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# Robust Adaptive Tracking with Pole Placement of First-Order Potentially Inversely Unstable Continuous-Time Systems

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**Abstract.** This note presents an indirect adaptive control scheme applicable to nominally controllable non-necessarily inversely stable first-order continuous linear time-invariant systems with unmodelled dynamics. The control objective is *to achieve a bounded tracking-error between the system output and a reference signal*. A least-squares algorithm with normalization is used to estimate the plant parameters by using two additional *design tools*, namely: 1) *a modification of the parameter estimates and* 2) *a relative adaptation dead-zone*. The modification is based on the properties of the inverse of the least-squares covariance matrix and it uses an hysteresis switching function. In this way, the non-singularity of the controllability matrix of the estimated model of the plant is ensured. The relative dead-zone is used to turn off the adaptation process when an absolute augmented error is smaller than the value of an available overbounding function of the unmodelled dynamics contribution plus, eventually, bounded noise.

**Key words:** Robust control, hysteresis, inversely unstable, controllability, dead-zone.

## 1. Introduction

Adaptive control of continuous linear time-invariant systems has been succesfully developed in the last decade (Feuer and Morse, 1978; Ioannou and Tsakalis, 1986; Morse, 1980; Narendra *et al.*, 1980). Most of the proofs of convergence of the adaptive control algorithms proposed in the literature are all based on a set of assumptions. Two of these assumptions are: the knowledge of the system order and the knowledge of the system relative degree. Besides, the approach of adaptive control via pole placement needs explicity the controllability of the estimated model of the plant. The hypothesis of inverse stability and the knowledge on the sign of the high-frequency gain of the plant are sufficient conditions to obtain a controllable estimated model. Works whose goal was to relax some of these aforementioned assumptions have been presented. In particular the studies in (Lozano *et al.*, 1994) and (Suarez and Lozano, 1996) relax the assumption of inverse stability of the plant. An adaptive control algorithm which stabilizes a first-order continuous time-invariant perfectly modelled regulator system has been designed in (Lozano *et* *al.*, 1994). Also, a global asymptotical stability for a robust adaptive pole placement control scheme for continuous time-invariant systems with bounded disturbances has been established in (Suarez and Lozano, 1996). The knowledge of an upper-bound of the disturbances was not assumed and it was estimated as well as the plant parameters. The stability has been established in both articles without either requiring persistent excitation probing signals into the system or assuming any prior knowledge of the plant parameters. However, there is not yet any work which studies the stability of a potentially inversely unstable system under the presence of unmodelled dynamics.

The presence of unmodelled dynamics and possibly bounded noise on a first-order continuous time-invariant non-necessarily inversely stable plant is considered in this work. In this way the assumptions of knowledge of the system order and inverse stability of the plant are relaxed. Besides, the knowledge of the sign of the high frequency gain will be also relaxed. This work differs from that proposed in (Suarez and Lozano, 1996) in that it adds the presence of unmodelled dynamics which complicates the stability analysis of the system. The tracking of a reference signal, given by a stable filter $W_m(s)$, with an additional pole placement is taken as control objective in the ideal case. This objective is relaxed to achieve a bounded tracking-error in the case of the presence of unmodelled dynamics, bounded disturbances and uncertainties in the plant parameters. The designed control law is such that the dynamics of the modelled part of the closed-loop system is defined by the zeros of an arbitrary Hurwitz polynomial. The controller takes directly the tracking-error as one of its inputs instead of the system output and it incorporates a new control parameter with respect to those considered in (Lozano *et al.*, 1994) and (Suarez and Lozano, 1996). This parameter makes equal the gain of the closed-loop system with that of $W_m(s)$.

In a non-inversely stable controllable system where the sign of the high frequency gain is unknown, the controllability of the estimated model of the plant has to be ensured at all time instant. There are basically two different approaches to circumvent the regions in the parameter space corresponding to uncontrollable models. One of them relies on the use of excitation probing signals (Anderson and Johnstone, 1985; Elliot *et al.*, 1985; Giri *et al.*, 1989; Goodwin and Teoh, 1985; Kreisselmeier and Smith, 1986; Polderman, 1989), while the other one is based on the use of a suitable modification of the plant parameter estimates as in (Larminat, 1984; Lozano *et al.*, 1994; Suarez and Lozano, 1996). In this paper, the latter approach is used. The modification is of the form $\overline{\theta} = \widehat{\theta} + \pi P \beta$, where $\overline{\theta}$ and $\widehat{\theta}$ are the modified and unmodified estimated parameters vectors, respectively. It is based in the properties of the matrix $P$, which is the inverse of the covariance matrix of the least-square algorithm. $\beta(t)$ is an hysteresis switching function and $\pi(t)$ is defined such that the estimated model of the plant is controllable for all time. This modification adds the function $\pi(t)$ with respect to that used in (Lozano *et al.*, 1994) and (Suarez and Lozano, 1996). The goal is to avoid the use of the modification procedure when the estimated model of the plant obtained with the unmodified estimates is far of the non-controllability domain. Such a function is nonzero only when the unmodified estimated model is close to losing its controllability domain. This technique is different from that of Landau, (Landau, 1979). In this paper the modification of the plant parameter estimates is analitic. Instead, in the Landau's techniques a standard estimation algorithm without modification of the plant parameter estimates is used. When the estimated model of the plant is close to the non controllability domain the value of the forgetting factor or any other free-design parameter of the estimation algorithm is modified on line within its admissibility domain for stabilization purposes.

A normalized least-squares algorithm with a relative adaptation dead-zone and the aforementioned modification is used to estimate the plant parameters. The above combined technique is the basis to prove the convergence of the estimated parameters and the boundedness of all the signals in the closed-loop system. The inclusion of forgetting factor in the parameter estimation algorithm is not obligatory since the plant is time-invariant and the controllability of the estimated model of the plant is ensured by means of the aforementioned modification of the estimates. The knowledge of an upper-bounding function for the contribution of the unmodelled dynamics and disturbances to the output is necessary to design the adaptive relative dead-zone (Feng, 1995). Besides, it is proved that the normalized modified prediction error belongs to a residual set defined by a normalized upper-bound for the contribution of the unmodelled dynamics, and bounded noise, to the output.

Section 2 of this paper presents the problem statement with the control objective and the control law. The pole placement problem of the tracking-error dynamics is solved for the ideal case, i.e., when the plant is known. Section 3 presents the parameters estimation algorithm that provides a controllable estimated model of the plant. Section 4 presents the main result of the stability analysis. Section 5 is devoted to a numerical simulation. Finally, conclusions are drawn at the end of the paper.

## 2. Problem Statement

Consider the following system under a time-differentiable input

$$\dot{y} = -ay + b_0\dot{u} + b_1 u + \eta \tag{1}$$

with $b_1 \neq 0$, where $\eta$ is the contribution of unmodelled dynamics and bounded noise of any order to the output. The argument $(t)$ has been omitted in the signals $u,\ y$ and $\eta$ and their time-derivatives. The following assumption is introduced.

ASSUMPTION 1. The modelled part of the true plant is controllable in the sense that $b_1 - ab_0 \neq 0$.

As in (Middleton *et al.*, 1988), the following filtered signals are introduced

$$\dot{y}_f = -qy_f + y; \quad \dot{u}_f = -qu_f + u; \quad \dot{\eta}_f = -q\eta_f + \eta \tag{2}$$

with any real constant $q > 0$. Then, the filtered plant output is given by

$$\dot{y}_f = -ay_f + b_0\dot{u}_f + b_1 u_f + \eta_f + \xi = \theta^T\phi + \eta_f, \tag{3}$$

with $\xi = \xi_0 e^{-qt}$ being an exponentially decaying term that depends on the parameterized initial conditions, denoted by $\xi_0$, and where

$$\theta^T = [b_0, b_1, a, \xi_0]; \quad \phi^T = [\dot{u}_f, u_f, -y_f, e^{-qt}] \tag{4}$$

are the true plant parameters and the filtered signals regression vectors, respectively. The following assumption is introduced in order to be able to calculate a worst-case contribution of the unmodelled dynamics and bounded noise to the plant output.

ASSUMPTION 2. There exists real constants $\sigma_0 \in (0,1)$, $\alpha_0 \geqslant 0$ and $\alpha \geqslant 0$ and a constant vector $v = [v_1, v_2]^T$, which are known, such that

$$\left|\eta_f(t)\right| \leqslant \overline{\eta}_f(t) = \alpha\rho(t) + \alpha_0 \quad \forall t \geqslant 0, \tag{5}$$

where $\rho(t) = \sup_{0 \leqslant \tau \leqslant t} \{|v_1 u_f(\tau) + v_2 \varepsilon_f(\tau)| e^{-\sigma_0(t-\tau)}\}$, $\varepsilon_f = y_f - y_{mf}$ is the filtered tracking-error with $y_{mf}$ being the filtered reference signal obtained from $\dot{y}_{mf} = -q y_{mf} + y_m$, $y_m(t) = W_m(s) r(t)$, $W_m(s) = \dfrac{b_0' s + b_1'}{s + a'}$ and $r(t)$ a bounded piecewise continuous signal.

REMARK 1. Assumption 2 is necessary to design a relative dead-zone for adaptation purposes in the parameter estimation shown in section 3. The width of the dead-zone is governed by the overbounding normalized function $\overline{\eta}_{fn} = \dfrac{\overline{\eta}_f}{1 + ||\phi||}$. This assumption is fulfilled by any system in which the signal $\eta_f(t)$ is the sum of a bounded term, plus a term related to $u(t)$ by a strictly proper exponentially stable transfer function, as shown in (Middleton *et al.*, 1988).

The control objective is *the tracking between the system output and the reference signal with a bounded tracking error*. The following adaptive control law is designed to meet the control objective:

$$\begin{aligned} u(t) &= r_0 r(t) - r_1 u_f(t) - r_2\big(y_f(t) - W_m r_f(t)\big) \\ &= \frac{r_0(s+q) + r_2 W_m}{s + q + r_1} r(t) - \frac{r_2}{s + q + r_1} y(t) \end{aligned} \tag{6}$$

with $r_f$ being such that $\dot{r}_f = -q r_f + r$. $\dfrac{y(t)}{u(t)} = W(s) = W_0(s)\big(1 + \nu\Delta_1(s)\big) + \nu\Delta_2(s)$ is the transfer function of the plant. $W_0(s) = \dfrac{b_0 s + b_1}{s + a}$ is the modelled part of the plant, $\nu\Delta_1(s)$ and $\nu\Delta_2(s)$ are the transfer functions of the multiplicative and additive unmodelled dynamics, respectively, and $\nu$ is a positive constant. $\Delta_1(s)$ and $\Delta_2(s)$ must be both strictly Hurwitz and strictly proper so that Assumption 2 is feasible. *The signal* $y_m(t) = W_m r(t)$ *is the reference signal to be asymptotically tracked by the plant output*

$y(t)$ *in the ideal case (i.e., when* $\nu = 0$*)*. The transfer function of the closed-loop system is:

$$\frac{y(s)}{r(s)} = \tag{7}$$
$$= \frac{\big[r_0(s+q)+r_2W_m(s)\big]\big((b_0s+b_1)+\nu\big((b_0s+b_1)\Delta_1(s)+(s+a)\Delta_2(s)\big)\big)}{(s+q+r_1)(s+a)+r_2(b_0s+b_1)+\nu r_2\big((b_0s+b_1)\Delta_1(s)+(s+a)\Delta_2(s)\big)}.$$

The closed-loop transfer function of the system has the pole of the filter $W_m$. This fact gives the possibility of achieving a slower or faster transient tracking-error, depending on the relative placement of the poles, than in the case of the system with non filtered reference signal ($W_m(s) = 1$) which only has two closed-loop poles. Two poles of the modelled part of the closed-loop system ($\nu = 0$ in (7)) are fixed according to the Hurwitz polynomial $C(s) = s^2 + c_1s + c_2$ by the control parameters $r_1$ and $r_2$, and its low-frequency gain is adjusted to that of $W_m$ by the control parameter $r_0$. From (7) the equations to calculate these parameters are

$$(s+q+r_1)(s+a) + r_2(b_0s+b_1) = C(s); \quad \frac{y(0)}{r(0)} = \frac{y_m(0)}{r(0)} = \frac{b_1'}{a'}. \tag{8}$$

Eqs. (8) can be written more compactly as

$$A[1\ r_0\ r_1\ r_2]^T = [1\ 0\ c_1 - q\ c_2]^T, \tag{9}$$

where $A$ is the non-singular $4 \times 4$ matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -qb_1'a & qa'b_1 & -b_1'a & 0 \\ a & 0 & 1 & b_0 \\ aq & 0 & a & b_1 \end{pmatrix}.$$

This equation is uniquely solvable if the determinant of $A$ is not zero, i.e.,

$$\big|\mathrm{Det}\,(A)\big| = \big|qa'b_1(b_1 - ab_0)\big| \geqslant \delta_0' > 0$$
$$\Rightarrow \big|b_1(b_1 - ab_0)\big| \geqslant \frac{\delta_0'}{qa'} = \delta_0 \quad \text{(Controllability condition)} \tag{10}$$

for some real small positive constant $\delta_0'$. Looking at Assumption 1, it can be seen that the nominal system is controllable and the control parameters $r_0$, $r_1$ and $r_2$ can be uniquely obtained from the algebraic system (9).

## 3. Adaptive Control

When the plant parameters are unknown an estimation algorithm calculates a controllable estimated model of the plant.

*Estimation Algorithm*

The algorithm has three steps. A least-square algorithm with normalization and a relative adaptation dead-zone is used to obtain an a-priori estimation of the plant parameters in *Step* 1. A suitable modification provides an a-posteriori estimation of the plant parameters in *Step* 2. This modification ensures the controllability of the estimated model of the plant obtained with the a-posteriori estimated parameters. It is based on the properties of the inverse of the covariance matrix of the least-square algorithm. Finally, the functions used in the modification of the parameters are described in *Step* 3. One of the functions is an hysteresis switching function which ensures the convergence of the a-posteriori estimates provided that the a-priori estimates converge.

*Step* 1 *(A-priori estimation)*

$e_n = \dfrac{e}{1+||\phi||}$ is used as adaptation error where $e = \dot{y}_f - \phi^T\widehat{\theta} = -\phi^T\widetilde{\theta} + \eta_f$ is the prediction error, $\widehat{\theta} = [\widehat{b}_0, \widehat{b}_1, \widehat{a}, \widehat{\xi}_0]^T$ the a-priori estimation vector of the plant parameters and $\widetilde{\theta} = \widehat{\theta} - \theta$ the parametrical error. The augmented error, $w = (e_n^2 + \phi_n^T P^2 \phi_n)^{1/2}$, is utilized in the design of a relative adaptation dead-zone, with $P(t) \geqslant 0$ being the inverse of the covariance matrix of the least-square algorithm and $\phi_n = \dfrac{\phi}{1+||\phi||}$ the regressor of normalized signals. The parameters a-priori estimation is obtained from

$$\dot{P} = -\frac{sP\phi_n\phi_n^T P}{1+\gamma\phi_n^T P\phi_n}; \quad \dot{\widehat{\theta}} = \frac{sP\phi_n e_n}{1+\gamma\phi_n^T P\phi_n}, \tag{11a}$$

where $P(0) = P^T(0) > 0$, $\gamma(t)$ is a real function chosen such that $0 < \gamma_1 \leqslant \gamma(t) \leqslant \gamma_2 < \infty$ with $\gamma_1$ and $\gamma_2$ being any predefined real constants, and $s(t)$ is the relative adaptation dead-zone defined as

$$s(t) = \begin{cases} 0, & \text{if} \quad w(t) \leqslant \mu\overline{\eta}_{fn}(t), \\ f(\mu\overline{\eta}_{fn}(t), w(t))/w(t), & \text{otherwise} \end{cases} \tag{11b}$$

for any constant $\mu > 1$, and

$$f(g, w) = \begin{cases} w - g, & \text{if} \quad w > g, \\ 0, & \text{if} \quad w \leqslant g. \end{cases} \tag{11c}$$

It will be proved in Lemma 3.1 of the current section that $\|P^{-1}(t)\widetilde{\theta}(t)\|$ is bounded. Then, a bounded function $\beta^*(t) \in \mathfrak{R}^{4\times 1}$ exists such that $\theta = \widehat{\theta}(t) + P(t)\beta^*(t)$. This fact provides the motivation for the following modification to obtain the a-posteriori estimates of the plant parameters.

*Step* 2 *(A-posteriori estimation)*

The modified estimates are calculated from

$$\overline{\theta}(t) = \widehat{\theta}(t) + \pi(t)P(t)\beta(t), \tag{12}$$

where $\overline{\theta}(t) = [\overline{b}_0, \overline{b}_1, \overline{a}, \overline{\xi}_0]^T$ is the a-posteriori estimation vector of the plant parameters, $\beta(t) = \mathfrak{R}^{4\times 1}$ and $\pi(t) \in \mathfrak{R}$. $\beta(t)$ is an hysteresis switching function which is equal to one of the following vectors

$$\begin{aligned} &\beta_1 = p_1; \quad \beta_2 = p_1 + p_3; \quad \beta_3 = p_1 + p_2 + p_3; \\ &\beta_4 = p_3; \quad \beta_5 = p_2 + p_3; \quad \beta_6 = p_2 \end{aligned} \tag{13}$$

at each time instant, with $p_i$ denoting the $i$-th column of the matrix $P$. In this way, $\beta(t)$ converges provided that $P(t)$ converges. This function tests the value of the functions $g(\beta_k) = |p_1^T\beta_k p_2^T\beta_k p_3^T\beta_k|$, for $k = 1,\ldots,6$, to choose the function $\beta_k$ among those in (13) as follows. Suppose that the current value for $\beta$ is $\beta_i$ and that $j$ is such that $g(\beta_j) \geqslant g(\beta_k)$ for $k = 1,\ldots,6$. Then, for some hysteresis width $\alpha^*,\ 1 \geqslant \alpha^* > 0,\ \beta$ will switch from $\beta_i$ to $\beta_j$ as

$$\beta = \begin{cases} \beta_j, & \text{if} \quad g(\beta_j) \geqslant (1+\alpha^*)g(\beta_i), \\ \beta_i, & \text{otherwise.} \end{cases} \tag{14}$$

$\pi(t)$ is chosen below in Step 3 such that the plant estimation model is controllable for all $t \geqslant 0$.

*Step* 3 *(Choice of $\pi$)*

As it will be shown below, in Remark 3, the controllability condition of the estimated model of the plant is

$$\psi(\overline{\theta}) = \left|\overline{b}_1(\overline{b}_1 - \overline{a}\overline{b}_0)\right| \geqslant \delta \tag{15}$$

for some real $0 < \delta \ll 1$. By introducing the Eq. (12) in Eq. (15), one obtains

$$\psi(\pi,\beta) = \left|f_3(\beta)\pi^3 + f_2(\beta)\pi^2 + f_1(\beta)\pi + f_0\right| \geqslant \delta, \tag{16}$$

where

$$\begin{aligned} &f_0 = \widehat{b}_1(\widehat{b}_1 - \widehat{b}_0\widehat{a});\ f_1(\beta) = \big((2\widehat{b}_1 - \widehat{b}_0\widehat{a})p_2 - \widehat{b}_1\widehat{a}p_1 - \widehat{b}_0\widehat{b}_1p_3\big)\beta \\ &f_2(\beta) = (p_2^T\beta)^2 - \widehat{a}p_1^T\beta p_2^T\beta - \widehat{b}_0p_2^T\beta p_3^T\beta - \widehat{b}_1p_1^T\beta p_3^T\beta; \\ &f_3(\beta) = p_1^T\beta p_2^T\beta p_3^T\beta. \end{aligned} \tag{17}$$

Then, $\pi(t)$ is defined as

$$\pi(t) = \big\{\min\{\pi_0 \geqslant 0,\ \pi_0 \in \mathfrak{R}\} \mid \psi(\pi_0,\beta) \geqslant \delta\big\} \tag{18}$$

REMARK 2. One way to compute the function $\pi(t)$ at each time instant is the following,

a) take $\pi_0 = 0$, if $\pi_0$ is such that $\psi(\pi_0,\beta) \geqslant \delta$, then the search is finished, otherwise,

b) $\pi_0 = 0$

while $\psi(\pi_0, \beta) < \delta$
do $\pi_0 = \pi_0 + \Delta(\pi_0)$
end

where $0 < \Delta\pi_0 \ll 1$ is a real constant. As it will be shown below, in Lemma 3.2, $f_3(\beta) \neq 0$ at all time instant. Then, $\psi(\pi, \beta)$ is the absolute value of a cubic function on $\pi$ which increases from a starting value $\overline{\pi}_0 \in \Re$. This procedure thus provides a finite value for the function $\pi(t)$ for all time.

REMARK 3. By introducing (12) in the expression $e = \dot{y}_f - \phi^T\widehat{\theta}$ the following plant estimated model

$$\dot{y}_f = -\overline{a}y_f + \overline{b}_0\dot{u}_f + \overline{b}_1 u_f + e + \overline{\xi}_0 e^{-qt} - \pi\beta^T P\phi = \phi^T\overline{\theta} + e_a \tag{19}$$

is obtained where $e_a = e - \pi\beta^T P\phi$ is referred to as *'a posteriori' identification error*, which has $u_f$ and $y_f$ as input and output, respectively, and $e + \overline{\xi}_0 e^{-qt} - \pi\beta^T P\phi$ as an external disturbance. The controllability condition of the plant estimated model is $\psi(\overline{\theta}) = |\overline{b}_1(\overline{b}_1 - \overline{a}\overline{b}_0 0| \geqslant \delta$. It is obtained from the determinant of the matrix $\overline{A}$ which is constructed from $A$, in Eq. (9) of Section 2, by replacing the parameters of the vector $\theta$ with those corresponding to the vector $\overline{\theta} = \overline{\theta}(\pi, \beta)$ through (12).

REMARK 4. If the modification of the parameters was not considered (i.e., $\beta' = \pi\beta = 0$ and $\overline{\theta} = \widehat{\theta}$) the controllability condition of the plant estimation model would be $\psi(\widehat{\theta}) = |f_0| \geqslant \delta > 0$. This condition cannot be ensured for all $t \geqslant 0$, thus the modification of the parameters, with a suitable choice of the functions $\pi(t)$ and $\beta(t)$, is crucial to guarantee the controllability of the plant estimated model.

REMARK 5. In the case of adaptive stabilization for a time-varying plant, the described algorithm would not ensure the exact identification of the plant parameters for all time. Then, an estimation algorithm with *forgetting factor* would be necessary to track time-varying parameters of the plant.

Let $L_2$ denote the space of square integrable functions. The convergence and stability scheme's properties are given in the following results. Step 1 posses the properties indicated in the following lemma:

**Lemma 3.1.**

1) $P(t)$ *is uniformly bounded and converges;*
2) $f(t) \in L_2 \cap L_\infty$ *and* $sw^2 \in L_1 \cap L_\infty$*;*
3) $\dot{\widehat{\theta}}(t) \in L_2 \cap L_\infty$ *and* $\widehat{\theta}(t)$ *is uniformly bounded and converges;*
4) $s(t) \in L_1 \cap L_\infty$ *and* $P(t) > 0 \quad \forall t \geqslant 0$*;*
5) $||P^{-1}(t)\widetilde{\theta}(t)||$ *is bounded.*

The proof is given in Appendix A. Steps 2 and 3 of the estimation algorithm have the following properties:

**Lemma 3.2.**

1) $g(\beta) = |p_1^T \beta p_2^T \beta p_3^T \beta| > 0$ *for all time;*
2) *There exists a finite number of switches in* $\beta(t)$*, and* $\beta(t)$ *and* $f_i(\beta)$*, for* $i = 0, \ldots, 3$*, are bounded and converge;*
3) $\pi(t)$ *and* $\overline{\theta}(t)$ *are piecewise continuous and bounded functions which converge;*
4) $\Pi(t) = \max\{1, z_1\chi(\delta, f_1, f_2, f_3)\}$ *is an upper-bound function for the absolute value of* $\pi(t)$*, where* $z_1 = \dfrac{-f_2 + \sqrt{f_2^2 + 4|f_3|(\delta - f_1 sgn(f_3) - f_0)}}{2|f_3|}$,
   $\chi = \begin{cases} 1, & \text{if } \delta \geqslant f_0 + f_1 sgn(f_3), \\ 0, & \text{otherwise,} \end{cases}$ *is a logical function which is zero when the value of* $z_1$ *is a non-positive real, and* $sgn(f_3) = \begin{cases} 1, & \text{if } f_3 > 0, \\ -1, & \text{if } f_3 < 0; \end{cases}$
5) *The controller parameters* $r_0(t),\ r_1(t)$ *and* $r_2(t)$ *are piecewise continuous and bounded functions with discontinuities at the discontinuity time instances of* $\beta'(t) = \pi(t)\beta(t)$ *which converge and besides they are time-differentiable functions except for the discontinuity time instances of* $\beta'(t)$*;*
6) $e_{an}(t) = \dfrac{e_a(t)}{1 + \|\phi(t)\|}$ *is such that* $\lim_{t\to\infty}\{e_{an}^2 - 2\beta_{\max}\mu^2\overline{\eta}_{fn}^2\} \leqslant 0$ *where* $\beta_{\max} = \max_{0 \leqslant t < \infty}\{1, \|\pi\beta\|^2\}$.

The proof is given in Appendix B.

REMARK 6. In view of Property 6 of the Lemma 3.2 the normalized 'a posteriori' identification error $e_{an}(t)$ belongs to the residual set $D$ given below

$$D = \Big\{ e_{an} \mid \lim_{t\to\infty}\{e_{an}(t)\} \leqslant \sqrt{2\beta_{\max}}\mu \lim_{t\to\infty}\{\overline{\eta}_{fn}(t)\} \Big\}.$$

If $\overline{\eta}_{fn}$ converges to zero, which can occur with $\alpha_0 = 0$ in (5) if $\rho(t)$ converges exponentially to zero, then the residual set $D$ converges to the zero equilibrium. Thus, the properties of the algorithm are the proper ones of the perfectly modelled case.

## 4. Convergence Analysis

The substitution of (2) and (6) into (19) leads to the following time-varying closed-loop system

$$\dot{x}(t) = \overline{A}_c(t)x(t) + \vartheta(t) \tag{20a}$$

with $x^T = [\varepsilon_f, u_f],\ \vartheta^T = [\vartheta_1, \vartheta_2]$ and

$$\overline{A}_c(t) = \begin{bmatrix} -(\overline{a} + \overline{b}_0 r_2) & \overline{b}_1 - \overline{b}_0(q + r_1) \\ -r_2 & -(q + r_1) \end{bmatrix}, \tag{20b}$$

$$\vartheta_1(t) = \overline{b}_0 r_0 r(t) + (q - \overline{a})y_{mf}(t) + \overline{\xi}_0 e^{-qt} - y_m(t) + e_a(t),$$

$$\vartheta_2(t) = r_0 r(t). \tag{20c}$$

The proposed modification procedure introduces commutations in the closed-loop system due to parametrical switchings in the adaptive controller. The issue of existence of solution of (20a) needs to be discussed before proceeding to the convergence analysis. Briefly, the solution obtained by linking the solutions between the time instants at which switchings occur. When there is a discontinuity in $\beta' = \pi\beta$ bounded jumps are produced in $\overline{\theta}, r_0, r_1$ and $r_2$. First, note that $\overline{A}_c(t)$ is uniformly bounded from Lemma 3.2 (Parts 3 and 5). If $u_f$ and $\varepsilon_f$ are bounded, then there is a bounded jump in $\dot{u}_f$ when $\beta'$ presents a discontinuity since $\vartheta_2(t)$ is bounded, from (20a) to (20c). The bounded jumps in $\dot{u}_f$ and $\beta'$ means that $e_a = -\widetilde{\theta}^T\phi + \eta_f - \pi\beta^T P\phi$ which also causes bounded jumps. This means that there is also a bounded jump in $\dot{\varepsilon}_f$, from (20a) to (20c). The signal $x(t)$ is therefore continuous if the signals $u_f$ and $\varepsilon_f$ are bounded, although $\dot{x}(t)$ is discontinuous at the discontinuity instants of $\beta'$.

The following theorem establishes the main result of the convergence analysis.

**Theorem 4.1** (main result). *The adaptive control law applied to the plant (1) is stable in the sense that $u(t)$ and $\varepsilon(t)$ are bounded for all finite initial states and any bounded reference signal $r(t)$, subject to Assumptions 1 and 2 and provided that $\alpha$ in (5) is sufficiently small so that $\frac{k\sqrt{2\beta_{\max}\mu\alpha\|v\|}}{\sigma} < 1$ for all $t \geqslant t_1$, where $t_1$ is a finite time instant, $k$ is an upper bound of the state transition matrix associated with $\overline{A}_c(t)$ and $\sigma$ is a lower bound of the absolute value of the real part of the closed-loop system poles.*

The proof is given in Appendix C.

## 5. Simulation Results

In this section, some simulation results are presented to illustrate the effectiveness of the robust adaptive control scheme described in Sections 2 and 3. The plant to be controlled is given by $W_0(s) = \frac{s-1}{s-2}$, $\Delta_1(s) = \frac{0.2}{s+3.5}$ and $\Delta_2(s) = \frac{0.2}{s+4.5}$. $W_m(s) = \frac{s-2}{s+3}$ and the external input signal is $r(t) = 0.5(\sin 3t + \sin 5t)$. The constants $q = 5$ in (2), $\alpha = 0.003$, $\alpha_0 = 0.25$, $\sigma_0 = 0.25$ and $v^T = [1, 0.6]$ in (5), the function $\gamma(t) = 0.05$ in (11a), $\mu = 1.1$ in (11b) and $\alpha^* = 0.05$ in (14), and $P(0) = 10000 \times \begin{bmatrix} 1 & 0.9 & 0.9 & 0.9 \\ 0.9 & 1 & 0.9 & 0.9 \\ 0.9 & 0.9 & 1 & 0.9 \\ 0.9 & 0.9 & 0.9 & 1 \end{bmatrix}$ and $\widehat{\theta}(0) = [-0.2,\ 0.2,\ 0.8\ 0.5]^T$ as initial conditions of the estimation algorithm are considered. Besides, $\delta = 0.01$ is chosen in (15) as the controllability lower bound. The procedure described above in Remark 2 of Section III is used to compute $\pi(t)$ with $\Delta\pi_0 = 10^{-6}$. The control objective is defined by

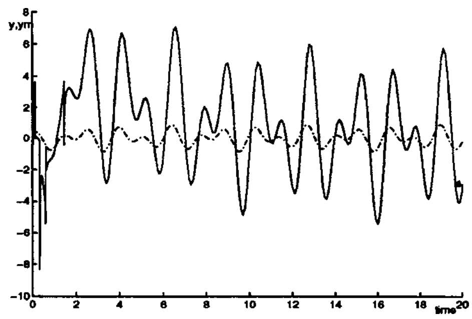


Fig. 1. Output signal in the interval $[0, 20]$.

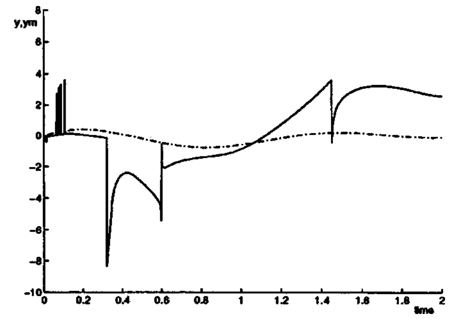


Fig. 2. Output signal in the interval $[0, 2]$.

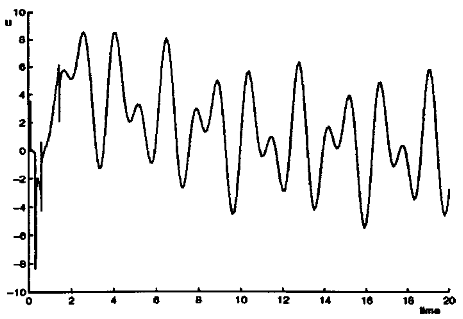


Fig. 3. Control signal in the interval $[0, 20]$.

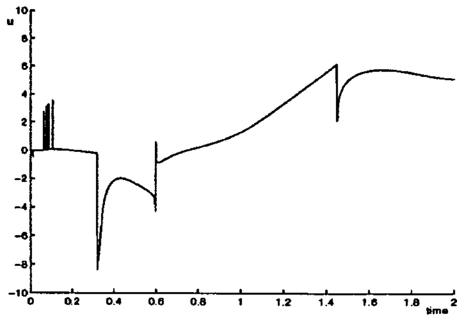


Fig. 4. Control signal in the interval $[0, 2]$.

the polynomial $C(s) = s^2 + 3s + 4$. The simulation results are shown in the Figures 1–8. Figures 1 and 2 show the output signals $y(t)$ (solid line) and $y_m(t)$ (dashdotted line). Figures 3 and 4 show the control signal $u(t)$. Figures 5 and 6 show the a-posteriori estimated parameters, with $\overline{b}_0$ given by the solid line, $\overline{b}_1$ given by the dashdotted line, $\overline{a}$ given by the dotted line and $\overline{\xi}_0$ given by the dashed line. Fig. 7 displays the index of the function $\beta$ chosen from those of (13) and finally, Fig. 8 shows the evolution of the function $\pi(t)$.

REMARK 7. Figures 2 and 4 present, respectively, bounded jumps in the output and control signals. These jumps occur at the time instants at which the function $\beta'(t) = \pi(t)\beta(t)$ presents a discontinuity. The a-posteriori estimated parameters also present discontinuities at those time instants. These discontinuities are crucial to ensure the controllability of the estimated model of the plant. The number of instants at which occur discontinuities in $\beta'(t)$ and the a-posteriori estimated parameters is finite as it is shown in the Figures 6, 7 and 8, respectively. Fig. 5 shows that the estimated parameters are very large when the estimation starts since the matrix $P(t)$ is very large at those time instants.

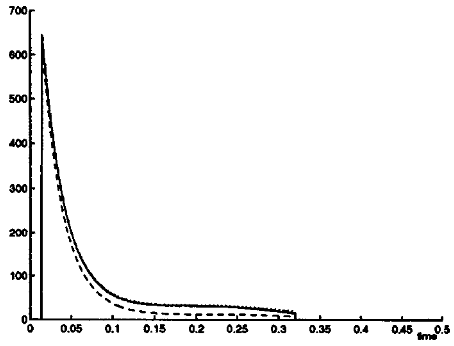


Fig. 5. 'A-posteriori' estimated parameters in the interval $[0, 0.5]$.

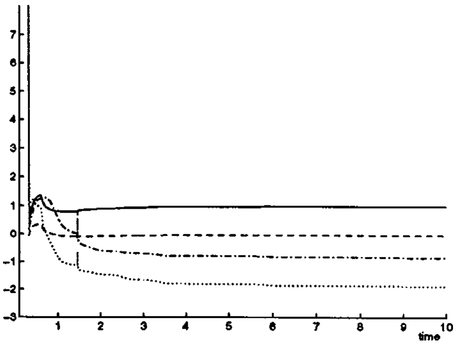


Fig. 6. 'A-posteriori' estimated parameters in the interval $[0.1, 10]$.

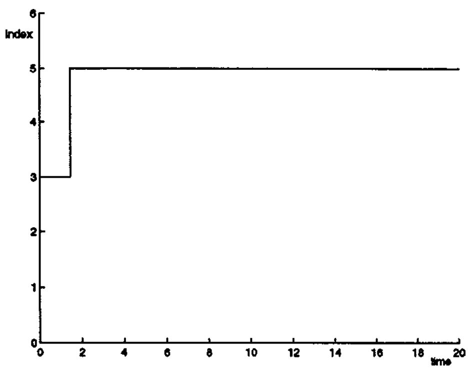


Fig. 7. Index of the chosen $\beta$-function in $[0, 20]$.

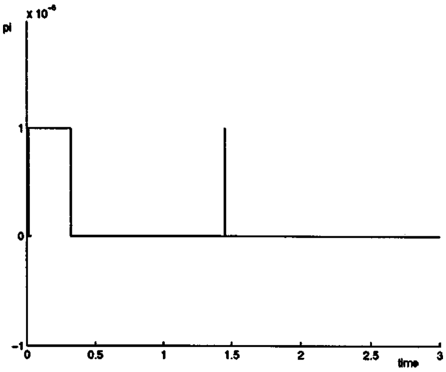


Fig. 8. Function $\pi(t)$ in the interval $[0, 3]$.

## 6. Conclusions

An adaptive control algorithm has been presented that stabilizes an, in general, inversely unstable first-order nominally controllable continuous-time system in the presence of unmodelled dynamics and, eventually, bounded noise. The algorithm includes the use of a relative adaptation dead-zone for the closed-loop robust adaptive stabilization. The controllability of the estimated model is ensured by incorporating an appropriate modification in the standard least-squares estimation algorithm. The boundedness of the tracking-error and all the remaining signals in the closed-loop control system is ensured provided that the nominal plant is controllable.

## Acknowledgements

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## A. Appendix

*Proof of Lemma* 3.1.

1) The matrix $P(t)$ is a continuous monotonic non-increasing function from Eq. (11a). Besides, $P(0)$ is positive and if $P(t_1) = 0$ at any $t_1 > 0$ then, $\dot{P}(t_1) = 0$ from (11a) and then, $P(t) = 0\ \forall t \geqslant t_1$. Thus, $P(t)$ is such that $0 \leqslant P(t) \leqslant P(0)$ and it also converges.

2) Consider the time function $V = \widetilde{\theta}^T P^{-1}\widetilde{\theta} + \operatorname{tr} P$ which is a non-negative function since $\operatorname{tr} P(t) \geqslant 0\ \forall t \geqslant 0$ because $P(t)$ is positive semidefinite. From Eqs. (11a) to (11c) and (5) it follows that

$$\begin{aligned}\dot{V} &= \frac{s(\eta_{fn}^2 - w^2)}{1 + \gamma\phi_n^T P\phi_n} \leqslant \frac{s(\overline{\eta}_{fn}^2 - w^2)}{1 + \gamma\phi_n^T P\phi_n} \\ &\leqslant -\Big(\frac{\mu^2 - 1}{\mu^2}\Big)\frac{sw^2}{1 + \gamma\phi_n^T P\phi_n} \leqslant -\Big(\frac{\mu^2 - 1}{\mu^2}\Big)\frac{f^2}{1 + \gamma\phi_n^T P\phi_n} \leqslant 0, \end{aligned} \tag{A.1}$$

where $sw^2 = fw \geqslant f^2$ has been used. From the integration of the inequality (A.1) it follows that

$$\begin{aligned}\int_0^\infty f^2\, dt &\leqslant -\Big(\frac{\mu^2}{\mu^2 - 1}\Big)\int_0^\infty (1 + \gamma\phi_n^T P\phi_n)\dot{V}\, dt \\ &\leqslant \Big(\frac{\mu^2}{\mu^2 - 1}\Big)\sup_{0\leqslant t<\infty}\{1 + \gamma\phi_n^T P\phi_n\}\big(V(0) - V(\infty)\big) \\ &\leqslant \Big(\frac{\mu^2}{\mu^2 - 1}\Big)MV(0) < \infty, \end{aligned} \tag{A.2}$$

where $M$ is a real constant since $P(t)$, $\phi_n(t)$ and $\gamma(t)$ are bounded. Thus, $f(t) \in L_2 \cap L_\infty$ from (A.2), (11c) and the fact that $w(t) = (e_n^2 + \phi_n^T P^2\phi_n)^{1/2}$ is bounded since $e_n(t)$ is also bounded. In the same way, $sw^2 \in L_1 \cap L_\infty$ can be also proved, since $s \in [0, 1]$ from (11b).

3) From Eqs. (11a) and (11b) one gets

$$\begin{aligned}\dot{\widehat{\theta}}^T\dot{\widehat{\theta}} &= \frac{s^2 e_n^2 \phi_n^T P^2\phi_n}{1 + \gamma\phi_n^T P\phi_n} \leqslant \frac{s^2 w^2 \phi_n^T\phi_n}{1 + \gamma\phi_n^T P\phi_n} \\ &\leqslant \lambda_{\max}^2\big(P(t)\big)\|\phi_n\|^2 sw^2 \leqslant k_0 sw^2 \end{aligned} \tag{A.3}$$

for some real constant $k_0$, where the facts that $s \in [0, 1]$and $\|\phi_n\|$ is bounded have been used and $\lambda_{\max}(P(t))$ denotes the maximum eigenvalue of matrix $P$ at time instant $t$. The integration of this inequality and $sw^2 \in L_1 \cap L_\infty$ leads to $\dot{\widehat{\theta}}(t) \in L_2 \cap L_\infty$. Thus, $\dot{\widehat{\theta}}(t)$ converges to zero as time tends to infinity.

The differentiation of the identity $PP^{-1} = I$ with respect to time and the Eq. (11a) leads to $\dfrac{d}{dt}(P^{-1}) = \dfrac{s\phi_n\phi_n^T}{1 + \gamma\phi_n^T P\phi_n} \geqslant 0$. Thus,

$$\lambda_{\min}\big(P^{-1}(t)\big) \geqslant \lambda_{\min}\big(P^{-1}(0)\big) \quad \forall t \geqslant 0. \tag{A.4}$$

From Eq. (A.1), it follows that

$$V(t) \leqslant V(0) \Leftrightarrow \widetilde{\theta}^T(t)P^{-1}(t)\widetilde{\theta}(t) + \operatorname{tr} P(t) \\ \leqslant \widetilde{\theta}^T(0)P^{-1}(0)\widetilde{\theta}(0) + \operatorname{tr} P(0) \quad \forall t \geqslant 0. \tag{A.5}$$

Eqs. (A.4) and (A.5) lead to

$$\lambda_{\max}\big(P^{-1}(0)\big)\big\|\widetilde{\theta}(0)\big\|^2 + \operatorname{tr} P(0) \geqslant \lambda_{\min}\big(P^{-1}(0)\big)\big\|\widetilde{\theta}(t)\big\|^2 + \operatorname{tr} P(t) \\ \Leftrightarrow \big\|\widetilde{\theta}(t)\big\|^2 \leqslant \frac{\lambda_{\max}\big(P(0)\big)}{\lambda_{\min}\big(P(0)\big)}\big\|\widetilde{\theta}(0)\big\|^2 + \frac{\operatorname{tr} P(0) - \operatorname{tr} P(t)}{\lambda_{\min}\big(P^{-1}(0)\big)} < \infty. \tag{A.6}$$

Hence $\widetilde{\theta}(t)$ and thus $\widehat{\theta}(t)$ are both bounded and they converge since $\dot{\widehat{\theta}}(t)$ converges to zero.

4) From Eq. (A.1), it follows that

$$\int_0^\infty \frac{s(w^2 - \eta_{fn}^2)}{1 + \gamma\phi_n^T P\phi_n}\, dt = V(0) - V(\infty) < \infty. \tag{A.7}$$

Let $I_1$ and $I_2$ be $I_1 = \{t \in [0,\infty) \mid s(t) = 0\}$ and $I_2 = \{t \in [0,\infty) \mid s(t) \neq 0\}$, respectively. At the time instants $t \in I_2,\ w > \mu\overline{\eta}_{fn} > |\eta_{fn}|$ and thus, $w^2 - \eta_{fn}^2 > 0$ are fulfilled. Then, (A.7) implies

$$\infty > \int_0^\infty \frac{s(w^2 - \eta_{fn}^2)}{1 + \gamma\phi_n^T P\phi_n}\, dt = \int_{I_2} \frac{s(w^2 - \eta_{fn}^2)}{1 + \gamma\phi_n^T P\phi_n}\, dt \\ \geqslant \inf_{t\in I_2}\left\{\frac{(w^2 - \eta_{fn}^2)}{1 + \gamma\phi_n^T P\phi_n}\right\}\int_{I_2} s\, dt = \inf_{t\in I_2}\left\{\frac{(w^2 - \eta_{fn}^2)}{1 + \gamma\phi_n^T P\phi_n}\right\}\int_0^\infty s\, dt. \tag{A.8}$$

Since $\inf_{t\in I_2}\left\{\frac{(w^2 - \eta_{fn}^2)}{1 + \gamma\phi_n^T P\phi_n}\right\} > 0$ so that $s(t) \in L_1 \cap L_\infty$. Then, from $\frac{d}{dt}(P^{-1}) = \frac{s\phi_n\phi_n^T}{1 + \gamma\phi_n^T P\phi_n}$, it follows that

$$P^{-1}(\infty) = P^{-1}(0) + \int_0^\infty \frac{s\phi_n\phi_n^T}{1 + \gamma\phi_n^T P\phi_n}\, dt \\ \leqslant P^{-1}(0) + \max_{0\leqslant t<\infty}\left\{\frac{\phi_n\phi_n^T}{1 + \gamma\phi_n^T P\phi_n}\right\}\int_0^\infty s\, dt < \infty, \tag{A.9}$$

since $\phi_n$ is bounded and $s(t) \in L_1$. Then, $P(t) > 0\ \forall t \in [0,\infty)$.

5) From Eqs. (11a) one can obtain,

$$\big\|P^{-1}(t)\widetilde{\theta}(t)\big\| - \big\|P^{-1}(0)\widetilde{\theta}(0)\big\| \leqslant \big\|P^{-1}(t)\widetilde{\theta}(t) - P^{-1}(0)\widetilde{\theta}(0)\big\| \\ = \left\|\int_0^t \frac{d}{dt}(P^{-1}\widetilde{\theta})\, d\tau\right\| = \int_0^t \left\|\Big(\frac{d}{dt}(P^{-1})\widetilde{\theta} + P^{-1}\dot{\widetilde{\theta}}\Big)\right\| d\tau$$

$$= \int_0^t \left\| \frac{s\phi_n \phi_n^T \widetilde{\theta} + s\phi_n e_n}{1+\gamma\phi_n^T P \phi_n} \right\| d\tau = \int_0^t \left\| \frac{s\phi_n \eta_{fn}}{1+\gamma\phi_n^T P \phi_n} \right\| d\tau$$
$$\leqslant \int_0^t \frac{s\|\phi_n\|\overline{\eta}_{fn}}{1+\gamma\phi_n^T P \phi_n}\, d\tau, \tag{A.10}$$

where $e_n = -\phi_n^T \widetilde{\theta} + \eta_{fn}$, and the fact that $|\eta_{fn}| \leqslant \overline{\eta}_{fn}$ have been used. Thus,

$$\left\|P^{-1}(t)\widetilde{\theta}(t)\right\| \leqslant \left\|P^{-1}(0)\widetilde{\theta}(0)\right\| + \int_0^t \frac{s\|\phi_n\|\overline{\eta}_{fn}}{1+\gamma\phi_n^T P \phi_n}\, dt$$
$$\leqslant \left\|P^{-1}(0)\widetilde{\theta}(0)\right\| + \max_{0\leqslant\tau\leqslant t}\left\{\frac{\|\phi_n\|\overline{\eta}_{fn}}{1+\gamma\phi_n^T P \phi_n}\right\} \int_0^t s\, dt < \infty, \tag{A.11}$$

since $s(t) \in L_1 \cap L_\infty$, from part 4 of this lemma, and $P$, $\phi_n$ and $\overline{\eta}_{fn}$ are bounded.

## B. Appendix

*Proof of Lemma* 3.2.

1) Since $P(t) >$ for all $t$, from Property 4 of the Lemma 3.1, $p_i(t) \neq [0\ 0\ 0\ 0]^T$ for all $t$ and $i = 1, \ldots, 4$. For $\beta = \beta_1 = p_1$, $g(\beta) = \|p_1\|^2 |p_1^T p_2||p_1^T p_3| > 0$ except at the time instants at which $p_1^T p_2 = 0$ and / or $p_1^T p_3 = 0$.

If $p_1^T p_2 = p_1^T p_3 = 0$ then, for $\beta = \beta_2 = p_1 + p_3$, $g(\beta) = \|p_1\|^2 |p_2^T p_3| \|p_3\|^2 > 0$ except at the time instants at which $p_2^T p_3 = 0$. If $p_1^T p_2 = p_1^T p_3 = p_2^T p_3 = 0$ then, for $\beta = \beta_3 = p_1 + p_2 + p_3$, $g(\beta) = \|p_1\|^2 \|p_2\|^2 \|p_3\|^2 > 0$ since $P(t) > 0$ for all $t$ implies that $\|p_i(t)\| > 0$ for all $t$ and $i = 1, \ldots, 4$.

If $p_1^T p_2 = 0$ but $p_1^T p_3 \neq 0$ then, for $\beta = \beta_4 = p_3$, $g(\beta) = |p_1^T p_3||p_2^T p_3| \|p_3\|^2 > 0$ except at the time instants at which $p_2^T p_3 = 0$. If $p_2^T p_3 = 0$ then, for $\beta = \beta_5 = p_2 + p_3$, $g(\beta) = |p_1^T p_3| \|p_2\|^2 \|p_3\|^2 > 0$.

If $p_1^T p_3 = 0$ but $p_1^T p_2 \neq 0$ then, for $\beta = \beta_6 = p_2$, $g(\beta) = |p_1^T p_2| \|p_2\|^2 |p_2^T p_3| > 0$ except at the time instants at which $p_2^T p_3 = 0$. If $p_2^T p_3 = 0$ then, for $\beta = \beta_5 = p_2 + p_3$, $g(\beta) = |p_1^T p_2| \|p_2\|^2 \|p_3\| > 0$. Thus, the Property 1 is proved since the modification algorithm takes the function $\beta(t)$ according to eqn. (14) and at every time instant at least one of the functions $g(\beta_i)$ for $i = 1, \ldots, 6$, is higher than zero.

2) Since $P(t)$ is bounded and converges from Part 1 of Lemma 3.1, all the functions $\beta_i(t)$ in (13) and thus $g(\beta_i)$, for $i = 1, \ldots, 6$, are bounded and converge. From the hysteresis rule (14), it can be seen that the number of switches in $\beta(t)$ is finite and $\beta(t)$ is bounded and converges. Besides, since $P$ and $\widehat{\theta}$ are bounded and converge from Parts 1 and 3 of Lemma 3.1, the functions $f_i(\beta)$, for $i = 0, \ldots, 3$, in (17) are also bounded and converge.

3) The function $\pi(t)$ takes values which depend on the functions $f_i(\beta)$, for $i = 0, \ldots, 3$, from (16) to (18). $\beta(t)$ has been constructed such that $\psi(\pi, \beta)$ in (16) is the absolute value of a third order function on $\pi$ since $g(\beta) = |f_3(\beta)| > 0\ \forall t \geqslant 0$. There exists a bounded function $\pi(t)$ such that $\psi(\pi, \beta) \geqslant \delta$. Besides, $\pi(t)$ is a piecewise continuous function and converges since the functions $\beta(t)$ and $f_i(\beta)$, for $i = 0, \ldots, 3$, are

also piecewise continuous functions and converge. Thus, $\overline{\theta}(t)$ is also bounded, piecewise continuous and converges from Eq. (12), since $P(t)$ and $\widehat{\theta}(t)$ are bounded and converge.

4) When $|f_0| \geqslant \delta$ the function $\pi(t) = 0$ fulfils the condition (16). Thus, in this case $|\pi(t)|$ is bounded. When $|f_0| < \delta$ then the function $\pi(t)$ that fulfils the condition

$$f_3\pi^3 + f_2\pi^2 + f_1\pi + f_0 \geqslant \delta \tag{B.1}$$

also verifies (16). By considering $\pi = |\pi|\text{sgn}\,(f_3)$ the function $\pi(t)$ such that

$$|f_3||\pi|^2 + f_2|\pi| + f_1\text{sgn}\,(f_3) \geqslant \frac{\delta - f_0}{|\pi|} \tag{B.2}$$

also fulfils the condition (B.1). If $|\pi(t)| \geqslant 1$ then the function $\pi(t)$ such that

$$|f_3||\pi|^2 + f_2|\pi| + f_1\text{sgn}\,(f_3) - \delta + f_0 \geqslant 0 \tag{B.3}$$

also verifies the condition (B.1). Any $\pi(t)$ such that $|\pi(t)| \geqslant z_1$, where $z_1 = \dfrac{-f_2 + \sqrt{f_2^2 + 4|f_3|(\delta - f_1\text{sgn}\,(f_3) - f_0)}}{2|f_3|}$ is the highest root of the function $G_1(|\pi|) = |f_3||\pi|^2 + f_2|\pi| + f_1\text{sgn}\,(f_3) - \delta + f_0$, fulfils (B.3). Then, the function $\pi(t)$ such that

$$\big|\pi(t)\big| \geqslant \max\big\{1, z_1\chi(\delta, f_1, f_2, f_3)\big\} \tag{B.4}$$

fulfils (B.3) and thus, (16), where $\chi = \begin{cases} 1, & \text{if } \delta \geqslant f_0 + f_1\text{sgn}\,(f_3) \\ 0, & \text{otherwise} \end{cases}$ is a logical function which is zero when the highest root of $G_1(|\pi|)$ is negative.

In any case, $\Pi(t) = \max\{1, z_1\chi\}$ is an upper-bound function for the absolute value of $\pi(t)$.

5) The estimated model of the plant is controllable since there exists a bounded function $\pi(t)$ which makes $\psi(\pi, \beta) \geqslant \delta$. Then, $r_0(t),\ r_1(t)$ and $r_2(t)$ are bounded while being piecewise continuous functions with a finite number of discontinuities from the resolution of the controller. Thus, they are time-differentiable functions except at the time instants of discontinuity of $\beta'(t) = \pi(t)\beta(t)$. Besides, $r_0(t),\ r_1(t)$ and $r_2(t)$ converge since $\overline{\theta}(t)$ converges.

6) From the definition of $e_a(t)$, $e_a = e - \pi\beta^T P\phi$, it follows that $e_{an} = e_n - \pi\beta^T P\phi_n$. From $(a + b)^2 \leqslant 2(a^2 + b^2)$ for any $a,\ b \in \mathfrak{R}$, one obtains that

$$\begin{aligned} e_{an}^2 &\leqslant 2\big(e_n^2 + \|\pi\beta\|^2\phi_n^T P^2\phi_n\big) \leqslant 2 \max_{0 \leqslant t < \infty} \big\{1, \|\pi\beta\|^2\big\}(e_n^2 + \phi_n^T P^2\phi_n) \\ &= 2\beta_{\max}w^2 \Rightarrow e_{an}^2 - 2\beta_{\max}\mu^2\overline{\eta}_{fn}^2 \leqslant 2\beta_{\max}(w^2 - \mu^2\overline{\eta}_{fn}^2) \\ &\Rightarrow \lim_{t\to\infty}\{e_{an}^2 - 2\beta_{\max}\mu^2\overline{\eta}_{fn}^2\} \leqslant \lim_{t\to\infty}\big\{2\beta_{\max}(w^2 - \mu^2\overline{\eta}_{fn}^2)\big\} \leqslant 0 \end{aligned} \tag{B.5}$$

since $s(t) \in L_1 \cap L_\infty$ implies that $w^2 \leqslant \mu^2\overline{\eta}_{fn}^2$ when $t$ tends to infinity.

## C. Appendix

*Proof of Theorem* 4.1.

$\overline{A}_c(t)$ is bounded since $\overline{\theta}(t)$ and the controller parameters $r_0,\ r_1(t)$ and $r_2(t)$ are bounded, from parts 3 and 5 of Lemma 3.2. The eigenvalues of $\overline{A}_c(t)$ are strictly inside the stability boundary for all $t$ because of the control objective. The elements of $\overline{A}_c(t)$ are bounded and time-differentiable functions in the open interval between any two consecutive discontinuity instant of $\beta'$, from Part 5 of Lemma 3.2. Then

$$\int_t^{t+T} \big\|\dot{\overline{A}}_c(\tau)\big\|\,d\tau = \int_t^{t_i^-} \big\|\dot{\overline{A}}_c(\tau)\big\|\,d\tau + \sum_{i=1}^{N}\int_{t_i^-}^{t_i^+} \big\|\Delta(\tau)\big\|\,d\tau + \sum_{i=1}^{N-1}\int_{t_i^+}^{t_{i+1}^-} \big\|\dot{\overline{A}}_c(\tau)\big\|\,d\tau + \int_{t_N^+}^{t+T} \big\|\dot{\overline{A}}_c(\tau)\big\|\,d\tau \leqslant k_0T(t)+k_1 \tag{C.1}$$

for all $t$ and $T$ and where $k_0$ is sufficiently small, with $t_i$, for $i \in \{1,\ldots,N\}$, being the switching time instants of the function $\beta'(t)$ in the open interval $(t, t+T)$ and $\Delta(\tau) = \begin{bmatrix}\delta(\tau - t_i) & (\tau - t_i)\\ \delta(\tau - t_i) & (\tau - t_i)\end{bmatrix}$ with $\delta(t)$ being the Dirac-delta function. $\int_{t_i^-}^{t_i^+} \|\Delta(t)\|\,d\tau$ is bounded since the interval of integration is of zero width. The condition (C.1) is verified since the number of switches of $\beta'(t)$ is finite and $\overline{A}_c(t)$ is time-differentiable in the open interval between any two consecutive discontinuity instants of $\beta'$. Then, the conditions of Lemma 3.1 of (Ioannou and Datta, 1991) are fulfilled and thus the homogeneous system $\dot{x}(t) = \overline{A}_c(t)x(t)$ is exponentially stable.

The algorithm ensures there are no finite escapes and the upper bound function for the absolute value $\pi(t),\ \Pi(t)$, converges asymptotically to a value such that $\frac{k\sqrt{2\beta_{\max}\mu\alpha\|v\|}}{\sigma} < 1$ is verified, provided that $\alpha$ in Assumption 2 is sufficiently small. We then redefine the time origin $t_1$ as the time instant such that for $t > t_1$ there are not switches in $\beta'(t)$ and the condition $\frac{k\sqrt{2\beta_{\max}\mu\alpha\|v\|}}{\sigma} < 1$ is verified. From (20a), it follows that

$$x(t) = \phi(t,t_1)x(t_1) + \int_{t_1}^{t} \phi(t,\tau)\vartheta(\tau)\,d\tau. \tag{C.2}$$

where $\phi(t,\tau)$ is the state transition matrix associated with $\overline{A}_c(t)$. Since the homogeneous part of (20a) is exponentially stable and $x(t_1)$ is bounded, (C.2) leads to

$$\big\|x(t)\big\| \leqslant k_1 + \int_{t_1}^{t} k_2 e^{-\sigma(t-\tau)}\big\|\vartheta(\tau)\big\|\,d\tau \tag{C.3}$$

for some $k_1,\ k_2,\ \sigma > 0$. The term $\|\vartheta(t)\|$ is upperly bounded by

$$\big\|\vartheta(t)\big\| \leqslant \big|\vartheta_1(t)\big| + \big|\vartheta_2(t)\big| \leqslant \big(1+|\overline{b}_0|\big)\big|r_0\big|\big|r(t)\big| + \big|q-\overline{a}\big|\big|y_{mf}(t)\big|\big|\overline{\xi}_0\big|\big|e^{-qt}\big| + \big|y_m(t)\big| + \big|e_a(t)\big| \leqslant k_3 + \big|e_a(t)\big|, \tag{C.4}$$

where the Eqs. (20c), the Schwarz's inequality and the fact that $\overline{b}_0(t),\ \overline{a}(t),\ \overline{\xi}_0(t)$ and $r_0(t)$ and the signals $y_{mf}(t),\ y_m(t),\ e^{-qt},\ r(t)$ are bounded have been used. The substitution of (C.3) yields

$$\|x(t)\| \leqslant k_1 + \int_{t_1}^{t} k_2 e^{-\sigma(t-\tau)}\big(k_3 + |e_a(\tau)|\big)\, d\tau. \tag{C.5}$$

From Eq. (B.5), $e_{an}^2 \leqslant 2\beta_{\max} w^2$. Thus, from (C.5), it follows that

$$\begin{aligned}
\|x(t)\| &\leqslant k_4 + \int_{t_1}^{t} k_2\sqrt{2\beta_{\max}}e^{-\sigma(t-\tau)}\big(1+\|\phi(\tau)\|\big)\big(w(\tau)-f(\tau)\big)\, d\tau \\
&\quad + \int_{t_1}^{t} k_2\sqrt{2\beta_{\max}}e^{-\sigma(t-\tau)} f(\tau)\big(1+\|\phi(\tau)\|\big)\, d\tau \\
&\leqslant k_5 + \int_{t_1}^{t} k_2\sqrt{2\beta_{\max}}e^{-\sigma(t-\tau)}\mu\alpha \sup_{t_1\leqslant\tau'\leqslant\tau}\big\{|v_1u_f(\tau')+v_2\varepsilon_f(\tau')|e^{-\sigma_0(\tau-\tau')}\big\}\, d\tau \\
&\quad + \int_{t_1}^{t} k_6 e^{-\sigma(t-\tau)} f(\tau)\big(1+\|\phi(\tau)\|\big)\, d\tau \\
&\leqslant k_5 + \frac{k_2\sqrt{2\beta_{\max}}\mu\alpha\|v\|}{\sigma}\sup_{t_1\leqslant\tau\leqslant t}\big\{\|x(\tau)\|\big\} \\
&\quad + \int_{t_1}^{t} k_6 e^{-\sigma(t-\tau)} f(\tau)\big(1+\|\phi(\tau)\|\big)\, d\tau,
\end{aligned} \tag{C.6}$$

where the term $w$ has been split into the terms $f$ and $w - f \leqslant \mu\overline{\eta}_{fn} = \mu(1+\|\phi\|)^{-1}\overline{\eta}_f$ from (11c) and (5) and the fact that $\|v\| \sup_{t_1\leqslant\tau\leqslant t}\{\|x(\tau)\|\} \geqslant \sup_{t_1\leqslant\tau\leqslant t}\big\{|v_1u_f(\tau) + v_2\varepsilon_f(\tau)|\big\}$ for a vector constant $v^T = [v_1,\ v_2]$ have been used. Since the right-hand side of (C.6) is monotonic non-decreasing in $t$, it follows that

$$\begin{aligned}
\sup_{t_1\leqslant\tau\leqslant t}\big\{\|x(\tau)\|\big\} &\leqslant k_5 + \frac{k_2\sqrt{2\beta_{\max}}\mu\alpha\|v\|}{\sigma}\sup_{t_1\leqslant\tau\leqslant t}\{\|x(\tau)\|\} \\
&\quad + \int_{t_1}^{t} k_6 e^{-\sigma(t-\tau)} f(\tau)\big(1+\|\phi(\tau)\|\big)\, d\tau.
\end{aligned} \tag{C.7}$$

Then, provided that $\alpha < \sigma/k_2\sqrt{2\beta_{\max}}\mu\|v\|$, one obtains

$$\|x(t)\| \leqslant k_7 + k_8\int_{t_1}^{t} f(\tau)\big(1+\|\phi(\tau)\|\big)\, d\tau. \tag{C.8}$$

From Eq. (4), one gets

$$\begin{aligned}
\|\phi\| &= \big(\dot u_f^2 + u_f^2 + y_f^2 + e^{-2qt}\big)^{1/2} \leqslant |\dot u_f| + |u_f| + |y_f| + |e^{-qt}| \\
&\leqslant |\dot u_f| + |u_f| + |\varepsilon_f| + |y_{mf}| + |e^{-qt}| \leqslant |\dot u_f| + |u_f| + |\varepsilon_f| + k_9.
\end{aligned} \tag{C.9}$$

Also, from Eqs. (2) and (6), if follows that

$$|\dot{u}_f| \leqslant |q + r_1||u_f| + |r_2||\varepsilon_f| + |r_0||r(t)| \leqslant k_{10} + k_{11}|u_f| + k_{12}|\varepsilon_f| \tag{C.10}$$

for some constants $k_{10},\ k_{11}$ and $k_{12}$ since $r_0(t),\ r_1(t)$ and $r_2(t)$ are bounded. The use of (C.10) in (C.9) leads to

$$\|\phi\| \leqslant k_{13} + k_{14}|u_f| + k_{15}|\varepsilon_f| \leqslant k_{13} + k_{16}\|x\|. \tag{C.11}$$

From (C.8) and (C.11), it follows that

$$\begin{aligned} \|x(t)\|^2 &\leqslant k_{17} + k_{18}\int_{t_1}^{t} f(\tau)^2\big(k_{19} + k_{20}\|x(\tau)\|^2\big)\,d\tau \\ &\leqslant k_{21} + k_{22}\int_{t_1}^{t} f(\tau)^2\|x(\tau)\|^2\,d\tau, \end{aligned} \tag{C.12}$$

where the fact that $f(t) \in L_2$ has been used. Gronwall's lemma (De Guzmán, 1982) in (C.12) leads to

$$\|x(t)\|^2 \leqslant k_{21}\exp\left(\int_{t_1}^{t} k_{22} f^2(\tau)\,d\tau\right) < \infty. \tag{C.13}$$

Thus, $\|x(t)\|$ is bounded and then, $u_f(t)$ and $\varepsilon_f(t)$ are bounded. From (C.11) and (5), $\phi(t),\ \rho(t)$ and $\overline{\eta}_f(t)$ are also bounded.

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## Robastinis adaptyvus pirmos eilės tolydinio laiko sistemos, galimai nestabilios, ją apvertus, sekimas su polių keitimu

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Darbe pasiūlyta ir tiriama nominaliai valdomų, nebūtinai jas apvertus stabilių pirmos eilės tolydinių tiesinių, nesikeičiančių laike sistemų su nemodeliujama dinamika netiesioginio adaptyvaus valdymo schema. Pateikti modeliavimo rezultatai.