INFORMATICA, 1999, Vol. 10, No. 3, 313–326

On the Existence of Optimal Control of Differential Systems with After-Effect

Manuel de la SEN

Departamento de Electricidad y Electrónica, Facultad de Ciencias Universidad del Pais Vasco, Aptdo. 644 de Bilbao Leioa. (Bizkaia), Spain
e-mail: msen@we.lc.ehu.es

Abstract. This paper deals with the study of the optimal control problem for the objective $F(x,u,v)=\int_0^T f(x(t),\ x(t-h),\ \zeta(t),\ u(t),\ v(t),\ t)\mathrm{d}t$, with $x\in X,\ u\in U,\ v\in V;\ X,\ U$ and V being vector spaces, and $\zeta(t)=\int_0^h R(t,\tau)x(t-\tau)\,\mathrm{d}\tau$, subject to the differential equation $\frac{d}{dt}x(t)=m(x(t),\ x(t-h),\ \zeta(t),\ u(t),\ v(t),t)\ (0\leqslant t\leqslant T)$, and the constraints $g_1(u(t),\ t)\in S_1,\ g_2(v(t),\ t)\in S_2;\ n_1(x(t),\ t)\in V_1,\ n_2(x(t-h),\ t)\in V_2;\ n_3(\zeta(t),t)\in V_3\ (0\leqslant t\leqslant T)$, where $x(t)\in\mathbb{R}^n;\ \zeta(t)\in\mathbb{R}^n;\ u(t)\in\mathbb{R}^k;\ f,\ m,\ g_i\ (i=1,2),\ n_i\ (1\leqslant i\leqslant 3)$ and the entries to $r(t,\tau):\mathbb{R}_+\times\mathbb{R}_+\to L(X,X)$ are continuously differentiable functions. It is assumed that boundary conditions $x(0)=x(T)=0$ are imposed. $S_i\ (i=1,2)$ and $V_i\ (1\leqslant i\leqslant 3)$ are convex cones. The existence of a time-optimal control in analytic linear systems is also investigated via an extension of the bang-bang principle.

Key words: differential systems with after-effect, optimal control.

1. Introduction. Notation and General Preliminaries on Optimal Control for Differential Systems with After-Effect

Linear systems with after effect have been studied recently (Minyuk, 1983; Churakova, 1969; Metel'skii, 1978; Kurzhanskii, 1966). The controllability and observability of such systems in Euclidean and Hilbert spaces have been studied in (Minyuk, 1983; Churakova, 1969). In (Metel'skii, 1978), a general analysis of the system's properties of such systems has been stated. These systems present more complicated theoretical analysis of such properties due to the delay. A vector of an Euclidean space is never a state for such systems and a function space is needed to establish the formalism. In (M. de la Sen, 1988), the properties of such systems in the linear and time-invariant case have been related to Algebraic Systems Theory of *standard* linear differential systems as given in (Kailath, 1980). This paper presents a formulation of the optimal control problem (OC) using a Lagrangean theory as stated in (Craven, 1978) for the linear case. The paper's body is organized in two sections. In Section 2, the optimization of a Hamiltonian functional is discussed by assuming for the optimal control for a convex cost functional to exist. Section 3 studies the conditions for the existence of a unique optimal control. Finally, conclusions end the paper.

1.1. *Notation*

The usual symbols for logical and set operations will be used, namely $\Rightarrow$ (implies), $\iff$ (if and only if, also written iff), $\forall$ (for all), $\exists$ (there exists); and $\in$ (belongs to), $\cup$ (union), $\cap$ (intersection); $\backslash$ (set difference; $S\backslash T$ is the set of elements in S but not in T); $\subset$ (inclusion), $\subseteq$ (inclusion allowing =), $\varnothing$ (empty set); $\times$ (cartesian product), int (S) and $\partial(S)$ denote, respectively, the interior and boundary of the set S.

- Superscripts T and $*$ indicate transposition and conjugate transposition of an operator.
- U, V, W, X, Y, Z are real vector spaces.
- $B(x, \varepsilon)$ is the n-open ball of center x and radius $\varepsilon > 0$ for $x \in X \subseteq \mathbb{R}^n$. $\bar{B}(x, \varepsilon)$ is the closure of $B(x, \varepsilon)$.
- $C(I)$: space of all continuous real functions on the interval I, with the uniform norm $||x||_\infty = \sup_{t \in I} |x(t)|$.
- $L^p(I)$: the space of functions whose pth powers are Lebesgue integrable on I, with finite seminorm $||x||_p = [\int_I |x|^p \, \mathrm{d}t]^p$.

When the space is required to be complete (as are $C(I)$ and $L^2(I)$), a Banach space will be specified.

For spaces of a vector space X and $\alpha \in \mathbb{R}$,
$\alpha S = \{\alpha s : s \in S\}$, and $S + T = \{s + t : s \in S,\ t \in T\}$

- A set $S \subset X$ is a convex cone if $S + S \subseteq S$ and $(\forall \alpha \in \mathbb{R}_+)\ \alpha S \subseteq S$.
- The vector space of all continuous functionals on X is the dual space of X, and denoted by X'.
- $L(X, Y)$ is the space of all continuous linear maps from X into Y.
- For the dual space X' of X, a weak $*$ neighborhood of $p \in X'$ is any set $N(p) = \{y \in X' : |y(x_i) - p(x_i)| < \varepsilon\ (i = 1, 2, \ldots, r)\}$ specified by finitely many points $x_1, x_2, \ldots, x_r$ in X and a positive real constant ε. $Q \subseteq X'$ is weak $*$ closed if every $p \in X' \backslash Q$ has a weak $*$ neighborhood $N(p)$ which does not meet Q.
- $< ., . >$ stands for the inner product in Euclidean spaces.

1.2. *Optimal Control*

Consider an optimal control problem, to minimize

$$F(x, u, v) = \int_0^T f(x(t),\ x(t-h),\ \zeta(t),\ u(t),\ v(t),\ t)\, \mathrm{d}t \tag{1.1}$$

$$\zeta(t) = \int_0^h R(t, \tau) x(t - \tau)\, \mathrm{d}\tau, \tag{1.2}$$

subject to the differential equation

$$\frac{d}{\mathrm{d}t} x(t) = \dot{x}(t) = m\,(x(t), x(t-h), \zeta(t), u(t), v(t), t) \quad (0 \leqslant t \leqslant T) \tag{1.3}$$

(A typical partucular case is (De la Sen, 1988) $\dot{x}(t) = Ax(t) + A_1x(t-h) + \int_{t-h}^{t} R(t,\tau)x(t-\tau)\,\mathrm{d}\tau + B(t)u(t) + v(t)$; $A, A_1, R(\cdot,\cdot)$ being real matrices of appropriate orders, and $u(\cdot)$, $v(\cdot)$ being piecewise real continuous control and disturbance vector functions), and the constraints:

$$\begin{aligned} &g_1(u(t),t) \in S_1;\;\; g_2(v(t),t) \in S_2;\\ &n_1(x(t),t) \in V_1;\;\; n_2(x(t-h),t) \in V_2;\;\; n_3(\zeta(t),t) \in V_3 \;\; (0 \leqslant t \leqslant T). \end{aligned} \tag{1.4}$$

Assume that $x(t), \zeta(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^k$; $f, m, g_i, n_j, \zeta(t)$ $(1 \leqslant i \leqslant 2; 1 \leqslant j \leqslant 3)$ are continuously differentiable functions in $\mathbb{R}_+$; and that boundary conditions $x(0) = 0$, $x(t) = \varphi(t)$, $\forall t \in [T, T+h)$ are imposed with φ: $\mathbb{R}_+ \to \mathbb{R}^n$ being an absolutely continuous differentiable function. Let $S_i \subseteq \mathbb{R}^{r_i}$ and $V_j \subseteq \mathbb{R}^{h_j}$ $(1 \leqslant i \leqslant 2;\ 1 \leqslant j \leqslant 3)$ be convex cones.

Appropriate vector spaces of functions must be specified for the subsequent formulation. Let $x \in X$, $u \in U$, $v \in V$, the spaces of piecewise continuous functions from $I = [0, T]$ into $\mathbb{R}^n$ or $\mathbb{R}^k$, with the uniform $\|\cdot\|_\infty$. Denote by W the space of piecewise continuous functions from I to $\mathbb{R}^n$, with the uniform norm. $x \in X$ is the Lebesgue integral of a function $x \in W$, and $x(0) = 0$, $x(t) = \varphi(t)$, $\forall t \in [T, T+h)$. Then, $x(t) = \int_0^t w(s)\,\mathrm{d}s$ is expressed as $w = Dx$, $D = d/dt$, except at discontinuities of w, the linear map $D{:}\, X \to W$ is made continuous, by giving X the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$.

The differential equation for $x(t)$, with initial condition, expressed as

$$x(t) = \int_0^t m\,(x(s), x(s-h), \zeta(s), u(s), v(s), s)\ \mathrm{d}s \;\; (x, \zeta \in X,\ u \in U), \tag{1.5}$$

may then be written as $Dx = M(x, u, v)$ where the map M: $X \times X \times X \times U \times \mathcal{V} \to L(\mathbb{R}, W)$ is defined by $M(x,u,v)(t) = m\,(x(t), x(t-h), \zeta(t), u(t), v(t), t)$; $\zeta(t) = \int_0^t r(t,\tau)x(t-\tau)\,\mathrm{d}\tau$. (The differential equation has now been seightly extended to allow a finite number of points where x is not differentiable).

Let Q_i, P_j denote the spaces of piecewise continuous functions from I into $\mathbb{R}^{r_i}$ resp. $\mathbb{R}^{h_j}$ $(1 \leqslant i \leqslant 2;\ 1 \leqslant j \leqslant 3)$. Define the convex cones $K_1 \subseteq Q_i$; $J_j \subseteq P_j$ as

$$\left.\begin{aligned} K_i &= \{q_i \in Q_i{:}\ (\forall t \in I) q_i(t) \in S_i\} \;\; (1 \leqslant i \leqslant 2)\\ J_j &= \{p_j \in P_j{:}\ (\forall t \in I) p_j(t) \in V_j\} \;\; (1 \leqslant j \leqslant 3) \end{aligned}\right\} \tag{1.6}$$

Define the maps G_1: $U \to Q_1$; $G_2{:}\, \nu \to Q_2$; and $N_j{:}\, X \to P_j$ $(1 \leqslant j \leqslant 3)$ by $G_1(u)(t) = g_1(u(t), t)$; $G_2(v)(t) = g_2(v(t), t)$; $N_1(x)(t) = n_1(x(t), t)$; $N_2(x)(t) = n_2(x(t-h), t)$; $N_3(x)(t) = n_3(\zeta(t), t)$ $(t \in I)$. Then, the constraints (1.4) are expressed as $G_i(u) \in K_i$; $N_j(x) \in I_j$ $(1 \leqslant i \leqslant 2;\ 1 \leqslant j \leqslant 3)$. Now, the problem (1.1)–(1.2) subject to (1.4) is expressible as

$$\text{(OC): Minimize} \quad \big\{F(x,u,v){:}\ Dx = M(x,u,v),\ G_i(u) \in K_i,\ N_j(x) \in J_j;\ x \in X,\ u \in U\ (1 \leqslant i \leqslant 2;\ 1 \leqslant j \leqslant 3)\big\}.$$

Since m is continuously Fréchet-differentiable,

$$\left.\begin{array}{l} m\left(x(t)+z(t),x(t-h),\zeta(t),u(t),v(t),t\right) \\ \quad -m\left(x(t),x(t-h),u(t),v(t),t\right) \\ = m_x\left(x(t),x(t-h),\zeta(t),u(t),v(t),t\right)z(t)+\sigma(t), \\ \\ m\left(x(t),x(t-h)+z(t),\zeta(t),u(t),v(t),t\right) \\ \quad -m\left(x(t),x(t-h),\zeta(t),u(t),v(t),t\right) \\ = m_{x(-h)}\left(x(t),x(t-h),\zeta(t),u(t),v(t),t\right)z(t)+\sigma_{-h}(t), \\ \\ m\left(x(t),x(t-h),\zeta(t)+z(t),u(t),v(t),t\right) \\ \quad -m\left(x(t),x(t-h),\zeta(t),u(t),v(t),t\right) \\ = m_\zeta\left(x(t),x(t-h),\zeta(t),u(t),v(t),t\right)z(t)+\sigma_\zeta(t), \end{array}\right\} \tag{1.7}$$

if $x(t), x(t-h), \zeta(t), x(t)+z(t), x(t-h)+z(t), \zeta(t)+z(t) \in X \times [0,T]$, where m_x, $m_{x(-h)}$, m_ζ denote partial derivatives, and $\|\sigma(t)\| \leqslant \varepsilon_1\|z(t)\| \leqslant \varepsilon_1\|z(t)\|_\infty$, $\|\sigma_{-h}(t)\| \leqslant \varepsilon_2\|z(t)\|$, $\|\sigma_3(t)\| \leqslant \varepsilon_3\|z(t)\| \leqslant \varepsilon_3\|z\|_\infty$ if $\|z\|_\infty < \delta(\varepsilon)$, $\varepsilon = \max(\varepsilon_i;\ 1 \leqslant i \leqslant 3)$.

Consequently, M is partially differentiable with respect to x, $x(t-h)$, $\zeta(t)$; with partial derivatives $M_x(x,u,v)$, $M_{x(-h)}(x,u,v)$, $M_\zeta(x,u,v)$ given by $M_x(x,u,v)z(t) = m_x(x(t)), x(t-h), \zeta(t), u(t), v(t), t)z(t)$; $M_{x(-h)}(x,u,v)(z(t)) = m_{x(-h)}(x(t), x(t-h), \zeta(t), u(t), v(t), t)z(t)$; $M_\zeta(x,u,v)(z(t)) = m_\zeta(x(t), x(t-h), \zeta(t), u(t), v(t), t)$ $(z(t))$. Also, $F(x,u,v)$ is Fréchet differentiable with respect to the above elements in X, with partial derivatives $F_{x,x(-h),\zeta}(x,u,v)$ given by $F_x(x,u,v)z = \int_0^T f_x(x(t), x(t-h), \zeta(t), u(t), v(t), t)z(t)\mathrm{d}t$; $F_{x(-h)}(x,u,v)z = \int_0^T f_{x(-h)}(x(t), x(t-h), \zeta(t), u(t), v(t), t)z(t)\mathrm{d}t$; $F_\zeta(x,u,v)z = \int_0^T f_\zeta(x(t), x(t-h), \zeta(t), u(t), v(t), t)z(t)\mathrm{d}t$; here

$$\left.\begin{array}{l} \|F(x+z,u,v)-F(x,u,v)-F_x(x,u,v)z\| \leqslant \varepsilon_1\|z\|_\infty, \\ \qquad\qquad\qquad\qquad \text{if} \quad \|z\|_\infty < \delta(\varepsilon_1/T); \\ \\ \left\|F(x_{(-h)}+z,u,v)-F(x_{(-h)},u,v)-F_{x(-h)}(x,u,v)z\right\| \leqslant \varepsilon_2\|z\|_\infty, \\ \qquad\qquad\qquad\qquad \text{if} \quad \|z\|_\infty < \delta(\varepsilon_2/T); \\ \\ \|F(\zeta+h,u,v)-F(x,u,v)-F_\zeta(x,u,v)z\| \leqslant \varepsilon_3\|z\|_\infty, \\ \qquad\qquad\qquad\qquad \text{if} \quad \|z\|_\infty < \delta(\varepsilon_3/T), \end{array}\right\} \tag{1.8}$$

and similarly for the other functions.

For the problem (OC), define a Lagrangean

$$\begin{aligned} & L\left(x,u,v,\tau,\overline{\lambda},\overline{\mu}_i,\overline{\nu}_j;\ 1 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant 3\right) \\ & = \tau F(x,u,v) - \overline{\lambda}\left(Dx - M(x,u,v)\right) - \overline{\mu}_1 G_1(u) - \overline{\mu}_2 G_2(u) \\ & \quad - \sum_{j=1}^{3} \overline{\nu}_j N_j(x), \end{aligned} \tag{1.9}$$

where $\tau \in \mathbb{R}_+$, $\overline{\lambda} \in W'$, $\overline{\mu}_i \in Q'_i$, $\overline{\nu}_j \in P'_j$ $(1 \leqslant i \leqslant 2;\ 1 \leqslant j \leqslant 3)$ are Lagrange multipliers not all zero.

DEFINITION 1.1 (Craven, 1978). The dual cone (or polar cone) of S is the convex cone

$$S^* = \{y \in X' \colon\ (\forall s \in S)\ y(s) \geqslant 0\}\,. \tag{1.10}$$

If $S \subseteq \mathbb{R}^n$, then $S^* \subseteq \mathbb{R}^n$.

Two standard abstract minimization problems are

$$\text{(P1)}:\ \underset{x\in X_0}{\text{Minimize}}\,\{f(x)\colon\ -g(x) \in S,\ -h(s) \in T\}\,,$$
$$\text{(P2)}:\ \underset{x\in X_0}{\text{Minimize}}\,\{f(x)\colon\ -h(x) \in T\}\,,$$

where X, Y, Z are Banach spaces; X_0 is an open subset of X; $S \subseteq Y$ is a convex cone; with int $S \neq \varnothing$; $T \subseteq Z$ is a closed convex cone; and the functions $f\colon\ X_0 \to \mathbb{R}$; $g\colon\ X_0 \to Y$; $h\colon\ X_0 \to Z$ are Fréchet differentiable.

The following Definition and Lemma are well-known (Craven, 1978).

DEFINITION 1.2. The system $-h(x) \in T$ is called locally solvable at the point a, if for some $\delta > 0$, whenever the direction d satisfies $h(a) + h'(a)d \in -T$ and $\|d\| < \delta$, there exists a solution $x = a + \alpha d + \eta(\alpha)$ to $-h(x) \in T$, valid for sufficiently small $\alpha > 0$, where $\|\eta(\alpha)\|/\alpha \to 0$ as $\alpha \downarrow 0$ (for brevity $\eta(\alpha) = 0(\alpha)$).

Lemma 1.1. (1) (Fritz – John theorem). *For* (P1)*, let the constraint* $-h(x) \in T$ *be locally solvable at* $a \in X_0$*; let* $[h'(a)|h(a)]^T(T^*)$ *be weak* $*$ *closed. Then, a necessary condition for* (P1) *to attain a local minimum at* $x = a$ *is*

$$\begin{aligned}\text{(FJ)}:\quad &\tau f'(a) + v\, g'(a) + w\, h'(a) = 0; \quad v\, g(a) = 0;\\ &w\, h(a) = 0; \quad \tau \in \mathbb{R}_+,\ v \in S^*,\ w \in T^*,\end{aligned}$$

where τ *and* v *are not both zero and define the modified Lagrangean* $\tau f(x) + v\, g(x) + w\, h(x)$*.*

(2) (Kuhn – Tucker theorem). *For* (P2)*, let the constraint* $-h(x) \in T$ *be locally solvable at* $a \in X_0$*; let* $[h'(a)|h(a)]^T(T^*)$ *be weak* $*$ *closed. Then, a necessary condition for* (P2) *to attain a local minimum at* $x = a$ *is*

$$\text{(KT)}: f'(a) + w\, h'(a) = 0; \quad w\, h(a) = 0,\ w \in T^*.$$

(3) (KT) *holds with the hypothesis of local solvability are placed by that of regularity at* a*.*

Representations will be required for the vectors in certain dual spaces. A $y \in [C(I)]'$ may be represented by a function $\lambda(\cdot)$ by $y(x) = \int_I x(t)\lambda(t)\,\mathrm{d}t$. Suppose that $\overline{\lambda}$ in (1.9) can be represented by a function $\lambda(\cdot)$, where

$$(\forall z \in X) \quad \overline{\lambda}(Dz) = \int_0^{T+h} \lambda(z) Dz(t)\, dt. \tag{1.11}$$

Assume that $\varphi(t) = 0$, $t \in [T, T+h)$. Integrating by parts, and using the boundary conditions $z(0) = z(T) = 0$,

$$Dz = -\int_0^{T+h} [D\lambda(t)]\, z(t)\, dt. \tag{1.12}$$

From Lemma 1.1 under the involved suitable hypothesis necessary conditions for (OC) to attain a minimum at $(x, x_{-h}), \zeta, u, v) = (x^*, x^*_{(-h)}, \zeta^*, u^*, v^*) = \underset{\sim}{q}^*$ are that the Fréchet derivative of L in (1.9) is zero at $\underset{\sim}{q}^*$ so that:

$$\left.\begin{aligned} &\tau F_x - \overline{\lambda}(0 - M_x) - \overline{\nu}_1 N_1(x) = 0,\\ &\tau F_{x(-h)} + \overline{\lambda} M_{x(-h)} - \overline{\nu}_2 N_2(x) = 0,\\ &\tau F_\zeta + \overline{\lambda} M_\zeta - \overline{\nu}_3 N_3(x) = 0,\\ &\tau F_u + \overline{\lambda} M_u - \overline{\mu}_1 G_1(v) = 0,\\ &\tau F_v + \overline{\lambda} M_v - \overline{\mu}_2 G_2(v) = 0. \end{aligned}\right\} \tag{1.13}$$

Using Eq. (1.12) into (1.13) for each $z \in X$, and since $z(t) = 0$, $t \in [T, T+h)$, one gets

$$\int_0^T \Big\{\tau f_x(\underset{\sim}{q}^*, t) + \lambda(t) m_x(\underset{\sim}{q}^*, t) + \lambda'(t) \\ -\nu(t) n_{1x}(\underset{\sim}{q}^*, t)\Big\} z(t)\, dt = 0, \tag{1.14}$$

where $\overline{\nu}$ is represented by the function $\nu(\cdot)$, and the Fréchet derivatives $F_{(\cdot)}$ are represented by integrals as above. Therefore $\{\cdot\} = 0$ in (1.14).

A minimum of (OC) at $\underset{\sim}{q}^*$ implies five transversality conditions, namely:

$$\overline{\mu}_1 G_1(u^*); \quad \overline{\mu}_2 G_2(v^*) = 0; \quad \overline{\nu}_1 N_1(x^*) = 0, \\ \overline{\nu}_2 N_2\left(x^*_{(-h)}\right) = 0; \quad \overline{\nu}_3 N_\zeta(\zeta^*) = 0. \tag{1.15}$$

The problem (OC) of Eqs. (1.1)–(1.4) has a local minimum $\underset{\sim}{q}^*$, under the necessary conditions of Lemma 1.1, with the transversality conditions (1.15). It is also useful to investigate sufficient conditions for the existence of a local minimum for the lagrangean theory for problem (OC). For this purpose, the following concept and result are useful. A function g: $X \to \mathbb{R}$ is convex if, for all x, y and $0 < \lambda < 1$, $\lambda g(x) + (1-\lambda)g(y) - g(\lambda x + (1-\lambda)y) \in \mathbb{R}_+$. g is convex iff its epigraph $\{(x, y) \in X \times \mathbb{R}\colon\ y \geqslant g(x)\}$ is a convex set. Let g: $X \to Y$, g is S-convex, where S is a convex cone in Y, if for all x, y and $0 < \lambda < 1$, $\lambda g(x) + (1-\lambda)g(y) - g(\lambda x + (1-\lambda)y) \in S$. If g is Fréchet differentiable, then g is S-convex (S being a closed convex cone in Y) iff, for all z, $x \in X$, $f(x) - f(z) - f'(z)(x - z) \in S$.

For sufficient Lagrangean conditions, the following result (Craven, 1978) is useful.

Lemma 1.2. (1) *Let* (FJ) *hold with* $\tau = 1$, f *convex,* g *S-convex, and* h *affine (linear plus a constant). Then* (P1) *is minimized at* $x = a$.

(2) *Let* (FJ) *hold with* τ, ν *not both zero,* f *convex,* $g(int\ U)$*-convex, where* $U \subset Y$ *is a convex cone such that* $S \subset U$ *and* $\mathcal{V} \subset U^*$, *and* h *T-convex (or affine). Then,* (P1) *is minimized at* a.

Lemmas 1.1 and 1.2 together with the transversality conditions (1.15) lead to the following main result for problems (OC) for system (1.1) to (1.4).

Theorem 1.1. (1) *Eqs.* (1.16) *below are necessary conditions for the differential system of the optimal control of problem* (OC)*, namely* (OCDE)*, if* $\mathcal{G}$ *being the set of systems* $G_i(u) \in K_i$, $N_j(x) \in J_j$ *(*$1 \leqslant i \leqslant 2$; $1 \leqslant j \leqslant 3$*) is locally solvable at a point* $\underset{\sim}{q}^* = (x^*, x^*_{(-h)}, \zeta^*, u^*, v^*) \in int\,(X \times X \times X \times U \times \mathcal{V})$*, the corresponding cone being a closed convex cone (and the others having nonempty interiors) with* $\rho(\underset{\sim}{q}^*) \in \mathcal{G}$ *fulfilling that* $[g'(a)|g(a)]^T(\mathcal{G}^*)$ *is weak* $*$ *closed.*

(2) *In proposition* (1) *the hypothesis of local solvability may be substituted by that of the regularity at* a.

(3) *Propositions* (1)–(2) *stand also under sufficient conditions if, furthermore, equations* (1.16) *hold with* $\tau = 1$, f *convex,* $\mathcal{G}$ *affine and the remaining* G_i, N_j *not being* $\mathcal{G}$*, being* G_i *or* N_j*-convex (*$1 \leqslant i \leqslant 2; 1 \leqslant j \leqslant 3$*), respectively. An alternative sufficient condition is* g, $\mathcal{G}$ *– convex,* f *convex and* $n_{(.)}$, $g_{(.)}$ *being, respectively, int* (N) *or int* (G)*-convex with either* τ *or one of* μ_1, ν_j *(*$1 \leqslant i \leqslant 2$; $1 \leqslant j \leqslant 3$*) being non zero.*

$$\left.\begin{aligned}
&\tau f_x(\underset{\sim}{q}^*, t) + \lambda(t) m_x(\underset{\sim}{q}^*, t) + \lambda'(t) - \nu_1(t) n_{1x}(q^*, t) = 0,\\
&\tau f_{x(-h)}(q^*, t) + \lambda(t) m_{x(-h)}(q^*, t) - \lambda'(t) - \nu_2(t) n_{2x(-h)}(q^*, t) = 0,\\
&\tau f_3(q^*, t) + \lambda(t) m_3(q^*, t) + \lambda'(t) - \nu_3(t) n_{3\zeta}(q^*, t) = 0,\\
&\tau f_u(q^*, t) + \lambda(t) m_u(q^*, t) - \mu_1(t) g_{1\mu}(q^*, t) = 0,\\
&\tau f_v(q^*, t) + \lambda(t) m_v(q^*, t) - \mu_2(t) g_{2_v}(q^*, t) = 0,\\
&\mu_1(t) g_1(q^*, t) = 0; \quad \mu_2(t) g_2(q^*, t) = 0;\\
&\nu_1(t) n_1(q^*, t) = 0; \quad \nu_2(t) n_2(q^*(-h), t) = 0; \quad \nu_3(t) n_3(q^*, t) = 0.
\end{aligned}\right\}\begin{array}{c}(1.16)\\ \text{(OCDE)}\end{array}$$

1.3. *Comments and Relationships with Standard Linear Systems*

The formulation of Section 1.2 has been generalized in a standard way that being applicable to stabdard linear systems (see Minyuk, 1983; Churakova, 1969; Metel'skii, 1978; Kurzanskii, 1966; De la Sen, 1988; Kailath, 1980; Craven, 1978) by including, using topological concepts, inequality constraints in the objective functional. The main conclusion is that the problem of equality (differential system equations) inequality (state, delayed-state, input and disturbance vectors) constraints may be formulated using the concept of S-convex functions in a S closed convex cone for constraints in the levels of the above magnitudes may be directly generalized from the standard linear cases using a Lagrangean theory for the existence of necessary conditions for convex objective functionals, for local solvability – and sufficient (Fritz-John/Kühn-Tücker) conditions for such a solvability, so that the problem reduces to the solution of a standard system of differential equations, including transversal differential constraints (OCDE).

2. Time-Optimal Control, Bang-Bang Principle

The general problem to be considered in this section is the following. Let a control system given by the differential equation:

$$\dot{x}(t) = A(t)x(t) + A_1(t)x(t-h) + \int_0^h R(t,\tau)x(t-\tau)\,\mathrm{d}\tau + B(t)u(t) + f(t) \quad (2.1)$$

subjected to the initial conditions $x_0(t) = x_s$ and $x(t) = \varphi(t)$, $t \in [t_0 - h, t_0)$. This system is a (linear) particular case with state convolution influencing the state time-derivative (see De Guzmán (1980) for the standard linear case).

The following assumptions are made.

(1) $A(t)$ is a real $n \times n$ matrix function, piecewise analytic in $[t_0, T]$, namely $[t_0, T]$ may be devided into intervals $[t_i, t_{i+1}]$ with $t_0 < t_1 < t_2 < \ldots < T$ and the entries of $A(t)$ are analytic in each (t_i, t_{i+1}). At the points t_i, $A(t)$ is continuous either on its left-side or in its right-side.

(2) $A_1(t)$ and $B(t)$ are, respectively, $n \times n$ and $n \times m$ real matrix functions being piecewise analytic in $[t_0, T]$.

(3) $R(t,\tau)$ is a $n \times n$ real matrix function being piecewise analytic in $[t_0, T] \times [t_0, T]$.

(4) $f(t)$ is a n-vector of disturbances being a set of piecewise continuous functions F on $[t_0, T]$.

(5) $u(t)$ is a m-vector piecewise continuous function defined in $[t_0, T]$ which takes values in a constraint compact subset of $\mathbb{R}^m$, $\mathcal{L} = \{u \in \mathbb{R}^m\colon\ |u_i| \leqslant 1;\ i = 1, 2, \ldots, m\}$.

The set $\mathcal{U}$ of such functions is the family of admissible controls.

(6) The pair $(x_0, \varphi(t))$ such that $\varphi\colon\ [t_0 - h, t_0] \times \mathbb{R}^n \to \mathbb{R}^n$ with $\varphi(\cdot)$ being piecewise continuous is the set of initial conditions in the initial-value problem (2.1).

(7) $x(t)$ is the state n-vector in a function space X of elements $x\colon\ [0,t] \times X \times X \times \mathcal{U} \times F \to X$.

An absolutely continuous solution (2.1) is in (De la Sen, 1988).

$$\begin{aligned} x(t) &= F(t,t_0)x_0 + \int_{t_0-h}^{t_0} F(t,s+h)A_1(s+h)\varphi(s)\,\mathrm{d}s \\ &\quad + \int_{t_0-h}^{t_0} \left[\int_{t_0-s}^{h} F(t,\tau+s)R(\tau+s,\tau)\,\mathrm{d}\tau\right]\varphi(s)\,\mathrm{d}s \\ &\quad + \int_{t_0}^{t} F(t,\tau)B(\tau)u(\tau)\,\mathrm{d}\tau + \int_{t_0}^{t} F(t,\tau)f(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \frac{\partial F(t,\tau)}{\partial\tau} &= -F(t,\tau)A(\tau) - F(t,\tau+h)A_1(\tau+h) \\ &\quad - \int_0^h F(t,s+\tau)R(s+\tau,s)\,\mathrm{d}s, \\ F(t,t) &= I; \quad F(t,\tau) \equiv 0, \ \ \tau > t. \end{aligned} \tag{2.3}$$

The attainable set $K(t)$ for $t \geqslant t_0$ is

$$K(t) = \{x(t,u)\colon\ u \in \mathcal{U}\}.$$

For each $t \geqslant t_0$, a compact set $G(t)$ which continuously varies with t is given. This set is the target set. It is assumed that $x_0 \notin G(t_0)$, $\varphi(t) \notin G(t)$, $t \in [t_0-h, t_0)$. If there is $t > t_0$ such that

$$K(t) \cap G(t) \neq \varnothing, \tag{2.4}$$

this control drives x_0, $\varphi(t)$, $t \in [t_0-h, t_0)$ to the target in time t. $u^* \in \mathcal{U}$ is a time-optimal control if there is $t^* > t_0$ such that $x(t^*,u) \in G(t^*)$ and $G(t) \cap K(t) = \varnothing$ for all t with $t_0 \leqslant t \leqslant t^*$.

The following result is useful (Craven, 1978) to establish the bang-bang principle for system (2.1).

Lemma 2.1 [(preliminary)]. (1) *Let* g: $[t_0,t] \to \mathbb{R}^n$ *a piecewise analytic function on* $[t_0,t]$. *Let* $\mathcal{V}$ *be the set of functions from* $[t_0,t]$ *to* $[-1,1]$ *nonoscillant* (see Craven, 1978) *and piecewise continuous. Define the sets*

$$\begin{aligned} &\mathcal{V}_0 = \{v \in \mathcal{V}\colon\ \forall s \in [t_0,t],\ |v(s)| = 1\}, \\ &K = \left\{\int_{t_0}^{t} g(s)v(s)\,\mathrm{d}s\colon\ v \in \mathcal{V}\right\}, \\ &K_0 = \left\{\int_{t_0}^{t} g(s)v(s)\,\mathrm{d}s\colon\ v \in \mathcal{V}_0\right\}. \end{aligned} \tag{2.5}$$

Then, $K = K_0$ *and it is a compact and convex set.*

(2) *Let* $Y(s)$ *be a piecewise analytic* $n \times m$*-matrix function defined on* $[t_0,t]$. *Let* $\mathcal{V}$ *be the set of piecewise* m*-vector functions on* $[t_0,t]$ *taking values in* $\mathcal{L}$ *and being*

nonoscillant. Define sets $\mathcal{V}_0$, K and K_0 as in proposition (1) *with the changes $g \to Y$ and $v \to v_1$ (in Definition of $\mathcal{V}_0$ only). Then, proposition 1 of the Lemma stands.*

(3) *Let $\mathcal{F}$ a family of compact sets $\mathbb{R}^n$. Introduce a metric ρ as follows:*

$$\rho(K_1, K_2) = \inf\{\varepsilon > 0\colon\ K_{1\varepsilon} \supset K_2,\ K_{2\varepsilon} \supset K_1\}$$

for each K_1, $K_2 \in \mathcal{F}$, where if A is a compact set, then

$$A_\varepsilon = U\left\{\overline{B}(x, \varepsilon)\colon\ x \in A\right\}.$$

Then if P and Q are two compact and convex sets $\rho(P, Q) = \rho(\partial(P), \partial(Q))$.

Now, Lemma 2.1 is used in the subsequent formulation to establish the bang-bang principle for the linear system with after-effect (2.1).

2.1. *Bang-Bang Principle*

$K(t) = K_0(t)$ is a compact and convex set is an immediate consequence of Lemma 2.1. The continuity is proved as follws. Let $t_2 \in [t_0, \infty)$. It is now proved that given a real constant $\varepsilon > 0$. There is $\delta > 0$ such that if $|t_3 - t_2| \leqslant \delta$, then $\rho(K(t_3), K(t_2)) \leqslant \varepsilon$; so that:

a) for all $x(t_3, u) \in K(t_3)$, there is $x(t_2; \overline{u})$ such that $|x(t_3, u) - x(t_2, \overline{u})| \leqslant \varepsilon$;

b) for all $x(t_2, \widetilde{u}) \in K(t_2)$, there is $x(t_3; \widehat{u}) \in K(t_3)$ such that $|x(t_2, u) - x(t_3, \widehat{u})| \leqslant \varepsilon$.

For some $\widetilde{u}, \widetilde{u} \in \mathcal{U}$ and each $u \in \mathcal{U}$. Assume now $t_3 > t_2$, then one gets from (2.2)

$$\begin{aligned}
|x(t_3, u) - x(t_2, \widetilde{u})| = \Bigg| \Bigg[& F(t_3, t_0)x_0 \\
& + \int_{t_0}^{t_3} F(t_3, s+h)A_1(s+h)\varphi(s)\,\mathrm{d}s \\
& + \int_{t_0-h}^{t_0} [F(t_3, \tau+s)R(s+\tau, \tau)\,\mathrm{d}\tau]\,\varphi(s)\,\mathrm{d}s \\
& + \int_{t_0}^{t_3} F(t_3, \tau)B(\tau)u(\tau)\,\mathrm{d}\tau + \int_{t_0}^{t_3} F(t_3, \tau)f(\tau)\,\mathrm{d}\tau \Bigg] \\
& - \Bigg[F(t_2, t_0) + \int_{t_0}^{t_2} F(t_2, s+h)A_1(s+h)\varphi(s)\,\mathrm{d}s \\
& + \int_{t_0-h}^{t_0} \left[\int_{t_0-s}^{h} F(t_2, \tau+s)R(\tau+s, \tau)\,\mathrm{d}\tau \right] \varphi(s)\,\mathrm{d}s \\
& + \int_{t_0}^{t_2} F(t_2, \tau)B(\tau)u(\tau)\,\mathrm{d}\tau + \int_{t_0}^{t_2} F(t_2, \tau)f(\tau)\,\mathrm{d}\tau \Bigg] \Bigg|
\end{aligned}$$

$$
\begin{aligned}
= \Bigg| & (F(t_3, t_0) - F(t_2, t_0))\, x_0 \\
& + \int_{t_0}^{t_2} [F(t_3, s+h) - F(t_2, s+h)]\, A_1(s+h)\varphi(s)\,\mathrm{d}s \\
& + \int_{t_0-h}^{t_0} \left[\int_{t_0-s}^{h} (F(t_3, \tau+s) - F(t_2, \tau+s))\, R(s+\tau, \tau)\,\mathrm{d}\tau \right] \\
& + \int_{t_0}^{t_2} [F(t_3, \tau)\,(B(\tau)u(\tau) + f(\tau)) - F(t_2, \tau)\,(B(\tau)\widetilde{u}(\tau) + f(\tau))]\,\mathrm{d}\tau \\
& + \int_{t_2}^{t_3} F(t_3, s+h) A_1(s+h)\varphi(s)\,\mathrm{d}s \\
& + \int_{t_2}^{t_3} F(t_3, \tau)\,(B(\tau)u(\tau) + f(\tau))\,\mathrm{d}\tau \Bigg|.
\end{aligned}
\tag{2.6}
$$

Since $x(t_3, u) \in K(t_3)$, take $\widetilde{u} = u$ in $[t_0, t_2]$. Since $F(t, \tau)$ is continuous and $B(t)$, $v(t)$ are piecewise continuous in $[t_0, T]$, and $\mathcal{U}$ is a compact set, it is clear that for each $u \in \mathcal{U}$, $|x(t_3, u) - x(t_2, u)| \leqslant \varepsilon$. To prove (a)–(b) for some $\widetilde{u}, \widehat{u} \in \mathcal{U}$, notice that since $\mathcal{U}$ is compact and t_2 is finite for each real constant $\varepsilon_1 > 0$, there is $\widetilde{u} \in \mathcal{U}$ such that $|x(t_2, u)| \leqslant |x(t_2, \widetilde{u})| + \varepsilon_1$. Let $\varepsilon_1' > 0$ an arbitrary constant $\varepsilon > 0$ verifying that $|x(t_3, u) - x(t_2, u)| \leqslant \varepsilon$, then

$$
\begin{aligned}
\varepsilon_1' & \geqslant |x(t_3, u) - x(t_2, u)| \geqslant |x(t_3, u)| - |x(t_2, u)| \\
& \geqslant |x(t_3, u)| - |x(t_2, \widetilde{u})| - \varepsilon_1 \\
& \geqslant |x(t_3, u) - x(t_2, \widetilde{u})| - \varepsilon_1,
\end{aligned}
\tag{2.7}
$$

so that $|x(t_3, u) - x(t_2, \widetilde{u})| \leqslant \varepsilon$ follows with $\varepsilon = \varepsilon_1 + \varepsilon_1'$. Result (b) follows using similar arguments.

The above results permit us to enounce the following result.

Theorem 2.1 (Bang-Bang principle)**.** *The function* $K\colon\ t \in [t_0, T] \to K(t) \in \mathcal{F}$ *(*$K(t)$ *being a compact and convex set) is continuous in the metric* ρ *defined in* $\mathcal{F}$ *(in Lemma* 2.1*). Furthermore, if* $\mathcal{U}_0 = \{u \in \mathcal{U}\colon\ \forall s \in [t_0, T],\ V_i = 1, 2, \ldots, m,\ |u_i(s)| = 1\}$, $K_0(t) = \{x(t, u)\colon\ u \in \mathcal{U}_0\}$*, then* $K(t) = K_0(t)$*.*

The above theorem states the reachable set may be generated continuously using bang-bang control in the same way as in the standard linear case without after-effect (De Guzmán, 1980). The following (result-) corollaries may be established without difficulty:

(A) If there is one $u_0^* \in \mathcal{U}_0$ which is optimal, then it is optimal in $\mathcal{U}$.

(B) If there is an optimal control in $\mathcal{U}_0$, then there is a bang-bang optimal control.

2.2. *Existence of the Optimal Control. Pontryagin's Maximum Principle*

The next result is reffered to the existence of optimal control assumed that the target is attainable. It follows by direct extension of results for the linear case by using Lemma 2.1.

Theorem 2.2. *Assume there is* $t \geqslant 0$ *such that* $K(t) \cap G(t) \neq \varnothing$ *for system* (2.1). *Then, there is an optimal control* u^* *which makes to verify* $x(t^*, u^*) \in \partial K(t^*)$.

$u \in \mathcal{U}$ is extremal iff $x(t, u) \in \partial K(t)$. Theorem 2.2 establishes that optimal control is necessarily extremal. Next theorem gives a necessary and sufficient condition for an optimal control to exist.

Theorem 2.3 (Pontryagin's principle)**.** *A control* u^* *is an extremal for system* (2.1) *in* $[t_0, t^*] \subset [t_0, T]$ *iff there is a* n*-vector function* $\lambda(t)$ *being identically zero whose time-derivative exists in* $[t_0, T^*]$ *and verifies:*
a) $\lambda'(t) = -\dfrac{\partial F^*(t,\tau)}{\partial \tau}\lambda(t)$; b) *for all* $t \in [t_0, t^*)$, *it follows that*

$$\langle \lambda(t), B(t)u^*(t)\rangle = \max\left\{\langle \lambda(t), B(t)\rangle \colon\ u \in \mathcal{U}\right\}.$$

Proof. Assume that $x(t^*, u) \in \partial K(t^*)$. Since $K(t^*)$ is convex and compact, there is a tangent hyperplane Π of $K(t^*)$ in $x(t^*, u^*)$ such that the outward normal $\lambda(t^*)$ in $x(t^*, u^*)$ verifies the equation of Π, $\langle \zeta - x(t^*, u^*), \lambda(t^*)\rangle = 0$. The points of $K(t^*)$ are in the affine half-space determined by Π given by $\langle \zeta - x(t^*, u^*), \lambda(t^*)\rangle \leqslant 0$. Define the function $\lambda\colon\ [t_0, t^*] \to \mathbb{R}^n$ by $\lambda(t) = F(t, t^*)\lambda(t^*)$ where $F(t,\tau)$ is given by Eq. 2.3. Assume that there is $t_1 \in [t_0, t^*)$ such that $\langle \lambda(t_1), B(t_1)u^*(t_1)\rangle < \max\{\langle \lambda(t_1), B(t_1)u\colon\ u \in \mathcal{U}\}$. By continuity, $\langle \lambda(s), B(s)u^*(s)\rangle < \max\{\langle \lambda(s), B(s)u\colon\ u \in \mathcal{U}\}$ for all s in an open interval of $[t_0, t^*)$. Define now a function $\overline{u}\colon\ [t_0, t^*] \to \mathcal{U}$ such that

$$\langle \lambda(s), B(s)\overline{u}(s)\rangle = \max\left\{\langle \lambda(s), B(s)u\rangle :\ u \in \mathcal{U}\right\} \tag{2.8}$$

which is defined for each $s \in [t_0, t^*)$ by taking the vector $\overline{u}(s)$ with the same direction as that of $B^*(s)\lambda(s)$ and maximum modulus with the constraint $\overline{u}(s) \in \mathcal{U}$. If $B^*(s)\lambda(s) = 0$, then $\overline{u}(s)$ is arbitrary piecewise continuous in $\mathcal{U}$. $\overline{u}$ is admissible. Then,

$$\begin{aligned}\langle \lambda(t^*), x(t^*, \overline{u})\rangle = \Big\langle &\lambda(t^*), F(t^*, t_0)x_0 \\ &+ \int_{t_0-h}^{t_0} F(t^*, s+h)A_1(s+h)\varphi(s)\,\mathrm{d}s \\ &+ \int_{t_0-h}^{t_0}\left[\int_{t_0-s}^{h} F(t^*, \tau+s)R(\tau+s,\tau)\,\mathrm{d}\tau\right]\varphi(s)\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+ \int_{t_0}^{t^*} F(t^*,\tau)\,(B(\tau)\overline{u}(\tau) + f(\tau))\,\mathrm{d}\tau\Big\rangle \\
&> \Big\langle \lambda(t^*), F(t^*,t_0)x_0 + \int_{t_0-h}^{t_0} F(t^*,s+h)A_1(s+h)\varphi(s)\,\mathrm{d}s \\
&+ \int_{t_0-h}^{t_0}\left[\int_{t_0-s}^{h} F(t^*,\tau+s)R(\tau+s,\tau)\,\mathrm{d}\tau\right]\varphi(s)\,\mathrm{d}s \\
&+ \int_{t_0}^{t^*} F(t^*,\tau)\,(B(\tau)u^*(\tau) + f(\tau))\,\mathrm{d}\tau\Big\rangle \\
&= \langle \lambda(t^*), x(t^*,u^*)\rangle\,. \qquad (2.9)
\end{aligned}
$$

Namely, $\langle \lambda(t^*), x(t^*,\overline{u}) - x(t^*,u^*)\rangle > 0$ so that $x(t^*,u^*) \notin K(t^*)$ which is a contradiction. Then (b) holds. By hypothesis, $x(t^*,u^*) \in \partial K(t^*)$ from Theorem 2.2. If $x(t^*,u^*) \in \operatorname{int}(K(t^*))$, consider $x(t^*,\overline{u}) \in K(t^*)$ such that $\langle \lambda(t^*), x(t^*,u^*) - x(t^*,\overline{u})\rangle < 0$. Such points exist since there is a ball of center $x(t^*,u^*)$ contained in $K(t^*)$. From hypothesis, $\langle \lambda(t), B(t)u^*(t)\rangle > \langle \lambda(t), B(t)\overline{u}(t)\rangle$, all $t \in [t_0,t^*)$. Then, $\langle \lambda(t^*), x(t^*,u^*)\rangle \geqslant \langle \lambda(t^*), x(t^*,\overline{u})\rangle$.

The above proof implies that once $\lambda(t^*)$ is fixed, $\lambda(s)$ results fixed, and condition b) fixes for $\forall s \in [t_0,t^*]$ such that $B^*(s)\lambda(s) = 0$.

3. Conclusion

This apper has dealt with the study of optimal control for systems with after-effect. The first part has studied the extention of an optimal control for the nonlinear case and arbitrary convex nonlinear functionals by extending Lagrangean theory. The second part has dealt with the derivation of the Pontryagin's maximum principle for time-optimal control of analytic linear systems with after-effect. In both cases, it has been found that well-known results for the standard linear case can be directly extended without difficulty.

4. Aknowledgments

The author is very grateful to Comisión Asesora de Investigación Cientifica y Técnica by its partial support of this work (Project 968/84). The author is also grateful to Mrs. M.J. González Gómez by her suggestion to improve the mathematical notation.

References

Churakova, S.V. (1969). *Differentsial'nye Uravneniya*, **5**(6), 1068–1075.
Craven, B.O. (1978). *Mathematical Programming and Control Theory*, John Wileyand Sons, New York.
De Guzmán, M. (1980). *Ordinary Differential Equations. Stability and Control*, Alhambra, Madrid (2nd Edition).

Kailath, M. (1980). *Linear Systems*, Prentice-Hall, Englewood Cliffs.
Kurzhanskii, A.B. (1966). *Prikl. Mat. Mekh.*, **30**(6).
Metel'skii, A.V. (1978). *An Investigation of Controllability and Observability of Certain Linear Stationary Systems with After-Effect*. Abstract of Candidate's Dissertation, Physico-Mathematical Sciences. Minsk.
Minyuk, S.A. (1983). The theory of the control of linear systems with time-lag. *Differentsial'nye Uravneniya*, **19**(4), 575–581.
De la Sen, M. (1988). *Math. Comput. Modelling*, **10**(7), part I, 473–489; part II, 491–502.

M. de la Sen obtained the Ph.D. degree in Applied Physics from the Basque Country University in 1979 and the degree of "Docteur d'Etat-ès-Scienes Physiques" (specialitè Automatique et Traitement du Signal) from the Universitè de Grenoble, France in 1987. He has had several teaching positions in the Basque Country University in Bilbao, Spain, where he is currently Proffessor of Systems and Control Engineering. He has also had the position of visiting Professor in the University of Grenoble, France and the University of Newcastle, New South Wales, Australia. He has been author or coauthor of number of papers in the field of adaptive systems, mathematical systems theory and ordinary differential equations, which are his research interest subjects. He acts or has acted as reviewer for several international journls of control theory and applied mathematics.

Apie diferencialinių sistemų su liekamuoju poveikiu optimalų valdymą

Manuel de la SEN

Straipsnyje nagrinėjamas sistemų su liekamuoju poveikiu optimalus valdymas. Pirmoje dalyje analizuojamas optimalus valdymas netiesiniu atveju panaudojant Lagranžo teoriją. Antroje dalyje Pontriagino maksimumo principas pritaikomas tiesinių sistemų su liekamuoju poveikiu optimaliam valdymui.