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Certain Subclasses of Analytic p-valent Functions with Negative Coefficients

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Abstract. The object of the present paper is to study certain subclasses $J_p^*(a,b,\sigma)$ and $C_p(a,b,\sigma)$ of analytic p-valent functions, and obtain coefficient bounds and distortion properties for functions belonging to these subclasses. Further results include distortion inequalities and radii of close-to-convexity, starlikeness and convexity for these classes of functions.

Key words: analytic p-valent functions, fractional derivative operator, Riemann-Liouville operator, close-to-convex functions, starlike functions, convex functions.

1. Introduction and Preliminaries

Let J_p denote the class of functions defined by

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \geqslant 0;\ p \in \mathbb{N}), \tag{1.1}$$

which are analytic and p-valent in the unit disk $U = \{z\colon |z| < 1\}$. A function $f(z) \in J_p$ is said to be in the class $J_p^*(a,b,\sigma)$ if and only if

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - p}{\dfrac{bzf'(z)}{f(z)} - ap} \right| < \sigma \quad (z \in U), \tag{1.2}$$

$$(-1 \leqslant a < b \leqslant 1 \quad \text{and} \quad 0 < \sigma \leqslant 1).$$

Also, we denote by $C_p(a,b,\sigma)$ the class of functions $f(z) \in J_p$, if and only if $\frac{zf'(z)}{p} \in J_p^*(a,b,\sigma)$. The classes $J_p^*(a,b,1) = \tau_p^*(a,b)$ and $C_p(a,b,1) = c_p(a,b)$ were studied in (Goel and Sohi, 1981). The extended form of fractional derivative operator $J_{0,z}^{\lambda,\mu,\eta}$ invoked in this paper is the one defined by (see, Raina and Srivastava (1996)):

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{d^k}{dz^k}\left\{ \frac{z^{\lambda-\mu}}{\Gamma(k-\lambda)} \cdot \int_0^z (z-t)^{k-\lambda-1} {}_2F_1\left(\mu-\lambda, k-\eta;\ k-\lambda;\ 1-\frac{t}{z}\right) f(t)dt \right\}, \tag{1.3}$$

$$(k-1 \leqslant \lambda < k;\ k \in \mathbb{N} \quad \text{and} \quad \mu,\eta \in \mathbb{R}),$$

where the function $f(z)$ is analytic in a simply connected region of the z-plane containing the origin, with the order

$$f(z) = O\left(|z|^r\right) \quad (z \to 0), \tag{1.4}$$

for

$$r > \max(0, \mu-\eta) - 1. \tag{1.5}$$

The multiplicity of $(z-t)^{k-\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0$. The operator $J_{0,z}^{\lambda,\mu,\eta}$ includes the well-known Riemann-Liouville and Erdélyi-Kober operators of fractional calculus (see (Raina and Saigo, 1993) and (Samko *et al.*, 1993)).

Indeed, we have

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) = \frac{d^k}{dz^k}\left\{ \frac{1}{\Gamma(k-\lambda)} \int_0^z (z-t)^{k-\lambda-1} f(t)dt \right\} = {}_0D_z^{\lambda} f(z), \tag{1.6}$$

$$(k-1 \leqslant \lambda < k;\ k \in \mathbb{N})$$

and

$$J_{0,z}^{\lambda,k,\eta} f(z) = \frac{d^k}{dz^k}\left\{ \frac{z^{\lambda-\eta}}{\Gamma(k-\lambda)} \int_0^z (z-t)^{k-\lambda-1} t^{\eta-k} f(t)dt \right\} = \frac{d^k}{dz^k}\left(E_{0,z}^{k-\lambda,\eta-\lambda} \right). \tag{1.7}$$

$$(k-1 \leqslant \lambda < k;\ k \in \mathbb{N}).$$

In the present paper we first establish some coefficient bounds and distortion properties for the functions belonging to the subclasses $J_p^*(a,b,\sigma)$ and $C_p(a,b,\sigma)$. Also, further distortion inequalities involving the fractional derivative operators of the function in the subclass $C_p(a,b,\sigma)$ are obtained. The radii of close-to-convexity, starlikeness and convexity for functions belonging to the classes $J_p^*(a,b,\sigma)$ and $C_p(a,b,\sigma)$ are also investigated.

2. Coefficient Bounds

In this section we prove two theorems giving the coefficient bounds for the function $f(z)$ belonging to classes $J_p^*(a,b,\sigma)$ and $C_p(a,b,\sigma)$.

Theorem 1. *The function* $f(z)$ *defined by* (1.1) *belongs to the class* $J_p^*(a,b,\sigma)$ *if and only if*

$$\sum_{n=1}^{\infty}\{(1+b\sigma)n+(b-a)p\sigma\}\,a_{p+n}\leqslant(b-a)p\sigma. \tag{2.1}$$

The result (2.1) *is sharp.*

Proof. Let $f(z)$ defined by (1.1) be in the class $J_p^*(a,b,\sigma)$. Then, in view of (1.2), we have

$$\left|\frac{\dfrac{zf'(z)}{f(z)}-p}{\dfrac{bzf'(z)}{f(z)}-ap}\right|$$

$$=\left|\frac{\sum\limits_{n=1}^{\infty}na_{p+n}z^{p+n}}{(b-a)pz^p-\sum\limits_{n=1}^{\infty}a_{p+n}\left[(b-a)p+bn\right]z^{p+n}}\right|<\sigma\quad(z\in U). \tag{2.2}$$

Since $|\mathrm{Re}(z)|\leqslant|z|$ for any z, we get from (2.2) that

$$\mathrm{Re}\left\{\frac{\sum\limits_{n=1}^{\infty}na_{p+n}z^{p+n}}{(b-a)pz^p-\sum\limits_{n=1}^{\infty}a_{p+n}\left[(b-a)p+bn\right]z^{p+n}}\right\}<\sigma. \tag{2.3}$$

Choosing values of z on the real axis, and letting $z\to1$ through real values, we arrive at the assertion (2.1) of Theorem 1. Conversely, we assume that the inequality (2.1) holds true. Then

$$\begin{aligned}&|zf'(z)-pf(z)|-\sigma\,|bzf'(z)-apf(z)|\\&<\sum_{n=1}^{\infty}na_{p+n}-\left\{(b-a)p-\sigma\sum_{n=1}^{\infty}a_{p+n}\left[(b-a)p+bn\right]\right\}\leqslant0,\end{aligned}$$

by assumption. This implies that $f(z) \in J_p^*(a,b,\sigma)$.

It may be noted that the assertion (2.1) of Theorem 1 is sharp, and the extremal function is given by

$$f(z) = z^p - \frac{(b-a)p\sigma}{(1+b\sigma)n + (b-a)p\sigma} z^{p+n}. \tag{2.4}$$

Theorem 2. *The function $f(z)$ defined by* (1.1) *belongs to the class $C_p(a,b,\sigma)$ if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{p+n}{p}\right) \{(1+b\sigma)n + (b-a)p\sigma\}\, a_{p+n} \leqslant (b-a)p\sigma. \tag{2.5}$$

The result (2.5) *is sharp.*

Proof. The desired assertion (2.5) follows easily on using the definition of $C_p(a,b,\sigma)$ and (2.1).

The assertion (2.5) of Theorem 2 is sharp, the extremal function being

$$f(z) = z^p - \frac{(b-a)p\sigma}{(p+n)\,[(1+b\sigma)n + (b-a)p\sigma]} z^{p+n}. \tag{2.6}$$

3. Distortion Properties

Next, we prove two results concerning distortion properties of $f(z)$ which give upper and lower bounds for the functions belonging to the class $J_p^*(a,b,\sigma)$ and $C_p(a,b,\sigma)$.

Theorem 3. *Let the function $f(z)$ defined by* (1.1) *belong to the class $J_p^*(a,b,\sigma)$. Then*

$$|f(z)| \geqslant |z|^p - \frac{(b-a)p\sigma}{(1+b\sigma) + (b-a)p\sigma} |z|^{p+1}, \tag{3.1}$$

and

$$|f(z)| \leqslant |z|^p + \frac{(b-a)p\sigma}{(1+b\sigma) + (b-a)p\sigma} |z|^{p+1}. \tag{3.2}$$

Proof. Since $f(z) \in J_p^*(a,b,\sigma)$, therefore in view of Theorem 1, we have

$$\{(1+b\sigma) + (b-a)p\sigma\} \sum_{n=1}^{\infty} a_{p+n} \leqslant \sum_{n=1}^{\infty} \{(1+b\sigma)n + (b-a)p\sigma\}\, a_{p+n} \leqslant (b-a)p\sigma.$$

This yields

$$\sum_{n=1}^{\infty} a_{p+n} \leqslant \frac{(b-a)p\sigma}{(1+b\sigma)+(b-a)p\sigma}. \tag{3.3}$$

On using (1.1) and (3.3), we easily arrive at the desired results (3.1) and (3.2).

Theorem 4. *Let the function $f(z)$ defined by* (1.1) *belong to the class $C_p(a,b,\sigma)$. Then*

$$|f(z)| \geqslant |z|^p - \frac{(b-a)p^2\sigma}{(p+1)\left[(1+b\sigma)+(b-a)p\sigma\right]}|z|^{p+1}, \tag{3.4}$$

and

$$|f(z)| \leqslant |z|^p + \frac{(b-a)p^2\sigma}{(p+1)\left[(1+b\sigma)+(b-a)p\sigma\right]}|z|^{p+1}. \tag{3.5}$$

Proof. Since $f(z) \in C_p(a,b,\sigma)$, then in view of Theorem 2, we have

$$\left(\frac{p+1}{p}\right) \quad \{(1+b\sigma)+(b-a)p\sigma\} \sum_{n=1}^{\infty} a_{p+n}$$

$$\leqslant \sum_{n=1}^{\infty} \left(\frac{p+n}{p}\right) \{(1+b\sigma)n+(b-a)p\sigma\}\, a_{p+n} \leqslant (b-a)p\sigma.$$

This yields

$$\sum_{n=1}^{\infty} a_{p+n} \leqslant \frac{(b-a)p^2\sigma}{(p+1)\{(1+b\sigma)+(b-a)p\sigma\}}. \tag{3.6}$$

On using (1.1) and (3.6), we immediately get the desired results (3.4) and (3.5).

4. Further Distortion Properties

Theorem 5. *Let the function $f(z)$ defined by* (1.1) *belong to the class $C_p(a,b,\sigma)$. Then, for $0<\lambda\leqslant\mu\leqslant 1,\ \eta\in\mathbb{R}_+,\ -1\leqslant a<b\leqslant 1,\ 0<\sigma\leqslant 1$, and $z\in U$:*

$$\left|J_{0,z}^{\lambda,\mu,\eta} f(z)\right| \geqslant \frac{|z|^{p-\mu}}{\phi_p(\lambda,\mu,\eta)} \left\{1 - \frac{(b-a)p\sigma}{(1+b\sigma)+(b-a)p\sigma}|z|\right\}, \tag{4.1}$$

and

$$\left|J_{0,z}^{\lambda,\mu,\eta} f(z)\right| \leqslant \frac{|z|^{p-\mu}}{\phi_p(\lambda,\mu,\eta)} \left\{1 + \frac{(b-a)p\sigma}{(1+b\sigma)+(b-a)p\sigma}|z|\right\}, \tag{4.2}$$

where

$$\phi_p(\lambda,\mu,\eta) = \frac{\Gamma(1-\mu+p)\Gamma(1+\eta-\lambda+p)}{\Gamma(1+p)\Gamma(1+\eta-\mu+p)}. \tag{4.3}$$

Proof. Let a function $H(z)$ be defined by

$$H(z) = \phi_p(\lambda,\mu,\eta) z^{\mu} J_{0,z}^{\lambda,\mu,\eta} f(z). \tag{4.4}$$

Then, in view of (1.1) and the formula (Raina and Srivastava 1996; see also Srivastava *et. al.*, 1988):

$$J_{0,z}^{\lambda,\mu,\eta} z^k = \frac{z^{k-\mu}}{\phi_k(\lambda,\mu,\eta)}, \tag{4.5}$$

$(\lambda \geqslant 0;\ \mu,\eta \in \mathbb{R};\ k > \max\{0,\mu-\eta\}-1)$ where $\phi_k(\lambda,\mu,\eta)$ is given by (4.3), we have

$$H(z) = z^p - \sum_{n=1}^{\infty} \delta_{p+n} z^{p+n}, \tag{4.6}$$

where

$$\delta_{p+n} = \frac{(p+n)(p)_n(1-\mu+p+\eta)_n}{p(1-\mu+p)_n(1-\lambda+p+\eta)_n} a_{p+n}, \tag{4.7}$$

$(p)_n$ etc. denote the usual factorial function.

It may be observed that

$$\frac{(p)_n(1-\mu+p+\eta)_n}{(1-\mu+p)_n(1-\lambda+p+\eta)_n} = 1, \quad \text{for } \lambda=\mu=1, \tag{4.8}$$

and

$$\frac{(p)_n(1-\mu+p+\eta)_n}{(1-\mu+p)_n(1-\lambda+p+\eta)_n} < 1, \qquad \text{for } 0<\lambda<\mu<1; \quad \eta \in \mathbb{R}_+;\ \forall n \in \mathbb{N}. \tag{4.9}$$

Hence

$$\frac{(p)_n(1-\mu+p+\eta)_n}{(1-\mu+p)_n(1-\lambda+p+\eta)_n} \leqslant 1, \text{ for } 0<\lambda\leqslant\mu\leqslant 1;\ \eta\in\mathbb{R}_+;\ \forall n\in\mathbb{N}. \tag{4.10}$$

From (4.7), it follows that

$$\delta_{p+n} \leqslant \frac{p+n}{p} a_{p+n}. \tag{4.11}$$

In view of (2.5) and (4.11), we have

$$\sum_{n=1}^{\infty}\{(1+b\sigma)n+(b-a)p\sigma\}\delta_{p+n}\leqslant(b-a)p\sigma. \tag{4.12}$$

This implies that $H(z)$ belongs to $J_p^*(a,b,\sigma)$ by virtue of Theorem 1. Therefore, using (3.1) and (3.2), we get

$$|H(z)|\geqslant|z|^p-\frac{(b-a)p\sigma}{(1+b\sigma)+(b-a)p\sigma}|z|^{p+1}, \tag{4.13}$$

and

$$|H(z)|\leqslant|z|^p+\frac{(b-a)p\sigma}{(1+b\sigma)+(b-a)p\sigma}|z|^{p+1}. \tag{4.14}$$

This leads to assertions (4.1) and (4.2) in conjunction with (4.4).

COROLLARY 1. Let $f(z)$ defined by (1.1) be in the class $C_p(a,b,\sigma)$. Then $J_{0,z}^{\lambda,\mu,\eta}f(z)$ is included in a disk with its centre at the origin and radius R given by

$$R=\frac{1}{\phi_p(\lambda,\mu,\eta)}\left\{1+\frac{(b-a)p\sigma}{(1+b\sigma)+(b-a)p\sigma}\right\}, \tag{4.15}$$

where $\phi_p(\lambda,\mu,\eta)$ is given by (4.3).

REMARK 1. On setting $\lambda=\mu=\alpha$ in Theorem 5, and noting the relation from (1.6) that

$$J_{0,z}^{\alpha,\alpha,\eta}f(z)={}_0D_z^{\alpha}f(z), \tag{4.16}$$

and

$$\phi_p(\alpha,\alpha,\eta)=\frac{\Gamma(1-\alpha+p)}{\Gamma(1+p)}, \tag{4.17}$$

where ${}_0D_z^{\alpha}f(z)$ is the Riemann-Liouville fractional derivative (Samko *et al.*, 1993), we get the known results (Srivastava and Owa, 1984, Theorem 1, Eqs. (2.4), (2.5), pp. 385–386).

5. Radii of Close-to-convexity, Starlikeness, and Convexity

A function $f(z)$ in J_p is said to be p-valently close-to-convex of order ρ in U if

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\}>\rho,\quad(0\leqslant\rho<p;\ z\in U). \tag{5.1}$$

A function $f(z)$ in J_p is said to be p-valently starlike of order ρ in U if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \rho, \quad (0 \leqslant \rho < p \quad \text{and} \quad z \in U). \tag{5.2}$$

A function $f(z)$ in J_p is said to be p-valently convex of order ρ in U if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho, \quad (0 \leqslant \rho < p \quad \text{and} \quad z \in U). \tag{5.3}$$

Theorem 6. *If $f(z) \in J_p^*(a, b, \sigma)$, then $f(z)$ is p-valently close-to-convex of order ρ $(0 \leqslant \rho < p)$ in $|z| < r_1$, where*

$$r_1 = \inf_{n \in \mathbb{N}} \left\{\frac{(p-\rho)[(1+b\sigma)n + (b-a)p\sigma]}{(p+n)(b-a)p\sigma}\right\}^{1/n}. \tag{5.4}$$

The result is sharp with the extremal function $f(z)$ given by (2.4).

Proof. Let $f(z) \in J_p^*(a, b, \sigma)$. Then, by virtue of (5.1), the function $f(z)$ is p-valently close-to-convex of order ρ $(0 \leqslant \rho < p)$ in U, provided that

$$\left| -\sum_{n=1}^{\infty}(p+n)a_{p+n}z^n \right| \leqslant \sum_{n=1}^{\infty}(p+n)a_{p+n}|z|^n \leqslant p - \rho. \tag{5.5}$$

In view of (2.1), the assertion (5.5) is true if

$$\frac{(p+n)|z|^n}{(p-\rho)} \leqslant \frac{[(1+b\sigma)n + (b-a)p\sigma]}{(b-a)p\sigma} \quad (\forall n \in \mathbb{N}). \tag{5.6}$$

On solving (5.6) for $|z|$, we get the desired result (5.4).

Theorem 7. *If $f(z) \in J_p^*(a, b, \sigma)$, then $f(z)$ is p-valently starlike of order ρ $(0 \leqslant \rho < p)$ in $|z| < r_2$, where*

$$r_2 = \inf_{n \in \mathbb{N}} \left\{\frac{(p-\rho)[(1+b\sigma)n + (b-a)p\sigma]}{(n+p-\rho)(b-a)p\sigma}\right\}^{1/n}. \tag{5.7}$$

The result is sharp with the extremal function $f(z)$ given by (2.4).

Proof. Let $f(z) \in J_p^*(a, b, \sigma)$. Then by virtue of (5.2), the function $f(z)$ is p-valently starlike of order ρ $(0 \leqslant \rho < p)$ in U, provided that

$$\left| \frac{-\sum\limits_{n=1}^{\infty} na_{p+n}z^n}{1 - \sum\limits_{n=1}^{\infty} a_{p+n}z^n} \right| \leqslant \frac{\sum\limits_{n=1}^{\infty} na_{p+n}|z|^n}{1 - \sum\limits_{n=1}^{\infty} a_{p+n}|z|^n} \leqslant p - \rho. \tag{5.8}$$

In view of (2.1), the assertion (5.8) is true if

$$\frac{(n+p-\rho)|z|^n}{(p-\rho)} \leqslant \frac{[(1+b\sigma)n+(b-a)p\sigma]}{(b-a)p\sigma} \quad (n \in \mathbb{N}). \tag{5.9}$$

On solving (5.9) for $|z|$, we get the desired result (5.7).

Similarly, by using the definition (5.3) of p-valently convex functions of order ρ, we easily arrive at the following result:

Theorem 8. *If $f(z) \in J_p^*(a,b,\sigma)$, then $f(z)$ is p-valently convex of order ρ $(0 \leqslant \rho < p)$ in $|z| < r_3$, where*

$$r_3 = \inf_{n\in\mathbb{N}} \left\{ \frac{p(p-\rho)[(1+b\sigma)n+(b-a)p\sigma]}{(p+n)(2+p-n-\rho)(b-a)p\sigma} \right\}^{1/n}. \tag{5.10}$$

The result is sharp with the extremal function $f(z)$ given by (2.4).

Proof. Let $f(z) \in J_p^*(a,b,\sigma)$. Then, by virtue of (5.3), the function $f(z)$ is p-valently convex of order ρ $(0 \leqslant \rho < p)$ in U, provided that

$$\left| \frac{-p + \sum\limits_{n=1}^{\infty} a_{p+n} z^n (p+n)(1-n)}{p - \sum\limits_{n=1}^{\infty} a_{p+n} z^n (p+n)} \right| \leqslant$$
$$\leqslant \frac{p + \sum\limits_{n=1}^{\infty} a_{p+n}(p+n)(1-n)|z|^n}{p - \sum\limits_{n=1}^{\infty} a_{p+n}(p+n)|z|^n} \leqslant 1+p-\rho. \tag{5.11}$$

In view of (2.1), the assertion (5.11) is true if

$$\frac{(p+n)(2+p-n-\rho)|z|^n}{p(p-\rho)} \leqslant \frac{[(1+b\sigma)n+(b-a)p\sigma]}{(b-a)p\sigma}, \quad n \in \mathbb{N}. \tag{5.12}$$

On solving (5.12) for $|z|$, we get the desired result (5.10).

We conclude this paper by remarking that several results giving the coefficient bounds, distortion inequalities, radii of close-to-convexity, starlikeness and convexity of functions which belong to various subclasses of J_p can be obtained by suitable choices of parameters a, b, σ and p, including some of the results obtained in (Goel and Sohi, 1981) and (Srivastava *et al.*, 1984).

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Kai kurios analizinių *p*-valenčių funkcijų su neigiamais koeficientais poklasės

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Analizinė vienetiniame skritulyje $|z|\leqslant 1$ funkcija $F(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^n$, $p = 1, 2, \ldots$, $a_{p+n} \geqslant 0$, vadinama p-valente. Gauti tokių funkcijų, priklausančių įvairiems p-valenčių funkcijų poklasiams, koeficientų ir jų deformacijų įverčiai. Atskirai įvertintos deformacijos, atsirandančios paveikus funkciją diferencialiniu operatoriumi. Nustatyti uždarojo iškilumo, iškilumo ir žvaigždėtumo spinduliai.