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ON THE LAW OF THE ITERATED LOGARITHM IN MULTIPHASE QUEUEING SYSTEMS. II

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Abstract. Queueing systems with a single device are well developed (see, for example, Borovkov, 1972; 1980). But there are only several works in the theory of multiphase queueing systems in heavy traffic (see Iglehart, Whitt, 1970b) and no proof of laws of the iterated logarithm for the probabilistic characteristics of multiphase queuing systems in heavy traffic. The law of the iterated logarithm for the waiting time of a customer is proved in the first part of the paper (see Minkevičius, 1995). In this work, theorems on laws of the iterated logarithm for the other main characteristics of multiphase queuing systems in heavy traffic (a summary queue length of customers, a queue length of customers, a waiting time of a customer) are proved.

Key words: multiphase queuing system, the law of the iterated logarithm, a summary queue length of customers, a queue length of customers, a waiting time of a customer.

1. Introduction. One can apply limit theorems to the waiting time of a customer and queue length of customers in order to get probabilistic characteristics of multiphase queueing systems (MQS) under various conditions of heavy traffic (see Borovkov, 1972; 1980). A single-phase case, when intervals of times between the arrival of customers to queue are independent identically distributed random variables, and there is a single device, working independently of output in heavy traffic, is competently investigated in many works (see, for example, Borovkov, 1980; Iglehart, 1973 and etc.). Iglehart (1971a) carefully investigated a single-device case and obtained the law of the iterated logarithm (LIL) for the single device case. It is a pity, that fundamental Iglehart's results in the queueing systems theory in heavy traffic are rarely used (see Iglehart, 1965–1973).

There are only a few works in the theory of MQS in heavy traffic (see, for example, Iglehart, Whitt, 1970b; Minkevičius, 1991; 1995) and no proof of LIL for the probabilistic characteristics of MQS in heavy traffic. LIL for a summary waiting time of a customer and a waiting time of a customer is proved in the first part of the paper (see Minkevičius, 1995).

In this work, theorems on laws of the iterated logarithm for the other main characteristics of MQS in heavy traffic (a summary queue length of customers, a queue length of customers, a waiting time of a customer) are proved.

The main tools for the analysis of MQS in heavy traffic are functional LIL for a Wiener process and a renewal process (proof can be found in Strassen, 1964 and Iglehart, 1971a).

We submit some definitions from the theory of metric spaces (see, for example, Billingsley, 1965, Chapter 2).

Let C is be a metric space consisting of real continous functions in $[0,1]$ with a uniform metric $\rho(x,y) = \sup_{0 \leqslant t \leqslant 1} \left|x(t) - y(t)\right|$, $x, y \in C$.

Also, let D be a space of all real-valued right-continuous functions in [0,1] having left limits and endowed with the Skorokhod topology induced by the metric d (under which D is complete and separable).

Define $\mathcal{D}$ as a Borel set in D. Also define $k(\delta)$ as a set of absolutely continous functions $x \in C$ such that $x(0) = 0$ and $\int_0^1 [\dot{x}(t)]^2\, dt \leqslant \delta^2$, where $\dot{x}$ is a derivative of x, which exists almost everywhere according to the Lebesque measure. Strassen (1964) showed that $k(\delta)$ was a compact set and for $x \in k(\delta)$ and $0 \leqslant a \leqslant b \leqslant 1$

$$\left|x(b) - x(a)\right| \leqslant \delta(b-a)^{1/2}.$$

2. Problem formulation. We investigate here a k-phase MQS (i.e., when a customer is served in the j-th phase of the MQS, he goes to the $j+1$st phase of the MQS, and after the customer is served in the k-th phase of the MQS, then he leaves the MQS). Let us note t_n as time of arrival of the n-th customer; $S_n^{(j)}$ as the service time of the n-th customer in the j-th phase of the MQS; $z_n = t_{n+1} - t_n$. Let us introduce mutually independent renewal processes $x_j(t) = \{\max k\colon \sum_{i=1}^k S_i^{(j)} \leqslant t\}$ (such a total number of customers can be served in the jth phase of the MQS until time t if devices are working without time wasted), $e(t) = \{\max k : \sum_{i=1}^k z_i \leqslant t\}$ (total number of customers which arrive to MQS until time moment t). Next, denote by $\tau_j(t)$ the total number of customers after service departure from the jth phase of the MQS until time t; $Q_j(t)$ as the queue length of customers in j-th phase of the MQS at time moment t; $v_j(t) = \sum_{i=1}^{j} Q_i(t)$ stands the summary queue length of customers until the j-phase of the MQS at time moment t, $j = 1, 2, \dots, k$ and $t > 0$.

Suppose that the queue length of customers and a virtual waiting time of a customer in each phase of the MQS are unlimited, the discipline service of customers is "first come, first served" (FCFS). All random variables are defined on one common probability space $(\Omega, \mathcal{F}, P)$.

Let interarrival times (z_n) to the MQS and service times $(S_n^{(j)})$ in every phase of the MQS for $j = 1, 2, \dots, h$ be mutually independent identically distributed random variables.

Let us define $\beta_j = (ES_1^{(j)})^{-1}$, $\beta_0 = (Ez_1)^{-1}$, $\alpha_j = \beta_0 - \beta_j$, $\alpha_0 = 0$, $\widehat{\sigma}_j^2 = DS_1^{(j)}(ES_1^j)^{-3} > 0$, $\widehat{\sigma}_0^2 = Dz_1(Ez_1)^{-3} > 0$, $\widetilde{\sigma}_j^2 = \widehat{\sigma}_0^2 + \widehat{\sigma}_j^2$, $\sigma_j^2 = \widehat{\sigma}_j^2 + \widehat{\sigma}_{j-1}^2$, $\widehat{x}_j(t) = e(t) - x_j(t)$, $j = 1, 2, \dots, k$.

Assume the following condition to be fulfilled $\beta_0 > \beta_1 > \dots > \beta_k > 0$.

Then

$$\alpha_k > \alpha_{k-1} > \dots > \alpha_1 > 0. \tag{1}$$

One of the main results of the work is a theorem on LIL for the summary length of customers.

Theorem 1. *If condition* (1) *is fulfilled, then*

$$P\Big(\varlimsup_{t\to\infty} \frac{v_j(t) - \alpha_j \cdot t}{\widetilde{\sigma}\cdot a(t)} = 1\Big) = 1$$

and

$$P\Big(\varliminf_{t\to\infty} \frac{v_j(t) - \alpha_j \cdot t}{\widetilde{\sigma}\cdot a(t)} = -1\Big) = 1$$

for $j = 1, 2, \dots, k$ *and* $a(t) = \sqrt{2t \ln\ln t}$.

Proof. In Minkevičius (1991) relations

$$Q_j(t) = \tau_{j-1}(t) - \tau_j(t), \tag{2}$$

$$Q_j(t) = f_t(\tau_{j-1}(\cdot) - x_j(\cdot)), \tag{3}$$

$$Q_j(t) = f_t\Big(\widehat{x}_j(\cdot) - \sum_{i=1}^{j-1} Q_i(\cdot)\Big) \tag{4}$$

are obtained for $j = 1, 2, \dots, k$ and $f_t(x(\cdot)) = x(t) - \inf_{0 \leqslant s \leqslant t} x(s)$.

In view of (4) we have $v_j(t) = \widehat{x}_j(t) - \inf_{0 \leqslant s \leqslant t}(\widehat{x}_j(s) - v_{j-1}(s))$ for $j = 1, 2, \ldots, k$ and $v_0(\cdot) \equiv 0$.

Next, using (2) and (3), we obtain $\tau_j(t) = \tau_{j-1}(t) - Q_j(t) = x_j(t) + \inf_{0 \leqslant s \leqslant t}(\tau_{j-1}(s) - x_j(s))$ for $j = 1, 2, \ldots, k$ and $\tau_0(t) = e(t)$.

Thus,

$$\begin{aligned} x_j(t) - \tau_j(t) &= \sup_{0 \leqslant s \leqslant t} \big(x_j(s) - \tau_{j-1}(s)\big) \\ &= \sup_{0 \leqslant s \leqslant t} \big(x_j(s) - x_{j-1}(s)\big) + \sup_{0 \leqslant v \leqslant s} \big(x_{j-1}(v) - \tau_{j-2}(v)\big) \\ &\leqslant \sum_{i=1}^{k} \Big\{ \sup_{0 \leqslant s \leqslant t} \big(x_i(s) - x_{i-1}(s)\big) \Big\} \\ &= \sum_{i=1}^{k} \Big\{ \sup_{0 \leqslant s \leqslant t} \big(\widehat{x}_{i-1}(s) - \widehat{x}_i(s)\big) \Big\} \end{aligned} \tag{5}$$

for $j = 1, 2, \ldots, k$.

From (2) and (5) we get

$$\begin{aligned} v_j(t) &= \sum_{i=1}^{j} Q_i(t) = \sum_{i=1}^{j} \{\tau_{i-1}(t) - \tau_i(t)\} \\ &= e(t) - \tau_j(t) = e(t) - x_j(t) + x_j(t) - \tau_j(t) \\ &\leqslant \widehat{x}_j(t) + \sum_{i=1}^{k} \Big\{ \sup_{0 \leqslant s \leqslant t} \big(\widehat{x}_{i-1}(s) - \widehat{x}_i(s)\big) \Big\}. \end{aligned} \tag{6}$$

Since for any j $(j = 1, 2, \ldots, k)$

$$\begin{aligned} v_j(t) &= e(t) - \tau_j(t) \geqslant e(t) - x_j(t) \\ &= \widehat{x}_j(t) \geqslant \widehat{x}_j(t) - \sum_{i=1}^{k} \Big\{ \sup_{0 \leqslant s \leqslant t} \big(\widehat{x}_{i-1}(s) - \widehat{x}_i(s)\big) \Big\}, \end{aligned}$$

we have from (5) following estimate

$$\big|v_j(t) - \widehat{x}_j(t)\big| \leqslant \sum_{i=1}^{k} \{ \sup_{0 \leqslant s \leqslant t} \big(\widehat{x}_{i-1}(s) - \widehat{x}_i(s)\big) \}. \tag{7}$$

Suppose $v_j^n(t) = \big(v_j(nt) - \alpha_j \cdot nt\big)/a(n)$ and $x_j^n(t) = \big(e(nt) - x_j(nt) - \alpha_j \cdot nt\big)/a(n)$ for $j = 1, 2, \ldots, k$.

By virtue Corollary 2.1 in Iglehart (1971a) for any fixed j $\{x_j^n,\ n \geqslant 3\}$ there is a relatively compact set in $(\mathcal{D}, D)$, and the set of its limit points is considers with $k(\widetilde{\sigma}_j)$. Then, in view of inequality (7) the family $\{v_j^n(t),\ n \geqslant 3\}$ is also a relatively compact set, and the set of its limit points is considers with $k(\widetilde{\sigma}_j)$.

Hence we prove

$$P\Big(\overline{\lim_{t\to\infty}} \frac{v_j(t) - \alpha_j \cdot t}{\widetilde{\sigma} \cdot a(t)} = 1\Big) = 1 \quad \text{and}$$
$$P\Big(\underline{\lim_{t\to\infty}} \frac{v_j(t) - \alpha_j \cdot t}{\widetilde{\sigma} \cdot a(t)} = -1\Big) = 1, \quad j = 1, 2, \ldots, k. \tag{8}$$

The proof is complete.

The theorem on LIL for the queue length of customers is proved similarly as Theorem 1.

Theorem 2. *If condition* (2) *is fulfilled, then*

$$P\Big(\overline{\lim_{t\to\infty}} \frac{Q_j(t) - (\alpha_j - \alpha_{j-1}) \cdot t}{\sigma_j \cdot a(t)} = 1\Big) = 1$$

and

$$P\Big(\underline{\lim_{t\to\infty}}\, Q_j(t) - (\alpha_j - \alpha_{j-1}) \cdot t = -1\Big) = 1, \quad \textit{for} \quad j = 1, 2, \ldots, k.$$

Proof. It follows from (7) that

$$\begin{aligned} \big|Q_j(t) - (\widehat{x}_j(t) - \widehat{x}_{j-1}(t))\big| &\leqslant \big|v_j(t) - \widehat{x}_j(t)\big| + \big|v_{j-1}(t) - \widehat{x}_{j-1}(t)\big| \\ &\leqslant 2\Big\{\sum_{i=1}^{k} \sup_{0 \leqslant s \leqslant t} (\widehat{x}_{i-1}(s) - \widehat{x}_i(s))\Big\} \quad \text{for} \quad j = 1, 2, \ldots, k. \end{aligned} \tag{9}$$

Define a family of random functions as $Q_j^n(t) = (Q_j(nt) - (\alpha_j - \alpha_{j-1}) \cdot nt)/a(n)$ and $\widetilde{x}_j^n(t) = \widehat{x}_j(nt) - \widehat{x}_{j-1}(nt)$ for $j = 1, 2, \ldots, k$ and $n \geqslant 3$.

Further proof of Theorem 2 is analogous to the proof of Theorem 1. The proof is complete.

REMARK. The results of Iglehart (1971a) in a single-device case follow from Theorem 2.

Finally, we will prove the theorem on LIL for the virtual waiting time of a customer.

Definitions of the random variables $t_n, z_n, S_n^{(j)}, e(t), x_j(t)$ and $\widehat{x}_j(t)$ for $j = 1, 2, \ldots, k$ are the same as in the proof of Theorem 1 and Theorem 2.

Let us define $\widehat{\alpha}_j = ES_1^{(j)}$, $\widehat{\alpha}_0 = Ez_1$, $\overline{\sigma}_j = DS_1^{(j)} > 0$, $\overline{\sigma}_0 = Dz_1 > 0$, $q_j^2 = \overline{\sigma}_j(\widehat{\alpha}_j)^{-1} + \overline{\sigma}_{j-1}(\widehat{\alpha}_{j-1})^{-3}\widehat{\alpha}_j^3 > 0$, $q_0^2 = 0$, $\widehat{\beta}_j = \widehat{\alpha}_j/\widehat{\alpha}_{j-1} - 1$ for $j = 1, 2, \ldots, k$.

Assume that condition (1) is fulfilled. Therefore, $\widehat{\beta}_j > 0$ for $j = 1, 2, \ldots, k$.

Also, let us define $w_j(t)$ as a virtual waiting time of a customer in the j-th phase of the MQS at time t (time, one must wait until a customer arrives to the j-th phase of the MQS to be served at time t); denote $S_j(t)$ as the time, which is the summary service of customers, arriving at customers, arriving at the jth phase of the MQS until time t for $j = 1, 2, \ldots, k$ and $t > 0$.

Note that $S_j(t) = \sum_{i=1}^{\tau_{j-1}(t)} S_i^{(j)}$ for $j = 1, 2, \ldots, k$ and $t > 0$.

Also, let

$$
\begin{aligned}
&y_j(t) = S_j(t) - t,\\
&f_t(y(\cdot)) = y(t) - \inf_{0 \leqslant s \leqslant t} y(s),\\
&\widehat{y}_j(t) = \sum_{i=1}^{x_{j-1}(t)} S_i^{(j)} - t,\\
&\widehat{w}_j(t) = f_t(\widehat{y}_j(\cdot)),\\
&x_0(t) = e(t) \quad \text{for} \quad j = 1, 2, \ldots, k \quad \text{and} \quad t > 0.
\end{aligned}
$$

If $S_j(0) = w_j(0) = 0$, then

$$
w_j(t) = f_t(y_j(\cdot)) \quad \text{for} \quad j = 1, 2, \ldots, k \quad \text{and} \quad t > 0 \tag{10}
$$

(see Borovkov, 1972, p. 41).

Theorem 3. *If condition* (1) *is fulfilled, then*

$$
P\Big(\overline{\lim_{t\to\infty}} \frac{w_j(t) - \widehat{\beta}_j t}{q_j \cdot a(t)} = 1\Big) = 1
$$

and

$$
P\Big(\underline{\lim_{t\to\infty}} \frac{w_j(t) - \widehat{\beta}_j t}{q_j \cdot a(t)} = -1\Big) = 1 \quad \textit{for} \quad j = 1, 2, \ldots, k.
$$

Proof. Denote a family of random functions as

$$
\begin{aligned}
y_j^n(t) &= \big(y_j(nt) - \widehat{\beta}_j \cdot nt\big)/a(n),\\
\widehat{y}_j^n(t) &= \big(\widehat{y}_j(nt) - \widehat{\beta}_j \cdot nt\big)/a(n),\\
w_j^n(t) &= \big(w_j(nt) - \widehat{\beta}_j \cdot nt\big)/a(n),\\
\widehat{w}_j^n(t) &= \big(\widehat{w}_j(nt) - \widehat{\beta}_j \cdot nt\big)/a(n),
\end{aligned}
\tag{11}
$$

for $j = 1, 2, \ldots, k$ and $n \geqslant 3$.

Analogously as Strassen (1964) and Iglehart (1971a) we prove the functional LIL for a compound renewal process: the family $\{\widehat{y}_j^n(t),\ n \geqslant 3\}$ is a relatively compact set, and the set of its limit points is consides with $k(q_j)$ for $j = 1, 2, \ldots, k$.

However,

$$\left|\widehat{w}_j(t) - w_j(t)\right| \leqslant 2 \sup_{0 \leqslant s \leqslant t} \left|\widehat{y}_j(s) - y_j(s)\right| \quad \text{for} \quad j = 1, 2, \ldots, k. \tag{12}$$

Therefore, making use of (12), we can get

$$\begin{aligned} d(w_j^n, \widehat{y}_j^n) &\leqslant \rho(w_j^n, \widehat{w}_j^n) + \rho(\widehat{w}_j^n, \widehat{y}_j^n) \leqslant 2\rho(y_j^n, \widehat{y}_j^n) + \rho(\widehat{w}_j^n, \widehat{y}_j^n) \\ &= 2\Big\{ \sup_{0 \leqslant s \leqslant n} |\widehat{y}_j(s) - y_j(s)|/a(n) \Big\} + \Big\{ \sup_{0 \leqslant s \leqslant n} (-\widehat{y}_j(s))/a(n) \Big\} \end{aligned} \tag{13}$$

for $j = 1, 2, \ldots, k$.

Note that $\left\{ \sup_{0 \leqslant s \leqslant n} (-\widehat{y}_j(s))/a(n) \right\} \geqslant 0$, and according to the law of large numbers for the compound renewal process $\lim_{t \to \infty}(-\widehat{y}_j(t)) = -\widehat{\beta}_j < 0$ almost everywhere for $j = 1, 2, \ldots, k$. Thus, similarly as in Iglehart (1971,a) we can prove that the second term in (13) also tends to zero.

Now we prove that the first term in (13) also tends to zero.

We get

$$\begin{aligned} \rho(y_j^n, \widehat{y}_j^n) &= \sup_{0 \leqslant s \leqslant n} \left|y_j(s) - \widehat{y}_j(s)\right|/a(n) = \sup_{0 \leqslant s \leqslant n} \sum_{l=\tau_{j-1}(s)}^{x_{j-1}(s)} S_l^{(j)}/a(n) \\ &\leqslant \sup_{0 \leqslant s \leqslant n} \Big\{ \sum_{l=\tau_{j-1}(s)}^{x_{j-1}(s)} (S_l^{(j)} - \widehat{\alpha}_j) \Big\} \Big/ a(n) \\ &\quad + \widehat{\alpha}_j \sup_{0 \leqslant s \leqslant n} \big(x_{j-1}(s) - \tau_{j-1}(s)\big)/a(n) \\ &= \sup_{0 \leqslant s \leqslant n} \Big\{ \sum_{l=x_{j-1}(s) - \sup_{0 \leqslant r \leqslant s} (x_{j-1}(r) - \tau_{j-2}(r))}^{x_{j-1}(s)} (S_l^{(j)} - \widehat{\alpha}_j) \Big\} \Big/ a(n) \\ &\quad + \widehat{\alpha}_j \cdot \sup_{0 \leqslant s \leqslant n} (x_{j-1}(s) - \tau_{j-1}(s))/a(n) \text{ for } j = 1, 2, \ldots, k. \end{aligned} \tag{14}$$

Estimate (5) implies that

$$0 \leqslant \sup_{0 \leqslant s \leqslant t} \big(x_{j-1}(s) - \tau_{j-1}(s)\big) \leqslant \sum_{i=1}^{k} \big\{ \sup_{0 \leqslant s \leqslant t} \big(\widehat{x}_{i-1}(s) - \widehat{x}_i(s)\big) \big\} \tag{15}$$

for $j = 1, 2, \ldots, k$.

Using estimate (15), we prove that the second term in (14) also tends to zero.

It follows from (5) that

$$0 \leqslant \sup_{0\leqslant s\leqslant t} \big(x_{j-1} - \tau_{j-2}(s)\big) \leqslant \sup_{0\leqslant s\leqslant t} \big(x_{j-1}(s) - x_{j-2}(s)\big)$$
$$+ \sup_{0\leqslant s\leqslant t} \big(x_{j-2}(s) - \tau_{j-2}(s)\big) \leqslant \sum_{j=2}^{k+1} \Big\{ \sup_{0\leqslant s\leqslant t} \big(x_{j-1}(s) - x_{j-2}(s)\big) \Big\}$$
$$= \sum_{j=1}^{k} \Big\{ \sup_{0\leqslant s\leqslant t} \big(\widehat{x}_{j-1}(s) - \widehat{x}_{j}(s)\big) \Big\} \quad \text{for} \quad j = 1, 2, \ldots, k. \tag{16}$$

Note that

$$\frac{1}{n} \cdot \sup_{0\leqslant t\leqslant 1} x_j(nt) \to \widehat{\alpha}_j < \infty \quad \text{for} \quad j = 0, 1, 2, \ldots, k$$

almost everywhere (see Borovkov, 1980, p. 100).

We will prove that

$$\frac{1}{n} \cdot \sup_{0\leqslant s\leqslant n} \big(x_{j-1}(s) - \tau_{j-2}(s)\big)$$

tends to zero almost everywhere. Then, in accordance with the proof of Theorem 3.3 in Iglehart (1971a) the first term in inequality (14) tends to zero almost everywhere.

Note that by (5)

$$0 \leqslant \sup_{0\leqslant s\leqslant n} \big(x_{j-1}(s) - \tau_{j-2}(s)\big)/n \leqslant \sup_{0\leqslant s\leqslant n} \big(x_{j-1}(s) - \tau_{j-2}(s)\big)/a(n)$$
$$\leqslant \sum_{j=1}^{k} \Big\{ \sup_{0\leqslant s\leqslant n} \big(\widehat{x}_{j-1}(s) - \widehat{x}_{j}(s)\big) \Big\} / a(n) \quad \text{for} \quad j = 1, 2, \ldots, k.$$

Using $\alpha_{j-1} - \alpha_j < 0$ for $j = 1, 2, \ldots, k$ and just like in (15), we prove that the first term in inequality (14) tends to zero.

Thus, $\rho(y_j^n, \widehat{y}_j^n)$ tends to zero for $j = 1, 2, \ldots, k$.

Finally, it follows from (13) that $d(w_j^n, \widehat{y}_j^n)$ also tends to zero (for $j = 1, 2, \ldots, k$).

Hence we obtain that family $\{w_j^n,\ n \geqslant 3\}$ is a relatively compact set, and, the set of its limit points is consides with $k(q_j)$ for $j = 1, 2, \ldots, k$.

Also similarly as in the proof of Theorem 1 we can prove

$$P\left(\overline{\lim_{t\to\infty}}\,\frac{w_j(t)-\widehat{\beta}_j\cdot t}{q_j\cdot a(t)}=1\right)=1$$

and

$$P\left(\underline{\lim}_{t\to\infty}\,\frac{w_j(t)-\widehat{\beta}_j\cdot t}{q_j\cdot a(t)}=-1\right)=1 \quad \text{for} \quad j=1,2,\dots,k.$$

The proof is complete.

3. Concluding remarks. The theorems of this work are proved for a class of MQS in heavy traffic with the service discipline "first come, first served", endless waiting time of a customer in each phase of the queue, when times between arriving customers to MQS are independent identically distributed random variables. However, similar limit theorems can be applied to a wider class of MQ S in heavy traffic: when arrival and service of customers in a queue is by group, when times between the arriving customers to the MQS are independent and weakly dependent random variables, etc.

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APIE KARTOTINIO LOGARITMO DĖSNĮ DAUGIAFAZĖSE MASINIO APTARNAVIMO SISTEMOSE. II

Saulius MINKEVIČIUS

Įrodyti kartotinio logaritmo dėsniai suminiam paraiškų ilgiui, paraiškų eilės ilgiui ir virtualiniam laukimo laikui daugiafazėse masinio aptarnavimo sistemose.