, u_i - L_i + S_i, \quad t > 0, \ \tau_1 \in Q, \\
&\partial u_{ij}^3/\partial t + \sum_{k=1}^3 \partial u_{ij}^3/\partial \tau_k = -(2\nu + \sigma)\, u_{ij}^3, \quad t > 0, \ (\tau_1, \tau_2, \tau_3) \in Q^3, \qquad (5.2.1) \\
&L_i = \begin{cases} 0, & \tau_1 < \tau, \\ m\, u_i, & \tau_1 > \tau, \end{cases} \quad S_i = \begin{cases} 0, & \tau_1 < \tau, \\ (\nu + \sigma) \int_0^{\tau_1 - \tau} \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \sum_{s=1}^n u_{is}^3\, \mathrm{d}\tau_2, & \tau_1 > \tau, \end{cases}
\end{aligned}
$$

subject to the conditions

$$
\begin{aligned}
&u_i|_{\tau_1=0} = \int_0^\infty \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int_{\tau_3+\tau}^\infty \sum_{s,j=1}^n u_{sj}^3\, b_{sj}\, \Omega_{sj}^i\, \mathrm{d}\tau_2, \quad t > 0, \\
&u_{ij}^3|_{\tau_3=0} = m\, u_i(t, \tau_1)\, u_j(t, \tau_2) \bigg(\sum_{s=1}^n \int_\tau^\infty u_s\, \mathrm{d}\xi \bigg)^{-1}, \quad t > 0, \ \tau_1, \tau_2 \in [\tau, \infty), \\
&u_i|_{t=0} = u_i^0, \quad \tau_1 \in [0, \infty), \qquad (5.2.2) \\
&u_{ij}^3|_{t=0} = u_{ij}^{30}, \quad (\tau_1, \tau_2, \tau_3) \in \overline{Q}^3, \\
&[u_i|_{\tau_1=\tau}] = 0, \quad t > 0,
\end{aligned}
$$

where

$$\sum_{i=1}^{n} \Omega_{sk}^{i} = 1 \text{ for all } s, k, \text{ and } t \geqslant 0, \ (\tau_1, \tau_2, \tau_3) \in \overline{Q}^3,$$

$$u_i^0(0) = \int_0^\infty \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1 \int_{\tau_3+\tau}^\infty \sum_{s,j=1}^{n} u_{sj}^{30} \left(b_{sj}\, \Omega_{sj}^{i}\right)|_{t=0} \,\mathrm{d}\tau_2, \quad [u_i^0(\tau)] = 0,$$

$$u_{ij}^{30}|_{\tau_3=0} = m\, u_i^0(\tau_1)\, u_j^0(\tau_2) \left(\sum_{s=1}^{n} \int_\tau^\infty u_s^0 \,\mathrm{d}\xi \right)^{-1}, \quad \tau_1, \ \tau_2 \in [\tau, \infty),$$

We first construct the formal solution of this system. From (5.2.1), for $\tau_1 \geqslant \tau$, we obtain the following expression

$$u_i(t, \tau_1) = \begin{cases} u_i^0(\tau_1 - t) \exp\{-t\nu\}, & 0 \leqslant t \leqslant \tau_1, \\ u_i(t - \tau_1, 0) \exp\{-\nu\tau_1\}, & \tau_1 \leqslant \min(\mathrm{t}, \tau). \end{cases} \tag{5.2.3}$$

Letting

$$z_i(t, \tau_1) = \int_0^{\tau_1-\tau} \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \sum_{j=1}^{n} u_{ij}^{3} \mathrm{d}\tau_2, \quad z_i^0(\tau_1) = \int_0^{\tau_1-\tau} \mathrm{d}\tau_3 \int_{\tau_3+\tau}^\infty \sum_{j=1}^{n} u_{ij}^{30} \mathrm{d}\tau_2,$$

and integrating Eq. (5.1.2), for $\tau_1 > \tau$, we derive the following system

$$\begin{aligned} &\partial u_i/\partial t + \partial u_i/\partial \tau_1 = -(\nu + m) u_i + (\nu + \sigma) z_i, \\ &\partial z_i/\partial t + \partial z_i/\partial \tau_1 = -(2\nu + \sigma)\, z_i + m\, u_i, \end{aligned} \tag{5.2.4}$$

with the conditions

$$u_i(0, \tau_1) = u_i^0, \quad z_i(0, \tau_1) = Z_i^0, \quad z_i(t, \tau) = 0,$$
$$u_i(t, \tau) = u_i^*(t) = \begin{cases} u_i^0(\tau - t) \exp\{-t\nu\}, & 0 \leqslant t \leqslant \tau, \\ u_i(t - \tau, 0) \exp\{-\nu\tau\}, & \tau < t. \end{cases}$$

Clearly,

$$\partial(u_i + z_i)/\partial t + \partial(u_i + z_i)/\partial \tau_1 = -\nu(u_i + z_i),$$
$$(u_i + z_i)|_{t=0} = u_i^0 + z_i^0, \quad (u_i + z_i)|_{\tau_1=\tau} = u_i^*.$$

Hence

$$\begin{aligned} u_i + z_i &= g_i(t, \tau_1) \\ &= \begin{cases} \left(u_i^0(\tau_1 - t) + z_i^0(\tau_1 - t)\right) \exp\{-\nu\, t\}, & 0 \leqslant t \leqslant \tau_1 - \tau, \\ u_i^0(\tau_1 - t) \exp\{-\nu\, t\}, & 0 \leqslant \tau_1 - \tau \leqslant t \leqslant \tau_1, \\ u_i(t - \tau_1, 0) \exp\{-\nu\, \tau_1\}, & t \geqslant \tau_1. \end{cases} \end{aligned} \tag{5.2.5}$$

This allows us to rewrite system (5.2.4) as follows

$$\begin{aligned}
&\partial u_i/\partial t + \partial u_i \partial \tau_1 = -(2\nu + m + \sigma)u_i + (\nu + \sigma)g_i,\\
&u_i|_{t=0} = u_i^0, \quad u_i|_{\tau_1=\tau} = u_i^*,\\
&\partial z_i/\partial t + \partial z_i/\partial \tau_1 = -(2\nu + m + \sigma)\, z_i + m\, g_i,\\
&z_i|_{t=0} = z_i^0, \quad z_i|_{\tau_1=\tau} = 0.
\end{aligned} \tag{5.2.6}$$

System (5.2.6) has the following solution

$$u_i = \begin{cases} p_{i1}(t)\, u_i^0(\tau_1 - t) + p_{i2}(t)\, z_i^0(\tau_1 - t), & 0 \leqslant t \leqslant \tau_1 - \tau,\\ p_{i3}(t, \tau_1)\, u_i^0(\tau_1 - t), & 0 \leqslant \tau_1 - \tau \leqslant t \leqslant \tau_1,\\ p_{i4}(\tau_1)\, u_i(t - \tau_1, 0), & t > \tau_1 \geqslant \tau, \end{cases} \tag{5.2.7}$$

$$z_i = \begin{cases} p_{i5}(t)\, u_i^0(\tau_1 - t) + p_{i6}(t) z_i^0(\tau_1 - t), & 0 \leqslant t \leqslant \tau_1 - \tau,\\ p_{i7}(t, \tau_1)\, u_i^0(\tau_1 - t), & 0 \leqslant \tau_1 - \tau \leqslant t \leqslant \tau_1,\\ p_{i8}(\tau_1)\, u_i(t - \tau_1, 0), & t > \tau_1 \geqslant \tau, \end{cases} \tag{5.2.8}$$

where

$$\begin{aligned}
p_{i1}(t) &= (\nu + \sigma + m \exp\{-(\nu + m + \sigma)t\}) \exp\{-t\nu\}/(\nu + \sigma + m),\\
p_{i2}(t) &= \frac{\nu + \sigma}{\nu + \sigma + m}\,(1 - \exp\{-(\nu + \sigma + m)t\}) \exp\{-t\nu\},\\
p_{i3}(t, \tau_1) &= \{\nu + \sigma + m \exp\{-(\tau_1 - \tau)(\nu + \sigma + m)\} \exp\{-t\nu\}/(\nu + \sigma + m),\\
p_{i4}(\tau_1) &= \{\nu + \sigma + m \exp\{-(\tau_1 - \tau)(\nu + \sigma + m)\} \exp\{-\tau_1\nu\}/(\nu + \sigma + m),\\
p_{i5}(t) &= \frac{m}{\nu + \sigma + m}\,(1 - \exp\{-(\nu + \sigma + m)t\}) \exp\{-t\nu\},\\
p_{i6}(t) &= \{m + (\nu + \sigma) \exp\{-(\nu + m + \sigma)t\}\} \exp\{-t\nu\}/(\nu + \sigma + m),\\
p_{i7}(t, \tau_1) &= \frac{m}{\nu + \sigma + m}\,(1 - \exp\{-(\tau_1 - \tau)(\nu + \sigma + m)\}) \exp\{-t\nu\},\\
p_{i8}(\tau_1) &= \frac{m}{\nu + \sigma + m}\,(1 - \exp\{-(\tau_1 - \tau)(\nu + \sigma + m)\}) \exp\{-\nu\tau_1\}.
\end{aligned}$$

Integrating $(5.2.1)_2$ with condition $(5.2.2)_4$, we find

$$u_{ij}^3 = \begin{cases} u_{ij}^{30}(\tau_1 - t, \tau_2 - t, \tau_3 - t) \exp\{-(2\nu + \sigma)t\}, & 0 \leqslant t \leqslant \tau_3,\\ u_{ij}^3(t - \tau_3, \tau_1 - \tau_3, \tau_2 - \tau_3, 0) \exp\{-(2\nu + \sigma)\tau_3\}, & 0 \leqslant \tau_3 \leqslant t. \end{cases} \tag{5.2.9}$$

Then from $(5.2.2)_1$ we obtain

$$\begin{aligned}
u_i(t, 0) = &\int_0^t \mathrm{d}\tau_3 \int_{\tau_3+\tau}^{\infty} \mathrm{d}\tau_1 \int_{\tau_3+\tau}^{\infty} \sum_{l,j=1}^{n} b_{lj}\, \Omega_{lj}^i\, u_{lj}^3 \mathrm{d}\tau_2\\
&+ \int_t^{\infty} \mathrm{d}\tau_3 \int_{\tau_3+\tau}^{\infty} \mathrm{d}\tau_1 \int_{\tau_3+\tau}^{\infty} \sum_{l,j=1}^{n} b_{lj}\, \Omega_{lj}^i u_{lj}^3\, \mathrm{d}\tau_2
\end{aligned}$$

$$= \int\limits_0^t \exp\{-(2\nu+\sigma)(t-\rho)\}\, \tilde{H}_i(t,\rho)\, d\rho + \tilde{f}_i(t), \tag{5.2.10}$$

where

$$\tilde{f}_i(t) = \exp\{-(2\nu+\sigma)\, t\} \times \int\limits_0^\infty d\rho_1 \int\limits_{\rho_1+\tau}^\infty \mathrm{d}\xi \int\limits_{\rho_1+\tau}^\infty \sum_{l,j=1}^n u_{lj}^{30}(\xi,\eta,\rho_1)\, (b_{lj}\Omega_{lj}^i)|_{(t,\xi+t,\eta+t,\rho_1+t)}\, d\eta,$$

$$\tilde{H}_i(t,\rho) = \int\limits_\tau^\infty \mathrm{d}\tau_1 \int\limits_\tau^\infty \sum_{l,j=1}^n \big(b_{lj}\Omega_{lj}^i\big)|_{(t,t+\tau_1-\rho,t+\tau_2-\rho,t-\rho)} u_{lj}^3(\rho,\tau_1,\tau_2,0)\mathrm{d}\tau_2,$$

$0 \leqslant \rho \leqslant t$, with $u_{lj}^3(t,\tau_1,\tau_2,0)$ defined by $(5.2.2)_2$ or,

$$\tilde{H}_i(t,\rho) = \sum_{s,k=1}^2 \int\limits_{I_s} \mathrm{d}\tau_1 \int\limits_{I_k} \sum_{l,j=1}^n \big(b_{lj}\Omega_{lj}^i\big)|_{(t,t+\tau_1-\rho,t+\tau_2-\rho,t-\rho)} u_{lj}^3(\rho,\tau_1,\tau_2,0)\mathrm{d}\tau_2,$$

$0 \leqslant \rho \leqslant \min(\mathrm{t},\tau)$, with $I_1 = [\tau, \tau+\rho]$, $I_2 = [\tau+\rho, \infty)$ and

$$\tilde{H}_i(t,\rho) = \sum_{s,k=1}^3 \int\limits_{I_s} \mathrm{d}\tau_1 \int\limits_{I_k} \sum_{l,j=1}^n \big(b_{lj}\Omega_{lj}^i\big)|_{(t,t+\tau_1-\rho,t+\tau_2-\rho,t-\rho)} u_{lj}^3(\rho,\tau_1,\tau_2,0)\mathrm{d}\tau_2,$$

$t \geqslant \rho > \tau$, with $I_1 = [\tau, \rho]$, $I_2 = [\rho, \rho+\tau]$, $I_3 = [\rho+\tau, \infty)$.

Now we examine Eqs. (5.2.7)–(5.2.10) going along the t axis by the step τ.

Let $t \in [0, \tau]$. Eqs. $(5.2.3)_1$, $(5.2.7)_{1,2}$, and $(5.2.8)_{1,2}$ determine u_i and z_i for $0 \leqslant t \leqslant \tau_1$, $\tau_1 \in (0, \infty)$ and $t \leqslant \tau_1$, $\tau_1 > \tau$, respectively. Then from $(5.2.2)_2$ and (5.2.10) we find $u_{sj}^3|_{\tau_3=0}$ and $u_i(t,0)$ for $t \leqslant \tau$. After that from Eq. $(5.2.9)_2$ we obtain u_{ij}^3 for $\tau_3 < t \leqslant \tau_3 + \tau$.

Let $t \in (\tau, 2\tau]$. By $(5.2.3)_2$, we can find u_i for $\tau_1 < t \leqslant \tau_1 + \tau$, $\tau_1 \in [0, \tau]$. Then Eqs. $(5.2.7)_3$ and $(5.2.8)_3$ determine u_i and z_i for $\tau_1 < t \leqslant \tau_1 + \tau$, $\tau_1 \geqslant \tau$. After that from $(5.2.2)_2$ and (5.2.10) we construct $u_{sj}^3|_{\tau_3=0}$ and $u_i(t,0)$ for $\tau < t \leqslant 2\tau$. Finally, from Eq. $(5.2.9)_2$ we find u_{ij}^3 for $\tau_3 + \tau < t \leqslant \tau_3 + 2\tau$.

Repeating this argument we can construct the formal solution of problem (5.2.1), (5.2.2) for any finite time t.

It remains to justify the formal analysis. Assume that:

(H.5.1) b_{ij}, $\Omega_{ij} \in C^1\big([0,\infty) \times \overline{Q}^3\big)\big)$ are positive,
(H.5.2) $u_i^0 \in C^1([0,\infty)) \cap L^1((0,\infty))$ is positive,
(H.5.3) $u_{ij}^{30} \in C^1(\overline{Q}^3) \cap L^1(Q^3)$ is positive,
(H.5.4) $\int\limits_\tau^\infty |\frac{\partial}{\partial \tau_s} u_{ij}^{30}(\tau_1,\tau_2,\tau_3)|\mathrm{d}\tau_2$, $s = 1, 3$ converges uniformly with respect to all τ_1, τ_3,

(H.5.5) u_i^0, u_{ij}^{30} satisfy compatibility conditions $(5.2.2)_{7,8}$.
Then Eqs. (5.2.3), (5.2.7)–(5.2.14) and $(5.2.2)_{2,7,8}$ show that:

$$u_i(t,\tau_1) \in C^0\big([0,T]\times[0,\infty)\big) \cap C^1\big([0,T]\times\big([0,\infty)\setminus\{\tau,t,t+\tau\}\big)\big) \text{ and}$$
$$u_i(t,\cdot),\ \partial u_i(t,\cdot)/\partial t \in L^1(0,\infty), \tag{5.2.11}$$
$$Z_i(t,\tau_1) \in C^0\big([0,T]\times[\tau,\infty)\big) \cap C^1([0,T]\times\big([\tau,\infty)\setminus\{t,t+\tau\}\big)\big) \text{ and}$$
$$Z_i(t,\cdot),\ \partial Z_i(t,\cdot)/\partial t \in L^1(0,\infty),$$
$$u_{ij}^3(t,\tau_1,\tau_2,\tau_3) \in C^0\big([0,T]\times\overline{Q}^3\big) \cap C^1\big([0,T]\times\big([\tau,\infty)\setminus\{t,t+\tau\}\big)$$
$$\times\big([\tau,\infty)\setminus\{t,t+\tau\}\big)\times\big((0,\ \min(\tau_1-\tau,\tau_2-\tau)]\setminus\{\mathrm{t}\}\big)\big) \text{ and}$$
$$u_{ij}^3(t,\cdot),\ \partial u_{ij}^3(t,\cdot)/\partial t \in L^1(Q^3). \tag{5.2.12}$$

Further justification of the derivation of Eq. $(5.2.4)_2$ is similar to that of Eq. $(4.2.10)_2$ and we omit it. As a result of the above consideration, we have

Theorem. *Assume hypotheses* (H.5.1)–(H.5.5) *hold and let* $m,\ \nu$, *and* σ *be some positive constants. Then, for any finite* $T>0$, *problem* (5.2.1)–$(5.2.2)_{1-6}$ *has a unique positive solution satisfying* (5.2.11), (5.2.12).

5.3. *The Asymptotic Analysis*

In this subsection we examine the longtime behavior of the total number of single individuals and pairs of each religion for problem (5.2.1), (5.2.2) in the case where $\Omega_{ii}^i=1,\ \Omega_{sk}^i=0$ if $i\neq s,k$, and both $\Omega_{sk}^s=\Omega_{sk}^k=1/2$ and either $\Omega_{sk}^s=1,\ \Omega_{sk}^k=0$ or $\Omega_{sk}^s=0,\ \Omega_{sk}^k=1$ if $s\neq k$.

Assume $\nu_i,\ \nu_{ij}=\nu_{ji},\ \sigma_{ij}=\sigma_{ji},\ m_{ij}=m_{ji}$ be positive and $b_{ij},\ \Omega_{sk}^i$ be arbitrary nonnegative constants. Then $u_{ij}=u_{ji}$. Letting

$$\varphi_i=\int_\tau^\infty u_i\mathrm{d}\tau_1,\quad \psi_{ij}=\psi_{ji}=\int_0^\infty \mathrm{d}\tau_3\int_{\tau_3+\tau}^\infty \mathrm{d}\tau_1\int_{\tau_3+\tau}^\infty u_{ij}^3\mathrm{d}\tau_2,$$

from (5.1.1), (5.1.2) we derive the following formal nonlinear system for $\varphi_i,\ \psi_{ij}$:

$$\varphi_i'=u_i(t,\tau)-\nu_i\varphi_i-\varphi_i\sum_{s=1}^n m_{is}\varphi_s\Big(\sum_{s=1}^n\varphi_s\Big)^{-1}+\sum_{s=1}^n(\nu_{is}+\sigma_{is})\psi_{is},$$
$$\psi_{ij}'=-(2\nu_{ij}+\sigma_{ij})+m_{ij}\varphi_i\varphi_j\Big(\sum_{s=1}^n\varphi_s\Big)^{-1}, \tag{5.3.1}$$
$$u_i(t,\tau)=\begin{cases} u_i^0(\tau-t)\exp\{-t\nu_i\}, & 0\leqslant t\leqslant\tau \\ \sum\limits_{s,k=1}^n \Omega_{sk}^i\, b_{sk}\psi_{sk}(t-\tau)\exp\{-\tau\nu_i\}, & t\geqslant\tau,\end{cases}$$

where the prime indicates differentiation. Assume φ_i^0, $\psi_{ij}^0 > 0$ and let $0 \leqslant u_i^0(\tau_1) \in C([0,\tau])$. By the argument applied to system (4.3.1) we can prove the existence of a unique positive solution of Eqs. (5.3.1) for any finite t. Using the formula

$$\sum_{s,k=1}^{n} \Omega_{sk}^i b_{sk}\psi_{sk} = \Omega_{ii}^i b_{ii}\psi_{ii} + \sum_{k\neq i}(\Omega_{ik}^i b_{ik} + \Omega_{ki}^i)\psi_{ik} + \sum_{s,k\neq i}\Omega_{sk}^i b_{sk}\psi_{sk}, \quad (5.3.2)$$

we analyze two particular cases of this system.

Let $\nu_{ij} = \nu_i = \nu$, $\sigma_{ij} = \sigma$, $m_{ij} = m$, $b_{ij} = b$, $\Omega_{ii}^i = 1$, $\Omega_{sk}^i = 0$ if $s,k \neq i$. Assume also: $\Omega_{sk}^s = \Omega_{sk}^k = 1/2$ if $s \neq k$, and either $\Omega_{sk}^s = 1$, $\Omega_{sk}^k = 0$, $s \neq k$, or $\Omega_{sk}^s = 0$, $\Omega_{sk}^k = 1$ if $s \neq k$. Then, for φ_i and $\psi_i = \sum_{j=1}^{n} \psi_{ij}$, from (5.3.1), (5.3.2) we obtain

$$\begin{aligned}
&\varphi_i' = u_i(t,\tau) - (\nu+m)\varphi_i + (\nu+\sigma)\psi_i,\\
&\psi_i' = -(2\nu+\sigma)\psi_i + m\varphi_i,\\
&u_i(t,\tau) = \begin{cases} u_i^0(\tau - t)\exp\{-t\nu\}, & 0 \leqslant t \leqslant \tau,\\ b\psi_i(t-\tau)\exp\{-\tau\nu\}, & t \geqslant \tau.\end{cases}
\end{aligned} \quad (5.3.3)$$

In the other case where $\nu_{ij} = \nu_i = \nu$, $\sigma_{ij} = \sigma$, $b_{ij} = b$, $m_{ij} = m$, and Ω_{sk}^i is arbitrary, for $\varphi = \sum_i \varphi_i$, $\psi = \sum_i \psi_i$, we obtain

$$\begin{aligned}
&\varphi' = \sum_i u_i(t,\tau) - (\nu+m)\varphi + (\nu+\sigma)\psi,\\
&\psi' = -(2\nu+\sigma)\psi + m\varphi,\\
&\sum_i u_i(t,\tau) = \begin{cases} \sum_i u_i^0(\tau - t)\exp\{-t\nu\}, & 0 \leqslant t \leqslant \tau,\\ b\psi(t-\tau)\exp\{-\tau\nu\}, & t \geqslant \tau.\end{cases}
\end{aligned} \quad (5.3.4)$$

Clearly, $\varphi_i \leqslant \varphi$, $\psi_i \leqslant \psi$. System (5.3.3) is a special case of (4.3.2), and system (5.3.4) is the same as (4.3.3), whose longtime behavior we have considered in 4.3.

6. Discussion

In this paper, we derived a general deterministic dynamics model for the interacting sociologically structured human communities. It is based on the Hoppensteadt-Staroverov-Hadeler pair formation model and describes dynamics of the interacting religions that tolerate both uniconfessional pairs and those with different religions. Two special models such as the uniconfessional pair model and that of the forbidden change of religions are investigated. In the case of constant vital functions, solutions of these two models are constructed, and the longtime behavior of the total numbers of single adults and pairs is obtained.

In the case where $\nu_i = \nu_i^3 = \nu$, b_i, σ_i, $m_{ij} = m$ are positive constants and $\omega_{sk}^i = 0$ if $s, k \neq i$, $\omega_{ik}^i = \omega_{ki}^i = 1/2$ if $k \neq i$, $\omega_{ii}^i = 1$, $\Omega_j^i = 0$ if $i \neq j$, $\Omega_i^i = 1$ the total numbers of the single adults φ_i and pairs ψ_i of the ith religion in the uniconfessional pair model (4.3.2) evolve independently of those of the other religions. The exponent λ_{i0} of the longtime behavior (4.3.8) depends only on the parameters of the ith religion. Hence, some of the religions may extinct or tend to the steady state, while the other ones may grow.

The exponent λ_0 of the largetime behavior of φ_i and ψ_i in the model of the forbidden change of religions (5.3.3) with the constant $\nu_{ij} = \nu_i = \nu$, $\sigma_{ij} = \sigma$, $m_{ij} = m$, $b_{ij} = b$ and $\Omega_{ii}^i = 1$, $\Omega_{sk}^i = 0$ if $s, k \neq i$, $\Omega_{sk}^s = \Omega_{ks}^s = 1/2$ if $s \neq k$ is the same for all the religions. Hence, all the religions extinct or grow with the same rate.

The total numbers of the single adults $\varphi = \sum_{i=1}^{n} \varphi_i$ and pairs $\psi = \sum_{i=1}^{n} \psi_i$ in the uniconfessional pair model (4.3.3) with constant $b_i = b$, $\nu_i^3 = \nu_i = \nu$, $m_{ij} = m$, $\sigma_i = \sigma$, ω_{lk}^i, Ω_j^i are the same as those in the model (5.3.4) for the forbidden change of religions with constant $\nu_{ij} = \nu_i = \nu$, $\sigma_{ij} = \sigma$, $b_{ij} = b$, $m_{ij} = m$, Ω_{sk}^i.

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Apie skirtingų religijų koegzistenciją

Vladas SKAKAUSKAS

Pasiūlytas bendras porų formavimo modelis saveikaujančiose žmonių bendrijose, įskaitant individų amžių, lytį ir religinę priklausomybę. Modelis aprašo religijų, toleruojančių vienos bei įvairių konfesijų poras, dinamiką. Sukonstruotas modelio sprendinys ir gauta jo asimptotika dviem specialiais atvejais: pirmuoju atveju modelis arašo vienos konfesijos porų dinamiką, o antruoju – toleruoja skirtingų konfesijų poras, bet draudžia bet kokį religijos pakeitimą.