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LIKELIHOOD INFERENCE ABOUT A CHANGE POINT IN SWITCHING AUTOREGRESSION

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Abstract. A likelihood approach is considered to the problem of making inferences about the point $t = \nu$ in a Gaussian autoregressive sequence $\{X_t,\ t = 1 \div N\}$ at which the underlying AR(p) parameters undergo a sudden change. The statistics of a loglikelihood function $L(n, \nu)$ is investigated over the admissible values $n \in (p + 1, \dots, N - 1)$ of a change point ν under validity of hypothesis of a change and no change. The expressions of $L(n, \nu)$ implying the loss of plausibility when moving away from the true change point ν are presented, and the probabilities $P\{\widehat{\nu}_N = \nu \pm r\}$, $r = 0, 1, 2, \dots$, where $\widehat{\nu}_N$ is the MLH estimate of a change point ν from the available realization $x_1, x_2, \dots, x_N$ of $\{X_t,\ t = 1 \div N\}$ are considered.

Key words: change point problem; likelihood inference; autoregressive sequences.

1. Introduction. There is a great number of situations when the assumed structure of a stochastic model at the beginning of an observation becomes unfit at the end. The most simple assumption is to set up that an abrupt change of parameters has occurred somewhere. Analogous examples can be found in quality control, econometrics, study of biomedical signals, etc.

To make a statistical inference about a change point, it is necessary: i) to determine if there exists any change point in the observed sequence – the output of a stochastic model; ii) to estimate the change point; iii) to investigate the statistical properties of the involved statistics and estimates. This paper is devoted to the last two goals but the crucial point is to evaluate the contribution to the inference of the magnitude of change in various parameters of the considered model before and after a change. As for detection of changes, there is a wide choice of approaches (see, e.g., surveys Basseville, 1988; Kligienė and Telksnys, 1983).

2. Model assumptions and notations. Let $\{X_t^{(i)}, t = 0, \pm 1, \ldots\}$, $i = 1, 2$, be two stationary, Gaussian autoregressive sequences of finite order p, i.e., $AR^{(i)}(p)$ described by the equations

$$\sum_{j=0}^{p} a_j^{(i)} (X_t^{(i)} - \mu^{(i)}) = \varepsilon_t, \qquad i = 1, 2, \tag{1}$$

where the sequence $\{\varepsilon_t,\ t = 0, \pm 1, \pm 2, \ldots\}$ consists of independent normally distributed variables $\varepsilon_t \sim \mathcal{N}(0, 1)$. The assumption of stationarity involves a condition that all the roots of the equation $\sum\limits_{j=0}^{p} a_j^{(i)} z^{p-j} = 0$ are less than 1 in the absolute value, i.e., the parameters $\theta^{(i)} = (\mu^{(i)}, a_0^{(i)}, \ldots, a_p^{(i)})$ of model (1) belong to the stationarity area $\Theta : \theta^{(i)} \in \Theta \subset R^k$, $k > p$.

Let us introduce a change point as an unknown time instant $t = \nu$ at which the process $\{X_t^{(1)}\}$ switches to $\{X_t^{(2)}\}$, resulting a new process:

$$X_t = \begin{cases} X_t^{(1)}, & t = \ldots, \nu - 1, \nu \\ X_t^{(2)}, & t = \nu + 1, \nu + 2, \ldots \end{cases} \tag{2}$$

represented by a single realization $x_{1N} = (x_1, x_2, \dots, x_N)$. We may set up the following two hypotheses of homogeneity: $H_1(\theta = \theta^{(1)})_N : \nu > N$; $H_2(\theta = \theta^{(2)})_N : \nu < 1$ against the alternative of a change: $H_{12}(\theta^{(1)}, \nu, \theta^{(2)})_N$: there exists an unknown change time ν, where $p < \nu \leqslant N - 1$. The alternative itself contains the multiple hypotheses $H_{12}(\theta^{(1)}, n, \theta^{(2)})_N$, $n = p+1, p+2, \dots, N-1$ with $N - p$ admissible values n of a change point ν.

Denote by $\ln p_{H_i}(x_{1N}; \theta^{(i)})$, $i = 1, 2$ and $\ln p_{H_{12}}(x_{1N}; \theta^{(1)}, n, \theta^{(2)})$, $n = p + 1, \dots, N$ the corresponding loglikelihood (LLH) functions. We shall investigate the LLH statistics $\ln p_{H_{12}}(X_{1N}; \theta^{(1)}, n, \theta^{(2)})$, $n = p + 1, \dots, N$ when the true hypothesis is $H_{12}(x_{1N}; \theta^{(1)}, \nu, \theta^{(2)})$ including special cases $\nu = N$ (H_1 is true) and $\nu = 0$ (H_2 is true).

The test statistics will be investigated under the assumptions:

(A1) a change point ν is an unknown non-random parameter with a set of admissible values $n \in \Lambda = (p+1, \dots, N-1)$;

(A2) $\theta^{(1)} \neq \theta^{(2)}$, and $\theta^{(1)}, \theta^{(2)}$ are known parameters belonging to the stationarity area Θ;

(A3) the sequences $\{X_t^{(1)}\}$ and $\{X_t^{(2)}\}$ are independent for every admissible value $n \in \Lambda$;

(A4) the order p of the equation (1) is a small number in comparison with ν and $N - \nu$.

Let for $i = 1, 2$

$$\begin{aligned} X_{1N}^{(i)} &= {}^T(X_1^{(i)}, X_2^{(i)}, \dots, X_N^{(i)}), \\ \mu_{1N}^{(i)} &= {}^T(\mu^{(i)}, \mu^{(i)}, \dots, \mu^{(i)}) \triangleq \mu^{(i)}, \end{aligned} \tag{3}$$

where ${}^T(\cdot)$ means transposed array, i.e, column vector and

$$A_N^{(i)} = \begin{pmatrix} & \ddots & & & & \mathbf{0} \\ a_p^{(i)} & \dots & a_1^{(i)} & a_0^{(i)} & & \\ & \ddots & & \ddots & \ddots & \\ \mathbf{0} & & a_p^{(i)} & \dots & a_1^{(i)} & a_0^{(i)} \end{pmatrix}. \tag{4}$$

Let $X_{1N}^{(i)}$ be a vector generated by model (1) which now can be rewritten as

$$A_N^{(i)}(X_{1N}^{(i)} - \mu_{1N}^{(i)}) = \varepsilon_{1N}. \tag{5}$$

Let us express for the sake of convenience the covariance matrix $\Sigma_N^{(i)}$ of $X_{1N}^{(i)}$ and its inverse $\overline{\Sigma}_N^{(i)}$ in terms of $A_N^{(i)}$:

$$X_{1N}^{(i)} - \mu_{1N}^{(i)} = \bar{A}_N^{(i)} \varepsilon_{1N}, \tag{6}$$

$$\begin{aligned} \Sigma_N^{(i)} &= E_{H_i}\Big\{ \left(X_{1N}^{(i)} - \mu_{1N}^{(i)}\right) \cdot {}^T\left(X_{1N}^{(i)} - \mu_{1N}^{(i)}\right) \Big\} \\ &= \bar{A}_N^{(i)} E_{H_i}\Big\{\varepsilon_{1N} \cdot {}^T\varepsilon_{1N}\Big\}\bar{A}_N^{(i)} = \bar{A}_N^{(i)} \cdot {}^T\bar{A}_N^{(i)}, \qquad (7) \\ \overline{\Sigma}_N^{(i)} &= {}^T A_N^{(i)} \cdot A_N^{(i)}. \end{aligned}$$

The model of change (2) can be transformed into the following one:

$$\begin{aligned} X_{1\nu} &= \mu_{1\nu}^{(1)} + \bar{A}_\nu^{(1)} \varepsilon_{1\nu}, \\ X_{\nu+1,N} &= \mu_{\nu+1,N}^{(2)} + \bar{A}_{N-\nu}^{(2)} \varepsilon_{\nu+1,N}. \end{aligned} \tag{8}$$

3. Test statistics. According to the assumptions (A1)–(A3) we can write out the expressions of LLH functions for each hypothesis

$$\begin{aligned} &\ln p_{H_i}(X_{1N}; \theta^{(i)}) = c_i + (N-p) \ln a_0^{(i)} \\ &- \frac{1}{2} \, {}^T(X_{1N} - \mu_{1N}^{(i)}) \, {}^T A_N^{(i)} A_N^{(i)} (X_{1N} - \mu_{1N}^{(i)}), \end{aligned} \tag{9}$$

$$\ln p_{H_{12}}(X_{1N};\theta^{(1)},n,\theta^{(2)}) = c_{12} + n\ln\frac{a_0^{(1)}}{a_0^{(2)}}$$
$$-\frac{1}{2}\,{}^T(X_{1n}-\mu_{1n}^{(1)})\,{}^TA_n^{(1)}A_n^{(1)}(X_{1n}-\mu_{1n}^{(1)}) \qquad (10)$$
$$-\frac{1}{2}\,{}^T(X_{n+1,N}-\mu_{n+1,N}^{(2)})\,{}^TA_{N-n}^{(2)}$$
$$\times A_{N-n}^{(2)}(X_{n+1,N}-\mu_{n+1,N}^{(2)}).$$

For every $n = p+1,\dots,N-1$ and fixed ν denote by

$$L(n,\nu) = \ln p_{H_{12}}(x_{1N};\theta^{(1)},n,\theta^{(2)}) - \ln p_{H_{12}}(x_{1N};\theta^{(1)},\nu,\theta^{(2)}) \qquad (11)$$

the difference between LLH values at the point n and the true change point ν. $L(n,\nu)$, as a function of n, represents the loss of plausibility when moving away from the point ν. Evidently: $L(\nu,\nu)=0$.

Lemma 1. *In the change point model (2), where the autoregressive process $\{X_t^{(1)}\}$ switches to $\{X_t^{(2)}\}$ at an unknown time instant $t=\nu$ under validity of the assumptions (A1)–(A4), the loss of plausibility $L(n,\nu)$ (11) has the expression*

$$L(n,\nu) = (n-\nu)\ln\frac{a_0^{(1)}}{a_0^{(2)}} - \frac{1}{2}\Big[\,{}^T(\mu^{(1)}-\mu^{(2)})\overline{\Sigma}_{\nu-n}^{(2)}$$
$$\times(\mu^{(1)}-\mu^{(2)}) + {}^T\varepsilon_{n+1,\nu}(R_{\nu-n}^{21}-I_{\nu-n})\varepsilon_{n+1,\nu} \qquad (12)$$
$$+2\,{}^T(\mu^{(1)}-\mu^{(2)})\overline{\Sigma}_{\nu-n}^{(2)}\bar{A}_{\nu-n}^{(1)}\varepsilon_{n+1,\nu}\Big], \quad \text{if } n<\nu;$$

$$L(n,\nu) = (n-\nu)\ln\frac{a_0^{(1)}}{a_0^{(2)}} - \frac{1}{2}\Big[\,{}^T(\mu^{(1)}-\mu^{(2)})\overline{\Sigma}_{n-\nu}^{(1)}$$
$$\times(\mu^{(1)}-\mu^{(2)}) + {}^T\varepsilon_{\nu+1,n}(R_{n-\nu}^{12}-I_{n-\nu})\varepsilon_{\nu+1,n} \qquad (13)$$
$$+2\,{}^T(\mu^{(1)}-\mu^{(2)})\overline{\Sigma}_{n-\nu}^{(1)}\bar{A}_{n-\nu}^{(2)}\varepsilon_{\nu+1,n}\Big], \quad \text{if } n>\nu,$$

where ε_{1N} is a vector of independent normally distributed $\varepsilon_t \sim \mathcal{N}(0,1)$ and I_k – the matrix of identity,

$$R_k^{ij} = {}^T\bar{A}_k^{(j)}\overline{\Sigma}_k^{(i)}\bar{A}_k^{(j)}, \quad i,j=1,2, \quad i \neq j. \tag{14}$$

The proof involves evident transformations of (10), (11) (different for $n < \nu$ and $n > \nu$), when the hypothesis $H_{12}(\theta^{(1)}, \nu, \theta^{(2)})_N$ is true and the model (8) is valid. The dimensions of matrices and vectors are diminished from N, $N-\nu$ to $|n-\nu|$ mainly due to the assumption (A4).

COROLLARY 1. If the hypothesis $H_1(\theta^{(1)})_N$ is true, the expression (12) is valid with $\nu = N$; if $H_2(\theta^{(2)})_N$ is true, the statistics $L(n,\nu)$ is expressed by (13) with $\nu = 0$.

Denote by $\lambda_{t,k}^{ij}$, $t = 1,2,\ldots,k$, the eigenvalues of matrix R_k^{ij} and by $\gamma_{t,k}^{ij}$, $t = 1,2,\ldots,k$, the eigenvalues of matrix $\overline{\Sigma}_k^{(i)} \cdot \bar{A}_k^{(j)}$. The results (12), (13) of Lemma 1 may be transformed into a non-vectorial from

$$\begin{aligned} L(n,\nu) = (n-\nu)\Big\{ \ln\frac{a_0^{(1)}}{a_0^{(2)}} + \frac{1}{2}(\mu^{(1)}-\mu^{(2)})^2\varkappa_0^{(2)} \\ + \frac{1}{2(\nu-n)}\sum_{t=n+1}^{\nu}(\lambda_{t,\nu-n}^{21}-1)\varepsilon_t^2 \\ + \frac{1}{\nu-n}(\mu^{(1)}-\mu^{(2)})\sum_{t=n+1}^{\nu}\gamma_{t,\nu-n}^{21}\varepsilon_t\Big\}, \quad \text{if } n<\nu; \end{aligned} \tag{15}$$

$$\begin{aligned} L(n,\nu) = (n-\nu)\Big\{ \ln\frac{a_0^{(1)}}{a_0^{(2)}} - \frac{1}{2}(\mu^{(1)}-\mu^{(2)})^2\varkappa_0^{(1)} \\ - \frac{1}{2(\nu-n)}\sum_{t=\nu+1}^{n}(\lambda_{t,n-\nu}^{12}-1)\varepsilon_t^2 \\ + \frac{1}{n-\nu}(\mu^{(1)}-\mu^{(2)})\sum_{t=\nu+1}^{n}\gamma_{t,n-\nu}^{12}\varepsilon_t\Big\}, \quad \text{if } n>\nu; \end{aligned} \tag{16}$$

where $\varkappa_0^{(i)}$ is the diagonal element of an asymptotically Toeplitz matrix $\overline{\Sigma}_k^{(i)}$. The matrix $\overline{\Sigma}_k^{(i)}$ is symmetric and it is not difficult to prove that $\frac{1}{k}\sum_t \lambda_{t,k}^{(i)} = \varkappa_0^{(i)}$, where $\lambda_{t,k}^{(i)}$, $t = 1, 2, \ldots, k$, are the eigenvalues of the matrix $\overline{\Sigma}_k^{(i)}$.

COROLLARY 2.

$$\begin{aligned} E_{H12}\{L(n,\nu)\} &= (n-\nu)\Big[\ln\frac{a_0^{(1)}}{a_0^{(2)}} \\ &\quad + \frac{1}{2}(\mu^{(1)} - \mu^{(2)})^2 \varkappa_0^{(2)} - \frac{1}{2}(1 - \sigma_{21}^2)\Big] \\ &\triangleq (n-\nu)m_1, \quad m_1 > 0, \quad \text{if } n < \nu; \end{aligned} \tag{17}$$

$$\begin{aligned} E_{H_{12}}\{L(n,\nu)\} &= (n-\nu)\Big[\ln\frac{a_0^{(1)}}{a_0^{(2)}} \\ &\quad - \frac{1}{2}(\mu^{(1)} - \mu^{(2)})^2 \varkappa_0^{(1)} + \frac{1}{2}(1 - \sigma_{12}^2)\Big] \\ &\triangleq (n-\nu)m_2, \quad m_2 < 0, \quad \text{if } n > \nu; \end{aligned} \tag{18}$$

where

$$\begin{aligned} \sigma_{ij}^2 = \sum_{k,l=0}^{p} a_k^{(i)} a_l^{(i)} E_{H_j}\Big\{(X_{t+k}^{(j)} - \mu^{(j)}) \\ \times (X_{t+l}^{(j)} - \mu^{(j)})\Big\}, \qquad i,j = 1,2, \quad i \neq j. \end{aligned} \tag{19}$$

The proof of (17), (18) is based essentially on the results of Kligienė (1989).

In statistics it is quite natural when a test statistics is investigated under the null hypothesis (H_1) and under the alternative one (H_2). However, in practice we can meet a non-standard situation when a hypothesis, different from H_1 and H_2, is true while the test statistics is fitted to distinguish between H_1 and H_2. It is interesting to investigate the behaviour of a test statistics under that other hypothesis which does not coincide with H_1 or H_2. In the change point problem very natural is to consider the likelihood ratio statistics (to distinguish between $H_1(\theta = \theta^{(1)})_N$ and $H_2(\theta = \theta^{(2)})_N$) while the hypothesis of change $H_{12}(\theta^{(1)}, \nu, \theta^{(2)})_N$ is true.

Lemma 2. *If the hypothesis of a change $H_{12}(\theta^{(1)}, \nu, \theta^{(2)})_N$ is true, the loglikelihood ratio equals to*

$$\ln \frac{p_{H1}(x_{1N}; \theta^{(1)})}{p_{H2}(x_{1N}; \theta^{(2)})} = L(N, \nu) - L(0, \nu) \triangleq \Delta \qquad (20)$$

and the inference in favour of H_1 or H_2 depends on a sign of the value Δ.

Proof. Using the expressions (9) and the analogous transformations as for deriving (12), (13), we have

$$\begin{aligned}
&\ln p_{H1}(x_{1N}; \theta^{(1)}) - \ln p_{H_2}(x_{1N}; \theta^{(1)}) = c_{12} + (N-p) \ln \frac{a_0^{(1)}}{a_0^{(2)}} \\
&- \frac{1}{2}\Big\{ {}^T\varepsilon_{1\nu} I_\nu \varepsilon_{1\nu} + {}^T(\mu^{(2)} - \mu^{(1)}) \overline{\Sigma}_{N-\nu}^{(1)} (\mu^{(2)} - \mu^{(1)}) \\
&+ {}^T\varepsilon_{\nu+1,N} R_{N-\nu}^{12} \varepsilon_{\nu+1,N} + 2\, {}^T(\mu^{(2)} - \mu^{(1)}) \overline{\Sigma}_{N-\nu}^{(1)} \\
&\times \bar{A}_{N-\nu}^{(2)} \varepsilon_{\nu+1,N} \Big\} + \frac{1}{2}\Big\{ {}^T(\mu^{(1)} - \mu^{(2)}) \overline{\Sigma}_\nu^{(2)} (\mu^{(1)} - \mu^{(2)}) \\
&+ 2\, {}^T(\mu^{(1)} - \mu^{(2)}) \overline{\Sigma}_\nu^{(2)} \bar{A}_\nu^{(1)} \varepsilon_{1\nu} + {}^T\varepsilon_{1\nu} R_\nu^{21} \varepsilon_{1\nu} \\
&+ {}^T\varepsilon_{\nu+1,N} I_{N-\nu} \varepsilon_{\nu+1,N} \Big\} = c_{12} + (N-p) \ln \frac{a_0^{(1)}}{a_0^{(2)}} \\
&+ \frac{1}{2}\, {}^T(\mu^{(1)} - \mu^{(2)}) \overline{\Sigma}_\nu^{(2)} (\mu^{(1)} - \mu^{(2)}) + \frac{1}{2}\, {}^T\varepsilon_{1\nu} (R_\nu^{21} - I_\nu) \varepsilon_{1\nu} \\
&+ {}^T(\mu^{(1)} - \mu^{(2)}) \overline{\Sigma}_\nu^{(2)} \bar{A}_\nu^{(1)} \varepsilon_{1\nu} - {}^T(\mu^{(1)} - \mu^{(2)}) \overline{\Sigma}_{N-\nu}^{(1)} \\
&\times (\mu^{(1)} - \mu^{(2)}) - \frac{1}{2}\, {}^T\varepsilon_{\nu+1,N} (R_{N-\nu}^{12} - I_{N-\nu}) \varepsilon_{\nu+1,N} -
\end{aligned}$$

$$- \ {}^{T}(\mu^{(1)} - \mu^{(2)})\overline{\Sigma}^{(1)}_{N-\nu}\bar{A}^{(2)}_{N-\nu}\varepsilon_{\nu+1,N}$$
$$= L(N,\nu) - L(0,\nu). \tag{21}$$

4. The MLH estimate of a change point. To obtain the maximum likelihood (MLH) estimate $\widehat{\nu}_N$ from $x_{1N} = (x_1, \dots, x_N)$, we have to maximize the loglikelihood function $\ln p_{H_{12}}(x_{1N}; \theta^{(1)}, n, \theta^{(2)})$ over the admissible values of n, namely $n \in \Lambda$ or do the same with $L(n,\nu)$, i.e., to find max $L(n,\nu)$. The expressions (12), (13), (15), (16) lead us to a statement that the MLH estimate $\widehat{\nu}_N$ is the value of n maximizing the sequences of partial sums

$$L(n,\nu) = \begin{cases} 0, & \sum\limits_{t=n+1}^{\nu} \xi_t = S^{\nu}_{n+1}, \ n = p+1 \ \div \nu - 1, \\ 0, & \sum\limits_{t=\nu+1}^{n} \zeta_t = S^{n}_{\nu+1}, n = \nu + 1 \ \div N - 1, \end{cases} \tag{22}$$

where

$$\xi_t = \ln \frac{a_0^{(1)}}{a_0^{(2)}} + \frac{1}{2}\delta^2 \varkappa_0^{(2)} + \frac{1}{2}(\lambda^{21}_{t,\nu-n} - 1)\varepsilon_t^2 + \delta\gamma^{21}_{t,\nu-n}\varepsilon_t, \tag{23}$$

$$\zeta_t = \ln \frac{a_0^{(1)}}{a_0^{(2)}} - \frac{1}{2}\delta^2 \varkappa_0^{(1)} - \frac{1}{2}(\lambda^{12}_{t,n-\nu} - 1)\varepsilon_t^2 + \delta\gamma^{12}_{t,n-\nu}\varepsilon_t,$$
$$\delta = \mu^{(1)} - \mu^{(2)}. \tag{24}$$

Evidently, ξ_t's and ζ_t's are independent and $L(n,\nu)$ (22) consist of two independent random walks (because of the independence of ε_t's).

The probability law of a random variable

$$\xi = a\varepsilon^2 + b\varepsilon \tag{25}$$

for any constant a, b and a standard normal variable ε should describe the probabilistic structure of random walks in (22).

Denote

$$F_{a\varepsilon^2+b\varepsilon}(x) = P\{a\varepsilon^2 + b\varepsilon < x\}, \quad D(x) = b^2 + 4ax \qquad (26)$$

and consider it separately for the cases $a < 0$ and $a > 0$. It is not difficult to prove that

$$\underset{a>0}{F_{a\varepsilon^2+b\varepsilon}(x)} =$$

$$= \begin{cases} 0, & \text{if } x \leqslant -\frac{b^2}{4a}, \\ \Phi\left(\frac{-b+D^{\frac{1}{2}}(x)}{2a}\right) - \Phi\left(\frac{-b-D^{\frac{1}{2}}(x)}{2a}\right), & \text{if } x > -\frac{b^2}{4a}, \end{cases} \qquad (27)$$

$$\underset{a<0}{F_{a\varepsilon^2+b\varepsilon}(x)} =$$

$$= \begin{cases} \Phi\left(\frac{-b+D^{\frac{1}{2}}(x)}{2a}\right) + 1 - \Phi\left(\frac{-b-D^{\frac{1}{2}}(x)}{2a}\right), & \text{if } x < -\frac{b^2}{4a}, \\ 1, & \text{if } x \geqslant -\frac{b^2}{4a}, \end{cases} \qquad (28)$$

where $\Phi(x)$ is a standard normal function. The expressions (27), (28) obviously lead us to the probability densities concentrated on $[-\frac{b^2}{4a}, \infty)$ if $a > 0$ and $(-\infty, -\frac{b^2}{4a})$ if $a < 0$.

Lemma 3. *The probability density function*

$$p_y(x) = \begin{cases} 0, & x \leqslant 0, \\ \frac{2a}{\sqrt{2\pi x}} e^{\frac{-b^2}{8a^2} - 2a^2 x} chb\sqrt{x}, & x > 0, \end{cases} \qquad (29)$$

describes the distribution of $Y = \frac{1}{4a^3}\xi + \frac{b^2}{16a^4}$, *where* ξ *is defined by (25),* $a > 0$.

The distribution function (28) leads us to the analogous probability density function, concentrated on $(-\infty, 0]$. Evidently

$$P\{\xi < 0\} = \Phi\left(\frac{b}{a}\right) - \frac{1}{2}, \quad \text{if} \quad a > 0,$$
$$P\{\xi < 0\} = \Phi\left(\frac{b}{a}\right) + \frac{1}{2}, \quad \text{if} \quad a < 0. \qquad (30)$$

The density function (29) is a natural generalization of χ_1^2 distribution law but a random walk S_{n+1}^{ν} or $S_{\nu+1}^{n}$, beginning its pattern at the point $t = \nu$ consists of the independent items ξ_t or ζ_t, each having a distribution function $F_{\xi_t}(x)$, $F_{\zeta_t}(x)$ of the same family as (27), (28) and with the individual parameters a_t, b_t, $t = 1, 2, 3, \ldots$, what makes the classical fluctuation theory unproper in this case.

To determine $\widehat{\nu}_N$, we must find the larger of two random walk (22) maxima $M_{\nu-p}$ and $M'_{N-\nu-1}$, where

$$\begin{aligned} M_{\nu-p} &= \max\{0,\ S_{n+1}^{\nu},\ n = p \div \nu - 1\}, \\ M'_{N-\nu-1} &= \max\{0,\ S_{\nu+1}^{n},\ n = \nu + 1 \div N - 1\}. \end{aligned} \tag{31}$$

We can express events involving $\widehat{\nu}_N$ in terms of events involving M_k and M'_k. Denote $p_{\nu\pm r} = P\{\widehat{\nu}_N = \nu \pm r\}$, $r = 0, 1, 2, \ldots$

$$\begin{aligned} p_{\nu} &= P\{M_{\nu-p} = M'_{N-\nu-1} = 0\} \\ &= P\{M_{\nu-p} = 0\} \cdot P\{M'_{N-\nu-1} = 0\}, \\ p_{\nu-r} &= P\{\widehat{\nu}_N \leqslant \nu\} \cdot P\{M_{\nu-p} = S_{\nu-r+1}^{\nu}\}, \\ p_{\nu+r} &= P\{\widehat{\nu}_N > \nu\} \cdot P\{M'_{N-\nu-1} = S_{\nu+1}^{\nu+r}\}. \end{aligned} \tag{32}$$

The probabilities $P\{\widehat{\nu}_N \leqslant \nu\}$ and $P\{\widehat{\nu}_N > \nu\}$ may be approximately evaluated with reference to the results of Corollary 2 and the concrete values m_1 and m_2, whereas $P\{M_{\nu-p} = S_{\nu-r+1}^{\nu}\}$ as well as $P\{M'_{N-\nu-1} = S_{\nu+1}^{\nu+r}\}$ are calculated in the following way:

$$\begin{aligned} &P\{M_{\nu-p} = S_{\nu-r+1}^{\nu}\} \\ &\quad = P\{S_{p+1}^{\nu-r} < 0, S_{p+2}^{\nu-r} < 0, \ldots, S_{\nu-r-1}^{\nu-r} < 0, \xi_{\nu-r} < 0, \\ &\quad \xi_{\nu-r+1} \geqslant 0,\ S_{\nu-r+1}^{\nu-r+2} \geqslant 0, \ldots, S_{\nu-r+1}^{\nu-1} \geqslant 0,\ S_{\nu-r+1}^{\nu} \geqslant 0\} \\ &\quad = P\{B_1\} \prod_{j=2}^{\nu-r-p} P\{B_j | B_{j-1} \cdot \ldots \cdot B_1\} \cdot P\{A_1\} \\ &\quad \times \prod_{j=2}^{r} P\{A_j | A_{j-1} \cdot \ldots \cdot A_1\}, \end{aligned} \tag{33}$$

where

$$B_k = \{S_{\nu-r-k+1}^{\nu-r} < 0\}, \quad A_k = \{S_{\nu-r+1}^{\nu-r+k} \geqslant 0\}. \tag{34}$$

The probabilities $P\{B_1\}$ or $P\{A_1\}$ follow from (30) while $P\{A_k|A_{k-1}\dots A_1\}$ as well as $P\{B_k|B_{k-1}\dots B_1\}$ may be calculated as:

$$\begin{aligned} P\{A_k|A_{k-1}\dots A_1\} &= P\{S_{\nu-r+1}^{\nu-r+k} \geqslant 0|S_{\nu-r+1}^{\nu-r+k-1} \geqslant 0, \\ &\dots, \xi_{\nu-r+1} \geqslant 0\} = P\{\xi_{\nu-r+k} \geqslant 0\} \\ &+ P\{\xi_{\nu-r+k} < 0\}\cdot P\{|\xi_{\nu-r+k}| < \sum_{\nu-r+1}^{\nu-r+k-1} \xi_t\}. \end{aligned} \tag{35}$$

The results (25)–(35) enable us to calculate the probabilities $p_{\nu\pm r}, r = 0,1,2\dots$ in terms of the parameters $\theta^{(1)}$, $\theta^{(2)}$ more explicitly as it was done by Kligienė (1989).

5. Conclusions. The function $L(n,\nu)$ representing the loss of plausibility, when moving away from the true change point ν, and its mean value $E_{H_{12}}L(n,\nu)$ are presented in terms of changing parameters. It is proved that $E_{H_{12}}\{L(n,\nu)\}$ is a polygon with a break point at ν, whereas its slopes are fully determined by $\theta^{(1)}$ and $\theta^{(2)}$. The results enable us to write out the exact values of probabilities $P\{\widehat{\nu}_N = \nu \pm r\}$, $r = 0,1,2,\dots$, derived for any N, r, ν. Asymptotic behaviour of $P\{\widehat{\nu}_N = \nu \pm r\}$ is a goal of future investigation.

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