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ON THE DESIGN OF UNIVERSAL STABILIZING CONTINUOUS LINEAR CONTROLLERS FOR TIME-DELAY SYSTEMS
Part I. Preliminary results

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Abstract. This paper deals with the design problem of generalized linear controllers for linear systems with after-effect so that the resulting closed-loop system is globally uniformly asymptotically stable in the Lyapunov's sense. The controllers are universal in the sense that they include the usual delays (namely, point, distributed and mixed point-distributed delays) which can be finite, infinite or even time-varying. In Part I of the paper, some preliminary concepts and results on stabilizability are given.

Key words: delay systems, point delays, distributed delays, Lyapunov's stability, stabilizing controllers.

1. Introduction. Delay differential systems are modelled by differential and difference functional equations (Burton, 1985; Hale, 1977; Pandolfi, 1975; Pandolfi, 90) and are often called hereditary systems or systems with after effect (De la Sen, 1988). Delay systems are useful to improve the modelling performance in problems of transportation, population growing and so on. Although these systems are described by functional equations, the problem of existence of unique solution can be extended from the standard Picard–Lindëloff and Cauchy–Peano Theorems for ordinary differential equations (De la Sen, 1988; Hale, 1977) under continuity assumptions for the time-varying coefficients in the differential system together with continuity or absolute continuity hypotheses on the function of initial conditions. Lyapunov's stability as well as orbital stability of periodic solutions in processess with delays are analyzed in (Burton, 1985) for open-loop differential systems. The relationships between generalized control systems, boundary control systems and delayed control systems are investigated in (Pandolfi, 1990). It is shown that a system with delay in control belongs to a special class of boundary control systems, in which the generalized control system is produced when it is projected onto its (unstable) eigenspace. The connections and equivalences between the various kinds of controllability and observability concepts (such as, spectral, initial and final observability, spectral controllability, reachability, etc.) are less direct than the delay-free case (Fiagbedzi and Pearson, 1990a,b; Lee and Olbrot, 1981; Watanable, 1986). In particular, questions of observability (measurement) have been of concern with the scientific community for a very long time since an authoritative history of observed data is crucial to any scheme to explain what did happen (predictive or retrospective rules). The classical point of view is that the universe can be divided into the observed objects and observers while there are a number of different concepts of observability and observers for the hereditary systems discussed in the literatures (Lee and Olbrot, 1981).

In this paper, a set of results on dynamic systems with commensurate point delays are presented. Most of them concerning with various observability concepts (namely, initial, $\mathbb{R}^n$ final, infinite-time, spectral, essential, etc.) can be generalized to the cases of distributed and commensurate delay systems. It is proved in (Watanable, 1986) that if the system is spectrally controllable, there is a delayed feedback matrix such that the closed-loop system is spectrally controllable through a single input. In (Fiagbedzi and Pearson, 1987; Fiagbedzi and Pearson, 1990a), the so-called left and right characteristic matrix equations are used to develop a stabilization/estimation theory for delay systems. This restriction was removed in (Fiagbedzi and Pearson, 1990) through a generalization of the characteristic matrix equations in a manner that allows the stabilization of an arbitrary but finite number of unstable modes which appear as modes of an associated delay-free system. The resulting feedback controller and estimator permit, under the minimal assumptions on stabilizability and detectability, the stabilization of general linear autonomous delay systems with output feedback.

The stabilizability and stabilization problems have received much attention in a set of papers (Agathoklis and Foda, 1989; Fiagbedzi and Pearson, 1990b; Mori *et al.*, 1983; Olbrot, 1978; Pandolfi, 1975; Tadmor, 1988) and more recently in (Alastruey *et al.*, 1995; De la Sen, 1988, 1993 and 1994; De la

Sen and Luo, 1994; Luo and De la Sen, 1995; Zheng *et al.*, 1994). The existing stability tests are basically of two types: the first one is based on the approach of locating the roots of a single-variable or a multi-variable characteristic polynomial of the delay differential system (Watanable, 1986), while the other treats the delay system as the one over a ring by using $1 - D$ models. Also, $n - D$ ($n \geqslant 2$-dimensional) systems have been used in (Agathoklis and Foda, 1989) to describe systems with commensurate and non-commensurate delays, respectively. Sufficient delay independent conditions for the asymptotic stability were derived in terms of frequency-dependent and constant parameter Lyapunov equations (Burton, 1985; Mori *et al.*, 1983). Related to linear autonomous systems with both point and distributed delays, it has been shown in (Olbrot, 1978; Pandolfi, 1975; Tadmor, 1988) that a simple algebraic rank condition, similar to the well-known Hautus condition (Hautus, 1969), is necessary and sufficient for stabilizability.

The detectability problem was proved to be dual to the state feedback stabilizability of a transposed system. If delays appear in the control variables only, then the state feedback spectral assignability is equivalent to the formal controllability of a certain pair of real matrices and also to the system state controllability. In (Tadmor, 1988), it was proved that systems with input (or external) delays can exhibit the trajectory stabilizability phenomenon, namely, the input can decay at a different rate from that of the state or even can diverge while the state trajectory asymptotically approaches the origin.

The objective of this paper is to study the problem of stabilization of linear systems involving, in general, several types (i.e., point, distributed and the mixed point-distributed) of delays in both state and input by the use of general controllers involving the same types of delays. In Part I, systems within the usual classes (i.e., point, distributed, mixed point-distributed commensurate and non-commensurate) of delays are presented. Some preliminary concepts and results on stabilizability are given. The appendices contain some mathematical developments and some proofs of the results presented in Sections 2–5.

2. Terminology and notation. Denote $\mathbf{R}$, $\mathbf{R}^+$, $\mathbf{R}_0^+$, $\mathbf{C}$, $\mathbf{C}^-$ and $\mathbf{C}^+$, respectively, the sets of real, positive real and non-negative real numbers, complex numbers, and left-half complex plane and right-half complex plane (both including the imaginary axis). $\pounds(\cdot)$ and $\pounds^{-1}(\cdot)$ stand for the Laplace and Laplace inverse transforms, respectively. $C^p(S; \mathbf{R}^q)$ is the set of functions $f\colon S \mapsto \mathbf{R}$ being p-continuously differentiable on S^0 (the interior of S) $\subset \mathbb{R}$. $\dot{f}(t)$ is the time derivative at t of $f \in C^1(\mathbb{R}_0^+;\mathbb{R}^q)$. $L^p(S;\mathbb{R}^q)$ is the set of functions $f\colon S \mapsto \mathbb{R}^q$ being p-integrable on S (i.e., $\int_S |f(t)|^p dt$ exists) whose L_p-norm on S is defined by $||f|| = [\int_S |f(t)|^p dt]^{1/p}$. In particular, the L_2-norm and supremum norm are denoted, respectively, by $|||(\cdot)|||$ and $||(\cdot)||$. Here, $|f(t)|$ denotes any well-posed norm of the q-vector $f(t)$, any $t \in S$. f_t is the segment of $f(s)$ for $t-\alpha \leqslant s \leqslant t$, with α being a positive constant, referred to as the maximum system delay. The set of uniformly bounded functions on S is denoted by $L_\infty(S;\mathbb{R}^{(\cdot)})$. All the above notations hold also for matrix functions. $BV(S;\mathbb{R}^{q\times m})$ is the set of $(q \times m)$ matrix measures of bounded variation and locally integrable (on S) entries. Similarly, $B(\cdot,\cdot)$, $PC(\cdot,\cdot)$ and $AC(\cdot,\cdot)$ denote the sets of bounded, piecewise continuous and absolutely continuous functions. Positive (negative) definite and semidefinite functions or matrices are denoted by "> 0" and "$\geqslant 0$" ("< 0", "$\leqslant 0$"), respectively. I_n and superscript T $(-T)$ denote, respectively, the identity n-matrix and transposition (inverse of the transpose). $\mathrm{Det}(\cdot)$, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the determinant, the minimum and maximum eigenvalues of the square $(\cdot)$-matrix. SP, SD, SED and SVD stand, respectively, for point, distributed, exponentially distributed and infinite-Volterra's type delay systems. SPD and $SPVD$ stand for systems with mixed point and distributed delays without or with Volterra's type terms, respectively. The set of eigenvalues of a matrix $(\cdot)$ (spectrum of $(\cdot)$) is denoted by $sp(\cdot)$. The stable (unstable) spectrum of the $(\cdot)$-matrix related to $\gamma \in \mathbb{R}$ (namely, the set of eigenvalues $s \in \mathbb{C}$ of $sp(\cdot)$) with $Re(s) < \gamma$ $(Re(s) \geqslant \gamma)$ is denoted by $sp_{s\gamma}(\cdot)$ $(sp_{u\gamma}(\cdot))$, respectively.

The complement to a set $(\cdot)$ (in $\mathbb{R}$ or $\mathbb{C}$ depending on the context) is denoted by $\overline{(\cdot)}$. $\emptyset$ denotes the empty set. $\boldsymbol{M}_{p\times q}$ means that $\boldsymbol{M}$ is of order $p \times q$. The left Kronecker product of the matrices $\boldsymbol{A}_{n\times m}$ and $\boldsymbol{B}_{p\times q}$ is the $np \times pq$ matrix $\boldsymbol{A} \otimes \boldsymbol{B} = (b_{ij}\boldsymbol{A})$; $(i = 1,2,...,p; j = 1,2,...,q)$. In particular, for any matrices $\boldsymbol{K}_{m\times p}$ and $\boldsymbol{C}_{n\times q}$ the linear system $AKB = C$ can be expressed in vector form as $(\boldsymbol{B}^T \otimes \boldsymbol{A})\boldsymbol{k} = \boldsymbol{c}$ where $\boldsymbol{k} = [k_{11},\ldots,k_{1p},\ldots,k_{m1},\ldots,k_{mp}]^T$ is a mp-vector formed by the entries to $\boldsymbol{K}$ in row's order.

$\exists$, $\forall$, ε are quantifiers for existence, "all" and member of a set. $W(r)$ denotes a continuous increasing scalar function satisfying $W(0) = 0$, $W(r) > 0$ if $r > 0$ and $W(r) \to \infty$ as $r \to \infty$. $Ker(\cdot)$ is the null space of the $(\cdot)$-matrix.

3. Delay systems. The following classes of delay systems are standard in the literature.

3.1. Systems with point delay.

$$(SP): \quad \dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{A}_0(t)\boldsymbol{x}(t-h) + \boldsymbol{B}(t)\boldsymbol{u}(t) + \boldsymbol{B}_0(t)\boldsymbol{u}(t-h'); \quad t \geqslant 0, \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is a trajectory value, $\boldsymbol{u}(t) \in \mathbb{R}^m$ $(m \leqslant n)$ is a control vector, $\boldsymbol{x}_t(\theta) = \boldsymbol{x}(t+\theta)$ and $\boldsymbol{u}_t(\theta') = \boldsymbol{u}(t+\theta')$ with $\theta \in [-h, 0)$, $\theta' \in [-h', 0)$ and the positive numbers h, h' are given and represent the internal (or state) and external (or input) delays in the system. $\boldsymbol{A}(\cdot)$ and $\boldsymbol{A}_0(\cdot)$ are $(n \times n)$-matrix functions and $\boldsymbol{B}(\cdot)$ and $\boldsymbol{B}_0(\cdot)$ are $(n \times m)$ matrix functions of real entries. The initial trajectory $\boldsymbol{x}(\theta) = \boldsymbol{x}_0(\theta)$ is a continuous or absolutely continuous bounded function on $\theta \in [-h, 0)$. An output equation can be defined together with Eq. 1.

3.2. Systems with distributed delays.

$$(SD): \qquad \dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \tilde{\boldsymbol{A}}\boldsymbol{x}_t + \boldsymbol{B}(t)\boldsymbol{u}(t) + \tilde{\boldsymbol{B}}\boldsymbol{u}_t; \quad t \geqslant 0, \tag{2}$$

where $\boldsymbol{x}(\cdot)$, $\boldsymbol{u}(\cdot)$, $A(\cdot)$, $B(\cdot)$ and $\boldsymbol{x}_0(\cdot)$ are defined as for SP. The operators $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{B}}$ are defined as follows: $\tilde{\boldsymbol{A}}\varphi = \int_{-\sigma}^{0}[d\alpha(\theta)]\varphi(\theta)$ and $\tilde{\boldsymbol{B}}\boldsymbol{\mu} = \int_{-\sigma'}^{0}[d\beta(\theta)]\boldsymbol{\mu}(\theta)$ with $\alpha(\cdot)$ and $\beta(\cdot)$ being two matrix valued (or scalar) finite measures in $BV([-\sigma, 0); \mathbb{R}^{n\times n})$ and $BV([-\sigma', 0); \mathbb{R}^{n\times m})$ respectively. A particular case of interest is that involving an exponential delay distribution (SED). Typically, $\alpha(\theta) =: \boldsymbol{A}_0 e^{A_0'\theta}$ (or $\lambda_0 e^{\lambda\theta}\boldsymbol{I}_n$; λ_0, $\lambda \in \mathbb{R}$) and $\beta(\theta) =: \boldsymbol{B}_0 e^{B_0'\theta}$ (or $b_0 e^{\lambda'\theta}$; b_0, $\lambda' \in \mathbb{R}$) which corresponds to matrix (or scalar) delay distribution. Another case of interest is that involving internal unbounded or Volterra equation (SVD) (Burton, 1985) with $\tilde{\boldsymbol{A}}$ defined by $\tilde{\boldsymbol{A}}_t\varphi = \int_0^t \boldsymbol{C}(t,\theta)\varphi(\theta)d\theta$ or $\tilde{\boldsymbol{A}}_t\varphi = \int_0^t \boldsymbol{C}(t-\theta)\varphi(\theta)d\theta$ (convolution Volterra equation), and $\tilde{\boldsymbol{B}}_t\boldsymbol{\mu} = \boldsymbol{\mu}(t)$. The entries to $\boldsymbol{C}(t,\theta)$ are continuous in θ for each $t \geqslant 0$. The function of initial conditions $\boldsymbol{x}_0(\theta)$ for SD and SED is of the same class as for SP, and it is of point-type (i.e., $\boldsymbol{x}_0(0) = \boldsymbol{x}_0$) for SVD, or it can be reduced to be of point-type by introducing in the forcing term the contribution of an interval of initial conditions (Burton, 1985).

REMARK 2.1. Note that the above systems can be extended without difficulty to include several combinations of delays. For systems with exponential distribution of delays, the integrals $\int_{-\sigma}^{0}$ and $\int_{-\sigma'}^{0}$ will be substituted by $\int_0^h$ and $\int_0^{h'}$ throughout this part for a more coherent notational similarity with the SP. On the other hand, note that $SED(s)$ have α, $\beta \in BV([-\sigma, 0); \mathbb{R}^{(\cdot)})$ in Eq. 2

provided that σ is finite. If the delay system is SVD (i.e., being of Volterra type and time-invariant with exponential distribution) then A_0' and B_0' have to be Hurwitz, or λ and λ' negative in order that α, $\beta \in BV([0,t);\mathbb{R}^{(\cdot)})$, all $t \geqslant 0$. Output equations for SD, SED, SVD can be defined as for SP. Some calculations related to the evaluations of the right-hand-side if (2) and their Laplace transforms for the various delay systems are given in Appendix A1.

The following result gives equivalent descriptions for time-invariant $SD(s)$ and $SED(s)$, through the reduction of them to $SP(s)$ and the definition of extended systems.

Lemma 1. (i) *Consider the following SED:*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)+\int_0^h \boldsymbol{A}_0 e^{A_0'\theta}\boldsymbol{x}(t-\theta)d\theta+\boldsymbol{B}\boldsymbol{u}(t)+\int_0^{h'} \boldsymbol{B}_0 e^{B_0'\theta}\boldsymbol{u}(t-\theta')d\theta \quad (3)$$

with $\boldsymbol{u}(t) = \mathbf{0}$; $t < 0$ and $\boldsymbol{x}_0(t) = \varphi(t)$ with $\varphi \in AC([-h,0];\mathbb{R}^n)$ is the function of initial conditions for $t \in [-h,0]$. System (3) can be equivalently described by the extended SP:

$$\dot{\bar{\boldsymbol{x}}}(t) = \bar{\boldsymbol{A}}\bar{\boldsymbol{x}}(t) + \bar{\boldsymbol{A}}_0\bar{\boldsymbol{x}}(t-h) + \bar{\boldsymbol{B}}\boldsymbol{u}(t) \quad (4)$$

with $\bar{\boldsymbol{x}}(t) = [\boldsymbol{x}^T(t) \vdots \boldsymbol{x}_1^T(t) \vdots \boldsymbol{u}_1^T(t)]^T$ and $\bar{\boldsymbol{x}}_0(t) = [\varphi^T(t) \vdots \mathbf{0}^T \vdots \mathbf{0}^T]^T$; $t \in [-h,0]$ where $\boldsymbol{x}_1(\cdot) \in \mathbb{R}^n$, $\boldsymbol{u}_1(\cdot) \in \mathbb{R}^m$ and $\bar{\boldsymbol{B}} = [B^T \vdots \mathbf{0}^T \vdots I_m]^T$ and

$$\bar{\boldsymbol{A}} =: \begin{bmatrix} A & A_0 & \boldsymbol{B}_0 \\ I_n & A_0' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{B}_0' \end{bmatrix}; \qquad \bar{A}_0 =: \begin{bmatrix} \mathbf{0} & -A_0e^{A_0'h} & -\boldsymbol{B}_0e^{B_0'h} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (5)$$

(Here, "equivalent" means that the $\boldsymbol{x}(\cdot)$ trajectories of (2) and (3) are identical on $[0,\infty)$ with the same initial conditions on $[-h,0)$ and the same control on $(0,\infty)$). In particular, the free system (i.e., $\boldsymbol{u} \equiv \mathbf{0}$ on $(0,\infty)$) is described by the extended state vector $\bar{\boldsymbol{x}} = [\boldsymbol{x}^T \vdots \boldsymbol{x}_1^T]^T \in \mathbb{R}^{2n}$ with

$$\bar{\boldsymbol{A}} =: \begin{bmatrix} A & A_0 \\ I_n & A_0' \end{bmatrix}; \qquad \bar{\boldsymbol{A}}_0 =: \begin{bmatrix} \mathbf{0} & -A_0e^{A_0'h} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}; \qquad \bar{\boldsymbol{B}} =: \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}, \quad (6)$$

and $\boldsymbol{x}_0(t) = [\varphi^T(t) \vdots \mathbf{0}^T]^T$ on $[-h,0]$.

(ii) *If the two integrals in the right-hand-side of* (3) *are changed respectively to* $\int_{-\sigma}^{0} A_0 e^{A_0'\theta} x(t+\theta)d\theta$ *and* $\int_{-\sigma'}^{0} B_0 e^{B_0'\theta} u(t+\theta)d\theta$ *then proposition* (i) *holds with* $\sigma = h$, $\sigma' = h'$.

The proof is given in Appendix A1. The above result can be directly particularized to the case of a scalar exponential distribution.

4. Stabilizability and related technical lemmas. An SP or SD ((1) or (2)) is γ-trajectory stabilizable if for some real constants γ and δ, there exists a control $u \in \mathbb{R}^m$ such that the functions $t \mapsto e^{-\gamma t} x(t)$ and $t \mapsto e^{-\delta t} u(t)$ are in $L^1((0,\infty), \mathbb{R}^m)$; i.e., $\int_0^\infty e^{-\gamma t}|x(t)|dt < \infty$ and $\int_0^\infty e^{-\delta t}|u(t)|dt < \infty$. It also follows that $\lim_{\tau\to\infty} \int_\tau^\infty e^{-\gamma t}|x(t)|dt = 0$; $\lim_{\tau\to\infty} \int_\tau^\infty e^{-\delta t}|u(t)|dt = 0$ and $\lim_{t\to\infty} e^{-\gamma t}|x(t)| = 0$; $\lim_{t\to\infty} e^{-\delta t}|u(t)| = 0$. If δ and γ are non-positive (which is the normal situation in practice), the relationships $\lim_{\tau\to\infty} \int_\tau^\infty e^{-\gamma t} \times |x(t)|dt = 0$ and $\lim_{\tau\to\infty} \int_\tau^\infty e^{-\delta t}|u(t)|dt = 0$ also hold, and, in this case, $|\delta| \leqslant |\gamma|$. If the above properties hold with $|\delta| = |\gamma|$, then the system is γ-state stabilizable; i.e., the system can be stabilized with a control input decreasing at the same rate as the generated state, which occurs always in the delay-free case. Thus, both stabilizability concepts coincide in the sense of state stabilizability. However, the γ-trajectory stabilizability does not always imply γ-state stabilizability under external delays (Tadmor, 1988) although the converse is always true. Moreover, γ-state stabilizability for any γ implies and is implied by spectrum assignability which is also guaranteed under spectral controllability (Olbrot, 1978; Watanable, 1986).

Note that stabilizability is a necessary condition for the existence of closed-loop stabilizing control laws. In the following, stabilizability refers to the wider concept of γ-trajectory stabilizability unless otherwise being stated. Also, a delay h referred to an SD denotes the delay interval $[0, h]$ defined through an integral (see Eq. 2). The next result focuses on some relationships between stabilizability of (1) – (2) related to linear delay-free systems. The inclusion of results for SED is trivial from the corresponding ones for SP and Lemma 1. It is essentially proved that a delay system, with its parameters weakly deviated respect to a linear stabilizable delay-free system, is stabilizable under a linear and time-invariant memoryless control.

Lemma 2. *Assume that the linear and time-invariant system* LS: $\dot{x}(t) = Ax(t) + Bu(t)$ *is* γ*-stabilizable (sufficient conditions are given in* Appendix A.2 – Lemma A.1) *for some* $\gamma \leqslant 0$. *Then, there exists a positive constant* ε *such that* SP (Eq. 1) *is* γ'*-trajectory stabilizable for* $\max(||A_0||, ||B_0||) < \varepsilon$, *all* $\gamma' \in [\gamma, 0]$ *and* $\hat{h} = \max(h, h')$, *with* ε *being, in general, dependent on* γ' *and* $\hat{h}$. *The value of an upper-bound* $\varepsilon_0 \geqslant \varepsilon > 0$ *can be calculated for each given parameters and delays in the* SP *and* γ'. *The results also hold for* SD (Eq. 2), SED *and* SVD *with sufficiently small parameters associated with delays. If the control law* $\boldsymbol{u}(t) = -\boldsymbol{K}_0\boldsymbol{x}(t)$ γ*-stabilizes* LS, *then it also* γ*-(trajectory) stabilizes any delay system with sufficiently small parameters associated with delays.*

The proof of Lemma 2 is given in the Appendix A3. Note that if γ is not constrained to the open left-half plane, then the stabilizability definition can become arbitrarily far from practical issues. This is addressed in the following lemma.

Lemma 3. *Assume that the* SP *(Eq. 1) is* γ*-trajectory-stabilizable for some* $\gamma < 0$ *and* $|\delta| \leqslant |\gamma|$. *Then, there exists (at least) a linear control* $\boldsymbol{u}_L(t)$ *which* γ_L*-(trajectory) stabilizes such a system for some pair of real constants* γ_L *and* δ_L. *The result also applies for the other types of delay systems (*SD, SVD, SED, *etc.).*

The proof is outlined in Appendix A4. This result indicates that there is a degree of ambiguity in the stabilization characterization. In fact, δ_L can be arbitrarily large and the control and state trajectory can be, in some cases, unstable (related to the left-half plane) so that the stabilizability under a linear controller losses its practical sense, since it cannot be achieved for $\gamma_L \leqslant 0$. This ambiguity is dealt with in the next section where uniform boundedness of the state trajectory is required for stability.

The general finite-delay system being considered can contain, in general, both point and distributed internal and/or external delays as follows:

$$(SPD): \quad \begin{aligned} \dot{\boldsymbol{x}}(t) =& \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{A}_0(t)\boldsymbol{x}(t-h) + \tilde{\boldsymbol{A}}\boldsymbol{x}_t + \boldsymbol{B}(t)\boldsymbol{u}(t) \\ &+ \boldsymbol{B}_0(t)\boldsymbol{u}(t-h') + \tilde{\boldsymbol{B}}\boldsymbol{u}_t; \quad t \geqslant 0. \end{aligned} \tag{7}$$

If $\tilde{\boldsymbol{A}} \to \tilde{\boldsymbol{A}}_t$ and $\tilde{\boldsymbol{B}} \to \tilde{\boldsymbol{B}}_t$, delays become in general time-dependent. A system ($SPVD$), involving a Volterra integro-differential system (i.e., subject to distributed infinite delays) can also be considered in (7) subject to initial conditions, satisfying similar conditions to those of SP/SD in (1) – (2), on

$[-\max(h, \sigma), 0)$, with h and σ charaterizing the internal point and finitely-distributed delays, respectively.

5. Preliminary stability definitions and results. The SP, SD or SPD Eqs. 1, 2 and 7 are (Burton, 1985): (a) *Stable* in the simple Lyapunov's sense if for each $\varepsilon > 0$ and $t_0 > 0$ there exists $\delta > 0$ such that $[|\boldsymbol{x}_0| < \delta, t \geqslant t_0]$ implies that $|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)| < \varepsilon_0$; (b) *Uniformly Stable* if it is stable and δ is independent of t_0; (c) *Asymptotically Stable* if it is stable and for each $t_0 \in \mathbb{R}_0^+$ there is an $\eta > 0$ such that $|\boldsymbol{x}_0| < \eta \Rightarrow \boldsymbol{x}(t, t_0, \boldsymbol{x}_0) \to 0$ as $t \to \infty$; (d) *Uniformly Asymptotically Stable* if it is uniformly stable and there is a $\eta > 0$ with the following property: for each $\mu > 0$ there exists $S > 0$ such that $[t_0 \in \mathbb{R}_0^+, |\boldsymbol{x}_0| < \eta, t \geqslant t_0 + S] \Rightarrow |\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)| < \varepsilon_0$. A weaker property than stability is *Ultimate Boundedness* for bound ε, namely, for each $\delta > 0$, there exists $L > 0$ such that $[t_0 \in \mathbb{R}_0^+, |\boldsymbol{x}_0| < \delta, t \geqslant t_0 + L] \Rightarrow |\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)| < \varepsilon$.

The following result applies to any SP/SD (and their particular cases) and SPD, Eqs. 1, 2 and 7, where $W_{(\cdot)}(r)$ are continuous increasing scalar functions with the properties given in Section 2.

Theorem 1 (Burton, 1985). *Let $V(t, \boldsymbol{x}_t)$ be a differential scalar function defined for $-\infty < t < \infty$ and $\boldsymbol{x}(\cdot)$ a continuous function into $\mathbb{R}^n$ with $|\boldsymbol{x}(t)| < D \leqslant \infty$.*

(a) *If $V(t, 0) = 0$, $W_1(|\boldsymbol{x}(t)|) \leqslant V(t, \boldsymbol{x}_t)$; $\dot{V}(t, \boldsymbol{x}_t) \leqslant 0$, then the zero solution is stable in the simple Lyapunov sense;*

(b) *if $W_1(|\boldsymbol{x}(t)|) \leqslant V(t, \boldsymbol{x}_t) \leqslant W_2(||\boldsymbol{x}_t||)$; $\dot{V}(t, \boldsymbol{x}_t) \leqslant 0$, then the zero solution is uniformly stable;*

(c) *if $F(t, \boldsymbol{x})$ is bounded for bounded $||\boldsymbol{x}_t||$ and $V(t, \mathbf{0}) = 0$, $W_1(|\boldsymbol{x}(t)|) \leqslant V(t, \boldsymbol{x}_t)$; $\dot{V}(t, \boldsymbol{x}_t) \leqslant -W_3(|\boldsymbol{x}_t|)$, then the zero solution is asymptotically stable;*

(d) *if $W_1(|\boldsymbol{x}(t)|) \leqslant V(t, \boldsymbol{x}_t) \leqslant W_3(|||\boldsymbol{x}_t|||) + W_2(|\boldsymbol{x}_t|)$ and $\dot{V}(t, \boldsymbol{x}_t) \leqslant -W_4(|\boldsymbol{x}_t|)$ then $\boldsymbol{x} = \mathbf{0}$ is uniformly asymptotically stable;*

(e) *if $D = \infty$ and there is a real constant $M_0 > 0$ with $W_1(|\boldsymbol{x}(t)|) \leqslant V(t, \boldsymbol{x}_t) \leqslant W_3(\int_{t-\sigma}^{t} W_4(|\boldsymbol{x}(\tau)|)d\tau) + W_2(|\boldsymbol{x}_t|)$ with σ being the maximum delay in the system, and $\dot{V}(t, \boldsymbol{x}_t) \leqslant -W_4(|\boldsymbol{x}_t|) + M_0$, then all the solutions are uniformly bounded and uniformly ultimately bounded for bound ε.*

For the SVD and $SPVD$ Eq. 7, propositions (a) – (c) remain valid. If there is a bounded continuous Φ: $[0, \infty) \mapsto \mathbb{R}^n$ which is in $L^1([0, \infty); \mathbb{R}^n)$

with $\Phi(t) \to 0$ as $t \to \infty$, then $W_3(\cdot)$ in (d) – (e) is changed into $W_3(\int_0^t \Phi(t-\tau)W_4(|\boldsymbol{x}(\tau)|)d\tau)$.

6. Appendices

A.1) Integral equalities useful for *SD* (Eq. 2), *SED* and *SVD*. The following results stand:

Propositions A.1:

(i.1) $\pounds[\int_{-\sigma}^{0} d\boldsymbol{v}(\theta)\boldsymbol{f}(t+\theta)] = [\int_{-\sigma}^{0} e^{\theta s} d\boldsymbol{v}(\theta)]\boldsymbol{F}(s)$ *and* $\pounds[\int_0^t d\boldsymbol{v}(\theta)\boldsymbol{f}(t-\theta)] = d\boldsymbol{V}(s)\boldsymbol{F}(s)$ *for all* $\boldsymbol{v} \in BV([-\sigma, 0]; \mathbb{R}^{p\times q})$ *and* $\boldsymbol{f} \in L^1([t-\sigma, t]; \mathbb{R}^q)$*;*

(i.2) *If, in addition,* $\boldsymbol{v}$ *is in* $C^1((0,t); \mathbb{R}^{p\times q})$ *for all* $t \geqslant 0$ *then* $\pounds[\int_0^t d\boldsymbol{v}(\theta) \boldsymbol{f}(t-\theta)] = \boldsymbol{V}(s)[s\boldsymbol{F}(s)] - \boldsymbol{v}(0)\boldsymbol{F}(s)$*. If, in addition,* $\boldsymbol{f} \in C^1((0,t); \mathbb{R}^{q\times m})$*, then* $\int_0^t d\boldsymbol{v}(\theta)\boldsymbol{f}(t-\theta) = \int_0^t \boldsymbol{v}(\theta)\dot{\boldsymbol{f}}(t-\theta)d\theta - \boldsymbol{v}(0)\int_0^t e^{-\theta s}\boldsymbol{f}(\theta)d\theta + \boldsymbol{f}(0)\int_0^t e^{-\theta s} \times \boldsymbol{v}(\theta)d\theta$*;*

(i.3) *If, in addition, both* $\boldsymbol{v}$ *and* $\boldsymbol{f}$ *are in* $C^1((0,t); \mathbb{R}^{p\times q})$*, all* $t \geqslant 0$*, then* $\int_{-\sigma}^{0} d\boldsymbol{v}(\sigma)\boldsymbol{f}(t+\theta) = \boldsymbol{v}(0)\boldsymbol{f}(t) - \boldsymbol{v}(-\sigma)\boldsymbol{f}(t-\sigma) - \int_{-\sigma}^{0} \boldsymbol{v}(\theta)\dot{\boldsymbol{f}}(t+\theta)d\theta$.

(ii) $\int_{-\sigma}^{0} d\boldsymbol{v}(\theta)e^{s\theta} = \boldsymbol{v}(0) - \boldsymbol{v}(-\sigma)se^{-s\sigma} - s\int_{-\sigma}^{0} \boldsymbol{v}(\theta)e^{s\theta}d\theta$ *under the assumptions of Proposition* (i1).

(iii) *Under the conditions of proposition* (i1) *and if, in addition, either* $\boldsymbol{v}$ *of* $\boldsymbol{f}$ *are in* $C^1((-\sigma, 0); \mathbb{R})$ *then* $\pounds[\int_{-\sigma}^{0} d\boldsymbol{v}(\theta)\boldsymbol{f}(t+\theta)] = [\boldsymbol{v}(0) - \boldsymbol{v}(-\sigma)e^{-s\sigma}]\boldsymbol{F}(s) - s\int_{-\sigma}^{0} \boldsymbol{v}(\theta)\boldsymbol{F}(s+\theta)d\theta$.

(iv) $\pounds[\int_{-\sigma}^{0} d\boldsymbol{v}(\theta)\boldsymbol{f}(t+\theta)] = [\boldsymbol{v}(0) - \boldsymbol{v}(-\sigma)e^{-s\sigma}]\boldsymbol{F}(s) - \int_{-\sigma}^{0} \boldsymbol{v}(\theta)\{\pounds[\dot{\boldsymbol{f}}(t+\theta)] - \boldsymbol{f}(\theta)\}d\theta$.

(v) *If* $\boldsymbol{v}(\theta) = e^{\lambda\theta}$ *then:*

(1) $\int_{-\sigma}^{0} d\boldsymbol{v}(\theta)e^{s\theta} = \lambda(\lambda+s)^{-1}[1 - e^{-(\lambda+s)\sigma}]$;

(2) $\pounds[\int_{-\sigma}^{0} d\boldsymbol{v}(\theta)\boldsymbol{f}(t+\theta)] = \lambda(\lambda+s)^{-1}[1 - e^{-(\lambda+s)\sigma}]\boldsymbol{F}(s)$;

(3) $\int_{-\sigma}^{0} d\boldsymbol{v}(\theta)\boldsymbol{f}(t+\theta) = \lambda[\int_0^t e^{-\lambda(t-\theta)}\boldsymbol{f}(\theta)d\theta - e^{-\lambda\sigma}\int_0^t e^{-\lambda(t-\theta)}\boldsymbol{f}(\theta-\sigma)d\theta]$.

(vi) *If the integrals* $\int_{-\sigma}^{0}$ *are changed into* $\int_0^h$ *then the above results remain valid with the changes in the right-hand-sides derived from the left-hand-side identities* $\int_{-\sigma}^{0} d\boldsymbol{v}(\theta)\boldsymbol{f}(t+\theta) = -\int_0^h d\boldsymbol{v}(-\theta)\boldsymbol{f}(t-\theta)$ *for* $\sigma = h$.

Proof. (i.1) follows directly from direct calculus and convolution theory. About (i.2), note that if $\boldsymbol{v} \in C^1((0,t); \mathbb{R}^n)$ then

$$\pounds\left[\int_0^t d\boldsymbol{v}(\theta)\boldsymbol{f}(t-\theta)\right] = [s\boldsymbol{V}(s) - \boldsymbol{v}(0)]\,\boldsymbol{F}(s). \tag{A.1}$$

If, in addition, $f \in C^1((0,t);\mathbb{R}^n)$ then, from (A.1)

$$£\left[\int_0^t dv(\theta)f(t-\theta)\right] = V(s)\left[sF(s)\right] - v(0)F(s) \tag{A.2}$$

$$= V(s)\left\{£\left[\dot{f}(t)\right] + f(0)\right\} - v(0)F(s), \tag{A.3}$$

and by taking inverse Laplace transforms

$$\int_0^t dv(\theta)f(t-\theta) = \int_{-\sigma}^0 v(\theta)\dot{f}(t-\theta) - v(0)\int_0^t e^{-\theta s}f(\theta)d\theta$$

$$+ f(0)\int_0^t e^{-\theta s}v(\theta)d\theta, \tag{A.4}$$

and thus (i.2) is proved. (i.3) follows directly from integration by parts. Propositions (iii) – (iv) follow directly from (i.1), the Laplace transform, differentiation and complex translation rules. Propositions (v) follows from the identity $\int_{-\sigma}^0 dv(\theta)e^{s\theta} = \lambda\int_{-\sigma}^0 e^{(\lambda+s)}\theta d\theta$ and the applications of (i.1) and its inverse Laplace transformation. (vi) follows from the given integral identities.

REMARK A.1. Extensions of Propostion A.1(v) are available for (matrix) delay exponential distributions (see Eq. A.6 in the proof of Lemma 1 below).

Proof of Lemma 1. (i) Since $\tilde{A}x_t = \int_0^h A_0 e^{A_0'\theta}x(t-\theta)d\theta$, it follows for $\bar{A}(s) =: A_0' - sI$ that

$$\int_0^h A_0 e^{A_0'\theta}e^{-\theta s}d\theta = A_0\bar{A}^{-1}(s)\sum_{k=0}^{\infty}\frac{\bar{A}^{k+1}(s)h^{k+1}}{(k+1)!}, \tag{A.5}$$

for all $s \in \mathbb{C}$ not being an eigenvalue of A_0' since the series $\sum_{k=0}^{\infty}\bar{A}^k(s)\theta^k/k!$ converges uniformly to the matrix function $e^{\bar{A}(s)\theta}$, for all $s \in \mathbb{C}$ and $\theta \in (-\infty,\infty)$. By using the variables $k' = k+1$, (A.5) can be written as

$$\int_0^h A_0 e^{A_0'\theta}e^{-\theta s}d\theta = A_0(A_0' - sI)^{-1}\left[e^{(A_0'-sI)h} - I\right]. \tag{A.6}$$

Then, the characteristic equation for the SED Eq. 3 is from (A.6) and Proposition A.1 (i.1):

$$\mathrm{Det}\left\{sI - A - A_0(A_0' - sI)^{-1}\left[e^{(A_0'-sI)h} - I\right]\right\} = 0; \quad \forall s \in \mathbb{C}, \quad (A.7)$$

with $s \notin sp(A_0')$. Now, the Laplace transform of $x(t)$, subject to zero initial conditions on $[-h, 0]$, satisfies from (A.7)

$$\left\{sI - A - A_0(A_0' - sI)^{-1}\left[e^{(A_0'-sI)h} - I\right]\right\} X(s) = 0, \quad (A.8)$$

for all $s \in \mathbb{C} \cap \overline{sp(A_0')}$. Define the auxiliary variable $X_1(s) =: (sI - A_0')^{-1}X(s)$ so that (A.8) can be expressed in the time domain as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_0\left[x_1(t) - e^{A_0'h}x_1(t-h)\right], \\ \dot{x}_1(t) &= A_0'x_1(t) + x(t), \end{aligned} \quad (A.9)$$

for all $t \geqslant 0$ and the initial conditions $[x^T(t) \vdots x_1^T]^T = [\varphi^T(t) \vdots 0^T]^T$, $t \in [-h, 0]$. The result follows directly for the free system (i.e., $B_0 = 0$ in (3)). For the forced SED, Eq. 3, define $\tilde{B}u_t = \int_0^{h'} B_0 e^{B_0'\theta} d\theta$. Eq. A.8 is modified as

$$\begin{aligned} &\left[sI - A - A_0(A_0' - sI)^{-1}\left(e^{(A_0'-sI)h} - I\right)\right] X(s) \\ &+ \left[B + B_0(B_0' - sI)^{-1}\left(e^{(B_0'-sI)h'} - I\right)\right] U(s) = 0, \end{aligned} \quad (A.10)$$

for all $s \in \mathbb{C} \cap \overline{sp(A_0')} \cap \overline{sp(B_0')}$. Define the auxiliary variable $U_1(s) =: (sI - B_0')^{-1}U(s)$. Thus, (A.9) is modified as

$$\begin{aligned} \dot{x}(t) =& Ax(t) + A_0\left[x_1(t) - e^{A_0'h}x_1(t-h)\right] \\ &+ Bu(t) + B_0\left[u_1(t) - e^{B_0'h'}u_1(t-h')\right], \quad (A.11a) \\ \dot{x}_1(t) =& A_0'x_1(t) + x(t); \quad \dot{u}_1(t) = B_0'u_1(t) + u(t), \quad (A.11b) \end{aligned}$$

for all $t \geqslant 0$ with $[x_0^T \vdots 0^T \vdots 0^T]^T = [\varphi_0^T \vdots 0^T \vdots 0^T]^T$ for $t \in [\max(h, h'), 0]$ and $\varphi(\cdot)$ being bounded (or absolutely) continuous on $[-h, 0]$. The result for the forced system follows directly from (A.1) and (i) is proved. (ii) follows from the use of Proposition A.1 (i.1) to yield $\pounds[\int_{-\sigma}^{0} A_0 e^{-A'_0\theta}\boldsymbol{x}(t+\theta)d\theta] = (\int_{-\sigma}^{0} A_0 e^{(sI-A'_0)\theta}d\theta)\boldsymbol{X}(s)$. Furthermore, $\int_{-\sigma}^{0} A_0 e^{-A'_0\theta}e^{\theta s}d\theta = \int_0^{\sigma} A_0 e^{\bar{A}(s)\theta_1}$ $d\theta_1$ which leads to an identical result to (A.5). This proves (ii) by considering similar identities for $\int_{-\sigma'}^{0} \boldsymbol{B}_0 e^{-B'_0\theta}\boldsymbol{u}(t+\theta)d\theta$.

A.2) Preliminaries for the proof of Lemma 2. Let $\boldsymbol{\Lambda}_s$ and $\boldsymbol{\Lambda}_u$ be the canonical real $n_s \times n_s$ and $(n-n_s)\times(n-n_s)$ matrices associated with $sp_{s\gamma}(\boldsymbol{A})$, $\boldsymbol{\Lambda}_a$ and $\boldsymbol{\Lambda}_{ua}$ be the canonical real $n_a \times n_a$ and $(n-n_a)\times(n-n_a)$ matrices associated with the assignable (or spectrally controllable) spectrum $sp_a(\boldsymbol{A})$ (i.e., the spectrum subset which can be arbitrarily re-allocated via linear feedback) and unassignable spectrum $sp_{ua}(A)$, respectively. Since the pairs $(sp_{s\gamma}(\boldsymbol{A}), sp_{u\gamma}(\boldsymbol{A}))$ and $(sp_a(\boldsymbol{A}), sp_{ua}(\boldsymbol{A}))$ are disjoint sets, there exist nonsingular $n\times n$ matrices $\boldsymbol{T}$ and L (where L is an unitary matrix describing an appropriate permutation of coordinates) of real entries such that

$$\begin{aligned}\boldsymbol{\Lambda} &=: \boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} = Block\ Diag[\boldsymbol{\Lambda}_s \vdots \boldsymbol{\Lambda}_u];\\ \boldsymbol{B}' &=: \boldsymbol{T}^{-1}\boldsymbol{B} = [(\boldsymbol{B}'_s)^T \vdots (\boldsymbol{B}'_u)^T]^T.\end{aligned} \tag{A.12}$$

$$\begin{aligned}\hat{\boldsymbol{\Lambda}} &= (\boldsymbol{L}\boldsymbol{T})^{-1}\boldsymbol{A}(\boldsymbol{L}\boldsymbol{T}) = Block\ Diag[\boldsymbol{\Lambda}_a \vdots \boldsymbol{\Lambda}_{ua}];\\ \hat{\boldsymbol{B}}' &=: (\boldsymbol{T}\boldsymbol{L})^{-1}\boldsymbol{B} = [(\boldsymbol{B}'_a)^T \vdots (\boldsymbol{B}'_{ua})^T]^T.\end{aligned} \tag{A.13}$$

Note from (A.12) that $\boldsymbol{LS}$ is γ-stabilizable (i.e., all the closed-loop modes can be allocated in $Re(s) < \gamma$ through the use of a time-invariant linear controller-see Lemma A.1 below) if and only if

$$rank[sI - \boldsymbol{A} \vdots \boldsymbol{B}] = rank[sI - \boldsymbol{\Lambda} \vdots \boldsymbol{B}'] = rank[sI - \hat{\boldsymbol{\Lambda}} \vdots \hat{\boldsymbol{B}}] = n, \tag{A.14}$$

for all $s \in \mathbb{C}$ with $Re(s) \geqslant \gamma$. From (A.13), the $\boldsymbol{LS}$ is completely controllable (or, equivalently, spectrum assignable) if and only if (A.14) holds for all $s \in \mathbb{C}$ (Popov–Belevitch–Hautus rank controllability test. Since (A.14) holds directly $\forall s \notin sp(\boldsymbol{A})$, it suffices to check it for $s \in sp(\boldsymbol{A})$. Eqs.(A.12) – (A.13) imply that

$$\begin{aligned}rank[sI - \boldsymbol{A} \vdots \boldsymbol{B}] &= rank\begin{bmatrix} sI - \boldsymbol{\Lambda}_s & \boldsymbol{0} & \boldsymbol{B}'_s\\ \boldsymbol{0} & sI - \boldsymbol{\Lambda}_u & \boldsymbol{B}'_u\end{bmatrix}\\ &= rank\begin{bmatrix} sI - \boldsymbol{\Lambda}_a & \boldsymbol{0} & \boldsymbol{B}'_a\\ \boldsymbol{0} & sI - \boldsymbol{\Lambda}_u & \boldsymbol{B}'_{ua}\end{bmatrix},\end{aligned} \tag{A.15}$$

for all $s \in \mathbb{C}$. From (A.12) to (A.15) the following result is proved.

Lemma A.1. *Assume that* $m \leqslant n$. *The following propositions are equivalent:*

(i) (i.1) *The* LS *is* γ*-stabilizable (or* $(\boldsymbol{A}, \boldsymbol{B})$ *is a* γ*-stabilizable pair) if and only if* $rank[s\boldsymbol{I} - \boldsymbol{\Lambda}_u \vdots \boldsymbol{B}'_u] = n - n_s$, $\forall s \in sp_{u\gamma}(A)$ *or, equivalently, if and only if* $rank[s\boldsymbol{I} - \boldsymbol{A} \vdots \boldsymbol{B}] = n$; $\forall s \in sp_{u\gamma}(\boldsymbol{A})$.

(i2) *There exists* $(\boldsymbol{K}_0)_{m\times n}$ *such that the control law* $\boldsymbol{u}(t) = -\boldsymbol{K}_0\boldsymbol{x}(t)$ γ*-stabilizes* $(\boldsymbol{A}, \boldsymbol{B})$*; i.e., the closed-loop system* $\dot{\boldsymbol{x}}(t) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_0)\boldsymbol{x}(t)$ *has all its eigenvalues in* $Re(s) < \gamma$.

(i3) *The* LS *is assignable (and thus spectrally controllable) if and only if the last rank condition holds for all* $s \in \mathbb{C}$ *and this implies that* $(\boldsymbol{A}, \boldsymbol{B})$ *is* γ*-stabilizable.*

(ii) $(\boldsymbol{A}, \boldsymbol{B})$ *is* γ*-stabilizable if* $sp_{ua}(A)$ *(empty or non-empty)* $\subset sp_{s\gamma}(\boldsymbol{A})$.

(iii) *For any full row rank matrix* $\boldsymbol{K}_{m\times n}$:

$$\begin{aligned} rank[s\boldsymbol{I} - \boldsymbol{A} \vdots \boldsymbol{B}] &= rank[s\boldsymbol{I} - \boldsymbol{A} \vdots s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}] \\ &= rank[s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K} \vdots \boldsymbol{B}\boldsymbol{K}] \leqslant n, \end{aligned} \tag{A.16}$$

for all $s \in \mathbb{C}$. *If* $(\boldsymbol{A}, \boldsymbol{B})$ *is* γ*-stabilizable then* (A.16) *holds with equality to* n *for all* $s \in \mathbb{C}$ *with* $Re(s) \geqslant \gamma$ *so that* $(\boldsymbol{A}, \boldsymbol{B}\boldsymbol{K})$ *and* $(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}, \boldsymbol{B}\boldsymbol{K})$ *are also* γ*-stabilizable pairs. In this case,* (A.16) *holds if and only if*

$$rank[s\boldsymbol{I} - \boldsymbol{\Lambda}_u \vdots \boldsymbol{B}'_u] = rank[s\boldsymbol{I} - \boldsymbol{\Lambda}_u + \boldsymbol{B}'_u\boldsymbol{K}\boldsymbol{T} \vdots \boldsymbol{B}'_u\boldsymbol{K}\boldsymbol{T}] = n - n_s, \tag{A.17}$$

for all $s \in sp_{u\gamma}(\boldsymbol{A})$.

(iv) *The spectral controllability of the unstable modes of* $\boldsymbol{A}$ *is a sufficient condition for* γ*-stabilizability of* $(\boldsymbol{A}, \boldsymbol{B})$. *This is guaranteed if*

$$rank[T^T \otimes \boldsymbol{E}_u T^{-1}\boldsymbol{B} \vdots \boldsymbol{\lambda}_u] = rank[T^T \otimes \boldsymbol{E}_u T^{-1}\boldsymbol{B}] = (n - n_s)m, \tag{A.18}$$

where $\boldsymbol{E}_u =: Diag(\boldsymbol{0}_{n-s} \vdots \boldsymbol{I}_{n-n_s})$ *for any* $(n - n_s)m$ *real vector* $\boldsymbol{\lambda}_u$.

There exists a (nonnecessarily unique and nonnecessarily full row rank) $(\boldsymbol{K}_0)_{m\times n}$ *such that* $\dot{\boldsymbol{x}}(t) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_0)\boldsymbol{x}(t)$ *has all its modes in* $Re(s) < \gamma$

(Proposition (i)*). A (more restrictive) sufficient condition for γ-stabilizability of $(\boldsymbol{A}, \boldsymbol{B})$ is the spectral controllability (assignability) of $\boldsymbol{LS}$ which is guaranteed if*

$$rank[T^T \otimes T^{-1}\boldsymbol{B} \vdots \boldsymbol{\lambda}] = rank[T^T \otimes T^{-1}\boldsymbol{B}] = nm, \qquad \text{(A.19)}$$

for any nm vector $\boldsymbol{\lambda}$ built by writing the $n \times n$ $(\Lambda^* - \Lambda)$-matrix ordered row-by-row and entry-by-entry where $\boldsymbol{\Lambda}^*$ is strictly Hurwitz. A stabilizing matrix K_0 supplying the control law $\boldsymbol{u}(t) = -\boldsymbol{K}_0\boldsymbol{x}(t)$ (Proposition (i)*) can be calculated in this case from the linear equation*

$$[T^T \otimes T^{-1}\boldsymbol{B}]\boldsymbol{k}_0 = \boldsymbol{\lambda}_0, \qquad \text{(A.20)}$$

with $\boldsymbol{\lambda}_0$ and $\boldsymbol{k}_0$ being obtained from $(\boldsymbol{\Lambda}_0^* - \boldsymbol{\Lambda})$ and K_0, as above, $\boldsymbol{\Lambda}_0^*$ being strictly Hurwitz. (At least) a solution $\boldsymbol{k}_0$ to (A.20) *is guaranteed to exist if* (A.19) *holds.*

(v) *If $sp_{ua}(\boldsymbol{A}) \cap sp_{u\gamma}(\boldsymbol{A}) \neq \emptyset$ then the pair $(\boldsymbol{A}, \boldsymbol{B})$ is not γ-stabilizable, and $\boldsymbol{A}_c =: \boldsymbol{A} - \boldsymbol{BK}$ is not γ-stable and then $(\boldsymbol{A}, \boldsymbol{B})$ not completely assignable (controllable) for any $\boldsymbol{K}_{m\times n}$-matrix (i.e., all the eigenvalues of $\boldsymbol{A}_c$ are not in $Re(s) < \gamma$). Also, there is at least one γ-unstable and uncontrollable (open-loop) mode (i.e., an eigenvalue of $\boldsymbol{A}$) which remains invariant under the control law $\boldsymbol{u}(t) = -\boldsymbol{Kx}(t)$ (i.e., it is also an eigenvalue of $\boldsymbol{A}_c$).*

Proof. (i.1) *Necessity.* If $rank[s_0\boldsymbol{I} - \boldsymbol{\Lambda}_u \vdots \boldsymbol{B}_u'] < n - n_s$ for some $s_0 \in sp_{u\gamma}(\boldsymbol{A})$ (i.e., for some $s_0 \in \mathbb{C}$ with $Re(s_0) \geqslant \gamma$), then $rank[s_0\boldsymbol{I} - \boldsymbol{A} \vdots \boldsymbol{B}] < n$ from (A.15). Since $Re(s_0) \geqslant \gamma$, the system is not γ-stabilizable.

Sufficiency. From (A.14) – (A.15), $rank[s\boldsymbol{I} - \boldsymbol{A} \vdots \boldsymbol{B}] < n$ with $Re(s) \geqslant \gamma$ is possible if and only if $\mathrm{Det}(s\boldsymbol{I} - \boldsymbol{\Lambda}_u) = 0$ (since $\mathrm{Det}(s\boldsymbol{I} - \boldsymbol{\Lambda}_s) \neq 0$ for $Re(s) \geqslant \gamma$) which implies that a rank test on $sp_{u\gamma}(\boldsymbol{A})$ is sufficient.

(i2) It is obvious since $(\boldsymbol{A}, \boldsymbol{B})$ can be γ-stabilized through a linear time-invariant controller.

(i3) The proof is similar to that of (i.1) by considering any $\gamma \in \mathbb{R}$.

(ii) Note that $sp_{ua}(\boldsymbol{A}) \subset sp_{s\gamma}(\boldsymbol{A}) \iff sp_{s\gamma}(\boldsymbol{A}) \supset sp_{u\gamma}(\boldsymbol{A})$. If $s_0 \in sp_{u\gamma}(\boldsymbol{A})$, then $s_0 \in sp_a(\boldsymbol{A})$ so that it is a controllable mode and $rank[s\boldsymbol{I} - \boldsymbol{A} \vdots \boldsymbol{B}] = n$, $\forall s \in \mathbb{C} \Longrightarrow rank[s\boldsymbol{I} - \boldsymbol{A} \vdots \boldsymbol{B}] = n$, $Re(s) \geqslant \gamma$ and $(\boldsymbol{A}, \boldsymbol{B})$ is γ-stabilizable.

(iii) There exist matrices of real entries $Q_1 \in \mathbb{R}^{(m+n)\times 2n}$, Q_2, $Q_3 \in \mathbb{R}^{2n\times 2n}$:

$$Q_1 =: \begin{bmatrix} I_n & 0_{n\times n} \\ K_{m\times n} & K_{m\times n} \end{bmatrix};$$

$$Q_2 =: \begin{bmatrix} & Q_1 & \\ (P_1)_{(n-m)\times n} & & (P_2)_{(n-m)\times n} \end{bmatrix}; \qquad (A.21)$$

$$Q_3 = Diag(I_n \vdots I_n),$$

for given K and P_1 and P_2 matrices such that

$$\begin{aligned} [sI - A + BK] &= [sI - A \vdots B]Q_1 \\ &= [sI - A \vdots B \vdots 0_{n\times(n-m)}]Q_2 = [sI - A \vdots BK]Q_3, \end{aligned} \qquad (A.22)$$

for all $s \in \mathbb{C}$. Note from (A.21) that Q_1 is full rank (i.e., $rank(Q_1) = n+m$) for any full row rank matrix K, and that there exist (in general, non-unique) matrices P_1 and P_2 with $P =: [P_1 \vdots P_2]$ and $rank(P) = n - m$ such that $rank(Q_2) = rank[Q_1^T \vdots P^T]^T = rank(Q_3) = 2n$. Since Q_2 and Q_3 are (square) $2n \times 2n$ nonsingular matrices, (A.22) implies (A.16) with the rank being n if (A, B) and then $(A - BK, BK)$ are also γ-stabilizable. (A.17) follows from (A.16) and Proposition (i.1) since:

$$rank \begin{bmatrix} sI - \Lambda_s + E_s T^{-1}BKT & 0 & E_s T^{-1}BKT \\ 0 & sI - \Lambda_u + E_u T^{-1}BKT & E_s T^{-1}BKT \end{bmatrix}$$

$$= n, \qquad (A.23)$$

for all $s \in \mathbb{C}$ with $Re(s) \geqslant \gamma \iff rank[sI - \Lambda_u + E_u T^{-1}BKT \vdots E_u T^{-1}BKT] = n - n_s$, all $s \in sp_{u\gamma}(A)$ (Proposition (i.1)), where $E_s =: Diag(I_{n_s} \vdots 0_{n-n_s})$; $E_u =: Diag(0_{n_s} \vdots I_{n-n_s})$. The equivalence in (A.23) for any full row rank matrix K implies and is implied by Proposition (i.1). Thus, (A, B) is γ-stabilizable if and only if (A.16) or, equivalently, (A.23) hold, with strict equality to n in (A.16).

(iv) The sufficient conditions (A.18) and (A.19) for γ-stabilizability implies the existence of $(K_0)_{m\times n}$ such that $E_u T^{-1} B K_0 T = \Lambda_u - \Lambda^*$ for any $(n - n_s)\times(n-n_s)$ predefined Hurwitz Λ^*-matrix. Using the (left) Kronecker product of matrices $P =: [T^T \otimes E_u T^{-1} B]\mathbf{k} = \lambda_u - \lambda^*$, where $\mathbf{k} = [\mathbf{k}_1^T \vdots \cdots \vdots \mathbf{k}_m^T]^T$ is a vector containing the entries to K arranged by rows, λ_u and λ^* are built in the same way from matrices Λ_u and Λ^*, with P being a $n(n - n_s) \times nm$-matrix. Thus, Froebenius theorem estabilishes that (at least) one solution exists to (A.24) if and only if (A.18) holds. (A.19) follows in the same way from (i.1) and (i.3) for solvability of (A.20) since γ-stabilizability of (A, B) is implied by its assignability.

(v) (*1st Proof*). Note that $rank[sI - A \vdots B] < n$, $\forall s \in sp_{ua}(A) \Longrightarrow rank[sI - A \vdots B] < n$, $\forall s \in \mathbb{C}$ with $Re(s) \geqslant \gamma$. From (i.1), the pair (A, B) is not γ-stabilizable so that it is not stabilizable under the linear and time-invariant control $\mathbf{u}(t) = -K\mathbf{x}(t)$, for $(K)_{m\times n}$ so that uncontrollable (or unassignable) modes being invariant under such a feedback law are present (otherwise, (A, B) would be γ-stabilizable).

2nd alternative proof. (A, B) is not γ-stabilizable$\Longrightarrow rank[sI - A \vdots B] < n$, $\forall s \in \mathbb{C}$ with $Re(s) \geqslant \gamma$ (Prop.(i)) $\Longrightarrow rank[s_0 I - A \vdots B] < n$, $\forall s_0 \in (sp_{ua}(A), sp_{u\gamma}(A))$. Note that $Re(s_0) \geqslant \gamma$ since $s_0 \in sp_{u\gamma}(A)$. From (iii), this implies $rank[sI - A \vdots B] = rank[sI - A + BK \vdots BK] < n$, for any full row rank K-matrix with all $s \in \mathbb{C}$. Applying this result to s_0, $\mathrm{Det}[s_0 I - A + BK] = 0$ with $Re(s_0) \geqslant \gamma$ and the $(A - BK)$-matrix is not γ-stable. The remaining of the proof concerning with the fact that s_0 is uncontrollable and invariant under linear and time-invariant feedback follows directly by using contradiction arguments.

REMARK A.2. Note from the proof of sufficiency in Lemma A.1(i) that the unstable spectrum related to γ can be always freely assigned since $sp_{u\gamma}(A) \subset sp_a(A)$. Note also that if $sp_{ua}(A) \cap sp_{u\gamma}(A) \neq \emptyset$ for some real γ, there are uncontrollable unstable (related to γ) open-loop modes such that the system is not γ-stabilizable.

A.3) Proof of Lemma 2. SP (Eq. 1) is γ'-stabilizable for all $\gamma' \in [\gamma, 0]$ if and only if $rank[S(s)] = n$, all $s \in \mathbb{C}$ with $Re(s) \geqslant \gamma'$ where $S(s) =:$

$[sI - A - e^{-hs}A_0 \vdots B + e^{-h's}B_0]$. This is equivalent to $W(s) =: S(s)S^T(s)$ being nonsingular for $Re(s) \geqslant \gamma'$, since $S(s)$ has less rows than columns. Direct calculus yields:

$$W(s) =: W_0(s) + \Delta W(s) = W_0(s)\left[I + W_0^{-1}(s)\Delta W(s)\right], \qquad \text{(A.24)}$$

with

$$W_0(s) =: (sI - A)(sI - A^T) + BB^T, \qquad \text{(A.25)}$$

$$\begin{aligned}\Delta W(s) =: & \, e^{-2h's}[e^{2(h'-h)s}A_0A_0^T + B_0B_0^T] \\ & + e^{-h's}\Big\{e^{(h'-h)s}\left[(sI - A)A_0^T + A_0(sI - A^T)\right] \\ & + BB_0^T + B_0B^T\Big\}. \end{aligned} \qquad \text{(A.26)}$$

Note that the last equality in (A.24) stands for $Re(s) \geqslant \gamma'$, $\forall \gamma' \in [\gamma, 0]$ with $\gamma \leqslant 0$, since LS being γ-stabilizable (Assumption A.1) implies that $W_0(s)$ is nonsingular for $Re(s) \geqslant \gamma$. A sufficient condition for $W(s)$ to be nonsingular for $Re(s) \geqslant \gamma'$ is, from (A.24), $||W_0^{-1}(s)\Delta W(s)|| < 1$ for $Re(s) \geqslant \gamma'$ and any matrix norm (Ortega, 1972). Note from (A.26) that

$$\begin{aligned}||\Delta W(s)|| \leqslant & \, e^{\hat{h}|\gamma'|}\Big\{e^{\hat{h}\gamma'}||A_0A_0^T + B_0B_0^T|| \\ & + ||(sI - A)A_0^T + A_0(sI - A^T) + BB_0^T + B_0B^T||\Big\} \\ < & \, ||\,[(sI - A)(sI - A^T) + BB^T]^{-1}||; \quad Re(s) \geqslant \gamma', \end{aligned} \qquad \text{(A.27)}$$

since $\gamma \leqslant 0$ for all $\hat{h} > 0$ and with A_0 and B_0 fulfilling $\max(||A_0||, ||B_0||) \leqslant \varepsilon(\gamma', \hat{h})$, for all $\gamma' \in [\gamma, 0]$ with some $\varepsilon \in \mathbb{R}$ being dependent on γ' and $\hat{h}$. The constant ε increases as $h|\gamma'|$ decreases. A sufficient condition to guarantee (A.27), independently of s, for given γ' and $\hat{h}$ is

$$\begin{aligned}& e^{2\hat{h}|\gamma'|}||A_0A_0^T + B_0B_0^T|| \\ & \quad + e^{\hat{h}|\gamma'|}\sup_{Re(s)\geqslant\gamma'}\left\{\left\|(sI - A)A_0^T + A_0(sI - A^T) + BB_0^T + B_0B^T\right\|\right\} \\ & \leqslant \inf_{Re(s)\geqslant\gamma'}\left\{\left\|\,\left[(sI - A)(sI - A^T) + BB^T\right]^{-1}\right\|\right\}. \end{aligned} \qquad \text{(A.28)}$$

Since $\inf_{Re(s)\geqslant\gamma'}\{\|[(sI-A)(sI-A^T)+BB^T]^{-1}\|\}>0$ and the involved matrix is nonsingular for $Re(s)\geqslant\gamma'$ from stabilizability of (A, B), a sufficient condition for (A.28) to hold is that the upper-bound $\varepsilon_0 \geqslant \varepsilon(\gamma', \hat{h}) > 0$ verifies

$$a\varepsilon_0^2 + b\varepsilon_0 - c < 0 \Longrightarrow \varepsilon_0 \in \left(0, \frac{1}{2a}\left[(b^2+4ac)^{\frac{1}{2}} - b\right]\right), \tag{A.29}$$

$$a =: e^{\hat{h}|\gamma'|};\quad b = \sup_{Re(s)\geqslant\gamma'}(\|sI-A\| + \|B\|) > 0, \tag{A.30a}$$

$$c =: \frac{1}{2}e^{-\hat{h}|\gamma'|}\inf_{Re(s)\geqslant\gamma'}\left(\left\|\left[(sI-A)(sI-A^T) + BB^T\right]^{-1}\right\|\right) > 0. \tag{A.30b}$$

This completes the proof of stabilizability for SP for all A_0 and B_0 such that $\max(\|A_0\|, \|B_0\|) \leqslant \varepsilon$. For SD, SED and SVD, the proof is outlined *mutatis-mutandis*. As a result of the above developments and Lemma A.1, if (A, B) is γ-stabilizable through the linear control $u_0(t) = -K_0x(t)$ then the delay systems of section 2 are γ'-trajectory stabilizable for all $\gamma' \in [\gamma, 0]$, for all delays and sufficiently small $\|A_0\|$ and $\|B_0\|$ dependent on h, h', and γ'. Also, $(A-BK, BK)$ and (A, BK) are γ-stabilizable and thus, the delay systems in Section 2 defined by A or $A_c =: A - BK$, $B_c = BK$ are also γ'-trajectory stabilizable ($\forall\gamma' \in [\gamma, 0]$) for some memoryless control $u_{0c}(t) = -K_{0c}x(t)$ γ-stabilizing (A, B_c) or (A_c, B_c). By a similar proof to the given one with $W_{0c}(s)$ redefined based on the (A_c, B_c)-parametrization, it follows that for sufficiently small $\|A_0\|$ and $\|B_0\|$, $u_0(t)$ also γ (trajectory)-stabilizes the SP.

A.4) Proof of Lemma 3 (Outline). Firstly, the following preliminary result is proved.

Lemma A.2. *Assume the SP (*Eq. 1*) under the linear and time-invariant control law $u_L(t) = -K_Lx_L(t)$ (generalizations for the various delay systems are immediate). Then, $\|u_L(t)\|$ and $\|x_L(t)\|$ are functions of exponential orders*[3] *for any $(K_L)_{m\times n}$.*

Proof. (*Outline*): Assume $t_0 = 0$ without loss of generality. Susbtitution of $u_L(t) = -K_Lx_L(t)$ in (1) yields the closed-loop system:

$$\dot{x}(t) = A_{cL}x(t) + A_0x(t-h) + B_{cL}x(t-h'), \tag{A.31}$$

[3] f: $(I)\mapsto\mathbf{R}$ is of exponential order if $f(t)=\alpha e^{\beta t}$, some α, $\beta\in\mathbf{R}$ and $\forall t\in I$.

with $A_{cL} =: A - BK_L$; $B_{cL} =: B_0 K_L$. Since the feature of having one or two delays in (A.31) is irrelavant to the subsequent developments, assume $B_0 = \mathbf{0}$ ($\Longrightarrow B_{cL} = \mathbf{0}$). Note that for each $t \geqslant -h$, there exists (at least) one $t' \in [-h, t]$ such that $v(t) = ||\boldsymbol{x}(t')|| =: \sup_{(\tau \leqslant t)}(||\boldsymbol{x}(\tau)||)$ with $\boldsymbol{x}(t) = \varphi(t)$ (i.e., the initial function), all $t \leqslant 0$. Then,

$$\begin{aligned} v(t) &= ||\boldsymbol{x}(t')|| =: \sup_{\tau \leqslant t}(||\boldsymbol{x}(\tau)||) \\ &\leqslant g(t)\left[\, ||\boldsymbol{x}_0|| + \int_0^t ||e^{-A_c\tau}||\,||B_c||\,||v(\tau)||d\tau\right], \end{aligned} \tag{A.32}$$

where

$$\begin{aligned} &g(t) =: ||e^{A_c t'_t}||; \\ &t'_t =: \{\min\ \tau \in [-h, t] : ||\boldsymbol{x}(t')|| = v(t)\ \}\,. \end{aligned} \tag{A.33}$$

Applying Gronwall's lemma (Bellman, 1970) to (A.32) leads to

$$v(t) \leqslant g_1(t) + \int_0^t g_1(\tau) g_2(t,\tau) e^{\int_\tau^t g_2(t,u)du} d\tau, \tag{A.34}$$

where

$$g_1(t) = g(t)||\boldsymbol{x}_0|| \ ; \ g_2(t,\tau) =: g(t)||e^{-A_c\tau}||\ ||B_c||. \tag{A.35}$$

Note from (A.33) and (A.35) that there are real constants $p_i \geqslant 0$ ($i = 1, 2, 3, 4$) and λ, such that

$$\begin{aligned} &g_1(t) \leqslant p_1 + p_2 e^{\lambda_1 t}; \\ &g_2(t,\tau) \leqslant K_1 + K_2 e^{\lambda_1 t} + K_3 e^{-\lambda_1 \tau} + K_4 e^{\lambda_1 (t-\tau)}, \end{aligned} \tag{A.36}$$

where $K_1 =: p_1 p_3$; $K_2 =: p_2 p_3$; $K_3 =: p_1 p_4$ and $K_4 =: p_2 p_4$. Substitution of (A.36) into (A.34) proves that $v(t)$, and then $||\boldsymbol{x}(t)||$, is of exponential order and the proof is complete. Thus, $||\boldsymbol{u}_L(t)|| \leqslant ||K_L||\ ||\boldsymbol{x}_L(t)||$ is also of exponential order.

Proof of Lemma 3 (Continued). Since the SP (Eq. 1) is γ-trajectory stabilizable, there are real constants δ and γ such that

$$\int_0^t e^{-\delta\tau}||\boldsymbol{u}(\tau)||d\tau < \infty \Longrightarrow \int_0^t e^{-\gamma\tau}||\boldsymbol{x}(\tau)||d\tau < \infty; \quad \forall t \geqslant 0, \tag{A.37}$$

for some control $\boldsymbol{u} : [0,\infty) \mapsto \mathbb{R}^m$. The existence of a linear and time-invariant control law $\boldsymbol{u}_L(t) = -\boldsymbol{K}_L\boldsymbol{x}_L(t)$ γ (trajectory)-stabilizing the SP is guaranteed from (A.37) if

$$\int_0^t e^{-\delta_L\tau}||\boldsymbol{u}_L(\tau)||d\tau \leqslant ||K_L|| \int_0^t e^{-\delta_L\tau}||\boldsymbol{x}_L(\tau)||d\tau$$

$$\leqslant \int_0^t e^{-\delta\tau}||\boldsymbol{u}(\tau)||d\tau \Longrightarrow \int_0^t e^{-\gamma_L\tau}||\boldsymbol{x}_L(\tau)||d\tau < \infty, \tag{A.38}$$

for some real constants δ_L and γ_L, where $\boldsymbol{x}_L(\cdot)$ is the state trajectory of Eq. 1 when submitted to the control law $\boldsymbol{u}_L(\tau)$. A sufficient condition for (A.38) to hold is

$$||\boldsymbol{x}_L(t)|| \leqslant (||K_L||)^{-1}\, e^{(\delta_L-\delta)t}||\boldsymbol{u}(t)||, \tag{A.39}$$

for all $t \geqslant 0$. Since the map $t \mapsto e^{-\delta t}||\boldsymbol{u}(t)||$ is in $L^1[(0,\infty);\mathbb{R}]$, $||\boldsymbol{u}(t)||$ is of exponential order. Then, (A.39) always holds, from Lemma A.2, for some real constant δ_L so that SP is γ (trajectory)-stabilizable through the linear control $\boldsymbol{u}_L(t)$ and the proof is complete.

REFERENCES

Agathoklis, P., and S. Foda (1989). Stability and the matrix Lyapunov equation for delay differential systems. *Int. J. Control,* **49**(2), 417–432.

Alastruey, C.F., M. De la Sen and J. R. González de Mendívil (1995). The stabilizability of integro-differential systems with two distributed delays. *Math. Comput. Modelling,* **21**(8), 85–94.

Bellman, R. (1970). *Methods of Nonlinear Analysis.* Academic Press, New York.

Burton, T.A. (1985). *Stability and Periodic Solutions of Ordinary and Functional Differential Equations.* Academic Press, New York.

De la Sen, M. (1988). Fundamental properties of linear control systems with aftereffect, Part I: The continuous case. Part II: Extensions including the discrete case. *Mathematical Computing and Modelling,* **10**(7), 473–489, 491–502.

De la Sen, M. (1993). On the stabilizability, controllability and observability of linear hereditary systems with distributed commensurate delays. *Int. J. of Systems Science,* **24**(1), 33–52.

De la Sen, M. (1994). On some structures of stabilizing control laws for linear and time-invariant systems with bounded point delays and unmeasurable states. *Int. J. of Control,* **59**(2), 529–541.

De la Sen, M., and N. Luo (1994). Discretization and FIR filtering of continuous linear systems with internal and external point delays. *Int. J. of Control,* **60**(6), 1223–1246.

Fiagbedzi, Y.A., and A.E. Pearson (1987). A multistage reduction technique for feedback stabilizing distributed time-lag systems. *Automatica,* **23**(3), 311–326.

Fiagbedzi, Y.A., and A.E. Pearson (1990a). Output stabilization of delay systems via generalization of the transformed method. *Int. J. Control,* **51**(4), 801–822.

Fiagbedzi, Y.A., and A.E. Pearson (1990b). A state observer for systems described by functional equations. *Automatica,* **26**(2), 321–331.

Hale, J. (1977). *Theory of Functional Differential Equations.* Springer–Verlag, New York.

Hautus, M.L.J. (1969). Controllability and observability conditions of linear autonomous systems. *Indag. Math.,* **31**, 443–448.

Luo, N., and M. De la Sen (1995). Output feedback stabilization of a class of internally delayed systems. *Int. J. of Systems Science,* **26**(2), 421–433.

Lee, E.B., and A. Olbrot (1981). Observability and related structural results for linear hereditary systems. *Int. J. of Control,* **34**(6), 1061–1078.

Mori, T., E. Noldus and M. Kuwahara (1983). A way to stabilize linear systems with delayed state. *Automatica,* **19**(5), 571–573.

Olbrot, A.W. (1978). Stabilizability, detectability and spectrum assignment for linear autonomous systems with general time delays. *IEEE Trans. Automat. Control,* **AC–23**(5), 887–890.

Ortega, T. (1972). *Numerical Analysis.* Academic Press, New York.

Pandolfi, L. (1975). On feedback stabilization of functional differential equations. *Bull. Un. Mat. Ital.,* **4**(11), Suppl. fasc. 3, 626–635.

Pandolfi, L. (1990). Generalized control systems boundary control systems and delayed control systems. *Math. Control Signal Systems,* **3**, 165–181.

Tadmor, G. (1988). Trajectory stabilizing controls in hereditary linear systems. *SIAM J. Control and Optimization,* **26**(1), 138–154.

Watanabe, K. (1986). Finite spectrum assignment and observer for multivariable systems with commesurate delays. *IEEE Trans. Autom. Control,* **AC–31**(6), 543–550.

Zheng, F., M. Cheng and W.B. Gao (1994). Feedback stabilization of linear systems with distributed delays in state and control variables. *IEEE Trans. Autom. Control,* **AC–39**(8), 1714–1718.

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APIE UNIVERSALŲ STABILIZUOJANČIŲ TOLYDINIŲ TIESINIŲ REGULIATORIŲ, SKIRTŲ SISTEMOMS SU VĖLINIMU, PROJEKTAVIMĄ. I dalis. Preliminariniai rezultatai

Manuel de la SEN ir Ningsu LUO

Straipsnyje nagrinėjami apibendrinti tiesiniai reguliatoriai, skirti tiesinėms sistemoms su liekamuoju poveikiu, kuriuos naudojant reguliavimo sisitemos uždarame kontūre, ši sistema tampa globaliai tolygiai ir asimptotiskai stabili Liapunovo prasme. Reguliatoriai yra universalūs ta prasme, kad jie turi įvairių tipų vėlinimus, kurie gali būti baigtiniai, neriboti arba net priklausyti nuo laiko. Straipsnio pirmoje dalyje aprašomos kai kurios įvadinės sąvokos ir stabilizuojamumo sąlygos.