INFORMATICA, 1997, Vol. 8, No. 3, 527–558

ON THE DESIGN OF UNIVERSAL STABILIZING CONTINUOUS LINEAR CONTROLLERS FOR TIME-DELAY SYSTEMS
Part II. Universal controller and main stability results

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Abstract. Part II deals with the design problem of generalized linear controllers for linear systems with after-effect so that the resulting closed-loop system is globally uniformly asymptotically stable in the Lyapunov's sense. The controllers are universal in the sense that they include the usual delays (namely, point, distributed and mixed point-distributed delays) which can be finite, infinite or even time-varying. The stability is formulated in terms of sufficient conditions depending, in general, on the system parameters and delays. It is shown that a stabilizing controller can be designed by using the well-known Kronecker product of matrices provided that a stabilizing controller exists in the absence of external (or, input) delay.

Key words: delay systems, point delays, distributed delays, Lyapunov's stability, stabilizing controllers.

1. Introduction. In Part II, several classes of systems involving combined point and distributed internal and external delays including the general $SPD's$ and $SPVD's$ (whose definitions are given in Sen and Luo (1997) are analyzed from a stability point of view by using Lyapunov's functions. The main stabilizability tool consists of the design of controllers containing the same types of delays as those of the controlled plant. Such a strategy is based on the well-known principle that the overall delay in the open-loop system is accumulative in the sense that each delay appearing in both plant and controller is a delay source for the closed-loop system. For generality purposes, the time-varying case for parametrization and delay is considered.

2. Delay-varying *SPVD* (Time-varying system with time-varying point and Volterra convolution type delays). The next developments can be particularized to SPD and SD (whose definitions are given in Sen and Luo, 1997). Consider the following controlled plant:

$$(SPVD): (\ \dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)x(t) + A_0(t)\boldsymbol{x}(t-h(t)) + \int_0^t \boldsymbol{B}(t-\tau)\boldsymbol{x}(\tau)\,d\tau$$
$$+ \boldsymbol{M}(t)\boldsymbol{u}(t) + \boldsymbol{E}(t)\boldsymbol{u}(t-h'(t)) + \int_0^t \boldsymbol{E}'(t-\tau)\boldsymbol{u}(\tau)\,d\tau;$$
$$\boldsymbol{y}(t) = \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)\boldsymbol{u}(t), \tag{1}$$

where the initial conditions for systems SP, SD, and SPD (whose definitions are given in Sen and Luo, 1997) and $\boldsymbol{u}(t) = \boldsymbol{0}$ for $t < 0$. $\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot)$ and $\boldsymbol{y}(\cdot)$ are n, m and p-vectors, respectively, and all the matrix functions in (1) are of appropriate orders. The controller is of a similar structure to (1) and of dynamic type as follows:

$$\dot{z}_f(t) = \boldsymbol{A}_f z_f(t) + \boldsymbol{A}_{0f}(t)z_f(t-h_f(t)) + \int_0^t \boldsymbol{B}_f(t-\tau)z_f(\tau)\,d\tau$$
$$+ \boldsymbol{M}_f(t)\boldsymbol{y}(t) + \boldsymbol{E}_f(t)\boldsymbol{y}(t-h'_f(t)) + \int_0^t \boldsymbol{E}'_f(t-\tau)\boldsymbol{y}(\tau)\,d\tau \tag{2a}$$
$$\dot{z}_p(t) = \boldsymbol{A}_p z_p(t) + \boldsymbol{A}_{0p}(t)z_p(t-h_p(t)) + \int_0^t \boldsymbol{B}_p(t-\tau)z_p(\tau)\,d\tau$$
$$+ \boldsymbol{M}_p(t)\boldsymbol{u}_r(t) + E_p(t)\boldsymbol{u}_r(t-h'_p(t)) + \int_0^t \boldsymbol{E}'_p(t-\tau)\boldsymbol{u}_r(\tau)\,d\tau, \tag{2b}$$
$$\dot{z}_c(t) = \boldsymbol{A}_c z_c(t) + A_{0c}(t)z_c(t-h_c(t)) + \int_0^t \boldsymbol{B}_c(t-\tau)z_c(\tau)\,d\tau$$
$$+ \boldsymbol{M}_c(t)[\boldsymbol{u}_p(t) - \boldsymbol{u}_f(t)] + \boldsymbol{E}_c(t)[\boldsymbol{u}_p(t-h'_c(t)) - \boldsymbol{u}_f(t-h'_c(t))]$$
$$+ \int_0^t \boldsymbol{E}'_c(t-\tau)[\boldsymbol{u}_p(\tau) - \boldsymbol{u}_f(\tau)]\,d\tau, \tag{2c}$$
$$\boldsymbol{u}_p(t) = \boldsymbol{C}_p(t)z_p(t) + \boldsymbol{D}_p(t)\boldsymbol{u}_r(t);\ \boldsymbol{u}_f(t) = \boldsymbol{C}_f(t)z_f(t) + \boldsymbol{D}_f(t)\boldsymbol{y}(t), \tag{2d}$$
$$\boldsymbol{u}(t) = \boldsymbol{u}_c(t) = \boldsymbol{C}_c(t)z_c(t) + \boldsymbol{D}_c(t)[\boldsymbol{u}_p(t) - \boldsymbol{u}_f(t)]. \tag{2e}$$

The nonnegative scalar functions $h_f(\cdot),\ h'_f(\cdot),\ h_p(\cdot)\ h'_p(\cdot),\ h_c(\cdot)$ and $h'_c(\cdot)$ represent delays and $z_f(\cdot),\ z_p(\cdot)$ and $z_c(\cdot)$ (namely, the state vectors of the feedback, precompensator and feedforward controller) are initialized on $[-h_f(0), 0]$,

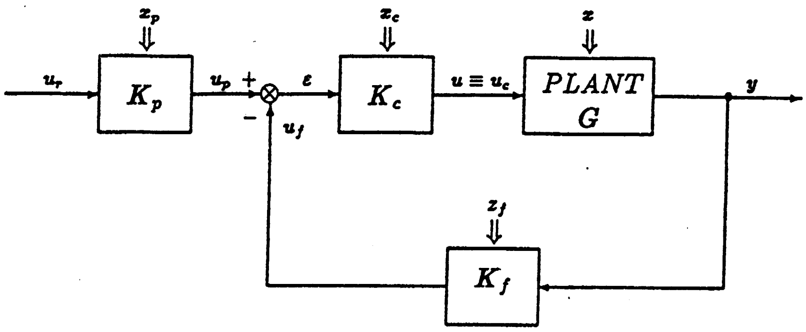

Fig. 1. Closedloop control system.

$[-h_p(0), 0]$ and $[-h_c(0), 0]$ with dimensions l_f, l_p, l_c, respectively. $\boldsymbol{u}_f(\cdot)$, $\boldsymbol{u}_p(\cdot)$ and $\boldsymbol{u}_c(\cdot)$ are outputs of the above compensators of dimensions $m_f = m_p$ and $m_c = m$. $\boldsymbol{u}_r(\cdot)$ is the reference signal of dimension m_q. All the time-varying matrices in (2) are of appropriate orders. The overall closed-loop system is shown in Fig. 1.

Substituting (2d) into (2e), one gets

$$\boldsymbol{u}(t) = \boldsymbol{u}_c(t) = \boldsymbol{H}(t)z(t) + \boldsymbol{K}(t)\boldsymbol{y}(t) + \boldsymbol{K}'(t)\boldsymbol{u}_r(t), \tag{3}$$

where $z(t) = [z_p^T \vdots z_c^T \vdots z_f^T]^T$ and

$$\begin{aligned}\boldsymbol{H}(t) =& [\boldsymbol{D}_c(t)\boldsymbol{C}_p(t) \vdots \boldsymbol{C}_c(t) \vdots -\boldsymbol{D}_c(t)\boldsymbol{C}_f(t)]; \; \boldsymbol{K}(t) = -\boldsymbol{D}_c(t)\boldsymbol{D}_f(t);\\ \boldsymbol{K}'(t) =& \boldsymbol{D}_c(t)\boldsymbol{D}_p(t).\end{aligned} \tag{4}$$

Substitution of (1) and (4) into (3) implies that $\boldsymbol{u}(t)$ can be, equivalently, calculated as

$$\begin{aligned}\boldsymbol{u}(t) =& [\boldsymbol{I} + \boldsymbol{D}_c(t)\boldsymbol{D}_f(t)\boldsymbol{D}(t)]^{-1}\{\boldsymbol{C}_c(t) + \boldsymbol{D}_c(t)[\boldsymbol{C}_p(t)z_p(t)\\ &- \boldsymbol{C}_f(t)z_f(t) - \boldsymbol{D}_f(t)\boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}_p(t)\boldsymbol{u}_r(t)]\},\end{aligned} \tag{5}$$

for all $t \geqslant 0$ *provided that the right-hand-side inverse matrix exists.* On the other hand, the substitution of (2d) – (2e) into (2a) – (2c) leads to a controller (3) – (4) (or (5)) together with the following dynamics:

$$\begin{aligned}\dot{z}(t) =& \boldsymbol{F}(t)z(t) + \boldsymbol{F}_1(t)z(t-h_p(t)) + \boldsymbol{F}_2 z(t-h_c(t)) \\ &+ \boldsymbol{F}_3(t)z(t-h_f(t)) + \boldsymbol{F}_4(t)z(t-h'_c(t)) + \boldsymbol{F}_5(t)\boldsymbol{y}(t-h'_f(t)) \\ &+ \int_0^t \boldsymbol{F}_6(t-\tau)z(\tau)d\tau + \boldsymbol{F}_7(t)\boldsymbol{u}_r(t-h_p(t)) \\ &+ \boldsymbol{F}_8(t)\boldsymbol{u}_r(t-h'_c(t)) + \boldsymbol{F}_9(t)\boldsymbol{y}(t-h'_c(t)) \\ &+ \int_0^t \boldsymbol{F}_{10}(t-\tau)\boldsymbol{u}_r(\tau)d\tau + \boldsymbol{F}_{11}(t)\boldsymbol{y}(t) \\ &+ \boldsymbol{F}_{12}(t)\boldsymbol{u}_r(t) + \int_0^t \boldsymbol{F}_{13}(t-\tau)\boldsymbol{y}(\tau)d\tau, \end{aligned} \tag{6}$$

where

$$\boldsymbol{F}(t) = \begin{bmatrix} A_p(t) & \mathbf{0} & \mathbf{0} \\ \boldsymbol{M}_c(t)\boldsymbol{C}_p(t) & A_c(t) & -\boldsymbol{M}_c(t)\boldsymbol{C}_f(t) \\ \mathbf{0} & \mathbf{0} & A_f(t) \end{bmatrix};$$

$$\boldsymbol{F}_4(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{E}_c(t)\boldsymbol{C}_p(t) & \mathbf{0} & -\boldsymbol{E}_c(t)\boldsymbol{C}_f(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}; \tag{7a}$$

$$\boldsymbol{F}_6(t-\tau) = \begin{bmatrix} \boldsymbol{B}_p(t-\tau) & \mathbf{0} & \mathbf{0} \\ \boldsymbol{E}'_c(t-\tau)\boldsymbol{C}_p(\tau) & \boldsymbol{B}_c(t-\tau) & -\boldsymbol{E}'_c(t-\tau)\boldsymbol{C}_f(t) \\ \mathbf{0} & \mathbf{0} & \boldsymbol{B}_f(t-\tau) \end{bmatrix}; \tag{7b}$$

$$\boldsymbol{F}_1(t) = Diag[\boldsymbol{A}_{0p}(t) \vdots \mathbf{0} \vdots \mathbf{0}];\ \boldsymbol{F}_2(t) = Diag[\mathbf{0} \vdots \boldsymbol{A}_{0c}(t) \vdots \mathbf{0}]; \tag{7c}$$

$$\boldsymbol{F}_3(t) = Diag[\mathbf{0} \vdots \mathbf{0} \vdots \boldsymbol{A}_{0f}(t)];\ \boldsymbol{F}_5(t) = Diag[\mathbf{0}^T \vdots \mathbf{0}^T \vdots \boldsymbol{E}_f^T(t)]; \tag{7d}$$

$$\boldsymbol{F}_7(t) = [\boldsymbol{E}_p^T(t) \vdots \mathbf{0}^T \vdots \mathbf{0}^T]^T;\ \boldsymbol{F}_8(t) = [\mathbf{0}^T \vdots \boldsymbol{D}_p^T(t)\boldsymbol{E}_c^T(t) \vdots \mathbf{0}^T]^T; \tag{7e}$$

$$\boldsymbol{F}_9(t) = [\mathbf{0}^T \vdots -\boldsymbol{D}_f^T(t)\boldsymbol{E}_c^T(t) \vdots \mathbf{0}^T]^T;$$

$$\boldsymbol{F}_{10}(t) = [\boldsymbol{E}'_p(t-\tau) \vdots \boldsymbol{D}_p^T(\tau)\boldsymbol{E}_c^T(t-\tau) \vdots \mathbf{0}^T]^T; \tag{7f}$$

$$\boldsymbol{F}_{11}(t) = [\mathbf{0}^T \vdots -\boldsymbol{D}_f^T(t)\boldsymbol{M}_c^T(t); \vdots \boldsymbol{M}_f^T(t)]^T;$$

$$\boldsymbol{F}_{12}(t) = [\boldsymbol{M}_p^T(t) \vdots \boldsymbol{D}_p^T(t)\boldsymbol{M}_p^T(t) \vdots \mathbf{0}^T]^T; \tag{7g}$$

$$\boldsymbol{F}_{13}(t-\tau) = [\boldsymbol{0}^T \vdots \boldsymbol{D}_f^T(\tau)\boldsymbol{E}_c^T(t-\tau) \vdots \boldsymbol{E}_f'(t-\tau)]^T; \tag{7h}$$

for all $t \geqslant 0$ and $\tau \leqslant t$.

Note that the finite distributed delays can be treated similarly to Volterra-type right-hand-side integrals by considering finite interval integrals.

3. Extended *SPVD* study of the regulator stability. Assume that the function matrices $\boldsymbol{H}(t)$ and $K(t)$ are continuously differentiable on $[0,\infty)$. The $SPVD$ and its controller (1) – (7) are equivalently described by the following extended system of state $\tilde{\boldsymbol{x}} =: [\boldsymbol{x}^T \vdots \boldsymbol{z}^T]^T$, control $\tilde{\boldsymbol{u}} =: [\boldsymbol{u}^T \vdots \dot{\boldsymbol{z}}^T]^T$ and output $\tilde{\boldsymbol{y}} =: [\boldsymbol{y}^T \vdots \boldsymbol{z}^T]^T$.

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}}(t) =& \tilde{\boldsymbol{A}}(t)\tilde{\boldsymbol{x}}(t) + \tilde{\boldsymbol{A}}_0(t)\tilde{\boldsymbol{x}}(t-h(t)) + \int_0^t \tilde{\boldsymbol{B}}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau + \tilde{\boldsymbol{M}}(t)\tilde{\boldsymbol{u}}(t)\\ &+ \tilde{\boldsymbol{E}}(t)\tilde{\boldsymbol{u}}(t-h'(t)) + \int_0^t \tilde{\boldsymbol{E}}'(t-\tau)\tilde{\boldsymbol{u}}(\tau)d\tau; \; \tilde{\boldsymbol{y}}(t) = \tilde{\boldsymbol{C}}(t)\tilde{\boldsymbol{x}}(t); \end{aligned}\tag{8a}$$

$$\begin{aligned}\tilde{\boldsymbol{u}}(t) =& \tilde{\boldsymbol{K}}(t)\tilde{\boldsymbol{x}}(t) + \tilde{\boldsymbol{F}}_1(t)\tilde{\boldsymbol{x}}(t-h_p(t)) + \tilde{\boldsymbol{F}}_2(t)\tilde{\boldsymbol{x}}(t-h_c(t))\\ &+ \tilde{\boldsymbol{F}}_3(t)\tilde{\boldsymbol{x}}(t-h_f(t)) + \tilde{\boldsymbol{F}}_4(t)\tilde{\boldsymbol{x}}(t-h_c'(t)) + \tilde{\boldsymbol{F}}_5(t)\tilde{\boldsymbol{x}}(t-h_f'(t))\\ &+ \int_0^t \tilde{\boldsymbol{F}}_6(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau + \tilde{\boldsymbol{K}}'(t)\boldsymbol{u}_r(t) + \tilde{\boldsymbol{K}}_c(t)\boldsymbol{u}_r(t-h_c'(t))\\ &+ \tilde{\boldsymbol{K}}_p(t)\boldsymbol{u}_r(t-h_p'(t)) + \int_0^t \tilde{\boldsymbol{K}}_d(t-\tau)\boldsymbol{u}_r(\tau)d\tau; \end{aligned}\tag{8b}$$

where

$$\tilde{K}(t) = \tilde{K}_0(t)\tilde{C} = \begin{bmatrix} K(t)\boldsymbol{C}(t) & \boldsymbol{H}(t) \\ \boldsymbol{F}_{11}(t)\boldsymbol{C}(t) & \boldsymbol{F}(t) \end{bmatrix}; \tag{9a}$$

$$\tilde{X}(t) = Diag[\boldsymbol{X}(t) \vdots \boldsymbol{0}]; \; \boldsymbol{X}(t) = A(t), A_0(t), \boldsymbol{B}(t), E(t) \text{ or } \boldsymbol{E}'(t); \tag{9b}$$

$$\tilde{\boldsymbol{M}}(t) = Diag[\boldsymbol{M}(t) \vdots I]; \; \tilde{\boldsymbol{F}}_i(t) = Diag[\boldsymbol{0} \vdots \boldsymbol{F}_i(t)]; \; 1 \leqslant i \leqslant 3; \tag{9c}$$

$$\tilde{\boldsymbol{F}}_4(t) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{F}_9(t)\boldsymbol{C}(t) & \boldsymbol{F}_4(t) \end{bmatrix}; \; \tilde{\boldsymbol{F}}_5(t) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{F}_5(t)\boldsymbol{C}(t) & \boldsymbol{0} \end{bmatrix}; \tag{9d}$$

$$\tilde{\boldsymbol{F}}_6(t-\tau) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{F}_{13}(t-\tau)\boldsymbol{C}(\tau) & \boldsymbol{F}_6(t-\tau) \end{bmatrix}; \tag{9e}$$

$$\tilde{K}'(t) = [K'^T(t) \vdots F_{12}^T(t)]^T;\ \tilde{K}_c(t) = [0^T \vdots F_8^T(t)]^T; \tag{9f}$$

$$\tilde{K}_p(t) = [0^T \vdots F_7^T(t)]^T;\ \tilde{K}_d(t) = [0^T \vdots F_{10}^T(t)]^T; \tag{9g}$$

for all $t \geqslant 0$ and $\tau \leqslant t$. Note that the extended system (8) – (9) is subject to an extended static control $\tilde{u}$. In the absence of an external input; i.e., $u_r \equiv 0$ on $[0, \infty)$, the closed-loop dynamics becomes from (8) – (9):

$$\begin{aligned}\dot{\tilde{x}}(t) =& [\tilde{A}(t) + \tilde{M}(t)\tilde{K}(t)]\tilde{x}(t) + \tilde{A}_0(t)\tilde{x}(t-h(t)) + \tilde{F}_1(t)\tilde{x}(t-h_p(t))\\ &+ \tilde{F}_2(t)\tilde{x}(t-h_c(t)) + \tilde{F}_3(t)\tilde{x}(t-h_f(t)) + \tilde{F}_4(t)\tilde{x}(t-h_c'(t))\\ &+ \tilde{F}_5(t)\tilde{x}(t-h_f'(t)) + \tilde{E}(t)\tilde{K}(t-h'(t))\tilde{x}(t-h'(t))\\ &+ \int_0^t [\tilde{B}(t-\tau) + \tilde{E}'(t-\tau)\tilde{K}(\tau)\tilde{F}_6(t-\tau)]\tilde{x}(\tau)d\tau. \end{aligned} \tag{10}$$

The cumbersome calculations leading to (10) are outlined by substitutting (8b) into (8a) with $u_r \equiv 0$ and noting from direct calculus through (9) that

$$\begin{aligned}&\tilde{M}(t)\tilde{F}_i(t) = \tilde{F}_i(t);\ \tilde{E}(t)\tilde{F}_i(t) = 0;\\ &\tilde{E}'(t)\tilde{F}_i(t) = 0;\ (i = 1, ..., 6);\end{aligned} \tag{11a}$$

$$\tilde{E}(t)\tilde{K}(t-h'(t)) = \begin{bmatrix} \tilde{E}K(t-h'(t)) & E(t)H(t-h'(t)) \\ 0 & 0 \end{bmatrix}; \tag{11b}$$

$$\tilde{E}'(t)\tilde{K}(\tau) = \begin{bmatrix} \tilde{E}'(t)K(\tau)C(\tau) & E'(t)H(\tau) \\ 0 & 0 \end{bmatrix}; \tag{11c}$$

For presentation simplicity, the next stability result is concerned with a particular $SPVD$ with the plant and controller matrices being required to satisfy a certain time-invariant constrain.

Theorem 1. *Assume that the closed-loop extended $SPVD$, Eqs. (8) – (9),satisfies the following assumptions:*

(1) $\int_0^\infty |\tilde{G}(\tau)| d\tau < 1$ *and* $\tilde{G}(t) \to 0$ *as* $t \to \infty$ *where* $\tilde{G}(t) = \tilde{G}(0) + (I + \tilde{D}_1^{-1}\tilde{D}) \int_0^t \tilde{C}_7(\tau) d\tau$; $\tilde{C}_7(t-\tau) = \tilde{B}(t-\tau) + \tilde{E}'(t-\tau)\tilde{K}(\tau) + \tilde{F}_6(t-\tau)$ *is a matrix function of appropriate order for some constant symmetric positive definite matrices* $\tilde{D}$ *and* $\tilde{D}_1$ *under the additional assumption that*

all the entries of $\tilde{C}_7(\cdot)$ are in $L^1([0,\infty);\mathbf{R})$ (this also implies that $\dot{\tilde{G}}(t) = (I + \tilde{D}_1^{-1}\tilde{D})\tilde{C}_7(t))$.

(2) The extended system and controller gain are chosen such that $\tilde{A}(t) + \tilde{M}(t)\tilde{K}(t)$ and $\sum_{i=0}^{6} \tilde{C}_i^T(t) + \tilde{C}_i(t)$ are constant matrices.

(3) $\tilde{G}(0) = \tilde{D}_1^{-1}(\tilde{A}(t) + \tilde{M}(t)\tilde{K}(t))$, with the $\tilde{D}_1$-matrix referred to in Assumption 1, and $I + D_c(t)D_f(t)D(t)$ is nonsingular for all $t \geqslant 0$.

(4) All delays in both plant and controller are, in general, time-functions of bounded time derivative of known upper-bound.

Then, the zero solution of the free closed-loop system (i.e., $u_r \equiv 0$ on $[0,\infty)$) is globally asymptotically stable if and if the following Lyapunov equation holds

$$
\begin{aligned}
[\tilde{A}^T + \tilde{K}^T\tilde{M}^T]\tilde{D} + \tilde{D}[\tilde{A} + \tilde{M}\tilde{K}] &= -qI + \sum_{i=0}^{6} \tilde{C}_i^T \tilde{C}_i \\
&\quad + \{\tilde{D}_1^{-1}[\tilde{A}^T + \tilde{K}^T\tilde{M}^T]\tilde{D}_1 - \tilde{A}^T - \tilde{K}^T\tilde{M}^T\}\tilde{D}_1 \\
&\quad + \tilde{D}_1\{\tilde{D}_1[\tilde{A} + \tilde{M}\tilde{K}]\tilde{D}_1^{-1} - \tilde{A} - \tilde{M}\tilde{K}\},
\end{aligned} \tag{12}
$$

with $\tilde{C}_i(t) =: \tilde{F}_i(t)$ $(i = 1, 2, ..., 5)$; $\tilde{C}_0(t) =: \tilde{A}_0(t)$ and $\tilde{C}_6(t) = \tilde{E}(t)\tilde{K}(t - h'(t))$, has a constant solution matrix $\tilde{D} = \tilde{D}^T > 0$ for some (sufficient large) $q \in \mathbf{R}^+$ and all constant $\tilde{D}_1 = \tilde{D}_1^T > 0$ which satisfy Assumptions 1 and 3. If Assumption 1 is changed to the less strong condition $\int_0^\infty |\tilde{G}(\tau)|d\tau < \infty$, then simple Lyapunov's stability is guaranteed.

The proof is given in Appendix A1. The sense of "global" stability is that stability holds for any admissible set of initial conditions. The lower treshold for q in Theorem 1 is calculated in the Appendix as well as rules to select the $(\tilde{D}, \tilde{D}_1)$-pair.

REMARK 1. (i) The above result and its proof can be trivially generalized to the modifications $qI \to \bar{Q} = \bar{Q}^T > 0$ ($\bar{Q}$ being a constant matrix) with $q = |\bar{Q}|$.

(ii) It is seen in the proof that $\tilde{D}$ can be chosen in (12) as being independent of q via a scalar normalization process of matrices $\tilde{D}$ and $\tilde{D}$. This process guarantees that q can be chosen independetly of the plant and controller parameters while accomplishing the statement of q being sufficiently large for a Lyapunov function candidate used in the proof, to result in a Lyapunov function for the closed-loop system Eqs. (8) – (9).

(iii) Note that the time invariance requirement on $\tilde{\boldsymbol{D}}_1$ and two matrices of parameters in (12) guarantees that $\tilde{\boldsymbol{D}}$ is constant. The sufficiency proof on stability could be extended to the time-varying case at the expense of more cumbersome proofs under lower and upper overbounding functions of the closed-loop matrices.

The following specializations of Theorem 1 are of interest.

3.1. *SVD* of time-invariant self dynamics. The extended closed-loop system Eq. 10 is defined by

$$\dot{\tilde{\boldsymbol{x}}}(t) = \tilde{\boldsymbol{A}}_c(t)\tilde{\boldsymbol{x}}(t) + \int_0^t \tilde{\boldsymbol{B}}_c(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau, \tag{13}$$

where

$$\tilde{\boldsymbol{A}}_c =: \tilde{\boldsymbol{A}}(t) + \tilde{\boldsymbol{M}}(t)\tilde{\boldsymbol{K}}(t);\ \tilde{\boldsymbol{B}}_c =: \tilde{\boldsymbol{B}}(t) + \tilde{\boldsymbol{E}}'(t)\tilde{\boldsymbol{K}}(t) + \tilde{\boldsymbol{F}}_6(t). \tag{14}$$

Although $\tilde{\boldsymbol{A}}_c$ is constant, it is not required that $\tilde{A}(\cdot)$, $\tilde{\boldsymbol{M}}(\cdot)$ and $\tilde{K}(\cdot)$ be individually time-invariant. Eq. 13 (subject to (14)) is a particular case of (10) with the constraints

$$\tilde{\boldsymbol{A}}_0(t) = \boldsymbol{0};\ \tilde{\boldsymbol{F}}_i(t) = \boldsymbol{0};\ (i = 1, 2, ..., 5);\ \tilde{\boldsymbol{E}}(t)\tilde{\boldsymbol{K}}(t - h'(t)) = \boldsymbol{0}, \tag{15}$$

arising from (9), which can be accomplished with a controller Eqs. (3) – (4) subject to

$$\boldsymbol{E}(t) = \boldsymbol{0};\ \boldsymbol{A}_{0p}(t) = \boldsymbol{0};\ \boldsymbol{A}_{0c}(t) = \boldsymbol{0};\ \boldsymbol{A}_{0f}(t) = \boldsymbol{0};\ \boldsymbol{E}_f(t) = \boldsymbol{0}; \tag{16a}$$

$$\boldsymbol{E}_c^T \in ker[\boldsymbol{C}_p^T \vdots \boldsymbol{C}_f^T(t)]; \tag{16b}$$

or, in particular, the constraint in (16b) can be substituted by $\boldsymbol{E}_c(t) = \boldsymbol{0}$ or $[\boldsymbol{C}_p(t) \vdots \boldsymbol{C}_f(t)] = \boldsymbol{0}$. Assume that $\tilde{\boldsymbol{B}}_c$ is continuous on $[0, \infty)$. Select an $(n + l_c + l_f + l_p) \times (n + l_c + l_f + l_p)$ matrix $\tilde{G}(t)$ with $\dot{\tilde{G}}(t) = \boldsymbol{B}_c(t)$ and set $\tilde{\boldsymbol{Q}} = \tilde{\boldsymbol{A}}_c - \tilde{\boldsymbol{G}}(0)$ so that (13) takes the form

$$\dot{\tilde{\boldsymbol{x}}}(t) = \tilde{\boldsymbol{Q}}(t)\tilde{\boldsymbol{x}}(t) + \frac{d}{dt}\int_0^t \tilde{G}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau. \tag{17}$$

Let $\tilde{\boldsymbol{D}}$ be a symmetric matrix satisfying

$$\tilde{\boldsymbol{Q}}^T\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{D}}\tilde{\boldsymbol{Q}} = -\boldsymbol{I}. \tag{18}$$

If $\tilde{D}$ is any positive definite matrix, then there is a positive constant k such that $k|\tilde{x}|^2 \leqslant \tilde{x}^T \tilde{D}\tilde{x}$. From Theorem 1, the next result follows immediately.

COROLLARY 1.1. Controller Eqs. (3) – (4), subject to constraints (16) is a stabilizing controller for the original free SVD, and the zero solution of the extended closed-loop free SVD, Eqs. (13) – (14), is stable in the sense of Theorem 1 in De la Sen and Luo, (1995) if and only if the solution $\tilde{D}$ to the Lyapunov equation (18) is positive definite provided that:

$$(1)\ 2|\tilde{D}\tilde{Q}| \int_0^\infty |\tilde{G}(\tau)|d\tau < 1;$$
$$(2)\ \tilde{G}(t) \to 0 \text{ as } t \to \infty;$$
$$(3)\ I + D_c(t)D_f(t)D(t);$$

is a nonsingular matrix for all $t \geqslant 0$.

The following results follow from Theorem 1 and Corollary 1.1.

COROLLARY 1.2. Assume the following: (1) The Lyapunov's function (18) holds and $(I+D_c(t)D_f(t)D(t))$ is nonsingular for all $t \geqslant 0$; (2) $|\int_0^\infty \tilde{B}_c(\tau)d\tau| < \infty$; $\tilde{G}(t) =: -\int_t^\infty \tilde{B}_c(\tau)d\tau$ and $\tilde{Q} =: \tilde{A}_c - \tilde{G}_0 = A_c + \int_0^\infty \tilde{B}_c(\tau)d\tau$; (3) $2|\tilde{D}\tilde{Q}|\int_0^\infty |\int_0^\infty \tilde{B}_c(\tau)d\tau|dt < 1$ and Corollary 1.1(1) holds. (i) Then, the zero solution of the extended closed-loop free SVD, Eqs. (13) – (14), is stable if and only if $\tilde{D}$ is positive definite. If, in addition, $A_c + \int_0^\infty \tilde{B}_c(\tau)d\tau$ is stable then all the solutions are in $L^2([0,\infty);\mathbb{R}^{n+l}) \cap L^\infty([0,\infty);\mathbb{R}^{n+l})$; $(l = l_c + l_p + l_f)$; (ii) If, in addition, $\int_0^\infty |B_c(\tau)|^2 d\tau < \infty$ or $\int_0^\infty |B_c(\tau)|d\tau < \infty$, then all the solutions of the extended SVD Eq. 13 tend to zero as $t \to \infty$ and hence, its zero solutions are asymptotically stable; (iii) If, in addition, $\int_0^\infty \int_t^\infty |B_c(\tau)|d\tau dt < \infty$, then all the solutions of the extended closed-loop free SVD, Eq. 13, are in $L^1([0,\infty);\mathbb{R}^{n+l})$ and its zero solution is uniformly asymptotically stable.

The proofs of Corollary 1.2(ii) – (iii) are given in Appendix A.1.

COROLLARY 1.3. Assume that the Lyapunov equation (18) holds, $(I + D_c(t)D_f(t)D(t))$ is nonsingular for all $t \geqslant 0$ and, in addition, (i) $\tilde{Q} = A_c - \tilde{G}(0) = A_c + \int_0^\infty \tilde{B}_c(\tau)d\tau$; (ii) $A_c + \int_0^\infty \tilde{B}_c(\tau)d\tau$ is a stable matrix; (iii) $2|\tilde{D}\tilde{Q}|\int_0^\infty \int_t^\infty |\tilde{B}_c(\tau)|d\tau dt < 1$. Then, the zero solution of the extended closed-loop free SVD (Eq. 13) is uniformly globally asymptotically stable.

The proofs follow after some calculations from Theorem 1 and Corollary 1.1 by applying the stability results in De la Sen and Luo (1997) (see proof of Theorem 1 in Burton, 1985).

COROLLARY 1.4. If there are no point delays in the plant and controller, then the extended free $SPVD$ (Eqs. (8)–(9)) reduces to the SVD (Eq. 13) which is only guaranteed to be globally Lyapunov's stable under the conditions of Theorem 1 and the changes in the Assumptions for $\tilde{G}(\cdot)$ and $\tilde{\boldsymbol{Q}}$ and the Lyapunov equation given in Corollary 1.1.

The proof is outlined in Appendix A1.

The use of the above results allows the design of universal controllers (Eq. 2) which are more general than the linear memoryless controllers referred to De la Sen and Luo (1997) for systems of "small" parameters associated with the delay influences.

4. Controller design. The controller design is inspired in Theorem 1 below. The philosophy is slightly changed with respect to the way about the statement of the theorem. Firstly, the calculation of the extended $(m+l) \times (n+l)$ matrix $\tilde{\boldsymbol{K}}(t)$ is done via the Kronecker product of matrices for some predefined symmetric positive definite matrices $\tilde{\boldsymbol{D}}$ and $\tilde{\boldsymbol{D}}_1$ and the scalar $q \in \mathbb{R}^+$ while maintaining all the remaining conditions from Theorem 1, As the second step, (A.30) in Appendix A is used to calculate the particular controller matrices to implement (2). The cumbersome matrix equations referred to in the following are written in detail in Appendix A.

a) Step 1 *(Calculation of $\tilde{\boldsymbol{K}}(t)$).* Eq. 12 is rewritten in vector form for the unknown matrix $\tilde{K}(t)$ for predefined (sufficiently large) $q \in \mathbb{R}^+$ and for matrices $\tilde{\boldsymbol{D}}$ and $\tilde{\boldsymbol{D}}_1$ by using the (left) Kronecker product of matrices (Barnett, 1971). Firstly, note that there exists a permutation (unitary) matrix U such that $\tilde{\boldsymbol{k}}^T = U\tilde{\boldsymbol{k}}$ where $\tilde{\boldsymbol{k}} = [\tilde{\boldsymbol{k}}_1^T \vdots \tilde{\boldsymbol{k}}_2^T \vdots \ldots \vdots \tilde{\boldsymbol{k}}_{n+l}^T]^T$; i.e., a vector containing the rows of the matrix $\tilde{K}$ while $\tilde{\boldsymbol{k}}^T$ is a column vector with the rows of the matrix $\tilde{K}^T$. Thus,

$$\tilde{\boldsymbol{\Delta}}\tilde{\boldsymbol{k}} =: \{[(\tilde{D} + \tilde{D}_1)\tilde{\boldsymbol{M}} \otimes I - \tilde{D}_1^2\tilde{M} \otimes \tilde{D}_1^{-1}]U + I \otimes (\tilde{D} + \tilde{D}_1)\tilde{\boldsymbol{M}} - \tilde{D}_1^{-1} \otimes \tilde{D}_1^2\tilde{\boldsymbol{M}}\}\tilde{\boldsymbol{k}} = \boldsymbol{v}, \qquad (19)$$

where $\boldsymbol{v} = [\boldsymbol{v}_1^T \vdots \boldsymbol{v}_1^T \vdots \cdots \vdots \boldsymbol{v}_{n+l}^T]^T$, obtained from the rows of V below, and

$$V = [v_1, v_2, \cdots, v_{n+l}]^T = -qI + \tilde{\Delta}' - \tilde{A}^T(\tilde{D} + \tilde{D}_1^{-1}) - (\tilde{D} + \tilde{D}_1^{-1})\tilde{A}$$
$$+ \tilde{D}_1^{-1}\tilde{A}^T\tilde{D}_1^2 + \tilde{D}_1^2\tilde{A}\tilde{D}_1^{-1}; \; \tilde{\Delta}' =: \sum_{i=0}^{6} \tilde{C}_i\tilde{C}_i^T. \tag{20}$$

Note that the coefficient matrix $\tilde{\Delta}$ in (19) is $(n+l)^2 \times (m+l)(n+l)$ and $m \leqslant n \Rightarrow \text{rank}(\tilde{\Delta}) \leqslant (m+l)(n+l)$. Note from (A.32) – (A.33) that the first row blocks of $\tilde{\Delta}'$ (Eq. 20) are dependent on $\tilde{\boldsymbol{k}}$ through the W_p-matrix equation (A.33). Two design strategies follow immediately.

Strategy 1 (*Time-invariant controller*). The controller parameters in $\tilde{\Delta}'$ are prefixed and the unknown vector $\tilde{\boldsymbol{k}}$ in (19) is changed into $\tilde{\boldsymbol{k}}'$ by deleting the corresponding components. Thus, $dim(\tilde{\boldsymbol{k}}') = dim(\tilde{\boldsymbol{k}}) = (m+l)(n+l) - m(p+l) = n(l+m) + l^2 - mp$. There exists a unitary matrix $\tilde{U}$ such that $\tilde{\boldsymbol{k}} = \tilde{U}\tilde{\boldsymbol{k}}^* = \tilde{U}[\,\tilde{\boldsymbol{k}}_1^T \vdots \tilde{\boldsymbol{k}}_2^T]$ where $\tilde{\boldsymbol{k}}_1$ is known and calculated from $\tilde{\Delta}'$, Eq. 20, by using the Kronecker product. Eq. 19 can be rewritten as

$$(\tilde{\Delta} - \tilde{\Delta}'')\tilde{\boldsymbol{k}} = (\tilde{\Delta} - \tilde{\Delta}')\tilde{U}[\tilde{\boldsymbol{k}}_1^T \vdots \tilde{\boldsymbol{k}}_2^T]^T = \tilde{\Delta}_1\tilde{\boldsymbol{k}}_1 + \tilde{\Delta}_2\tilde{\boldsymbol{k}}_2; \; \tilde{\Delta} - \tilde{\Delta}''$$
$$= [\tilde{\Delta}_1 \vdots \tilde{\Delta}_2], \tag{21}$$

so that $\tilde{\Delta}''\tilde{\boldsymbol{k}}$ are the components of $\tilde{\Delta}'$ arranged by rows. From (21), one obtains

$$\tilde{\Delta}_2\tilde{\boldsymbol{k}}' = \boldsymbol{v} - \tilde{\Delta}_1\tilde{\boldsymbol{k}}_1, \tag{22}$$

with $\text{rank}(\tilde{\Delta}_2) \leqslant n(l+m) + l^2 - mp$. (At least) a solution $\tilde{\boldsymbol{k}}'$ to (17) exists if and only if $\text{rank}(\tilde{\Delta}_2) = \text{rank}[\tilde{\Delta}_2 \vdots \boldsymbol{v} - \tilde{\Delta}_1\tilde{\boldsymbol{k}}_1]$ (Froebenius theorem) which holds typically if $m > n$. Assume that $m \leqslant n$. The above rank condition can be always satisfied by choosing $\boldsymbol{v}$ (see (19) and (22)) through the design of the $(n+l) \times (n+l)$ matrix $\tilde{D}_1$ satisfying $\tilde{D}_1 = \tilde{D}_1^T > 0$, so that $\boldsymbol{v} - \tilde{\Delta}_1\tilde{\boldsymbol{k}}_1 = \sum_{i=1}^{l+n} \lambda_i\tilde{\Delta}_{2i}$ where $\tilde{\Delta}_{2(\cdot)}$ denotes the columns of the $l \times (l+n)$ matrix $\tilde{\Delta}_2$ and $\lambda(\cdot)$ are scalars not all being zero. Note that the application of the Kronecker product in (21) makes the $\tilde{D}_1$-matrix to generate an $(n+l)^2$ vector $\boldsymbol{d}_1$ with its components depending on the set $\lambda(\cdot)$ to accomplish the Froebenius condition. $\boldsymbol{d}_1$ can be chosen to satisfy the $(n+l)$ constraints for $\tilde{D}_1 = \tilde{D}_1^T > 0$ (i.e., all principal minors are positive) by using inequality type constraints on the $\lambda_{(\cdot)}$.

The same tool can be applied in the case of Remark 1, namely, qI is replaced by $\bar{Q} > \mathbf{0}$ in the Lyapunov equation of Theorem 1. In this case, $\boldsymbol{v} = \boldsymbol{q} + \boldsymbol{q}_1$ where $\boldsymbol{q}$ is a vector obtained from the rows of the $\bar{Q}$-matrix by using the Kronecker product. In this case, the solvability constraint is $\boldsymbol{q}+\boldsymbol{q}_1-\tilde{\boldsymbol{\Delta}}_1\tilde{\boldsymbol{k}}_1 = \sum_{i=1}^{\delta}\lambda_i\tilde{\boldsymbol{\Delta}}_{2i}$; $\delta = n(l+m)+l^2-mp$. Since the numbers of equations are not less than those of unknowns involved, the constraint can be satisfied by a set of $\lambda_{(\cdot)}$ such that $\boldsymbol{q}$ generates a positive definite matrix.

Similar algebraic equations are used when a part of $\tilde{\boldsymbol{\Delta}}'$ is used as predesigned one while the other has to be found in the design procedure. In this case, $\tilde{\boldsymbol{\Delta}}''$ is redesigned and, in order to satisfy Frobenius theorem, $(\tilde{\boldsymbol{\Delta}} - \tilde{\boldsymbol{\Delta}}'')$ has to fulfill the necessary condition of having no less columns than rows; i.e., $l_p^2 + l_c^2 + l_f^2 - mp \geqslant n(l+n-m)$; i.e., at least one of the orders of the procompensator, feedward or feedback controllers is sufficiently large related to the plant dimension. In this case, the rank condition for solvability, i.e., $\operatorname{rank}(\tilde{\boldsymbol{\Delta}}_2) = \operatorname{rank}[\boldsymbol{v} - \tilde{\boldsymbol{\Delta}}_1\tilde{\boldsymbol{k}}_1] = l_p^2 + l_c^2 + l_f^2 - mp + n(l+m) + l^2$, is typically fulfilled.

REMARK 2. Note from Theorem 1 that Assumption 1 can be relaxed to have the first integral being bounded while maintaining the stability property. Thus, the design of a stabilizing controller does not imply a possible normalization procedure on $\tilde{G}$. If the upper-bound unity is required to guarantee the asymptotitc stability, $\boldsymbol{v}$ in (19) can be of a sufficiently small norm by choosing $\tilde{D}$, $\tilde{D}_1$ and $(\tilde{\boldsymbol{\Delta}}' - qI)$ of sufficiently small norms. The requirement of $\tilde{\boldsymbol{\Delta}}'$ being of a sufficiently small norm can be fulfilled by choosing appropriately the controller matrices provided that $||A_0||$ is sufficiently small (see (A.32)–(A.34) in Appendix A). The condition of q being sufficiently large can be satisfied through the normalization procedure described in the proof of Theorem 1 (Appendix A) which leads to a successive application of the given design method in case of failure. The same method is applicable to any particular extended subsystem of that given in (10) including the SVD under the stability conditions of the corollaries to Theorem 1. If such corollaries are used, the various conditions related to upper-bounds being unity can be guaranteed by choosing $|\tilde{D}\tilde{Q}|$ sufficiently small and, as before, failures in checking such conditions can be overcome by successive applications of the proposed algebraic controller design method.

Strategy 2 (*Time-varying controller*). Note from Remark 1 that time-invariance of the closed-loop system matrices is unessential for the stability proof. In this case, the submatrix of $\tilde{\Delta}'$ referred to the controller can be known at time $t-\sigma$ (some $\sigma > 0$) to design $\tilde{\boldsymbol{k}}$ at time t (see (A.32) – (A.34)). Thus, the design can be done again from Strategy 1 by estabilishing a set of time-dependent equations. From continuity with respect to the time arguments, the entries to $\tilde{\boldsymbol{k}}$ do not present unbounded variations versus time.

b) Step 2. The controller matrices of (2) are calculated from $\tilde{\boldsymbol{k}}$, which has been calculated in Step 1 by using the Kronecker product. For simplicity of the subsequent discussion, assume that the system is time-varying so that the controller matrices in $\tilde{\Delta}'$ are available at $t' < t$ for each time t. Denote by pairs (i, j); $i, j = 1, 2, 3, 4$ the block matrices in (A.30). From the values of $\tilde{\boldsymbol{k}}$, the $\tilde{\boldsymbol{K}}$-matrix is calculated from blocks (2,2) (3,3), (4,4) (1,3) and (4,1), respectively, in (A.30). Subsequently, the products $\boldsymbol{D}_c\boldsymbol{D}_f$, $\boldsymbol{D}_c\boldsymbol{C}_p$, $\boldsymbol{D}_c\boldsymbol{C}_f$, $\boldsymbol{M}_c\boldsymbol{D}_f$, $\boldsymbol{M}_c\boldsymbol{C}_p$ and $\boldsymbol{M}_c\boldsymbol{C}_f$ are obtained from the blocks (1,1), (1,2), (1,4), (3,1), (3,2) and (3,4), respectively. In compact form, $[\boldsymbol{D}_c^T \vdots \boldsymbol{M}_c^T]X = [\boldsymbol{M}_1^T \vdots \boldsymbol{M}_2^T]$; $X = [\boldsymbol{D}_f \vdots \boldsymbol{C}_p \vdots \boldsymbol{C}_f]$ where $\boldsymbol{D}_c$, $\boldsymbol{M}_c$, $\boldsymbol{X}$, $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are, respectively, of orders $m \times m_p$, $l_c \times m_p$, $m_p \times (m_f + l_p + p)$ (since $m_p = m_f$ and $p = l_f$), $m \times (m_p + l_p + p)$ and $l_c \times (m_p + l_p + p)$. Since $\boldsymbol{M}_{1,2}$ are given from the block matrices in $\tilde{K}$, a (in general non-unique) solution $\boldsymbol{X}$ exists if the pair $(\boldsymbol{D}_c, \boldsymbol{M}_c)$ is chosen *a priori* such that $\text{rank}[\boldsymbol{D}_c^T \vdots \boldsymbol{M}_c^T] = \text{rank}[\boldsymbol{D}_c^T \vdots \boldsymbol{M}_c^T \vdots \boldsymbol{M}_1^T \vdots \boldsymbol{M}_2^T]$. A unique solution $X = [\boldsymbol{D}_f \vdots \boldsymbol{C}_p \vdots \boldsymbol{C}_f] = [\boldsymbol{D}_c^T \vdots \boldsymbol{M}_c^T]^{-T}[\boldsymbol{M}_1^T \vdots \boldsymbol{M}_2^T]^T$ exists provided that $m_c + l_c = m + l_c = m_p^2$ and $(\boldsymbol{D}_c, \boldsymbol{M}_c)$ is chosen such that $Det([\boldsymbol{D}_c^T \vdots \boldsymbol{M}_c^T]) \neq 0$. If $\boldsymbol{M}_{1,2}$ are zero at time t then the set $(\boldsymbol{D}_c, \boldsymbol{M}_c, \boldsymbol{D}_f, \boldsymbol{C}_p, \boldsymbol{C}_f)$ can be chosen identically zero. Continuity with respect to time of the entries of the controller matrices follows from the continuity of the entries to $\tilde{K}$. If the plant is time-invariant and stabilizable, then a time-invariant controller can be designed by extending slightly the above arguments and by using the modified $\tilde{\boldsymbol{k}}'$ of Step 1.

5. Stabilization of the time-invariant *SP* with time-invariant controller. Now, Eqs. (1) – (2) and then (10) are particularized to the time-invariant case and the Volterra-type terms are deleted. The stability of the resulting SP (1) can be studied via transform domain methods. The results are directly extendable to the SED by converting it to an extended SP (see Lemma 1 in De la Sen and Luo, 1997) and with further generalization to the $SD's$ by using Proposition

A.1 in De la Sen and Luo (1997). The closed-loop extended system is stable iff its closed-loop poles lie within a prescribed stability constraint, namely,

$$Det[\Delta(s)] \neq 0;\ \Delta(s) = \Delta_0(s) - \tilde{\Delta}(s); \quad \forall s \in \mathbb{C};\ \text{with } Re(s) \geqslant -\gamma, \tag{23}$$

for some $\gamma \geqslant 0$, where one gets from particularization of (1) – (2) to (1) to constant matrices by zeroing the appropriate ones when building the dynamics of the extended vector $[\boldsymbol{x}^T, z_p^T, z_p^T, z_f^T]^T$:

$$\Delta_0(s) = \begin{bmatrix} sI - A + (M - Ee^{-sh'})D_cD_fC & -MD_cC_p & -MC_c & MD_cC_f \\ 0 & sI - A_p & 0 & 0 \\ M_cD_fC & -M_cC_p & sI - A_c & M_cC_f \\ -M_fC & 0 & 0 & sI - A_f \end{bmatrix} \tag{24a}$$

$$\tilde{\Delta}(s) = \begin{bmatrix} A_0e^{-sh} & ED_cC_pe^{-sh'} & EC_ce^{-sh'} & -ED_cC_fe^{-sh'} \\ 0 & A_{0p}e^{-sh_p} & 0 & 0 \\ -E_cD_fe^{-sh'_c} & E_cC_pe^{-sh'_c} & A_{0c}e^{-sh_c} & -E_cC_fe^{-sh'_c} \\ E_fCe^{-sh'_f} & 0 & 0 & A_{0f}e^{-sh_f} \end{bmatrix} \tag{24b}$$

The following observation follows from (23) – (24). If $Det[\Delta_0(s)] \neq 0$ for all complex s with $Re(s) \geqslant -\gamma_0$, some real $\gamma_0 > 0$, then there exists a real ϖ such that if all delays are within a prescribed positive interval $[0, \delta]$ then, if $||\tilde{\Delta}(s)|| \leqslant \varpi$, $\forall s \in \mathbb{C}_{\gamma_0} =: \{s \in \mathbb{C} : Re(s) \geqslant -\gamma_0\}$, there exists $\gamma \in (0, \sigma_0) \cap \mathbb{R}$ such that all the roots of $Det(\Delta_0(s) - \tilde{\Delta}(s)) = 0$ are in $\mathbb{C}_\gamma = \{s \in \mathbb{C} : Re(s) \geqslant -\gamma\}$. Since $\tilde{\Delta}(s) = \Delta_0(s)[I - \Delta_0^{-1}(s)\tilde{\Delta}(s)]$, provided that $\Delta_0(s)$ is invertible in $\mathbb{C}_{\gamma_0}$, it suffices to fulfill $||\tilde{\Delta}(s)|| \leqslant 1/||\Delta_0^{-1}(s)||$ on $\mathbb{C}_\gamma$ in order to guarantee that $\Delta(s)$ is nonsingular in $\mathbb{C}_\gamma$ (Banach perturbation lemma (Ortega, 1972)). Both conditions related to $\Delta_0(s)$ and $||\tilde{\Delta}(s)||$ can be simultaneously guaranteed with $\Delta_0(s)$ being diagonally dominant, and lower or upper triangular, in particular, and $A + (Ee^{-sh'} - M)D_cD_fC$, A_p, A_c and A_f being Hurwitz related to $\mathbb{C}_{\gamma_0}$; i.e., with their eigenvalues being in $\bar{\mathbb{C}}_{\gamma_0}$

and the norm of remaining controller matrices in $\tilde{\Delta}(s)$ not being greater than the value ϖ (being dependent on γ) in $\mathbb{C}_\gamma$. Particular cases of interest are the following.

Case A. $\Delta_0(s)$ is upper-triangular. For instance, we choose $\boldsymbol{D}_f$, $\boldsymbol{C}_p$ and $\boldsymbol{M}_f$ being zero, which means that the feedback controller is strictly proper, the precompensator has pure dynamics and the feedback controller does not involve the undelayed output. A second choice leading to a triangular $\Delta_0(s)$ is $\boldsymbol{M}_f = \boldsymbol{0}$, $\boldsymbol{M}_c = \boldsymbol{0}$. With the first choice, $Det[\Delta_0(s)] = Det(s\boldsymbol{I} - \boldsymbol{A})Det(s\boldsymbol{I} - \boldsymbol{A}_p)Det(s\boldsymbol{I} - \boldsymbol{A}_c)Det(s\boldsymbol{I} - \boldsymbol{A}_f)$. Then, the overall system is asymptotically stable related to $\mathbb{C}_{\gamma_0}$, and $||\tilde{\Delta}(s)|| \leqslant \varpi$ on $\mathbb{C}_\gamma$, some nonnegative γ, within a neighborhood of γ_0, and ϖ depending on γ. Some results of De la Sen and Luo (1997) appear again since, in fact, there exists always a delay-free controller which stabilizes the closed-loop system of a plant with point delay provided that $||A_0||$ is sufficiently small. In the second situation, modify $\Delta_0(s) \rightarrow \Delta_0'(s) = \Delta_0(s) + Block\{Diag[\boldsymbol{E}\boldsymbol{D}_c\boldsymbol{D}_f\boldsymbol{C}e^{-sh'} \vdots \boldsymbol{0} \vdots \boldsymbol{0} \vdots \boldsymbol{0}]\}$, so that, for analysis simplicity, $\Delta(s) = \Delta_0(s) - \tilde{\Delta}(s)$ remains identical. Choose $\boldsymbol{A}_p$, $\boldsymbol{A}_c$ and $\boldsymbol{A}_f$ strictly Hurwitz related to $\mathbb{C}_\gamma$. Then, if $(A, \boldsymbol{M})$ is output-stabilizable in the sense that there exists a pair $(\boldsymbol{K}, \boldsymbol{K}_1)$ of appropriate dimensions such that $\boldsymbol{K} = -\boldsymbol{K}_1\boldsymbol{C}$ and $(A - \boldsymbol{M}\boldsymbol{K}_1\boldsymbol{C})$ is strictly Hurwitz related to $\mathbb{C}_\gamma$ (see Lemma 2 in De la Sen and Luo, 1997). Thus, $A^* = A - \boldsymbol{M}\boldsymbol{K}_1\boldsymbol{C}$ with A^* having its eigenvalues in $Re(s) < -\gamma$. Considering $\boldsymbol{K}_1$ as unknown for a given A^*, the problem can be algebraically solved by using the Kronecker product, as in Section 4, leading to $(\boldsymbol{C}^T \otimes \boldsymbol{M})\boldsymbol{k}_1 = \boldsymbol{a} - \boldsymbol{a}^*$ with $\boldsymbol{a}$ and $\boldsymbol{a}^*$ arising from $\boldsymbol{A}$ and A^* by ordering their rows in a vector. Output-stabilizability of $(\boldsymbol{A}, \boldsymbol{M})$ is equivalent to rank$[\boldsymbol{C}^T \otimes \boldsymbol{M}]$ = rank$[\boldsymbol{C}^T \otimes \boldsymbol{M} \vdots \boldsymbol{a} - \boldsymbol{a}^*]$ since a solution $\boldsymbol{k}_1$ has to exist for some $\boldsymbol{A}^*$ of eigenvalues in $\mathbb{C}_\gamma$. Moreover, if $(\boldsymbol{A}, \boldsymbol{M})$ is output-assignable, the above reasoning holds for all matrix $\boldsymbol{A}^*$ of eigenvalues in $\mathbb{C}_\gamma$. Under output-assignability, the above (Froebenius) rank condition holds for all $\boldsymbol{A}^*$ of spectrum in $\mathbb{C}_\gamma$ defined *a priori* while under output-stabilizability, $\boldsymbol{A}^*$ has to be with its spectrum in $\mathbb{C}_\gamma$ satisfying the necessary solvability condition from Froebenius theorem. Once $\boldsymbol{K}_1$ has been calculated under output assignability or stabilizability, we choose $\boldsymbol{D}_c$ to fulfil rank$(\boldsymbol{D}_c)$ = rank$(\boldsymbol{D}_c \vdots \boldsymbol{K}_1)$ so that at least a solution $\boldsymbol{D}_f$ exists to $\boldsymbol{K}_1 = \boldsymbol{D}_c\boldsymbol{D}_f$. It is required for coherence that rank$(\boldsymbol{M}) \leqslant dim(sp_{s\gamma}(\boldsymbol{A}))$. This can be seen from $\boldsymbol{M}\boldsymbol{D}_c\boldsymbol{D}_f\boldsymbol{C} = A - A^* =$

$Diag(\lambda_i - \lambda_i^*;\ i = 1, 2, ..., n)$ by taking a diagonal matrix $(\boldsymbol{A} - \boldsymbol{A}^*)$ (this is sufficient to prove the above necessary condition). This condition can be verified as follows. Assume that rank$(\boldsymbol{M}) = m_0$ and there are $(m_0 + \tilde{m})$ nonzero entries to $Diag(\lambda_i - \lambda_i^*;\ i = 1, 2, ..., n)$; i.e., $dim[sp_{s\gamma}(\boldsymbol{A})] = m_0 + \tilde{m}$. Thus, there are at least $(m_0 + \tilde{m})$ nonzero entries to $Diag(\lambda_i - \lambda_i^*)$ for any choice of λ_i^* $(i = 1, 2, ..., n)$ being less than zero so that rank$(\boldsymbol{M}) <$ rank$[\boldsymbol{M} \vdots Diag(\lambda_i - \lambda_i^*);\ i = 1, 2, ..., n]$ and the algebraic problem has no solution.

Case B. Choose $\Delta_0(s)$ as being lower-triangular with $Det[\Delta_0(s)] \neq 0$ for all $s \in \mathbb{C}$ with $Re(s) \geqslant -\gamma_0$, some real positive constant γ_0. Procedures to achieve this requirement are the particular choices ($\boldsymbol{D}_c = \boldsymbol{0}$, $\boldsymbol{C}_c = \boldsymbol{0}$, $\boldsymbol{M}_c = \boldsymbol{0}$); ($\boldsymbol{C}_p = \boldsymbol{0}$, $\boldsymbol{C}_f = \boldsymbol{0}$); ($\boldsymbol{D}_c = \boldsymbol{0}$, $\boldsymbol{C}_c = \boldsymbol{0}$, $\boldsymbol{C}_f = \boldsymbol{0}$) etc. Similar conclusions as in **Case A** remain valid.

6. Nonzero convolution terms. ***SPD* systems.** In the presence of nonzero time-invariant terms, the stabilizability conditions of Section 5 will be changed. If the extended system is γ (state or trajectory)-stabilizable, then

$$\begin{aligned}\text{rank}\{sI &- \tilde{\boldsymbol{A}} - \tilde{\boldsymbol{M}}\tilde{\boldsymbol{K}} - \tilde{\boldsymbol{A}}_0 e^{-sh} - \tilde{\boldsymbol{F}}_1 e^{-sh_p} - \tilde{\boldsymbol{F}}_2 e^{-sh_c} - \tilde{\boldsymbol{F}}_3 e^{-sh_f} \\ &- \tilde{\boldsymbol{F}}_4 e^{-sh_c'} - \tilde{\boldsymbol{F}}_5 e^{-sh_f'} - \tilde{\boldsymbol{E}}\tilde{\boldsymbol{K}} e^{-sh'} - s^{-1}[\boldsymbol{B} + \tilde{\boldsymbol{E}}'\tilde{\boldsymbol{K}} + \tilde{\boldsymbol{F}}_6]\} \\ =& n + l_p, \end{aligned} \tag{25}$$

all $s \in \mathbb{C}$ with $Re(s) \geqslant -\gamma$. Since (25) must hold for $s = 0$, then the current particular case of the extended system (10) has to be satisfy the following integration condition compatible with (10)

$$\begin{aligned}\tilde{\boldsymbol{C}}_7 =& \tilde{\boldsymbol{B}} + \tilde{\boldsymbol{E}}'\tilde{\boldsymbol{K}} + \tilde{\boldsymbol{F}}_6 \\ =& \begin{bmatrix} \boldsymbol{B} - \boldsymbol{E}'\boldsymbol{D}_c\boldsymbol{D}_f\boldsymbol{C} & \boldsymbol{E}'\boldsymbol{D}_c\boldsymbol{C}_p & \boldsymbol{E}'\boldsymbol{C} & -\boldsymbol{E}'\boldsymbol{D}_c\boldsymbol{C}_f \\ \boldsymbol{0} & \boldsymbol{B}_p & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{E}_c\boldsymbol{D}_f\boldsymbol{C} & \boldsymbol{E}_c'\boldsymbol{C}_p & \boldsymbol{B}_c & -\boldsymbol{E}_c'\boldsymbol{C}_f \\ \boldsymbol{E}_f'\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & B_f \end{bmatrix} = \boldsymbol{0}. \end{aligned} \tag{26}$$

Thus, $\boldsymbol{B}_p$, $\boldsymbol{B}_c$ and $\boldsymbol{B}_f$ have to be zero while $\boldsymbol{C} \in Ker(\boldsymbol{E}_c\boldsymbol{D}_f) \cap Ker(\boldsymbol{E}_f')$; $\boldsymbol{C}_p \in Ker(\boldsymbol{E}'\boldsymbol{D}_c) \cap Ker(\boldsymbol{E}_c')$; $\boldsymbol{C}_f \in Ker(\boldsymbol{E}_c'\boldsymbol{D}_c) \cap Ker(\boldsymbol{E}_c')$; $\boldsymbol{C}_c \in Ker(\boldsymbol{E}')$. Furthermore, $\boldsymbol{C}^T\boldsymbol{D}_f^T\boldsymbol{D}_c^T(E')^T = \boldsymbol{B}^T$ so that rank$[\boldsymbol{D}_c\boldsymbol{D}_f\boldsymbol{C}]$ = rank$[\boldsymbol{D}_c\boldsymbol{D}_f \boldsymbol{C} \vdots \boldsymbol{B}]$ = rank$[\boldsymbol{D}_c\boldsymbol{D}_f\boldsymbol{C} \vdots \boldsymbol{E}'\boldsymbol{D}_c\boldsymbol{D}_f\boldsymbol{C}]$ = rank$\{[I \vdots \boldsymbol{E}']\boldsymbol{D}_c\boldsymbol{D}_f\boldsymbol{C}\}$ for some $\boldsymbol{E}'$ of appropriate order. If $\gamma > 0$, then the constraint (26) is not required for constant matrices.

REMARK 3. The convolution terms in $\tilde{C}_7$ include a pure time derivation; i.e., $\tilde{B} = s\tilde{B}'$, $\tilde{E}'\tilde{K} = s(\tilde{E}'\tilde{K})'$, $\tilde{F}_6 = s\tilde{F}_6'$ with $\tilde{B}'$, $(\tilde{E}'\tilde{K})'$ and $\tilde{F}_6'$ being constant. This is the trivial case of zero matrices $\tilde{B}$, $\tilde{E}'\tilde{K}$, $\tilde{F}_6$. The constraint $\tilde{C}_7 \equiv \mathbf{0}$ is unnecessary to accomplish with the stabilizability condition (25).

For the case of SPD-systems, two cases are of interest, namely, the distributed delays have an arbitrary distribution function. Condition (25) is modified for stabilizability according to Proposition A1. The above infinite (Volterra-type) integrals change into finite-interval integrals $\int_{-\sigma(\cdot)}^{0}(\cdot)$ for each time t. No pure integration appears so that the term $s^{-1}[\tilde{B} + \tilde{E}'\tilde{K} + \tilde{F}_6] \to [\sum_\alpha \int_{-\sigma_\alpha}^{0} d\tilde{B}(\theta)e^{\theta s}d\theta + \sum_\alpha \int_{-\sigma_\alpha}^{0} d(\tilde{E}'\tilde{K})e^{\theta s}d\theta + \sum_\gamma \int_{-\sigma_\gamma}^{0} d(\tilde{F}_6)e^{\theta s}d\theta]$ for appropriate matrix-valued finite measures of bounded variations $\tilde{B}$, $\tilde{E}'\tilde{K}$, $\tilde{F}_6$ (see Olbrot (1978), Tadmor (1988)). The stabilizability condition could be satisfied without relaxing the integrability condition (26). The second case is related to $SED's$. The above considerations can be dealt with by using Lemma 1 in De la Sen and Luo (1997) to reduce the distributed delays to point delays or by using the related results of Propositions A1 of that paper. In particular, the following result follows.

Theorem 2. *The next propositions hold:*

(i) *Assume the particular SP closed-loop regulator (i.e., $\mathbf{u}_r \equiv \mathbf{0}$ obtained from (1) – (2) with all the right-hand-side integrals being zero). Thus, the extended closed-loop system Eqs. (10) and (1) is globally asymptotically stable if all delay functions in both plant and controller have a bounded time-derivative of known upper-bound for all $t \geqslant 0$ (namely, condition (iv) of Theorem 1 holds) and, furthermore, the Lyapunov equation $[\tilde{A}^T + \tilde{K}^T\tilde{M}^T]\tilde{D} + \tilde{D}[\tilde{A} + \tilde{M}\tilde{K}] = -qI + \sum_{i=0}^{6}\tilde{C}_i\tilde{C}_i^T$ has a constant solution $\tilde{D} = \tilde{D}^T > 0$ for all $q \in \mathbb{R}^+$ provided that $(\tilde{A} + \tilde{M}\tilde{K})$ and $\sum_{i=0}^{6}\tilde{C}_i\tilde{C}_i^T$ are constant matrices subject to (4) – (7) and (9) with $\mathbf{F}_6(\cdot)$, $\mathbf{F}_8(\cdot)$ and $\mathbf{F}_{12}(\cdot)$ being zero matrix functions.*

(ii) *The global asymptotic stability of the zero solution of the SED regulator (3) holds (in a sufficiency sense) under the conditions of proposition (i) if, in addition, $\mathbf{E}(\cdot) \equiv \mathbf{0}$, and the parametrical changes $\mathbf{A}(t) \to \bar{\mathbf{A}}$, $A_0(t) \to \bar{\mathbf{A}}_0$ (Eq. 5), $\tilde{\mathbf{B}} = \mathbf{0}$, $\mathbf{B}(t) \to \bar{\mathbf{B}} =: [B^T \vdots \mathbf{0} \vdots I_m]^T$ (Eqs. 2 and 5) are made prior to the calculation of the extended system Eqs. (10) – (11).*

(iii) *Consider the $S\dot{D}$ of constant upper-bound distribution obtained* *from (10) by changing Volterra-type right-hand-side integral term by finite integrals associated with distributed delays (see* (2)*) of arbitrary distribution functions and assume that the controller is modified mutatis-mutandis. Then, the global asymptotic stability of the zero solution of the closed-loop system* (10) *holds if the conditions of Theorem* 1 *hold for arbitrary* $q \in \mathbb{R}^+$ *with* $\tilde{G}(0) = \mathbf{0}$. *This result applies, as a particular case, to the* SED *of proposition* (ii).

Proof (outline): The assertion (i) follows from Theorem 1 by omitting $\tilde{\boldsymbol{D}}_1$ since (A.6) with $\tilde{\boldsymbol{D}}_1 \equiv \mathbf{0}$ and no minimum lower-bound constraint on q is required. The assertion (ii) follows from Lemma 1 in Sen and Luo (1997) since the stability of augmented system implies that of the SD under the stabilizing controller. The assertion (iii) holds since (10) describes the extended SD if the infinite right-hand-side integrals are changed by the corresponding finite ones. The term $\tilde{\boldsymbol{G}}(0)$ is deleted since it appears by taking the derivatives of the integral symbol when obtaining $\dot{V}(\tilde{\boldsymbol{x}})$ ((A.7) – (A.9)) from $V(\tilde{\boldsymbol{x}})$ ((A.6)) with $\tilde{\boldsymbol{Q}} = \mathbf{0}$. No constraint on q is required since $\tilde{\boldsymbol{Q}} = \mathbf{0}$ and the delays associated with limits of the right-hand-side integrals are constant.

The controller design can be done by following steps similar to those in Section 4.

7. Stability for the non-regulator case. Now, the reference signal $\boldsymbol{u}_r$ in some of the systems and controller relationships (2) to (8) is nonzero. Gronwall's lemma or the solution through the use of a fundamental matrix can be used. As an illustrative example, take the SVD (Eq. 1)

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \int_0^t \boldsymbol{B}(t-\tau)\boldsymbol{x}(\tau)d\tau + \boldsymbol{p}(t); \ \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{27}$$

where $\boldsymbol{A}(\cdot)$ is an $n \times n$-matrix of continuous functions on $[0, \beta)$, $\boldsymbol{B}(\cdot)$ is an $n \times n$-matrix of functions continuous for $0 \leqslant \tau \leqslant t \leqslant \beta$ and $\boldsymbol{p}(\cdot)$ contains the influence of $\boldsymbol{u}_r(\cdot)$ and is continuous on $[0, \beta)$, with $\beta \leqslant \infty$. Eq. 23 is converted to an integral equation, involving integrated integrals, by integration from zero to t:

$$\begin{aligned}\boldsymbol{x}(t) &= \boldsymbol{x}_0 + \int_0^t \boldsymbol{A}(\tau)\boldsymbol{x}(\tau)d\tau + \int_0^t \int_0^\mu \boldsymbol{B}(\mu-\tau)\boldsymbol{x}(\tau)d\tau d\mu + \int_0^t \boldsymbol{p}(\tau)d\tau \\ &= \boldsymbol{x}_0 + \int_0^t \boldsymbol{p}(\tau)d\tau + \int_0^t [\boldsymbol{A}(\tau) + \int_\tau^t \boldsymbol{B}(\mu-\tau)d\mu]\boldsymbol{x}(\tau)d\tau. \end{aligned} \tag{28}$$

If $\boldsymbol{x}_0$ is bounded and $\boldsymbol{p} \in L^1([0,\infty);\mathbb{R}^n)$, then, $|\boldsymbol{x}(0) + \int_0^t \boldsymbol{p}(\tau)d\tau| \leqslant K_0$ (real constant) for all $t \geqslant 0$ so that $|\boldsymbol{x}(t)| \leqslant K_0 + \int_0^t g(t,\tau)|\boldsymbol{x}(\tau)|d\tau$ where $g(t,\tau) = |A(\tau) + \int_\tau^t \boldsymbol{B}(u-\tau)|d\tau|$. Application of Gronwall's inequality to (27) yields $|\boldsymbol{x}(t)| \leqslant K_0\ exp[\int_0^t g(t,\tau)d\tau]$ (Bellman, 1970), which is bounded for all $t \geqslant 0$ provided that $g(t,\tau)$ is in $L^1([t,\infty);\mathbb{R}^n)$ for all $0 \leqslant \tau \leqslant t \leqslant \infty$. Then, by rearranging terms in (31) below, the next result follows:

PROPOSITION 1. The free (i.e., $\boldsymbol{p} \equiv \mathbf{0}$) SVD system (27) is stable in the sense that $\boldsymbol{x} \in \boldsymbol{B}([0,\infty);\mathbb{R}^n)$ provided that $|\boldsymbol{x}_0| < \infty$ and $|\int_0^t [A(\tau) + \int_\tau^t \boldsymbol{B}(\mu-\tau)d\mu|]d\tau < \infty$ for all $t \geqslant 0$. If, in addition, $\boldsymbol{p} \in L^1([0,\infty);\mathbb{R}^n)$ then the corresponding forced system is also stable.

Note that the above stability concept implies and is implied by Lyapunov's stability. The same sufficient conditions are deduced by writing the solution to (27) as $\boldsymbol{x}(t) = Z(t)\boldsymbol{x}_0 + \int_0^t Z(t-\tau)\boldsymbol{p}(\tau)d\tau$ where $Z(\cdot)$ is a fundamental matrix with $Z(0) = I$ satisfying (27) for $\boldsymbol{p} \equiv 0$, (Burton, 1985). This follows from $|Z(t)| \leqslant |I| + \int_0^t g(t,\tau)|Z(\tau)|d\tau$ and $|\boldsymbol{x}(t)| \leqslant |Z(t)\boldsymbol{x}_0| + sup_{t \geqslant 0} |Z(t)|\ |\int_0^t \boldsymbol{p}(\tau)d\tau|$. The same analysis techniques can be used for other delay systems even in the case of presence of delay in the external precompensator which can be included in h_p'.

8. Further results about stability of Volterra equations. In some cases, $SVD's$ can be reduced to ordinary differential systems provided that the signals satisfy some regularity assumptions and the coefficient functions satisfy an ordinary differential system (Burton, 1985). Consider the SVD:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \int_0^t \boldsymbol{B}(t-\tau)\boldsymbol{x}(\tau)d\tau + \boldsymbol{B}_0(t)\boldsymbol{u}(t)\\ &= \boldsymbol{A}\boldsymbol{x}(t) + \int_0^t \boldsymbol{M}(t-\tau)\boldsymbol{x}(\tau)d\tau + \boldsymbol{p}(t),\end{aligned} \tag{29}$$

provided that

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{C}(t-\tau)\boldsymbol{x}(\tau)d\tau + \boldsymbol{D}(t)\boldsymbol{u}_r(t);\ \boldsymbol{p}(t) = \boldsymbol{B}_0(t)\boldsymbol{D}(t)\boldsymbol{u}_r(t) \quad (30a)$$

$$\boldsymbol{M}(t-\tau) = \boldsymbol{B}(t-\tau) + \boldsymbol{B}_0(t)\boldsymbol{C}(t-\tau), \quad (30b)$$

where $\boldsymbol{A}$ is a constant $n \times n$-matrix, $\boldsymbol{u}_r \in C^2([0,\infty);\mathbb{R}^m)$ and $\boldsymbol{B}_0$ and $\boldsymbol{D}$ have also entries in $C^2([0,\infty);\mathbb{R})$ so that $\boldsymbol{p}$ is twice differentiable on $[0,\infty)$;

$n \geqslant m \geqslant m'$, for all $0 \leqslant \tau \leqslant t$. Taking time-derivatives twice in (29), one gets directly:

$$\frac{d^3}{dt^3}\boldsymbol{x}(t) + \dot{\boldsymbol{x}}(t) = \boldsymbol{A}(\boldsymbol{x}(t) + \ddot{\boldsymbol{x}}(t)) + \boldsymbol{M}(0)\dot{\boldsymbol{x}}(t) + \dot{\boldsymbol{M}}(0)\boldsymbol{x}(t) + \int_0^t [\boldsymbol{M}(t-\tau) + \ddot{\boldsymbol{M}}(t-\tau)]\boldsymbol{x}(\tau)d\tau + \boldsymbol{p}(t) + \ddot{\boldsymbol{p}}(t) \quad (31)$$

Assume that the matrix functions $\boldsymbol{B}(\cdot)$, $\boldsymbol{B}_0(\cdot)$ and $\boldsymbol{C}(\cdot)$ satisfy the constraint

$$\boldsymbol{B}(t-\tau) + \boldsymbol{B}_0\boldsymbol{C}(t-\tau) + \ddot{\boldsymbol{B}}(t-\tau) + \ddot{\boldsymbol{B}}_0(t)\boldsymbol{C}(t-\tau) + \dot{\boldsymbol{B}}_0(t)\dot{\boldsymbol{C}}(t-\tau) + \boldsymbol{B}_0(t)\boldsymbol{C}(t-\tau) + \boldsymbol{B}_0(t)\ddot{\boldsymbol{C}}(t-\tau) = \boldsymbol{0}, \quad (32)$$

which implies from (30b) that $\boldsymbol{M}(t) + \ddot{\boldsymbol{M}}(t) = \boldsymbol{0}$, $\forall\ t \geqslant 0$ so that (31) is reduced to the ordinary differential system

$$\frac{d^3}{dt^3}\boldsymbol{x}(t)+\dot{\boldsymbol{x}}(t) = A[\boldsymbol{x}(t)+\ddot{\boldsymbol{x}}(t)]+\boldsymbol{M}(0)\dot{\boldsymbol{x}}(t)+\dot{\boldsymbol{M}}(0)\boldsymbol{x}(t)+\boldsymbol{p}(t)+\ddot{\boldsymbol{p}}(t). \quad (33)$$

This is also guaranteed under the stronger condition $\boldsymbol{M}(t) = \ddot{\boldsymbol{M}}(t) = \boldsymbol{0}$ which is fulfilled if $\operatorname{rank}[\boldsymbol{B}(t-\tau) \vdots \boldsymbol{B}_0(t)] = \operatorname{rank}[\boldsymbol{B}_0(t)]$, almost all (τ, t), $0 \leqslant \tau \leqslant t$ for the existing solution $\boldsymbol{C}(t-\tau)$ to $\boldsymbol{M}(t-\tau) = \boldsymbol{0}$ in (30b) for almost all $\tau \in [0,t]$. This specifies a control law type in (30a). The equivalence between (31) and (33) is then guaranteed if $(m_{ij}(0)) = \boldsymbol{0} \Rightarrow b_{ij}(0) + (b_0)_{ij}(-\tau)c_{ij}(0) = 0$, $\forall\ \tau \geqslant 0$ which is guaranteed if $\boldsymbol{B}(0)$ and $\boldsymbol{C}(0)$ are zero which can be accomplished by designer's choice. Also, since $\boldsymbol{p} \in C^2([0,\infty);\mathbf{R})$, direct calculus with the definition of $\boldsymbol{p}(t)$ in (30a) and its two first time derivatives yields:

$$\boldsymbol{p}(t) + \ddot{\boldsymbol{p}}(t) = [\boldsymbol{B}_0(t)\boldsymbol{D}(t) + \ddot{\boldsymbol{B}}_0(t)\boldsymbol{D}(t) + 2\dot{\boldsymbol{B}}_0(t)\dot{\boldsymbol{D}}(t) + \boldsymbol{B}_0(t)\ddot{\boldsymbol{D}}(t)]\boldsymbol{u}_r(t) + 2[\dot{\boldsymbol{B}}_0(t)\boldsymbol{D}(t) + \boldsymbol{B}_0(t)\dot{\boldsymbol{D}}(t)]\dot{\boldsymbol{u}}_r(t) + \boldsymbol{B}_0(t)\boldsymbol{D}(t)\ddot{\boldsymbol{u}}_r(t);\ t \geqslant 0. \quad (34)$$

<u>Particular situation 1</u>: $\boldsymbol{v} \equiv 0 \Rightarrow \boldsymbol{p} + \ddot{\boldsymbol{p}} \equiv \boldsymbol{0}$ in (34). The particular control loop $\boldsymbol{u}(t) = \int_0^t \boldsymbol{C}(t-\tau)\boldsymbol{x}(\tau)d\tau$ from (30a) leads to the closed-loop system

$$\frac{d^3}{dt^3}x(t) + [I - \boldsymbol{M}(0)]\dot{\boldsymbol{x}}(t) - [A + \dot{\boldsymbol{M}}(0)]\boldsymbol{x}(t) - \boldsymbol{A}\ddot{\boldsymbol{x}}(t) = \boldsymbol{0}, \quad (35)$$

provided a solution $\boldsymbol{C}(\cdot)$ exists to (32).

Particular situation 2: Matrix functions of continuous entries α, β and γ of appropriate orders are designed so that

$$\ddot{\boldsymbol{p}}(t) + \boldsymbol{p}(t) = \alpha(t)\ddot{\boldsymbol{x}}(t) + \beta(t)\dot{\boldsymbol{x}}(t) + \gamma(t)\boldsymbol{x}(t). \tag{36}$$

Now, if a $\boldsymbol{u}_r(\cdot)$ (non external) input exists (generarated in a closed-loop fashion from equalizing the right-hand-sides of (34) and (36)) and $\boldsymbol{C}(\cdot)$, then the closed-loop system obtained from (29) – (30) is given by

$$\frac{d^3}{dt^3}\boldsymbol{x}(t) - [A + \alpha I]\ddot{\boldsymbol{x}}(t) + [I - \boldsymbol{M}(0) - \beta I]\dot{\boldsymbol{x}}(t) \\ - [A + \dot{\boldsymbol{M}}(0) + \gamma I]\boldsymbol{x}(t) = \mathbf{0}. \tag{37}$$

If α, β and γ are real constants, asymptotic stability of (37) and, thus, that of (29) – (30) follow from applying the Routh-Hurwitz criterion. For situation 1, the stability conditions hold by choosing α, β and γ as zero. However, in Situation 2, the system can be stabilized by appropriate choice of such coefficient matrices.

9. Conclusions. This part of the paper has dealt with in a unified way the stability problem of a wide class (namely, point, distributed and infinite Volterra-type) of time-varying delay systems. The study of a general stabilizing controller and some particular versions for plants involving combined delays has been given through the use of Lyapunov functions. The universal stabilizing regulator contains as many delay types as the controlled plant since all open-loop delays have an accumulative effect in the closed-loop behaviour which cannot be eliminated but compensated. However, it has been proved that a memoryless linear controller stabilizes a delay system provided it stabilizes a nominal controller, defined by the same system in the absence of delays, provided that the parameters affected by delays are sufficiently small.

11. Appendices.

A.1) Proof of Theorem 1. For simplicity in the subsequent mathematical developments, the following notation is used:

$$\tilde{\boldsymbol{C}}_0 =: \tilde{A}_0;\ \tilde{\boldsymbol{C}}_i =: \tilde{\boldsymbol{F}}_i\ (i = 1, 2, ..., 5); \\ \tilde{\boldsymbol{C}}_6 =: \tilde{\boldsymbol{E}}(t)\tilde{\boldsymbol{K}}(t - h'(t)) =: \tilde{\boldsymbol{E}}(t)\tilde{\boldsymbol{K}}(h'(t)), \tag{A.1}$$

$$r_0(t) =: h(t);\ r_1(t) =: h_p(t);\ r_2(t) =: h_c(t);\ r_3(t) =: h_f(t); \quad (A.2)$$

$$r_4(t) =: h_c'(t);\ r_5(t) =: h_f'(t);\ r_6(t) =: h'(t);$$

$$h_i(t) =: t - r_i(t);\ (i = 0, 1, ..., 6), \quad (A.3)$$

with all the delays being, in general, time-functions and $f(t-\alpha)$ being denoted by $f(\alpha)$ (see (A.3)) for each $t \geqslant 0$. The time-dependence is only explicited in the developments below when the argument is different from "t". Using (A.1) – (A.3), Eq. 10 can be written as

$$\dot{\tilde{\boldsymbol{x}}}(t) = [\tilde{\boldsymbol{A}}(t) + \tilde{\boldsymbol{M}}(t)\tilde{\boldsymbol{K}}(t)]\tilde{\boldsymbol{x}}(t) + \sum_{i=0}^{6} \tilde{\boldsymbol{C}}_i(t)\tilde{\boldsymbol{x}}(h_i) + \int_0^t \tilde{\boldsymbol{C}}_7(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau, \quad (A.4)$$

where

$$\tilde{\boldsymbol{C}}_7(t-\tau) =: \tilde{\boldsymbol{B}}(t-\tau) + \tilde{\boldsymbol{E}}'(t-\tau)\tilde{\boldsymbol{K}}(\tau) + \tilde{\boldsymbol{F}}_6(t-\tau). \quad (A.5)$$

The stability proof is devided into two parts. Firstly, simple Lyapunov's stability is proved and then extended to asymptotic stability. Define a Lyapunov's function candidate as follows

$$\begin{aligned} V(\tilde{\boldsymbol{x}}(t), t) =: {} & \tilde{\boldsymbol{x}}^T(t)\tilde{\boldsymbol{D}}\tilde{\boldsymbol{x}}(t) + \sum_{i=0}^{6} \int_{h_i(t)}^t \tilde{\boldsymbol{x}}^T(\tau)\tilde{\boldsymbol{K}}_i(\tau)\tilde{\boldsymbol{x}}(\tau)d\tau \\ & + |\tilde{\boldsymbol{D}}_1\tilde{\boldsymbol{Q}}| \int_0^t \int_t^\infty |\tilde{\boldsymbol{G}}(\tau-\tau')d\tau| \tilde{\boldsymbol{x}}(\tau')d\tau' + \Big[\tilde{\boldsymbol{x}}^T(t) \\ & - \int_0^t \tilde{\boldsymbol{G}}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau\Big]^T \tilde{\boldsymbol{D}}_1 \Big[\tilde{\boldsymbol{x}}(t) - \int_0^t \tilde{\boldsymbol{G}}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau\Big], \end{aligned} \quad (A.6)$$

where $\tilde{\boldsymbol{D}} = \tilde{\boldsymbol{D}}^T$ and $\tilde{\boldsymbol{D}}_1 = \tilde{\boldsymbol{D}}_1^T$ are $(n+l)\times(n+l)$ positive definite matrices, $\tilde{\boldsymbol{G}}(t,\tau) = \tilde{\boldsymbol{G}}(t-\tau) \in C^1((0,\infty); \mathbf{R}^{n\times n})$ and $\tilde{\boldsymbol{Q}} = \tilde{\boldsymbol{Q}}^T$ is constant arbitrary and compatible for right-multiplication with $\tilde{\boldsymbol{D}}_1$. Taking time-derivatives in (A.6), one gets:

$$\begin{aligned} \dot{V}(\tilde{\boldsymbol{x}}) = {} & \dot{\tilde{\boldsymbol{x}}}^T\tilde{\boldsymbol{D}}\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{x}}^T\tilde{\boldsymbol{D}}\dot{\tilde{\boldsymbol{x}}} + \sum_{i=0}^{6} \tilde{\boldsymbol{x}}^T\tilde{\boldsymbol{K}}_i\tilde{\boldsymbol{x}} + (1-\dot{r}_i)\tilde{\boldsymbol{x}}^T(h_i)\tilde{\boldsymbol{K}}_i(h_i)\tilde{\boldsymbol{x}}(h_i) \\ & + \Big[\dot{\tilde{\boldsymbol{x}}} - \tilde{\boldsymbol{G}}(0)\tilde{\boldsymbol{x}} - \int_0^t \dot{\tilde{\boldsymbol{G}}}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau\Big]^T \tilde{\boldsymbol{D}}_1 \Big[\tilde{\boldsymbol{x}} - \int_0^t \tilde{\boldsymbol{G}}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau\Big] \\ & + \Big[\tilde{\boldsymbol{x}} - \int_0^t \tilde{\boldsymbol{G}}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau\Big]^T \tilde{\boldsymbol{D}}_1 \Big[\dot{\tilde{\boldsymbol{x}}} - \tilde{\boldsymbol{G}}(0)\tilde{\boldsymbol{x}} - \int_0^t \dot{\tilde{\boldsymbol{G}}}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau\Big] \\ & + |\tilde{\boldsymbol{D}}_1\tilde{\boldsymbol{Q}}| \int_t^\infty |\tilde{\boldsymbol{G}}(\tau-t)d\tau|\, |\tilde{\boldsymbol{x}}|^2 - |\tilde{\boldsymbol{D}}_1\tilde{\boldsymbol{Q}}| \int_0^t |\tilde{\boldsymbol{G}}(t-\tau)|\, |\tilde{\boldsymbol{x}}(\tau)|^2 d\tau. \end{aligned} \quad (A.7)$$

Denote

$$f(t) = : \tilde{\boldsymbol{x}}^T[\sum_{i=0}^{6} \tilde{\boldsymbol{K}}_i - \tilde{\boldsymbol{G}}^T(0)\tilde{\boldsymbol{D}}_1 - \tilde{\boldsymbol{D}}_1\tilde{\boldsymbol{G}}(0)]\tilde{\boldsymbol{x}} + \sum_{i=0}^{6} \tilde{\boldsymbol{x}}^T(h_i)[(1-\dot{r}_i)$$
$$\times \tilde{\boldsymbol{K}}_i(h_i)]\tilde{\boldsymbol{x}}(h_i) + 2\tilde{\boldsymbol{x}}^T\tilde{\boldsymbol{D}}_1\tilde{\boldsymbol{G}}^T(0) + 2\int_0^t \tilde{\boldsymbol{x}}^T(\tau)\dot{\tilde{\boldsymbol{G}}}^T(t-\tau)d\tau\tilde{\boldsymbol{D}}_1$$
$$\times \int_0^t \tilde{\boldsymbol{G}}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau - 2\int_0^t \tilde{\boldsymbol{x}}^T(\tau)\dot{\tilde{\boldsymbol{G}}}^T(t-\tau)d\tau\tilde{\boldsymbol{D}}_1\tilde{\boldsymbol{x}}$$
$$+ \int_0^t \tilde{\boldsymbol{G}}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau + |\tilde{\boldsymbol{D}}_1\tilde{\boldsymbol{Q}}|[\int_0^\infty |\tilde{G}(\tau-t)d\tau|\tilde{\boldsymbol{x}}|^2$$
$$- \int_0^t |\tilde{G}(t-\tau)|\,|\tilde{\boldsymbol{x}}(\tau)|^2 d\tau]. \qquad (A.9)$$

Since $\tilde{\boldsymbol{G}}(t,t) = \tilde{\boldsymbol{G}}(0)$ and $\dot{\tilde{\boldsymbol{G}}}(t) = (\boldsymbol{I} + \tilde{\boldsymbol{D}}_1^{-1}\tilde{\boldsymbol{D}})\tilde{\boldsymbol{C}}_7(t)$ for all $t \geqslant 0$ from Assumption 1, one gets from the substitution of (A.1) – (A.4) and (A.9) into (A.7) after some rutinary cumbersome calculations involving grouping terms of zero total contribution

$$\dot{V}(\tilde{\boldsymbol{x}}) =: \dot{V}_1(\tilde{\boldsymbol{x}}) + \dot{V}_2(\tilde{\boldsymbol{x}}), \qquad (A.12)$$

where

$$\dot{V}_1(\tilde{\boldsymbol{x}}) = \tilde{\boldsymbol{x}}^T\Big\{[\tilde{A}^T + \tilde{\boldsymbol{K}}^T\tilde{\boldsymbol{M}}^T]\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{D}}[\tilde{A} + \tilde{\boldsymbol{M}}\tilde{\boldsymbol{K}}] + [\tilde{A}^T + \tilde{\boldsymbol{K}}^T\tilde{\boldsymbol{M}}]\tilde{\boldsymbol{D}}_1$$
$$+ \tilde{\boldsymbol{D}}_1[\tilde{A} + \tilde{\boldsymbol{M}}\tilde{\boldsymbol{K}}] + \sum_{i=0}^{6}\tilde{\boldsymbol{K}}_i - \tilde{\boldsymbol{G}}^T(0)\tilde{\boldsymbol{D}}_1 - \tilde{\boldsymbol{D}}_1\tilde{\boldsymbol{G}}(0)$$
$$+ \int_0^\infty |\tilde{\boldsymbol{D}}_1\boldsymbol{Q}|\,|\tilde{G}(\tau-t)|d\tau\}\tilde{\boldsymbol{x}} + \sum_{i=0}^{6}\tilde{\boldsymbol{x}}^T(h_i)[(\dot{r}_i - 1)\tilde{\boldsymbol{K}}_i(h_i)]\tilde{\boldsymbol{x}}(h_i)$$
$$+ 2\sum_{i=0}^{6}\tilde{\boldsymbol{x}}^T(h_i)[\tilde{\boldsymbol{C}}_i^T(\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{D}}_1)]\tilde{\boldsymbol{x}}, \qquad (A.13a)$$

$$\dot{V}_2(\tilde{\boldsymbol{x}}) = -2\sum_{i=0}^{6}\tilde{\boldsymbol{x}}^T(h_i)\tilde{\boldsymbol{C}}_i^T\tilde{\boldsymbol{D}}_1\int_0^t \tilde{G}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau - |\tilde{\boldsymbol{D}}_1\tilde{\boldsymbol{Q}}|$$
$$\times \left[\int_0^\infty |\tilde{G}(\tau-t)d\tau|\tilde{\boldsymbol{x}}|^2 + \int_0^t |\tilde{G}(t-\tau)|\,|\tilde{\boldsymbol{x}}(\tau)|^2 d\tau\right], \qquad (A.13b)$$

for all pair $(\tilde{\boldsymbol{x}}(\cdot), t)$, $t \geqslant 0$. Note that $\dot{V}_2(\tilde{\boldsymbol{x}})$ is always nonpositive provided that

$$\lambda_{\min}(|\boldsymbol{Q}|) \geqslant 2\lambda_{\min}^{-1}(\tilde{\boldsymbol{D}}_1) \times \sup_{t \geqslant t_0} \frac{|\sum_{i=0}^{6} \tilde{\boldsymbol{x}}^T(h_i)\tilde{\boldsymbol{C}}_i^T \tilde{\boldsymbol{D}}_1 \int_0^t \tilde{\boldsymbol{G}}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau|}{|\int_0^{\infty} |\tilde{G}(\tau - t)d\tau|\tilde{\boldsymbol{x}}|^2 + \int_0^t |\tilde{G}(t-\tau)|\,|\tilde{\boldsymbol{x}}(\tau)|^2 d\tau|}, \quad (A.14)$$

and $|\tilde{\boldsymbol{x}}(t)|$ is uniformly bounded for $t \geqslant t_0$, some finite t_0 and all the entries to $\tilde{\boldsymbol{G}}(\cdot)$ are in $L^1([0,\infty); \mathbf{R})$. Now, note that for any arbitrary and sufficiently large T, (A.13b) is uniformly bounded on $[0, T]$ so that $\boldsymbol{Q}$ can be chosen constant while verifying (A.13) – (A.14). Thus, if (A.13a) is proved to be negative for all pair $(\tilde{\boldsymbol{x}}, t)$ with $\tilde{\boldsymbol{x}} \neq \mathbf{0}$ and $t \in [0, T]$, it follows from the definition of the asymptotic stability that there exists $\mu = \mu(T) > 0$ such that $|\tilde{\boldsymbol{x}}(t)| \leqslant \mu$, all $t \geqslant T$ (for every prefixed μ, there always exists a T verifying this property) with μ approaching zero as T increases. Since $\tilde{\boldsymbol{x}}(\cdot)$ cannot be zero on $[0, t_0]$ (unless the equilibrium has been reached), $\tilde{\boldsymbol{x}}(\cdot)$ is uniformly bounded on $[0, \infty)$ (provided $\dot{V}(t) \leqslant 0$, $t \geqslant 0$, $V(\cdot)$ bounded on $[0, \infty)$) and $\int_0^{\infty} \boldsymbol{G}(\tau)d\tau$ is bounded, it follows that the right-hand-side of (A.14) is upper-bounded by a finite constant whose value is irrelevant since $\boldsymbol{Q}$ is not used in the controller design. Then, the only required property in the proof is that there exist $\tilde{\boldsymbol{D}}$ and $\tilde{\boldsymbol{D}}_1$ which make (A.13a) to be negative definite. This implies, from Theorem 1 in De la Sen and Luo (1997), (global) uniform stability since $\dot{V}_1(\tilde{\boldsymbol{x}}) + \dot{V}_2(\tilde{\boldsymbol{x}}) < 0 \Rightarrow \dot{V}(\tilde{\boldsymbol{x}}) < 0$. Thus, it suffices to guarantee that $\dot{V}_1(\tilde{\boldsymbol{x}}) < 0$. According to (A.1), we choose $\tilde{\boldsymbol{K}}_i(t)$ as follows

$$\tilde{\boldsymbol{K}}_i(t) =: [\tilde{\boldsymbol{C}}_i(h_i^{-1}(t))]^T[\tilde{\boldsymbol{C}}_i(h_i^{-1}(t))] \geqslant 0; \; (i = 0, 1, ..., 6), \quad (A.15)$$

so that $\tilde{\boldsymbol{K}}_i(h_i) = \tilde{\boldsymbol{K}}_i(t - r_i(t)) = \tilde{\boldsymbol{C}}_i^T(t)\tilde{\boldsymbol{C}}_i(t)$ $(i = 0, 1, ..., 6)$. Since, by hypothesis, $\dot{r}_i(t) \leqslant \gamma_i < 1$ $(i = 0, 1, ..., 6)$, it follows from substitution of (A.15) into (A.13a) that

$$\begin{aligned}\dot{V}_1(\tilde{\boldsymbol{x}}) \leqslant & -\tilde{\boldsymbol{x}}^T[\tilde{\boldsymbol{\Pi}} - \sum_{i=0}^{6} \tilde{\boldsymbol{C}}_i^T(h_i^{-1})\tilde{\boldsymbol{C}}(h_i^{-1})]\tilde{\boldsymbol{x}} + \sum_{i=0}^{6} \tilde{\boldsymbol{x}}^T(h_i) \\ & \times (\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{D}}_1)\tilde{\boldsymbol{C}}_i\tilde{\boldsymbol{x}}(h_i) + 2\sum_{i=0}^{6} \tilde{\boldsymbol{x}}^T(h_i)\tilde{\boldsymbol{C}}_i^T(\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{D}}_1)\tilde{\boldsymbol{x}} \\ & - \sum_{i=0}^{6}(1 - \gamma_i)\tilde{\boldsymbol{x}}^T(h_i)\tilde{\boldsymbol{C}}_i^T\tilde{\boldsymbol{C}}_i\tilde{\boldsymbol{x}}(h_i), \end{aligned} \quad (A.16)$$

where

$$\tilde{\mathbf{\Pi}} =: -\Big\{[\tilde{\boldsymbol{A}}^T + \tilde{\boldsymbol{K}}^T\tilde{\boldsymbol{M}}^T][\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{D}}_1 - \tilde{\boldsymbol{G}}(0)] + [\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{D}}_1 - \tilde{\boldsymbol{G}}(0)] \times [\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{M}}\tilde{\boldsymbol{K}}] + \int_0^\infty |\tilde{\boldsymbol{D}}_1\tilde{\boldsymbol{Q}}|\,|\tilde{\boldsymbol{G}}(\tau - t)|d\tau\Big\}. \qquad (A.17)$$

Since $\tilde{\boldsymbol{G}}(0) = [\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{M}}\tilde{\boldsymbol{D}}_1\tilde{\boldsymbol{K}}]\tilde{\boldsymbol{D}}_1^{-1}$, consider the following Lyapunov's equation in the unknown $\tilde{\boldsymbol{D}}_0 = \tilde{\boldsymbol{D}}_0^T$.

<u>"A priori" Lyapunov's equation.</u>

$$[\tilde{\boldsymbol{A}}^T + \tilde{\boldsymbol{K}}^T\tilde{\boldsymbol{M}}^T]\tilde{\boldsymbol{D}}_0 + \tilde{\boldsymbol{D}}_0[\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{M}}\tilde{\boldsymbol{K}}] = -q_0'\boldsymbol{I} + \sum_{i=0}^{6}\tilde{\boldsymbol{C}}_i^T\tilde{\boldsymbol{C}}_i + \Big\{[\tilde{\boldsymbol{D}}_{10}^{-1}(\tilde{\boldsymbol{A}}^T + \tilde{\boldsymbol{K}}^T\tilde{\boldsymbol{M}}^T)\tilde{\boldsymbol{D}}_{10} - \tilde{\boldsymbol{A}}^T - \tilde{\boldsymbol{K}}^T\tilde{\boldsymbol{M}}^T]\tilde{\boldsymbol{D}}_{10} + \tilde{\boldsymbol{D}}_{10}[\tilde{\boldsymbol{D}}_{10}(\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{M}}\tilde{\boldsymbol{K}})\tilde{\boldsymbol{D}}_{10}^{-1} - \tilde{\boldsymbol{A}} - \tilde{\boldsymbol{M}}\tilde{\boldsymbol{K}}]\Big\}, \qquad (A.18)$$

any $\tilde{\boldsymbol{D}}_{10} = \tilde{\boldsymbol{D}}_{10}^T > 0$. Since the entries of $|\tilde{\boldsymbol{G}}(\cdot)|$ are in $L^1([0,\infty);\mathbb{R})$, a positive real q exists such that $q_0' =: q - \int_0^\infty |\tilde{\boldsymbol{D}}_{10}\tilde{\boldsymbol{Q}}_0|\,|\tilde{\boldsymbol{G}}(\tau - t)|d\tau$ is positive in (A.18). Note that, if $\tilde{\boldsymbol{D}}_0 = \tilde{\boldsymbol{D}}_0^T$ is a solution for (A.18), then $\tilde{\boldsymbol{D}} = \lambda\tilde{\boldsymbol{D}}_0$ is also a solution for (A.18) with $q' = \lambda q_0'$, $\tilde{\boldsymbol{D}}_1 = \lambda\tilde{\boldsymbol{D}}_{10}$ and $\tilde{\boldsymbol{Q}} = \lambda\tilde{\boldsymbol{Q}}_0$, replacing to q_0', $\tilde{\boldsymbol{D}}_{10}$ and $\tilde{\boldsymbol{Q}}_0$, respectively, for any $\lambda \in \mathbb{R}$; i.e., λ is a "normalization factor" for (A.18). For such a solution, the substitution of (A.18) into (A.16) yields

$$\begin{aligned}\dot{V}_1(\tilde{\boldsymbol{x}}) &\leqslant -[q - \sum_{i=0}^{6}\alpha_i\lambda^2|\tilde{\boldsymbol{D}}_0 + \tilde{\boldsymbol{D}}_{10}|^2 - \lambda^2\int_0^\infty |\tilde{\boldsymbol{D}}_{10}\tilde{\boldsymbol{Q}}_0|\,|\tilde{\boldsymbol{G}}(\tau - t)|d\tau]|\tilde{\boldsymbol{x}}|^2 \\ &- \sum_{i=0}^{6}[\alpha_i^{1/2}(\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{D}}_1)\tilde{\boldsymbol{x}} - \beta_i^{1/2}\tilde{\boldsymbol{C}}_i\tilde{\boldsymbol{x}}(h_i)]^T[\alpha_i^{1/2}(\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{D}}_1)\tilde{\boldsymbol{x}} - \beta_i^{1/2}\tilde{\boldsymbol{C}}_i\tilde{\boldsymbol{x}}(h_i)] \\ &- \leqslant [q - \lambda^2\sum_{i=0}^{6}(\gamma_i - 1)^{-1}|\tilde{\boldsymbol{D}} + \tilde{\boldsymbol{D}}_{10}|^2 - \lambda^2\int_0^\infty |\tilde{\boldsymbol{D}}_{10}\tilde{\boldsymbol{Q}}_0| \\ &\times |\tilde{\boldsymbol{G}}(\tau - t)|d\tau]|\tilde{\boldsymbol{x}}|^2 \leqslant -\mu_0|\tilde{\boldsymbol{x}}|^2;\ \mu_0 > 0, \end{aligned} \qquad (A.19)$$

since the inequalities hold if $\beta_i = \gamma_i - 1$, $\alpha_i = \beta_i^{-1} = (\gamma_i - 1)^{-1}$ ($\gamma_i > 1$; $i = 0, 1, ..., 6$). Then, $\dot{V}_1(\tilde{\boldsymbol{x}})$, and thus, $\dot{V}(\tilde{\boldsymbol{x}})$ is negative definite if (A.18) has a solution $\tilde{D}_0 = \tilde{D}_0^T > 0$ for $q_0' = (\lambda_0' - 1)\int_0^\infty |\tilde{D}_{10}\tilde{Q}_0|\,|\tilde{G}(\tau - t)|d\tau$, any real constant $\lambda_0' > 1$ such that

$$\begin{aligned} q &= q_0' + \int_0^\infty |\tilde{D}_{10}\tilde{Q}_0|\,|\tilde{G}(\tau - t)|d\tau = \lambda_0' \int_0^\infty |\tilde{D}_{10}\tilde{Q}_0|\,|\tilde{G}(\tau - t)|d\tau \\ &> \lambda^2 Max(\{\sum_{i=0}^{6}(\gamma_i - 1)^{-1}|\tilde{D}_0 + \tilde{D}_{10}|^2 + \int_0^\infty |\tilde{D}_{10}\tilde{Q}_0|\,|\tilde{G}(\tau - t)|d\tau\}, \\ &\int_0^\infty |\tilde{D}_{10}\tilde{Q}_0|\,|\tilde{G}(\tau - t)|d\tau) = \lambda^2 \sum_{i=0}^{6}(\gamma_i - 1)|\tilde{D}_0 + \tilde{D}_{10}|^2 \\ &+ \int_0^\infty |\tilde{D}_{10}\tilde{Q}_0|\,|\tilde{G}(\tau - t)|d\tau, \end{aligned} \tag{A.20}$$

which always holds, since $\lambda_0 > 1$, for

$$\begin{aligned} \lambda <& (\lambda_0')^{\frac{1}{2}}\Big[\sum_{i=0}^{6}(\gamma_i - 1)^{-1}|\tilde{D}_0 + \tilde{D}_{10}|^2 + \int_0^\infty |\tilde{D}_{10}\tilde{Q}_0|\,|\tilde{G}(\tau - t)d\tau\Big]^{-\frac{1}{2}} \\ &\times \Big[\Big|\int_0^\infty |\tilde{D}_{10}\tilde{Q}_0|\,|\tilde{G}(\tau - t)d\tau|\Big]^{\frac{1}{2}}. \end{aligned} \tag{A.21}$$

<u>"A posteriori" Lyapunov equation.</u> (A.18) with the changes $(\tilde{D}_0, \tilde{D}_{10}, q_0') \to (\lambda\tilde{D}_0, \lambda\tilde{D}_{10}, \lambda q_0')$.

<u>Remark in the proof</u>. Note that the λ-normalization in $\tilde{D}$, $\tilde{D}_1$ and $\tilde{Q}$ is necessary since, from (A.18) – (A.19), q must satisfy (A.20) for $\lambda = 1$. That means that $\dot{V}_1(\tilde{x}) < 0$, corresponding to the solution $\tilde{D}_0$, $\tilde{D}_{10}$ of (A.18), is guaranteed under values of q_0' and q' which are *a priori* dependent on such a solution. Then, the pair $(\tilde{D}, \tilde{D}_1)$ is calculated as follows:

Step 1. For each given $\tilde{K}$, fix $\tilde{D}_{10} = \tilde{D}_{10}^T > 0$, and $q_0' \in \mathbb{R}^+$ as $(\lambda_0' - 1)\int_0^\infty |\tilde{D}_{10}\tilde{Q}_0|\,|\tilde{G}(\tau - t)|d\tau$, $\lambda_0' > 1$, and then solve (A.18) in $\tilde{D}_0$. If $\tilde{D}_0$ is nonpositive definite, then the theorem fails and the asymptotic stability is not guaranteed.

Step 2. If $\tilde{D}_0 = \tilde{D}_0^T$, check if the right-hand-side of (A.21) is not less than unity. Otherwise, modify $\tilde{D}_{10} \to \tilde{D}_1 = \lambda\tilde{D}_{10}$; $\tilde{D}_0 \to \tilde{D} = \lambda\tilde{D}_0$ for some λ fulfilling (A.21) and so that (A.18) is then guaranteed.

Alternative strategy for calculation of the pair $(\tilde{D}, \tilde{D}_1)$. The two above steps can be compated into one by noting that if the theorem does not fail then $|\tilde{D}_0 + \tilde{D}_{10}| \geqslant |\tilde{D}_0|^2$ for any $\tilde{D}_0 > 0$. Thus, if suffices to choose

$$\lambda_0 \geqslant Max\Big\{1, \Big(\int_0^{\infty} |\tilde{D}_{10}\tilde{Q}_0|\,|\tilde{G}(\tau-t)|d\tau\Big)^{-\frac{1}{2}} \Big(\sum_{i=0}^{6}(\gamma_i-1)^{-1}|\tilde{D}_{10}|^2 + \int_0^{\infty} |\tilde{D}_{10}\tilde{Q}_0|\,|\tilde{G}(\tau-t)|d\tau\Big)^{\frac{1}{2}}\Big\}, \tag{A.22}$$

so that $q_0' = (\lambda_0' - 1)\int_0^{\infty} |\tilde{D}_{10}\tilde{Q}_0|\,|\tilde{G}(\tau-t)|d\tau$ allows to find $\tilde{D}_0$ in (A.18) in one step. If $\tilde{D}_0 = \tilde{D}_0^T > 0$, then $\tilde{K}$ is a stabilizing extended controller gain.

It has been proved from (A.6) and (A.10) that since $V(0, \varphi(\cdot))$ is bounded on $[-d, 0]$; $d = max(r_i; i = 1, 2, ..., 6)$, $V(t, \boldsymbol{x}(\cdot)) \in B([0,\infty); \mathbf{R})$ and, furthermore, provided that $\tilde{D}_0$ is positive definite, $|\tilde{\boldsymbol{x}}| \in B([0,\infty); \mathbf{R}) \cap L^2([0,\infty); \mathbf{R})$ so that Lyapunov's simple stability follows. Now, it is proved that if the finiteness constraint on $\int_0^{\infty} |\tilde{G}(\tau)d\tau|$ is strengthened to be upper-bounded by unity, then $\tilde{D}_0$ is always positive definite. Firstly, suppose that $\tilde{D}$ is positive definite. Since $V(t, \tilde{\boldsymbol{x}}(\cdot)) \in B([t_0,\infty); \mathbf{R})$, all $t_0 > 0$, irrespective to the positive definiteness of $\tilde{D}$, it is obvious from (A.6) that

$$\begin{aligned} K^2\Big[|\tilde{\boldsymbol{x}}| - |\int_0^t \tilde{G}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau|\Big] + \tilde{\boldsymbol{x}}^T\tilde{D}\tilde{\boldsymbol{x}} \leqslant V(t, \tilde{\boldsymbol{x}}(\cdot)) \\ \leqslant \Big[\tilde{\boldsymbol{x}} - \int_0^t \tilde{G}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau\Big]^T \tilde{D}_1\Big[\tilde{\boldsymbol{x}} - \int_0^t \tilde{G}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau\Big] + \tilde{\boldsymbol{x}}^T\tilde{D}\tilde{\boldsymbol{x}} \\ \leqslant V(t_0, \tilde{\varphi}(\cdot)) \leqslant |\tilde{D}|\,|\tilde{\boldsymbol{x}}|^2 + \sum_{i=0}^{6}\int_{h_i(t)}^{t} \tilde{\boldsymbol{x}}^T(\tau)\tilde{K}_i(\tau)\tilde{\boldsymbol{x}}_i(\tau)d\tau \\ + |\tilde{D}_1|\,|\tilde{\varphi}(t_0)| + [\int_0^{t_0} |\tilde{G}(t_0-\tau)|\,|\tilde{\varphi}(\tau)|d\tau]^2 \\ + |\tilde{D}_1\tilde{Q}|\int_0^{t_0}\int_{t_0}^{\infty} |\tilde{G}(\tau-\tau')|d\tau|\tilde{\varphi}(\tau')|d\tau z \leqslant \delta^2 N^2; \end{aligned} \tag{A.23}$$

for some $K \neq 0$ and $N > 0$ if $\delta > 0$ exists such that for $|\boldsymbol{\phi}(t)| < \delta$ on $[0, t_0]$ and any given $\varepsilon > 0$ and $t_0 \geqslant 0$, then $|\tilde{\boldsymbol{x}}(t, t_0, \tilde{\varphi})| < \varepsilon$ for $t \geqslant t_0$. The existence of such a δ is proved below. Comparing the first and the last term of (A.23), it follows, since $(a^2+b^2)^{\frac{1}{2}} \geqslant (\sqrt{2})^{-1}(a+b)$, that

$$\begin{aligned} |\tilde{\boldsymbol{x}}(t)| \leqslant (\delta N/K') + \varepsilon\int_0^{\infty} |\tilde{G}(\tau)|\,|\tilde{\boldsymbol{x}}(\tau)|d\tau; \\ K' = (\sqrt{2})^{-1}[\lambda_{min}(\tilde{D}) + K]. \end{aligned} \tag{A.24}$$

So long as $|\tilde{\boldsymbol{x}}(t)| < \varepsilon$, we have $|\tilde{\boldsymbol{x}}(t)| < (\delta N/K') + \varepsilon \int_0^\infty |\tilde{G}(\tau)|\,|\tilde{\boldsymbol{x}}(\tau)|d\tau < \varepsilon$ for all $t \geqslant t_0$, provided $\delta < (K'/N)[1 - \int_0^\infty |\tilde{G}(\tau)|d\tau]\varepsilon$. From Assumption 1, the right-hand-side of the above inequality is positive. Hence, $\tilde{\boldsymbol{x}} = \mathbf{0}$ is stable. Now, suppose that $\tilde{\boldsymbol{x}} = \mathbf{0}$ is stable but $\tilde{\boldsymbol{D}}$ is not positive definite that will then lead to a contradiction. Then, it can be shown that there is an $|\tilde{\boldsymbol{x}}_0| < \delta$ implies $|\tilde{\boldsymbol{x}}(t, 0, \tilde{\boldsymbol{x}}_0)| < 1$ for all $t \geqslant 0$. Letting $\tilde{\boldsymbol{x}}(t) = \tilde{\boldsymbol{x}}(t, 0, \tilde{\boldsymbol{x}}_0)$, one gets from (A.19) $V(t, \tilde{\boldsymbol{x}}(\cdot)) \leqslant V(0, \tilde{\boldsymbol{x}}_0) - \mu \int_0^t |\tilde{\boldsymbol{x}}(\tau)|^2 d\tau = -\eta - \mu \int_0^t |\tilde{\boldsymbol{x}}(\tau)|^2 d\tau$ where $\eta =: -\tilde{\boldsymbol{x}}_0^T \tilde{\boldsymbol{D}} \tilde{\boldsymbol{x}}_0 > 0$. Thus, from (A.23),

$$\tilde{\boldsymbol{x}}^T \tilde{\boldsymbol{D}} \tilde{\boldsymbol{x}} + \Big[\tilde{\boldsymbol{x}} - \int_0^t \tilde{G}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau\Big]^T \tilde{\boldsymbol{D}}_1 \Big[\tilde{\boldsymbol{x}} - \int_0^t \tilde{G}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau\Big] \leqslant -\eta - \mu \int_0^t |\tilde{\boldsymbol{x}}(\tau)|^2 d\tau \leqslant 0. \qquad (A.25)$$

Using the Schwartz inequality, one concludes that

$$\Big[\int_0^t |\tilde{G}(t-\tau)|\,|\tilde{\boldsymbol{x}}(\tau)|d\tau\Big]^2 \leqslant \int_0^t |\tilde{G}(t-\tau)|d\tau \times \int_0^t |\tilde{G}(t-\tau)|\,|\tilde{\boldsymbol{x}}(\tau)|^2 d\tau. \qquad (A.26)$$

As $|\tilde{\boldsymbol{x}}(t)| < 1$ and $\int_0^t |\tilde{G}(t-\tau)|d\tau$ is in $L^\infty([0,t];\mathbb{R})$, all $t \geqslant 0$, then $\int_0^t \tilde{G}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau$ is uniformly bounded. Thus, (A.25) – (A.26) imply

$$\eta + \mu \int_0^t |\tilde{\boldsymbol{x}}(\tau)|^2 d\tau \leqslant \tilde{\boldsymbol{x}}^T \tilde{\boldsymbol{G}} \tilde{\boldsymbol{x}} + |\tilde{\boldsymbol{D}}_1|\,[|\tilde{\boldsymbol{x}}(t)| + |\int_0^t \tilde{G}(t-\tau)\tilde{\boldsymbol{x}}(\tau)d\tau|] \leqslant K_1, \qquad (A.27)$$

for some constant K_1. Thus, $|\tilde{\boldsymbol{x}}(t)|^2$ is in $L^1([0,\infty);\mathbb{R})$. Now, $\tilde{G}(t) \to 0$ as $t \to \infty$ and $|\tilde{\boldsymbol{x}}(t)|^2$ in $L^1(\cdot,\cdot)$ imply that $\int_0^t |\tilde{G}(t-\tau)|\,|\tilde{\boldsymbol{x}}(\tau)|^2 d\tau \to 0$ as $t \to \infty$. Thus, by the Schwartz inequality argument, $\int_0^t \tilde{G}(t-\tau)\tilde{x}(b)d\tau \to 0$ as $t \to \infty$. From (A.25), it is seen that for large t, $\tilde{\boldsymbol{x}}^T(t)\tilde{\boldsymbol{D}}\tilde{\boldsymbol{x}}(t) \leqslant -\eta/2$. Moreover, as $\tilde{\boldsymbol{x}} \to 0$. Hence, $|\tilde{\boldsymbol{x}}|^2 \geqslant \gamma$ for some $\gamma > 0$ and all t sufficiently large. Thus, $\int_0^t |\tilde{\boldsymbol{x}}(\tau)|^2 d\tau \to \infty$ as $t \to \infty$, contradicting $|\tilde{\boldsymbol{x}}(t)|^2$ being in $L^1([0,\infty);\mathbb{R})$. Therefore, the assumption that $\tilde{\boldsymbol{D}}$ is not positive definite is false and the proof of simple stability is complete. Since $V(\tilde{\boldsymbol{x}}, t) \to 0$ as $t \to \infty$, it follows that $\lambda_{min}(\tilde{\boldsymbol{D}})|\tilde{\boldsymbol{x}}|^2 \leqslant \tilde{\boldsymbol{x}}^T(t)\tilde{\boldsymbol{D}}\tilde{\boldsymbol{x}}(t) \to 0$ as $t \to \infty$ from (A.6), since all the additive terms are nonnegative so that the extended $SPVD$ is asymptotically stable since $\lambda_{min}(\tilde{D}) > 0$. This completes the sufficiency part of the theorem. The "only if" statement follows from the fact that the system given by matrices $(\tilde{A} + \tilde{M}\tilde{K})$ and $\tilde{C}(\cdot)$ is time-invariant from Assumption 2.

Outline of proof of Corollary 1.2 (ii) – (iii). Note that $\dot{V}(t, \tilde{x}(\cdot)) \leqslant -\mu \times |\tilde{x}(t)|^2$. If $|B_c(\cdot)|^2$ is in $L^1([0, \infty); \mathbb{R})$ then, $|\dot{\tilde{x}}(t)| \leqslant |\tilde{A}_c|\,|\tilde{x}(t)| + \int_0^t |\tilde{B}_c(t - \tau)||\tilde{x}(\tau)|d\tau \leqslant |\tilde{A}_c|\,|\tilde{x}(t)| + \frac{1}{2}\int_0^t |\tilde{B}_c(t-\tau)|^2 d\tau + \frac{1}{2}\int_0^t |\tilde{x}(\tau)|^2 d\tau$. Thus, $|\dot{\tilde{x}}(t)|$ and, then, $|\tilde{x}(t)|^2$ are in $L^\infty([0, \infty); \mathbb{R})$; for $\frac{d}{dt}(|\tilde{x}(t)|^2) = \frac{d}{dt}[\tilde{x}^T(t)\tilde{x}(t)] \leqslant 2|\tilde{x}(t)|\,|\dot{\tilde{x}}(t)|$. Hence, $|\tilde{x}(t)| \to 0$ as $t \to \infty$ and $\tilde{x} = \mathbf{0}$ is asymptotically stable (for details and the proof of (iii), see Burton, (1985) and Theorem 1 in De la Sen and Luo, 1997).

Outline of proof of Corollary 1.4 In the absence of point delays,

$$V(t, \tilde{x}(\cdot)) = \Big[\tilde{x} - \int_0^t \tilde{G}(t-\tau)\tilde{x}(\tau)d\tau\Big]^T \tilde{D}_1 \Big[\tilde{x} - \int_0^t \tilde{G}(t-\tau)\tilde{x}(\tau)d\tau\Big] + |\tilde{D}\tilde{Q}| \int_0^t \int_t^\infty |\tilde{G}(u-\tau)|du|\tilde{x}(\tau)|^2 d\tau, \qquad (A.28)$$

and $\dot{V}(t, \tilde{x}(\cdot))$, under simialr calculations as in the proof of Theorem 1 leads to

$$\begin{aligned}\dot{V}(t, \tilde{x}(\cdot)) \leqslant& -|\tilde{x}|^2 + |\tilde{D}\tilde{Q}| \int_0^t |\tilde{G}(t-\tau)|[|\tilde{x}|^2 + |\tilde{x}(\tau)|^2]d\tau \\ &+ |\tilde{D}\tilde{Q}| \int_0^\infty |\tilde{G}(\tau - t)|d\tau|\tilde{x}|^2 - |\tilde{D}\tilde{Q}| \int_0^t |\tilde{G}(t-\tau)|\,|\tilde{x}(\tau)^2 d\tau \\ \leqslant& [-1 + 2|\tilde{D}\tilde{Q}| \int_0^\infty |\tilde{G}(\tau)|d\tau]|\tilde{x}|^2 = -\mu|\tilde{x}|^2, \qquad (A.29)\end{aligned}$$

and this implies $\dot{V}(t, \tilde{x}(\cdot)) \leqslant 0 \Rightarrow V(t, \tilde{x}(\cdot)$ tends to a constant from (A.28). Because of the form of $V(t, \tilde{x}(\cdot))$, this does not imply $|\tilde{x}(t)| \to 0$ as $t \to \infty$ but only $|\tilde{x}(\cdot)| \in L^1([0, \infty); \mathbb{R})$ (for details, see Burton 1985).

A.2) Expressions for the matrices of the extended system (10). The time arguments are only expressed when different from t. From (4a), (7a) and (7h), one gets

$$\tilde{K} =: \begin{bmatrix} -D_cD_fC & D_cC_p & C_c & -D_cC_f \\ \mathbf{0} & A_p & \mathbf{0} & \mathbf{0} \\ -M_cD_fC & M_cC_p & A_c & -M_cC_f \\ M_fC & \mathbf{0} & \mathbf{0} & A_f \end{bmatrix}. \qquad (A.30)$$

Also, from (7) and (9), one gets:

$$\tilde{A}+\tilde{M}\tilde{K} =: \begin{bmatrix} A-MD_cD_fC & MD_cC_p & MC_c & -MD_cC_f \\ 0 & A_p & 0 & 0 \\ -M_cD_fC & M_cC_p & A_c & -M_cC_f \\ M_fC & 0 & 0 & A_f \end{bmatrix} \quad (A.31)$$

$$\tilde{\Delta}' =: \sum_{i=0}^{6} \tilde{C}_i\tilde{C}_i^T =: Block\ Diag\ [\ -A_0A_0^T + W_p \vdots A_{0p}A_{0p}^T \vdots A_{0c}A_{0c}^T$$

$$+ W_c(t,\tau) \vdots A_{0f}A_{0f}^T + W_f\ ] \quad (A.32)$$

$$W_p =: E[D_c(h')D_f(h')C(h')C^T(h')D_f^T(h')D_c^T(h') + D_c(h')C_p(h')C_p^T(h')D_c^T(h') + C_c(h')C_c^T(h') + D_c(h')C_f(h')C_f^T(h')D_c^T(h')]E^T, \quad (A.33)$$

with $W_c(t,\tau) =: E_c(t-\tau)D_fCC^TD_f^TE_c^T(t-\tau)$ for $t \geqslant \tau$ and $W_c(t,\tau) = 0$, $t < \tau$, and

$$\tilde{C}_7(t-\tau) =: \tilde{B}(t-\tau) + \tilde{E}'(t-\tau)\tilde{K}(\tau) + \tilde{F}_6(t-\tau)$$

$$= \begin{bmatrix} B(t-\tau) - E'(t-\tau)D_c(\tau)D_f(\tau)C(b) & E'(t-\tau)D_c(\tau)D_f(\tau) & E'(t-\tau)C_c(\tau) & -E'(t-\tau)D_c(\tau)D_f(\tau) \\ 0 & B_p(t-\tau) & 0 & 0 \\ E_c(t-\tau)D_f(\tau)C(\tau) & E_c'(t-\tau)C_p(\tau) & B_c(t-\tau) & -E_c'(t-\tau)C_f(\tau) \\ E_f'(t-\tau)C(\tau) & 0 & 0 & B_p(t-\tau) \end{bmatrix} \quad (A.34)$$

for any $t \geqslant \tau \geqslant 0$ and $\tilde{C}_7(t-\tau) = 0$ for $t < \tau$.

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Received June 1997

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APIE UNIVERSALIŲ STABILIZUOJANČIŲ TOLYDINIŲ TIESINIŲ REGULIATORIŲ, SKIRTŲ SISTEMOMS SU VĖLINIMU, PROJEKTAVIMĄ. II dalis. Universalūs reguliatoriai ir pagrindiniai stabilumo rezultatai

Manuel De la SEN ir Ningsu LUO

Antroje straipsnio dalyje nagrinėjamas apibendrintų tiesinių reguliatorių, skirtų tiesinėms sistemoms su liekamuoju poveikiu, kuriuos naudojant reguliavimo sistemos uždarame kontre, ši sistema tampa globaliai tolygiai ir asimptotiškai stabili Liapunovo prasme, projektavimo uždavinys. Reguliatoriai yra universalūs ta prasme, kad jie turi vairių tipų vėlinimus, kurie gali būti baigtiniai, neriboti arba net priklausyti nuo laiko. Stabilumo pakankamos sąlygos priklauso nuo sistemos parametrų ir vėlinimų. Parodyta, kad stabilizuojantis reguliatorius gali būti suprojektuotas naudojant gerai žinomas Kronekerio matricų sandaugas su sąlyga, jei stabilizuojantis reguliatorius egzistuoja, kai nėra išorinio (arba ėjimo) vėlinimo.