INFORMATICA, 1991, Vol.2, No.1, 003–032

FINITE TIME RUIN PROBABILITIES AND MARTINGALES

Jan GRANDELL

Department of Mathematics, Royas Institute of Technology, S–10044 Stockholm, Sweden

Abstract. In this paper we give an introduction to collective risk theory in its simplest form. Our aims are to indicate how some basic facts may be obtained by martingale methods and to point out some open problems

Key words: risk theory, ruin probabilities, martingales.

1. Introduction. Collective risk theory, as a part of insurance – or actuarial – mathematics, deals with stochastic models of an insurance business. In such a model the occurrence of the claims is described by a point process and the amounts of money to be paid by the company at each claim by a sequence of random variables. The company receives a certain amount of premium to cover its liability. The company is furthermore assumed to have a certain initial capital at its disposal. An important problem in collective risk theory is to investigate the "ruin probability", i.e., the probability that the risk business ever – or before some finite time – becomes negative.

Risk theory, which mathematically has many similarities to queueing theory, has been, more or less, systematically studied for a long time. In fact, calculation of the distribution of waiting times is equivalent with calculation of ruin probabilities. The traditional way of studying risk theory – like queueing theory – is by the use of analytical methods. One purpose of this paper is to give an account of a different approach, due to Gerber (1973), which uses martingales. We will only apply those methods to the simplest risk theoretic model.

This paper is not a scientific paper in that sense that new results are derived. In fact, we will often only give weak and incomplete versions of old results, because of the sole reason that the "better" versions require analytical methods. Generally, the martingale approach is very powerful for proving inequalities and exact results in special cases, while asymptotic relations seem to be better proved by analytical methods. The important merit of the martingale approach is, however, that it is often very well suited for handling more general models.

The first impression may, very naturally, be that risk theory is a very special branch of applied probability and with no interest to anyone outside a small group of mathematically initiated actuaries. A second purpose of this paper is to try to convince the reader that risk theory is a fruitful research area with interesting open problems. In Section 7 two open problems are described.

In order to facilitate the reading, most references are given in Section 8. Any reader who has the slightest idea to go deeper into risk theory, will certainly find Section 8 to be the most useful section.

1.1. The risk process. We start with formulating the simplest risk model. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space carrying the following independent objects:

(i) a Poisson process $N = \{N(t);\ t \geq 0\}$ with $N(0) = 0$ and $E[N(t)] = \alpha t$;

(ii) a sequence $\{Z_k\}_1^\infty$ of i.i.d. random variables, having the common distribution function F, mean value μ and variance σ^2.

DEFINITION 1. The *risk process*, X, is defined by

$$X(t) = ct - \sum_{k=1}^{N(t)} Z_k, \quad \left(\sum_{k=1}^{0} Z_k \stackrel{\text{def}}{=} 0 \right) \tag{1}$$

where c is a positive real constant.

This is the classical model of the risk business of an insurance company, where $N(t)$ is to be interpreted as the number of claims on the company during the interval $(0, t]$. At each point of N the company has to pay out a stochastic amount of money, and the company receives (deterministically) c units of money per unit time. The constant c is called the *gross risk premium* rate.

In risk theory the most interesting situation is when $c > 0$ and $F(0) = 0$. This case is generally called *positive risk sums*, and includes most non-life branches and also the ordinary types of life insurance, where a certain amount of money is paid at the death of a policyholder.

There are, however, situations where the circumstances are reversed, i.e., where $c < 0$ and $F(0) = 1$. The typical example is life annuity, or pension, insurance, where a life annuity rate $-c$ is paid from the company to the policyholder and where the claim, i.e., the death of the policyholder, will place an amount of money corresponding to the "expected pension to be paid" at the company's free disposal. Thus the claim means an income, or a negative cost, for the company. This situation is generally called *negative risk sums*.

The profit of the risk business over the interval $(0, t]$ is $X(t)$ and thus the expected profit is

$$E[X(t)] = ct - E[N(t)]E[Z_k] = (c - \alpha\mu)t.$$

The relative *safety loading* ρ is defined by

$$\rho = \frac{c - \alpha\mu}{\alpha\mu} = \frac{c}{\alpha\mu} - 1.$$

The risk process X is said to have *positive* safety loading if $\rho > 0$. Then $X(t)$ has a drift to $+\infty$.

Instead of describing the risk business by a risk process one may sometimes use a *Wiener process* with positive drift. Then we put

$$X(t) = \beta t + \delta W(t), \qquad \beta > 0, \tag{2}$$

where W is a standard Wiener process, i.e., $W(0) = 0$, $W(t)$ has independent and normally distributed increments such that

$$E[W(t) - W(s)] = 0 \quad \text{and} \quad \mathrm{Var}[W(t) - W(s)] = t - s$$

for $t > s$ and its realizations are continuous.

One motivation for the Wiener process is the diffusion approximation which works – although numerically rather bad – if u and ρ^{-1} are of the same large order. Here we consider the Wiener process as an alternative to the risk process merely for mathematical reasons.

In order to simultaneously consider risk processes and Wiener prosesses with positive drift we introduce the following class of processes.

DEFINITION 2. An *additive* process X is a process such that:

(i) $X(t)$ has right continuous process trajectories;

(ii) $X(0) = 0 \quad P$-a.s.;

(iii) X has stationary and independent increments;

(iv) $E[X(t)] = t\beta \quad$ where $\beta > 0$;

(v) $E[e^{-rX(t)}] < \infty$ for some $r > 0$.

Then

$$E[e^{-rX(t)}] = e^{tg(r)} \qquad \text{for some function } g(\cdot).$$

For technical reasons we slightly extend condition (v) above.

ASSUMPTION 3. *We assume that there exists* $r_\infty > 0$ *such that* $g(r) < \infty$ *for* $r \in [0, r_\infty)$ *and* $g(r) \to +\infty$ *when* $r \uparrow r_\infty$ *(we allow for the possibility* $r_\infty = +\infty$*).*

It is easily seen that $g(0) = 0$, $g'(0+) < 0$ and that g is convex and continuous on $[0, r_\infty)$.

If X is a risk process with positive safety loading we have $\beta = c - \alpha\mu$. Put

$$h(r) = \int_{-\infty}^{\infty} e^{rz}\, dF(z) - 1.$$

Then we have

$$E[e^{-rX(t)}] = e^{-rct} \sum_{k=0}^{\infty} \frac{(\alpha t)^k}{k!} e^{-\alpha t}(h(r)+1)^k$$
$$= e^{-rct+\alpha t(h(r)+1)-\alpha t} = e^{t(\alpha h(r)-rc)}$$

and thus $g(r) = \alpha h(r) - rc$.

REMARK 5. Obviously h fulfils Assumption 3. The important part of Assumption 3 is that $h(r) < \infty$ for some $r > 0$, i.e., (v) in Definition 2. This means that the right tail of dF decreases at least exponentially fast, and thus for example the lognormal and the Pareto distributions are not allowed. Further the rather pathological case when $h(r_\infty-) < \infty$ and $h(r) = \infty$ for $r > r_\infty$ is excluded.

If X is a Wiener process with positive drift we have

$$E[e^{-rX(t)}] = E[e^{-r\beta t - r\delta W(t)}] = e^{t(-\beta r + \delta^2 r^2/2)}$$

and thus $g(r) = -\ \beta r + \delta^2 r^2/2$.

We can now define the *ruin probabilities* $\Psi(u)$ and $\Psi(u,t)$, of a company facing an additive process and having initial capital u.

DEFINITION 5. $\Psi(u) = P\{u + X(s) < 0 \text{ for some } s > 0\}$ is called the *infinite time* ruin probability. $\Psi(u,t) = P\{u + X(s) < 0 \text{ for some } s \in (0,t]\}$ is called the *finite time* ruin probability.

Let T_u be the time of ruin, i.e., $T_u = \inf\{t \geq 0 \mid u + X(t) < 0\}$. We have

$$\Psi(u,t) = P\{T_u \leq t\} \qquad \text{and} \qquad \Psi(u) = P\{T_u < \infty\}.$$

1.2. Basic facts about martingales

DEFINITION 6. A *filtration* $\mathbf{F} = (\mathcal{F}_t;\ t \geq 0)$ is a non-decreasing family of sub-σ-algebras of $\mathcal{F}$.

DEFINITION 7. Let for any process $X = \{X(t);\ t \geq 0\}$, the filtration $\mathbf{F}^X = (\mathcal{F}_t^X;\ t \geq 0)$ be defined by $\mathcal{F}_t^X = \sigma\{X(s);\ s \leq t\}$.

Thus $\mathcal{F}_t^X$ is the σ-algebra generated by X up to time t, and represents the *history* of X up to time t. X is *adapted* to $\mathbf{F}$, i.e., X is $\mathcal{F}_t$-measurable for all $t \geq 0$, if and only if $\mathcal{F}_t^X \subseteq \mathcal{F}_t$ for all $t \geq 0$.

DEFINITION 8. An $\mathbf{F}$-*martingale* $M = \{M(t);\ t \geq 0\}$ is a real valued process such that:

(i) $M(t)$ is $\mathcal{F}_t$-measurable for $t \geq 0$;

(ii) $E[|M(t)|] < \infty$ for $t \geq 0$;

(iii) $E^{\mathcal{F}_s}[M(t)] = E[M(t) \mid \mathcal{F}_s] = (\ \leq\ )\ M(s)$ P-a.s. for $t \geq s$.

An $\mathbf{F}$-martingale M is called *right continuous* if:

(i) The trajectories $M(t)$ are right continuous;

(ii) The filtration $\mathbf{F}$ is right continuous, i.e., $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for $t \geq 0$.

All processes which we consider have right continuous trajectories and the filtrations are so simple that the condition of right continuity is of no problem.

DEFINITION 9. A random variable T taking values in $[0,\infty]$, is an $\mathbf{F}$-*stopping* time if $\{T \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$.

This means that one, knowing the history up to time t, can decide if $T \leq t$ or not. Note that the outcome $T = \infty$ is allowed. If T is a stopping time, so is $t \wedge T = \min(t, T)$ for each t.

The following simplified version of the "Optional Stopping Theorem" is essential for our applications.

Theorem 10. *Let T be a bounded stopping time, i.e., $T \leq t_0 < \infty$, and M a right continuous $\mathbf{F}$-martingale. Then*

$$E^{\mathcal{F}_0}[M(T)] = M(0) \qquad P\text{-a.s.}$$

2. Infinite time ruin probability. Now we consider the *"martingale approach"*, due to Gerber (1973), applied to the infinite time ruin probability.

The time of ruin T_u is a $\mathbf{F}^X$-stopping time and recall that $\Psi(u) = P\{T_u < \infty\}$. Put

$$M_u(t) = \frac{e^{-r(u+X(t))}}{e^{tg(r)}}.$$

M_u is an $\mathbf{F}^X$-martingale, since

$$\begin{aligned} E^{\mathcal{F}_s^X}[M_u(t)] &= E^{\mathcal{F}_s^X}\left[\frac{e^{-r(u+X(t))}}{e^{tg(r)}}\right] \\ &= E^{\mathcal{F}_s^X}\left[\frac{e^{-r(u+X(s))}}{e^{sg(r)}} \cdot \frac{e^{-r(X(t)-X(s))}}{e^{(t-s)g(r)}}\right] \qquad (3) \\ &= M_u(s) \cdot E^{\mathcal{F}_s^X}\left[\frac{e^{-r(X(t)-X(s))}}{e^{(t-s)g(r)}}\right] = M_u(s). \end{aligned}$$

Choose $t_0 < \infty$ and consider $t_0 \wedge T_u$ which is a bounded $\mathbf{F}^X$-stopping time. Since $\mathcal{F}_0^X$ is trivial, i.e., $\mathcal{F}_0^X$ consists of only $\emptyset$ and Ω, and since M_u is positive, it follows from Theorem 10 that

$$\begin{aligned} e^{-ru} &= M_u(0) = E[M_u(t_0 \wedge T_u)] \\ &= E[M_u(t_0 \wedge T_u) \mid T_u \le t_0]P\{T_u \le t_0\} \\ &+ E[M_u(t_0 \wedge T_u) \mid T_u > t_0]P\{T_u > t_0\} \\ &\ge E[M_u(t_0 \wedge T_u) \mid T_u \le t_0]P\{T_u \le t_0\} \\ &= E[M_u(T_u) \mid T_u \le t_0]P\{T_u \le t_0\} \end{aligned} \tag{4}$$

and thus, since $u + X(T_u) \le 0$ on $\{T_u < \infty\}$,

$$\begin{aligned} P\{T_u \le t_0\} &\le \frac{e^{-ru}}{E[M_u(T_u) \mid T_u \le t_0]} \\ &\le \frac{e^{-ru}}{E[e^{-T_u g(r)} \mid T_u \le t_0]} \le e^{-ru} \sup_{0 \le t \le t_0} e^{tg(r)}. \end{aligned} \tag{5}$$

Let $t_0 \to \infty$ in (5). Then we get

$$\Psi(u) \le e^{-ru} \sup_{t \ge 0} e^{tg(r)}. \tag{6}$$

In order to get this inequality as good as possible, we shall choose r as large as possible under the restriction $\sup_{t \ge 0} e^{tg(r)} < \infty$. Let R denote that value. Obviously this means that

$$R = \sup\{r \mid g(r) \le 0\}. \tag{7}$$

In the risk process case this gives R as the positive solution of $h(r) = cr/\alpha$, i.e., R is the *Lundberg exponent.* In the Wiener process case we get $R = 2\beta/\delta^2$ Thus we have

$$\Psi(u) \le e^{-Ru} \tag{8}$$

which is the *"Lundberg inequality"*.

A refinement of (8) is the *"Cramér–Lundberg approximation"*

$$\lim_{u\to\infty} e^{Ru}\Psi(u) = C, \quad 0 < C < \infty, \tag{9}$$

where, in the case of positive risk sums, $C = \frac{\rho\mu}{h'(R)-c/\alpha}$. From this it follows that R is the "right" exponent in (8).
Let us go back to (4). Then we have, with $r = R$,

$$\begin{aligned} e^{-Ru} &= E[e^{-R(u+X(T_u))} \mid T_u \le t_0]P\{T_u \le t_0\} \\ &\quad + E[e^{-R(u+X(t_0))} \mid T_u > t_0]P\{T_u > t_0\}. \end{aligned} \tag{10}$$

Let $I\{A\}$ denote the indicator function for the set A. Then we have

$$\begin{aligned} 0 &\le E[e^{-R(u+X(t_0))} \mid T_u > t_0]P\{T_u > t_0\} \\ &= E[e^{-R(u+X(t_0))}I\{T_u > t_0\}] \\ &\le E[e^{-R(u+X(t_0))}I\{u + X(t_0) \ge 0\}]. \end{aligned}$$

Since $0 \le e^{-R(u+X(t_0))}I\{u+X(t_0) \ge 0\} \le 1$ it follows, due to the drift of $X(t)$ to $+\infty$, by dominated convergence that

$$\lim_{t_0\to\infty} E[e^{-R(u+X(t_0))} \mid T_u > t_0]P\{T_u > t_0\} = 0$$

and thus we get from (10) that

$$\Psi(u) = \frac{e^{-Ru}}{E[e^{-R(u+X(T_u))} \mid T_u < \infty]}. \tag{11}$$

When $X(t)$ is continuous at the time of ruin we have $u + X(T_u) = 0$ on $\{T_u < \infty\}$ and thus $\Psi(u) = e^{-Ru}$. This holds for the risk process with negative risk sums and for the Wiener process.

When $X(t)$ is a risk process with positive risk sums there is, in general, difficult to apply (11), since the "overshot" $B(u) \stackrel{\text{def}}{=} -(u + X(T_u))$ on $\{T_u < \infty\}$ is difficult to handle. It is, however, well-known, that (since $\Psi(0) = \alpha\mu/c$)

$$E[e^{-RX(T_0)} \mid T_0 < \infty] = \frac{c}{\alpha\mu} = 1 + \rho$$

and that, due to the Cramér–Lundberg approximation (9),

$$\lim_{u\to\infty} E[e^{-R(u+X(T_u))} \mid T_u < \infty] = \frac{h'(R) - c/\alpha}{\rho\mu} = \frac{g'(R)}{\rho\alpha\mu}.$$

EXAMPLE 11. EXPONENTIALLY DISTRIBUTED CLAIMS. One case where the overshot is easy to handle also for risk process with positive risk sums is when Z_k is exponentially distributed with mean μ. Then

$$h(r) = \frac{1}{\mu}\int_0^\infty e^{rz}e^{-z/\mu}dz - 1 = \frac{\mu r}{1 - \mu r}$$

and thus R is the positive solution of $\frac{\mu r}{1-\mu r} = \frac{cr}{\alpha}$, i.e.,

$$R = \frac{\rho}{\mu(1+\rho)}.$$

The exponential distribution is characterized by its "lack of memory", i.e., that

$$P\{Z_k > z + x \mid Z_k > x\} = P\{Z_k > z\}.$$

This implies that $B(u)$ is exponentially distributed with mean μ and independent of T_u and thus

$$\begin{aligned}\Psi(u) &= \frac{e^{-Ru}}{E[e^{-R(u+X(T_u))} \mid T_u < \infty]} \\ &= \frac{e^{-Ru}}{E[e^{RB(u)} \mid T_u < \infty]} = \frac{e^{-Ru}}{h(R)+1}.\end{aligned}$$

Since

$$h(R)+1=\frac{cR}{\alpha}+1=\frac{c\rho}{\alpha\mu(1+\rho)}+1=\rho+1$$

we get the well-known result

$$\Psi(u)=\frac{1}{1+\rho}e^{-\frac{\rho u}{\mu(1+\rho)}}. \tag{12}$$

3. The time-dependent Lundberg inequality.
Recall from (5) that

$$\Psi(u,t)\le e^{-ru}\sup_{0\le s\le t}e^{sg(r)}.$$

Obviously we can always choose $r=R$, but it might be possible – at least sometimes – to choose a better, i.e., a larger, value of r.

Put $t=yu$. Then (5) yields

$$\Psi(u,yu)\le\max(e^{-ru},e^{-u(r-yg(r))})=e^{-u\min(r,r-yg(r))}$$

and it seems natural to define the *"time-dependent" Lundberg exponent* R_y by

$$R_y=\sup_{r\ge0}\min(r,r-yg(r))$$

and we have the *"time-dependent" Lundberg inequality* (Gerber 1973, p. 208)

$$\Psi(u,yu)\le e^{-R_y u}. \tag{13}$$

Put

$$f(r)=r-yg(r)$$

and note that $f(R) = R$, $f(r) < r$ for $r > R$ and that $f(r)$ is concave. Thus we have, since $R_y \geq R$,

$$R_y \begin{matrix} = \\ > \end{matrix} R \quad \text{according as} \quad f'(R) \begin{matrix} \leq \\ > \end{matrix} 0.$$

Since $f'(R) = 1 - yg'(R)$ it follows that

$$R_y \begin{matrix} = \\ > \end{matrix} R \quad \text{according as} \quad y \begin{matrix} \geq \\ < \end{matrix} \frac{1}{g'(R)} .$$

The value $y_0 \stackrel{\text{def}}{=} \frac{1}{g'(R)}$ is called the *critical value.* For $y < y_0$ we have

$$R_y = f(r_y) \text{ where } r_y \text{ is the solution of } f'(r) = 0.$$

For $t \geq y_0 u$ (13) is just the "ordinary" Lundberg inequality. It is then natural to look for an inequality for $\Psi(u) - \Psi(u,t)$. We will derive such an inequality by a slight extension of (4).

Let t be given, choose $\tilde{t} \in (t, \infty)$ and consider $\tilde{t} \wedge T_u$. Then

$$\begin{aligned} e^{-ru} &= M_u(0) = E[M_u(\tilde{t} \wedge T_u)] \\ &= E[M_u(\tilde{t} \wedge T_u) \mid T_u \leq t]P\{T_u \leq t\} \\ &+ E[M_u(\tilde{t} \wedge T_u) \mid t < T_u \leq \tilde{t}\,]P\{t < T_u \leq \tilde{t}\} \\ &+ E[M_u(\tilde{t} \wedge T_u) \mid T_u > \tilde{t}\,]P\{T_u > \tilde{t}\} \\ &\geq E[M_u(\tilde{t} \wedge T_u) \mid t < T_u \leq \tilde{t}]P\{t < T_u \leq \tilde{t}\} \\ &= E[M_u(T_u) \mid t < T_u \leq \tilde{t}]P\{t < T_u \leq \tilde{t}\} \end{aligned}$$

and thus

$$\begin{aligned} P\{t < T_u \leq \tilde{t}\} &\leq \frac{e^{-ru}}{E[M_u(T_u) \mid t < T_u \leq \tilde{t}\,]} \\ &\leq \frac{e^{-ru}}{E[e^{-T_u g(r)} \mid t < T_u \leq \tilde{t}\,]} \leq e^{-ru} \sup_{t \leq s \leq \tilde{t}} e^{sg(r)}. \end{aligned} \tag{14}$$

As (6) followed from (5) we get

$$\Psi(u) - \Psi(u,t) \le e^{-ru} \sup_{s \ge t} e^{sg(r)}. \tag{15}$$

Put, as above, $t = yu$ and $f(r) = r - yg(r)$. Then

$$\Psi(u) - \Psi(u,yu) \le e^{-ru} \sup_{x \ge y} e^{xug(r)} = \begin{cases} e^{-f(r)u} & \text{if } g(r) \le 0 \\ \infty & \text{if } g(r) > 0. \end{cases}$$

Since $\Psi(u,yu) \ge 0$ we always have $\Psi(u) - \Psi(u,yu) \le e^{-Ru}$. Further $g(r) \le 0$ if and only if $r \le R$. This implies that

$$\Psi(u) - \Psi(u,yu) \le e^{-R^y u} \tag{16}$$

where

$$R^y = \max(R, \sup_{0 \le r \le R} f(r)) = \sup_{0 \le r \le R} f(r).$$

The last equality follows since $f(R) = R$. Thus $R^y > R$ if and only if $f'(R) < 0$ which, see above, holds if and only if $y > y_0$. Thus

$$R^y \begin{array}{c} = \\ > \end{array} R \quad \text{according as} \quad y \begin{array}{c} \le \\ > \end{array} \frac{1}{g'(R)} .$$

For $y > y_0$ we have

$$R^y = f(r_y) \text{ where } r_y \text{ is the solution of } f'(r) = 0.$$

EXAMPLE 12. EXPONENTIALLY DISTRIBUTED CLAIMS. We have

$$\begin{aligned} g'(R) &= \alpha h'(R) - c = \alpha \frac{\mu}{(1-\mu R)^2} - c \\ &= \alpha\mu(1+\rho)^2 - \alpha\mu(1+\rho) = \alpha\mu\rho(1+\rho) \end{aligned}$$

and thus

$$y_0 = \frac{1}{\alpha\mu\rho(1+\rho)}.$$

Since

$$f'(r) = r - y\left(\alpha\frac{\mu}{(1-\mu r)^2} - c\right)$$

we get

$$\mu r_y = 1 - \sqrt{\frac{\alpha\mu y}{1+\alpha\mu y(1+\rho)}} \quad \text{and} \quad f(r_y) = \alpha y\left(\frac{\mu r_y}{1-\mu r_y}\right)^2 .$$

Using that $B(u)$ is exponentially distributed we can replace the second inequality in (14) with the equality

$$\frac{e^{-ru}}{E[M_u(T_u) \mid t < T_u \le \tilde{t}\,]} = \frac{e^{-ru}}{(1+\rho)E[e^{-T_u g(r)} \mid t < T_u \le \tilde{t}\,]}$$

we get the strengthened versions of (13) and (16)

$$\Psi(u, yu) \le \frac{e^{-R_y u}}{1+\rho} \quad \text{and} \quad \Psi(u) - \Psi(u, yu) \le \frac{e^{-R^y u}}{1+\rho} \qquad (17)$$

In Table 1 the upper bounds in (17) are compared with exact values. The exact values are taken from Wikstad (1971, p. 149). It is seen that (17) gives rather crude bounds.

EXAMPLE 13. WIENER PROCESS WITH DRIFT. Since $g(r) = -\beta r + \delta^2 r^2/2$ and $R = 2\beta/\delta^2$ we get

$$y_0 = \frac{1}{\delta^2 R - \beta} = \frac{1}{\beta}\,, \; r_y = \frac{1+\beta y}{y\delta^2} \quad \text{and} \quad f(r_y) = \frac{(1+\beta y)^2}{2y\delta^2} .$$

4. The time of ruin. If we rewrite (13) and (16) in terms of T_u we have

$$P\left\{\frac{T_u}{u} \le y\right\} \le e^{-R_y u} \quad \text{and} \quad P\left\{y < \frac{T_u}{u} < \infty\right\} \le e^{-R^y u}$$

Table 1. Exponentially distributed claims. The upper bounds are given (17).

u	t	ρ	$\frac{t}{u}$	y_0	relevant probability	upper bound	exact value
10	10	5%	1	19.05	$\Psi(10,10)$	0.1476	0.0367
10	10	10%	1	9.09	$\Psi(10,10)$	0.1209	0.0319
10	10	15%	1	5.80	$\Psi(10,10)$	0.0989	0.0277
10	10	20%	1	4.17	$\Psi(10,10)$	0.0807	0.0241
10	100	5%	10	19.05	$\Psi(10,100)$	0.5640	0.3464
10	100	10%	10	9.09	$\Psi(10)-\Psi(10,100)$	0.3656	0.1058
10	100	15%	10	5.80	$\Psi(10)-\Psi(10,100)$	0.2159	0.0440
10	100	20%	10	4.17	$\Psi(10)-\Psi(10,100)$	0.1168	0.0175
10	1000	5%	100	19.05	$\Psi(10)-\Psi(10,1000)$	0.3974	0.0243
10	1000	10%	100	9.09	$\Psi(10)-\Psi(10,1000)$	0.0516	0.0014
10	1000	15%	100	5.80	$\Psi(10)-\Psi(10,1000)$	0.0023	0.0001
10	1000	20%	100	4.17	$\Psi(10)-\Psi(10,1000)$	0.0000	0.0000
100	100	5%	1	19.05	$\Psi(100,100)$	0.0000	0.0000
100	1000	5%	10	19.05	$\Psi(100,1000)$	0.0051	0.0019
100	1000	10%	10	9.09	$\Psi(100)-\Psi(100,1000)$	0.0001	0.0000

where $R_y > R$ for $y < y_0$ and $R^y > R$ for $y > y_0$.
For any $\epsilon > 0$ we have

$$P\left\{\left|\frac{T_u}{u} - y_0\right| > \epsilon \mid T_u < \infty\right\} \le \frac{e^{-R_{(y_0-\epsilon)}u} + e^{-R^{(y_0+\epsilon)}u}}{P\{T_u < \infty\}} \tag{18}$$

From the Cramér–Lundberg approximation, i.e.,

$\lim_{u\to\infty} e^{Ru}P\{T_u < \infty\} = C$, it follows that

$$\lim_{u\to\infty} P\left\{\left|\frac{T_u}{u} - y_0\right| > \epsilon \mid T_u < \infty\right\} = 0$$

or, where $\xrightarrow{P}$ means "convergence in probability", that

$$\frac{T_u}{u} \xrightarrow{P} y_0 \quad \text{on } \{T_u < \infty\} \text{ as } u \to \infty. \tag{19}$$

(Strictly speaking (19) means $\mathcal{L}(\frac{T_u}{u} \mid T_u < \infty) \to \delta_{y_0}$ as $u \to \infty$.) Consider again the martingale $M_u(t) = e^{-r(u+X(t))-tg(r)}$. The funtion $g(r)$ is non-negative and increasing for $r \in [R, r_\infty)$. For $r \in [R, r_\infty)$

$$e^{-ru} = E[e^{-r(u+X(T_u))-T_u g(r)} \mid T_u < \infty]P\{T_u < \infty\} \tag{20}$$

follows with the same arguments as (11). From Neveu (1972, p. 81) it follows that (20), in fact, holds for $r \in [r^*, r_\infty)$, where r^* is the solution of $g'(r) = 0$.

Consider now the restriction of g to $[r^*, r_\infty)$ and its inverse $\gamma : [g(r^*), \infty) \to [r^*, r_\infty)$. Note that

$$\gamma(0) = R, \quad \gamma'(0) = \frac{1}{g'(R)} = y_0, \quad \gamma''(0) = -\frac{g''(R)}{g'(R)^3} \stackrel{\text{def}}{=} -v_0.$$

Thus (20) leads to

$$E[e^{-\gamma(v)(u+X(T_u))-vT_u} \mid T_u < \infty] = \frac{e^{-\gamma(v)u}}{\Psi(u)} . \tag{21}$$

When $B(u)$ is independent of T_u we put $c(r,u) = E[e^{-r(u+X(T_u))} \mid T_u < \infty]$. Then (21) reduces to

$$E[e^{-vT_u} \mid T_u < \infty] = \frac{e^{-\gamma(v)u}}{c(\gamma(v), u)\Psi(u)} . \tag{22}$$

For risk processes with negative risk sums and for the Wiener process we have $c(r,u)=1$ and for exponentially distributed claims we have $c(r,u)=\frac{1}{1-\mu r}$. For simplicity we consider first the case $c(r,u)=1$. Then

$$E[e^{-vT_u} \mid T_u<\infty] = e^{-(\gamma(v)-R)u} = e^{-vy_0u+v^2v_0u/2+O(v^3)}$$

and thus

$$E[T_u \mid T_u<\infty] = y_0u \quad \text{and} \quad \mathrm{Var}[T_u \mid T_u<\infty] = v_0u. \tag{23}$$

EXAMPLE 14. WIENER PROCESS WITH DRIFT. In this case we have $g(r) = -\beta r + \delta^2 r^2/2$ and thus

$$R=\frac{2\beta}{\delta^2} \quad \text{and} \quad \gamma(v)=\frac{\beta+\sqrt{\beta^2+2\delta^2 v}}{\delta^2}\,.$$

Then $E[e^{-vT_u} \mid T_u<\infty] = e^{u\left(\beta-\sqrt{\beta^2+2\delta^2 v}\right)/\delta^2}$ which leads to

$$\begin{aligned} P\{T_u\le t \mid T_u<\infty\} &= \int_0^t \tfrac{u}{\delta\sqrt{s^3}}\varphi\left(\tfrac{\beta s-u}{\delta\sqrt{s}}\right)\,ds \\ &= \left(1-\Phi\left(\tfrac{\beta t+u}{\delta\sqrt{t}}\right)\right)e^{Ru}+\Phi\left(\tfrac{\beta t-u}{\delta\sqrt{t}}\right) \end{aligned} \tag{24}$$

where, as usual, $\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$ and $\Phi(x)=\int_{-\infty}^{x}\varphi(y)\,dy$.

EXAMPLE 15. EXPONENTIALLY DISTRIBUTED CLAIMS. We have $g(r)=\frac{\alpha\mu r}{1-\mu r}-rc$ and thus

$$R=\frac{\rho}{\mu(1+\rho)}\,,\qquad y_0=\frac{1}{\alpha\mu\rho(1+\rho)} \quad \text{and} \quad v_0=\frac{2}{\alpha^2\mu\rho^3}\,.$$

From (12) we then get

$$\begin{aligned} \tilde\gamma_u(v) &\stackrel{\text{def}}{=} E[e^{-vT_u} \mid T_u<\infty] = (1-\mu\gamma(v))(1+\rho)e^{-(\gamma(v)-R)u} \\ &= \big(1-\mu(1+\rho)(\gamma(v)-R)\big)e^{-(\gamma(v)-R)u} \\ &= \tilde\gamma_0(v)e^{-u(1-\tilde\gamma_0(v))/(\mu(1+\rho))}. \end{aligned}$$

This implies that

$$\mathcal{L}(T_u \mid T_u < \infty) = \mathcal{L}\left(\sum_{k=1}^{\widetilde{N}_u+1} \widetilde{T}_k\right) \tag{25}$$

where $\widetilde{N}_u$ is Poisson-distributed with mean $u/(\mu(1+\rho))$, $\mathcal{L}(\widetilde{T}_k) = \mathcal{L}(T_0 \mid T_0 < \infty)$ and $\widetilde{N}_u$, $\widetilde{T}_1$, $\widetilde{T}_2 \dots$ are independent. Now we have $E[\widetilde{T}_1] = -\tilde{\gamma}_0'(0) = \mu(1+\rho)y_0$ and $E[\widetilde{T}_1^2] = \tilde{\gamma}_0''(0) = \mu(1+\rho)v_0$ and thus

$$E[T_u \mid T_u < \infty] = y_0 u + \tfrac{1}{\alpha\rho}$$

and (26)

$$\mathrm{Var}[T_u \mid T_u < \infty] = v_0 u + \tfrac{2+\rho}{\alpha^2\rho^3} \ .$$

Formula (26) can, of course, also be obtained by derivation of $\tilde{\gamma}_u(v)$.

5. The claim cousing ruin. Consider now positive risk sums, so that ruin can occur only at the times of claims. Put $N_u = N(T_u)$, i.e., N_u is the "number" of the claim causing ruin. Let S_1, S_2, $S_3, \dots$ denote the claim times, and put $X_k = X(S_k)$. Since the Poisson process is a renewal process it follows that $\{X_k\}_{k=1}^{\infty}$ forms a random walk. Note that $\Psi(u) = P\{N_u < \infty\}$. Now we have

$$e^{\tilde{g}(r)} \stackrel{\text{def}}{=} E[e^{-rX_1}] = E[e^{-r(cS_1 - Z_1)}] = \frac{\alpha}{\alpha + cr} \cdot (h(r)+1).$$

Consider now

$$M_u(n) = \frac{e^{-r(u+X_n)}}{e^{n\tilde{g}(r)}}$$

which is the discrete time martingale corresponding to $M_u(t)$. Exactly as (6) was proved, it follows that

$$\Psi(u) \le e^{-ru} \sup_{n\ge 0} e^{n\tilde{g}(r)}$$

and thus $R = \sup\{r \mid \tilde{g}(r) \leq 0\}$. Now $\tilde{g}(r) = 0 \Leftrightarrow \frac{\alpha}{\alpha+cr} \cdot (h(r)+1) = 1 \Leftrightarrow \alpha h(r) = cr$ so we have, of course, not proved anything new.

The arguments in Section 3 go through, and, exactly as (19) was derived, we get

$$\frac{N_u}{u} \xrightarrow{P} \frac{1}{\tilde{g}'(R)} = (\alpha + cR)y_0 \quad \text{on } \{N_u < \infty\} \text{ as } u \to \infty. \quad (27)$$

6. Generalizations of the risk model. The classical risk model, discussed in this paper, can be generalized in many ways.

A. The premiums may depend on the result of the risk business. It is natural to let the safety loading at a time t be "small" if the risk business, at that time, attains a large value and vice versa.

B. Inflation and interest may be included in the model.

C. The occurrence of the claims may be described by a more general point process than the Poisson process.

Dassios and Embrechts (1989) and Delbaen and Haezendonck (1987) are very readable studies focusing mainly on generalizations A and B, while generalization C is considered by Grandell (1990). In all these studies most results are derived with the help of martingales.

There are, at last, two very different reasons for using other models for the claim occurrence than the Poisson process. *Firstly* the Poisson process is stationary, which – among other things – implies that the number of policy-holders involved in the portfolio can not increase (or decrease). Few insurance managers would accept a model where the possibility of an increase of the business is not taken into account. We shall refer to this case as *size fluctuation*. *Secondly* there may be fluctuation in the underlying risk. Typical examples are automobile insurance and fire insurance. We shall refer to this as *risk fluctuation.* Consider shortly the first case.

6.1 Models allowing for size fluctuation. The simplest way to take size fluctuation into account, is to let N be a non-homogeneous Poisson process. Let $A(t)$ be a continuous non-decreasing function with $A(0) = 0$ and $A(t) < \infty$ for each $t \leqslant \infty$.

DEFINITION 16. A point process N is called a (non-homogeneous) *Poisson process with intensity measure* A if

(i) $N(t)$ has independent increments;

(ii) $N(t) - N(s)$ is Poisson distributed with mean $A(t) - A(s)$.

REMARK 17. The function $A(t)$ can be looked upon as the distribution function corresponding to the measure A. The continuity of $A(\cdot)$ guarantees that N is *simple,* i.e., that $N(\cdot)$ increases exactly one unit at its epochs of increase.

Define the inverse A^{-1} of A by

$$A^{-1}(t) = \sup(s|A(s) \leq t). \tag{28}$$

A^{-1} is always right-continuous. Since $A(\cdot)$ is continuous, A^{-1} is (strictly) increasing and

$$A \circ A^{-1}(t) \stackrel{\text{def}}{=} A(A^{-1}(t)) = t \quad \text{for } t < A(\infty).$$

DEFINITION 18. A Poisson process $\tilde{N}$ with $\alpha = 1$ is called a *standard* Poisson process.

The following obvious results are, due to their importance, given as lemmata.

Lemma 19. *Let N be a Poisson process with intensity measure A such that $A(\infty) = \infty$. Then the point process $\tilde{N} \stackrel{\text{def}}{=} N \circ A^{-1}$ is a standard Poisson process*

Proof. Since A^{-1} is increasing it follows that $\tilde{N}$ has independent increments. Further $\tilde{N}(t) - \tilde{N}(s) = N(A^{-1}(t)) - N(A^{-1}(s))$ is Poisson distributed with mean $A \circ A^{-1}(t) - A \circ A^{-1}(s) = t - s$.

Lemma 20. *Let $\tilde{N}$ be a standard Poisson process. Then the point process $N \stackrel{\text{def}}{=} \tilde{N} \circ A$ is a Poisson process with intensity measure A.*

The proof is omitted.

Without much loss of generality we may assume, although it is not at all necessary, that A has the representation

$$A(t) = \int_0^t \alpha(s)\, ds \tag{29}$$

where $\alpha(\cdot)$ is called the *intensity function.* It is natural to assume that $\alpha(s)$ is proportional to the number of policy-holders at time s. When the premium is determined individually for each policy-holder it is also natural to assume the gross risk premium to be proportional to the number of policy-holders. If the relative safety loading ρ is constant we get $c(t) = (1 + \rho)\mu\alpha(t)$ and the corresponding risk process is given by,

$$X(t) = (1 + \rho)\mu A(t) - \sum_{k=1}^{N(t)} Z_k. \tag{30}$$

where N is a Poisson process with intensity measure A such that $A(\infty) = \infty$.

Consider now the process $\tilde{X}$ defined by

$$\tilde{X}(t) \stackrel{\text{def}}{=} X \circ A^{-1}(t) = (1 + \rho)\mu t - \sum_{k=1}^{\tilde{N}(t)} Z_k. \tag{31}$$

Thus $\tilde{X}$ is a classical risk process with $\alpha = 1$. Recall that

$$\Psi(u) = P\{\inf_{t \geq 0} X(t) < -u\}.$$

If $A(\cdot)$ is increasing, or if $\alpha(t) > 0$, A^{-1} is continuous and it is obvious that $\inf_{t \geq 0} X(t) = \inf_{t \geq 0} \tilde{X}(t)$. Here it would only be a minor restriction to assume that $A(\cdot)$ is increasing, but for the further discussion we do not want to make that restriction. Suppose that A^{-1} has a jump at t. In the time interval $(A^{-1}(t-), A^{-1}(t))$ no claims occur, since $N(A^{-1}(t)) - N(A^{-1}(t-))$ is Poisson distributed with mean $A \circ A^{-1}(t) - A \circ A^{-1}(t-) = t - (t-) = 0$, and no premiums are recieved. Thus $\inf_{t \geq 0} X(t) = \inf_{t \geq 0} \tilde{X}(t)$ and the problem of calculating the ruin probability is brought back to the classical situation.

The time scale defined A^{-1} is generally called *the operation time scale*, see e.g. Cramér (1955, p. 19).

We have referred to this generalization as "size fluctuations", only because then the gross risk premium rate $c(t) = (1 + \rho)\mu\alpha(t)$ is very natural. Obviously it is mathematically irrelevant *why* $\alpha(\cdot)$ fluctuates, as long as those fluctuations are compensated by the premium in the above way. We shall now see that a kind of operational time scale can be defined for a very wide class of point processes. Those processes may very well more naturally correspond to "risk fluctuation" than to "size fluctuation".

7. Two open problems

7.1. Generalizations of the Poisson process. The, at least mathematically, most natural generalization of the classical risk model is probably to assume that the occurence of the claims is described by a renewal process. The first treatment of this generalization is due to Sparre Andersen (1957). After the publication of his paper this model has been considered in several works. In a series of papers Thorin, see the review Thorin (1982), has carried through a systematic study based on Wiener-Hopf methods. It is rather natural that analytical methods work well for the renewal process.

Björk and Grandell (1988) considered the Lundberg inequality when the occurence of the claims are described by a Cox process. Intuitively we shall think of a Cox process N as generated in the following way: First a realization A of a random measure Λ is generated and then, conditioned upon that realization, the point process N is a Poisson process with intensity measure A. This indicates that Cox processes are very natural as models for "risk fluctuation". It is natural to consider the filtration $(\mathcal{F}_\infty^\Lambda \vee \mathcal{F}_t^X;\ t \geq 0)$ and the martingale

$$M(t) = \frac{e^{-r(u+X(t))}}{e^{\Lambda(t)h(r)-trc}}\ .$$

A Lundberg inequality will be of the form:
For every $\epsilon > 0$ such that $0 < \epsilon < R$ we have

$$\Psi(u) \leq C(R-\epsilon)e^{-(R-\epsilon)u},$$

where $C(R-\epsilon) < \infty$.

The problem is not to get this general inequality, but to find interesting special cases where R can be explicitely determined. The condition $\epsilon > 0$ is unpleasant, but it is quite possible – and really natural – that $C(R) = +\infty$. If

$$\Lambda(t) = \int_0^t \lambda(s)\ ds,$$

where $\lambda(s)$ is a Markov process, it is natural to consider the filtration $(\mathcal{F}_t^\lambda \vee \mathcal{F}_t^X;\ t \geq 0)$. By using a natural martingale, the condition $\epsilon > 0$ can, in certain cases, be removed. The most interesting case is probably when $\lambda(s)$ is a finite state Markov process. In that case Asmussen (1989) proved a Cramér–Lundberg approximation with analytical methods.

We shall not discuss Cox models any more, although much more certainly can be done. We shall just point out that $\Lambda(t)$ is the compensator of N relative $\mathcal{F}_\infty^\Lambda \vee \mathcal{F}_t^X$. A natural question is now, if there exists interesting non-Cox processes whose compensators relative some filtration can be used in order to obtain results about the ruin probability.

7.2. Inverse thinning of point processes. Let N be a point process, and denote by N_p the point process obtained by retaining (in the same location) every point of the process with probability p and deleting it with probability $q = 1 - p$, independently of everything else. N_p is called the *p-thinning* of N and N is called the *p-inverse* of N_p. It is natural to regard the claims as as caused by "risk situations" or *incidents.* To each incident we associate a *claim probability* p and we assume that incidents become claims independently of each other. Under these assumptions the point process describing the incidents is the p-inverse of the "claim process" N. It follows from Mecke (1968) that the claim process is a Cox process if and only if a *p-inverse* exists for all p.

Let us now consider the "opposite" class.

DEFINITION 21. A point process N is called a *top process* if it cannot be obtained by p-thinning for any $p \in (0,1)$.

Let, as an example, N be a renewal process with Γ-distributed inter-occurence times with form parameter γ. Yannaros (1988a) has shown that N is

a *Cox process if* $0 < \gamma \leq 1$ and *a top process if* $\gamma > 1$.

This result is – in our opinion – very interesting since it concerns an important renewal process and since it illustrates that transition between the "extreme" classes of Cox and top processes is not "continuous". Further it was very surprising

– at least to the author – that such a simple renewal process can be a Cox process.

It would be interesting to have a description of the class of top processes, or, at least, to understand what kind of properties such a description ought to be based on. A sufficient condition for a renewal process to be a top process is given by Yannaros (1988b).

8. Remarks on the literature

1. Several main results in risk theory, as the Lundberg inequality and the Cramér–Lundberg approximation, are due to Lundberg (1926) and Cramér (1930), while the general ideas underlying the collective risk theory go back as far as to Lundberg (1903). These works appeared before the theory of stochastic processes was developed and are therefore not mathematically quite stringent. They are pioneering works, not only in risk theory, but also in the development of the general theory of stochastic processes. The development of risk theory using rigorous methods is to a large extent due to Arfwedson, Cramér, Saxén, Segerdahl and Täcklind. For a survey of their contribution we refer to Cramér (1955, pp. 48–51), where a stringent presentation of risk theory, based on Wiener–Hopf methods, is given.

Diffusion approximations was first applied to risk theory by Iglehart (1969). As mentioned, its numerical accuracy is not very good. Asmussen (1984) applied the diffusion approximation to the so-called conjugate process and achieved much better accuracy. De Vylder (1978) has proposed an approximation, based on the simple, but ingenious, idea to replace the risk process X with a risk process $\tilde{X}$ with exponentially distributed claims such that the first three moments coincide. Numerical comparison indicate that its numerical accuracy is very good.

A discussion about the relation between risk theory and queueing theory is found in Grandell (1990, pp. 123–128)[1)]

2. The results (8), (9), (12) and $\Psi(0) = \alpha\mu/c$ go all back to Lundberg (1926) and Cramér (1930). A nice proof of (9) is given by Feller (1966, pp. 363–364). $\Psi(0) = \alpha\mu/c$ is an insensitivity or robustness result, since $\Psi(0)$ only depends on ρ and thus on F only through its mean. It does, in fact, hold for any stationary and ergodic point process, see Björk and Grandell (1985).

3. Martin–Löf (1986) has proved (13) and (16) with different methods. Arfwedson (1955) has given "Cramér–Lundberg approximation" corresponcences to (13) and (16). It follows from Arfwedson (1955, pp. 58 and 78) that

$$\Psi(u, yu) \sim \begin{cases} \frac{C_y}{\sqrt{u}} e^{-R_y u} & \text{if } y < y_0 \\ \frac{C}{2} e^{-Ru} & \text{if } y = y_0 \\ Ce^{-Ru} & \text{if } y > y_0 \end{cases} \quad \text{as } u \to \infty. \tag{32}$$

The "$\frac{1}{\sqrt{u}}$" may explain why (17) gives rather crude bounds. Höglund (1990) has generalized (32) to include the case when the occurence of the claims is described by a renewal process.

4. Segerdahl (1955, p. 34) showed that

$$P\{T_u \le t \mid T_u < \infty\} \sim \Phi\left(\frac{t - uy_0}{\sqrt{uv_0}}\right) \tag{33}$$

as $u, t \to \infty$ and $\frac{t-uy_0}{\sqrt{u}}$ is bounded. The exact results (23) and (26) are also due to Segerdahl (1955). From (25) it follows that (33) holds in the case of exponentially distributed claims. It is also easy to realize that (33) holds when $c(r, u) = 1$. It

[1)] References to pages in Grandell (1990) may be not quite accurate, since the book is in the stage of production.

seems difficult to prove (33) by martingale methods. von Bahr (1974) has extended (33) to the case when the occurence of the claims is described by a renewal process.

Formula (24) follows from Feller (1966, p. 439) and Skorohod (1965, p. 171). The idea to derive (21) with martingale methods, and especially Example 15, is due to Asmussen (1984, pp. 37–42).

6. Early studies of generalizations A and B are Davidson (1946) and Segerdahl (1942) respectively.

7.2. Let N be a renewal process with inter-occurence time distribution K. (We denote by K^{2*} the convolution of K with itself and understand that K has finite, positive mean.) Yannaros (1988b) has shown that N is a top process if

$$\lim_{t\to\infty} \frac{1-K^{2*}(t)}{t\cdot(1-K(t))} = \infty. \tag{34}$$

If K has density k, the sufficient condition (34) holds if

$$\lim_{t\to\infty} \frac{k^{2*}(t)}{t\cdot k(t)} = \infty.$$

If K is a Γ-distribution with form parameter γ we have $\frac{k^{2*}(t)}{tk(t)} = t^{\gamma-1}\frac{\Gamma(\gamma)}{\Gamma(2\gamma)}$.

REMARK 22. Condition (34) can be generalized by taking suprema instead of limits. For example, if

$$\sup_t \frac{1-K^{2*}(t)}{t\cdot(1-K(t))} = \infty,$$

then N is a top process. Thus, if K has compact support, N is a top process.

REFERENCES

Andersen, E. Sparre (1957). On the collective theory of risk in the case of contagion between the claims. In *Transactions XVth International Congress of Actuaries,* New York, II. pp. 219–229.

Arfwedson, G. (1955). Research in collective risk theory. Part 2. *Skand. AktuarTidskr.,* 53–100. Part 1 in *SAT* (1954, pp.191–223.)

Asmussen, S. (1982). Conditioned limit theorems relating a random walk to its associate with applications to risk reserve process and the G1/G/1 queue. *Adv. Appl. Prob.,* 143–170

Asmussen, S. (1984). Approximations for the probability of ruin within finite time. *Scand. Actuarial J.,* 31–57. Erratum in *SAJ* (1985, p. 64).

Asmussen, S. (1989). Risk theory in a markovian environment. *Scand. Actuarial J.,* 66–100.

von Bahr, B. (1974). Ruin probabilities expressed in terms of ladder height distributions. *Scand. Actuarial J.,* 190–204.

Björk, T., and J.Grandell (1985). An insensitivity property of the ruin probability. *Scand. Actuarial J.,* 148–56.

Björk, T., and J.Grandell (1988). Exponential inequalities for ruin probabilities in the Cox case. *Scand. Actuarial J.,* 77–111.

Cramér, H. (1930). *On the mathematical theory of risk.* Skandia Jubilee Volume, Stockholm.

Cramér, H. (1955). *Collective risk theory.* Skandia Jubilee Volume, Stockholm.

Dassios, A., and P.Embrechts (1989). Martingales and insurance risk. *Commun. Statist. – Stochastic models,* **5**, 181–217.

Davidson, Å. (1946). Om ruinproblemet i den kollektiva riskteorin under antagande av variabel säkerhetsbelastning. *Försäkringsmatematiska studier tillägnade Filip Lundberg,* Stockholm, 32–47. English translation: – (1969). On the ruin problem in the collective theory of risk under the assumption of variable safety loading. *Skand. AktuarTidskr. Suppl.*, 70–83.

Delbaen, F., and J.Haezendonck (1987). Classical risk theory in an economic environment. *Insurance: Mathematics and Economics,* **6**, 85–16.

De Vylder, F. (1978). A practical solution to the problem of ultimate ruin probability. *Scand. Actuarial J.,* 114–119.

Feller, W.1966 *An introduction to probability theory and its applications,* Vol.II, 1st ed. John Wiley & Sons, New York.

Gerber, H. U. (1973). Martingales in risk theory. *Mitt. Ver. Schweiz. Vers. Math.,* **73**, 205–216.

Grandell, J. (1990). *Aspects of risk theory.* Springer-Verlag, New York (to appear).

Höglund, T. (1990). An asymptotic expression for the probability of ruin within finite time. *Ann. Prob. 18,* **18**(1), (to appear).

Iglehart, D.L. (1969). Diffusion approximations in collective risk theory. *J. Appl. Prob.,* **6**, 285–292.

Lundberg, F. (1903). *I. Approximerad framställning av sannolikhetsfunktionen. II. Återförsäkring av kollektivrisker.* Almqvist & Wiksell, Uppsala.

Lundberg, F. (1926). *Försäkringsteknisk riskutjämning.* F. Englunds boktryckeri AB, Stockholm.

Martin–Löf, A. (1986). Entropy, a useful concept in risk theory. *Scand. Actuarial J.,* 223–235.

Mecke, J. (1968). Eine charakteristische Eigenschaft der doppelt stochastischen Poissonschen Prozesse. *Z. Wahrschein. und Verw. Geb.,* **11**, 74–81.

Neveu, J. (1972). *Martingales a temps discret.* Masson, Paris. English translation: – (1975). *Discrete-parameter martingales.* North-Holland, Amsterdam.

Segerdahl, C.–O. (1942). Über einige risikotheoretische Fragestellungen. *Skand. AktuarTidskr.,* 43–83.

Segerdahl, C.–O. (1955). When does ruin occur in the collective theory of risk. *Skand. AktuarTidskr.,* 22–36.

Skorohod, A.V. (1965). *Studies in the theory of random processes.* Addison–Wesley (in Russian).

Thorin, O. (1982). Probabilities of ruin. *Scand. Actuarial J.,* 65–102.

Wikstad, N. (1971). Examplification of ruin probabilities. *Astin Bulletin VI,* 147–152.

Yannaros, N. (1988a). On Cox processes and gamma renewal processes. *J. Appl. Prob.,* **25**, 423–427.

Yannaros, N. (1988b). The inverses of thinned renewal processes. *J. Appl. Prob.,* **25**, 822–828.

Received June 1990

J. Grandell is associate professor in matematical statistics at the Royal Institute of Technology in Stockholm. He became Ph.d. and docent in mathematical statistics at the University of Stockholm, Sweden, in 1976 and 1981. His research interests are in the area of random processes.