

STATISTICAL PROPERTIES OF PIPELINED-BLOCK LINEAR TIME-VARYING DISCRETE-TIME SYSTEMS IN STATE SPACE

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Abstract. Statistical properties are examined for a class of pipelined-block linear time-varying (LTV) and linear time-invariant (LTI) discrete-time systems. Pipelined-block equations are derived, using the general solution of LTV discrete-time system in state space. Afterwards, we analysed the state covariance and output covariance matrices of pipelined-block LTV and LTI discrete-time systems in state space. For this class of pipelined-block realizations expressions are found for calculation of characteristics of the roundoff noise. Finally, scaling in the pipelined LTV discrete-time systems in state space is considered.

Key words: linear, discrete systems, pipelining, block processing, state space, statistical properties.

1. Introduction. Pipelining and block processing are two of several algorithmic transformation techniques that can be used to exploit the concurrency within a digital signal processing algorithm to improve its operating speed (Lucke and Parhi, 1994). Pipelining (Chung and Parhi, 1994; Parhi and Messerschmitt, 1989b; Lucke and Parhi, 1994; Parhi and Messerschmitt, 1989c; Jump and Ahuja, 1978; Cappello and Steiglitz, 1983; Lim and Lui, 1992) increases the speed of a system at the expense of latency. Block processing (Parhi and Messerschmitt, 1989a; Burrus, 1971; Barnes and Shinnaka, 1980a; Azimi-Sadjadi and King, 1986; Azimi-Sajadi and Rostampour, 1989; Nikias, 1984) is a form of parallel processing which transforms a scalar system into a block system. Block processing has been applied to numerous areas in digital signal processing (Burrus, 1971; Meyer and Burrus, 1976; Lu *et al.*, 1985; Barnes and Shinnaka, 1980a) and control (Khorasani and Azimi-Sadjadi, 1987). This is, to a great extent, due to its advantages in performing parallel processing

with an increased throughput rate, increased computation efficiency, reduced roundoff error, and sensitivity performance (Burrus, 1971; Meyer and Burrus, 1976; Barnes and Shinnaka, 1980a).

When systems are implemented on general-purpose computers or with special-purpose hardware, numbers are represented by a sequence of finite word-length binary digits. Finite wordlength representation of numbers causes inaccuracies of system coefficients, rounding or truncation after multiplication, and overflow of addition. Furthermore, rounding, truncation, and overflow cause undesirable oscillations in recursive digital systems. The effects of the quantization of digital systems can be classified as roundoff noise, coefficient sensitivity, and limit cycles, all of which deteriorate system performance. The problems of quantization effects have received a great attention for the second-order direct form realization, since it is the basic section used in high-order cascade and parallel forms with low roundoff noise and low coefficient sensitivity. Quantization effects are well reviewed in (Liu, 1971; Oppenheim, 1972; Claassen *et al.*, 1976).

Synthesis of digital systems with respect to quantization effects is an important problem, since quantization effects depend on system structures. It is well known that the state space approach is the most effective method that can be used to find the optimum structures for narrow-bandwidth filters (Hwang, 1976, 1977; Mullis and Roberts, 1976; Jackson *et al.*, 1979; Barnes, 1979).

One disadvantage of the state space approach is that the resultant structures require more multipliers than the direct forms. However, this does not always mean a demerit of the state space approach in practical applications (Kawamata and Higuchi, 1985). While most of the literature dealing with quantization effects in state space digital systems have studied roundoff noise and limit cycles, the output error due to the coefficient quantization has not yet been analysed in a state space formulation for pipelined-block systems.

In Section 2, using the general solution of LTV discrete-time systems in state space, we got pipelined-block equations. In Section 3, we derived general expressions of state covariance and output covariance matrices for pipelined-block LTV discrete-time systems in state space. Roundoff noise in pipelined-block LTV and LTI discrete-time systems in state space is considered in Section 4. Finally, in Section 5, we have analysed scaling in the LTV discrete-time systems in state space.

2. Pipelined-block LTV discrete-time state space model. We derive here equations for pipelining and block processing of LTV, LPTV, and LTI discrete-time systems in state space.

Consider an LTV discrete-time system described by

$$x(k+1) = A(k)x(k) + b(k)u(k), \quad (1a)$$

$$y(k) = c^T(k)x(k) + d(k)u(k), \quad k = 0, 1, 2, \dots, \quad (1b)$$

where the state $x(k)$ is $N \times 1$, the state update matrix $A(k)$ is $N \times N$, $b(k)$ and $c(k)$ are $N \times 1$; $d(k)$, the input sample $u(k)$ and the output sample $y(k)$ are scalars, and N is the order of the system.

The solution of the dynamic equation (1a) is given by Chui and Chen (1991)

$$x(n) = F(n, k)x(k) + \sum_{j=k}^{n-1} F(n, j+1)b(j)u(j), \quad n = 0, 1, 2, \dots, \quad (2)$$

where for the $N \times N$ state transition matrix $F(n, k)$ the following relationships

$$F(n, n) = I_N, \quad \text{and} \quad F(n+1, k) = A(n)F(n, k) \quad (3)$$

hold.

From Eq. 3, we get

$$F(k+1, k) = A(k), \quad F(k+2, k) = A(k+1)A(k),$$

so

$$F(n, k) = \prod_{i=1}^{n-k} A(n-i), \quad \text{and} \quad F(n, j+1) = \prod_{i=1}^{n-j-1} A(n-i). \quad (4)$$

Substituting $x(n)$ from (2) into (1b), we have

$$y(n) = c^T(n)F(n, k)x(k) + \sum_{j=k}^{n-1} c^T(n)F(n, j+1)b(j)u(j) + d(n)u(n), \quad n = 0, 1, 2, \dots \quad (5)$$

Substituting $n = kM + M$ and $k = kM$, $k = 0, 1, 2, \dots$, where M is the pipelining level, into (2), we obtain

$$\begin{aligned}
 x(kM + M) &= F(kM + M, kM)x(kM) \\
 &+ \sum_{j=kM}^{kM+M-1} F(kM + M, j+1)b(j)u(j) \\
 &= F(kM + M, kM)x(kM) \\
 &+ \sum_{j=1}^M F(kM + M, kM + j)b(kM + j - 1)u(kM + j - 1), \quad (6)
 \end{aligned}$$

where

$$\begin{aligned}
 F(kM + M, kM) &= \prod_{i=1}^M A(kM + M - i), \\
 F(kM + M, kM + j) &= \prod_{i=1}^{M-j} A(kM + M - i).
 \end{aligned}$$

Then we get from (6) the state equation of a pipelined LTV discrete-time system in matrix form:

$$\bar{x}(k+1) = \bar{A}(k)\bar{x}(k) + \bar{B}(k)\bar{u}(k), \quad k = 0, 1, 2, \dots, \quad (7)$$

where the $N \times N$ matrix $\bar{A}(k)$ is defined by

$$\bar{A}(k) = \prod_{i=1}^M A(kM + M - i). \quad (8)$$

The $N \times M$ matrix $\bar{B}(k)$ is defined by

$$\bar{B}(k) = [B_1, \dots, B_j, \dots, B_M], \quad (9)$$

in which

$$\begin{aligned}
 B_j &= \prod_{i=1}^{M-j} A(kM + M - i)b(kM + j - 1), \quad j = 1, 2, \dots, M - 1, \\
 B_M &= b(kM + M - 1), \\
 \bar{u}(k) &= [u(kM), u(kM + 1), \dots, u(kM + M - 1)]^T, \\
 \bar{x}(k) &= x(kM), \quad \bar{x}(k+1) = x[(k+1)M].
 \end{aligned}$$

Substituting $n = kM + i$, and $k = kM, k = 0, 1, 2, \dots, i = 0, 1, \dots, M - 1$ into (5), we obtain

$$\begin{aligned}
 y(kM + i) &= c^T(kM + i)F(kM + i, kM)x(kM) \\
 &\quad + \sum_{j=kM}^{kM+i-1} c^T(kM + i)F(kM + i, j + 1)b(j)u(j) \\
 &\quad + d(kM + i)u(kM + i), \quad i = 0, 1, \dots, M - 1,
 \end{aligned}$$

or

$$\begin{aligned}
 y(kM + i - 1) &= c^T(kM + i - 1)F(kM + i - 1, kM)x(kM) \\
 &\quad + \sum_{j=1}^i c^T(kM + i - 1)F(kM + i - 1, kM + j)b(kM + j - 1) \\
 &\quad \times u(kM + j - 1) + d(kM + i - 1)u(kM + i - 1), \quad (10) \\
 &\quad i = 1, 2, \dots, M,
 \end{aligned}$$

where

$$F(kM + i - 1, kM) = \prod_{j=1}^{i-1} A(kM + i - j - 1) = \prod_{j=2}^i A(kM + i - j),$$

and

$$F(kM + i - 1, kM + j) = \prod_{l=1}^{i-j-1} A(kM + i - l - 1) = \prod_{l=2}^{i-j} A(kM + i - l).$$

Then from (10) we get the output equation of pipelined-block LTV discrete-time system in matrix form:

$$\bar{y}(k) = \bar{C}(k)\bar{x}(k) + \bar{D}(k)\bar{u}(k), \quad k = 0, 1, 2, \dots, \quad (11)$$

where the $M \times N$ matrix $\bar{C}(k)$ is defined by

$$\bar{C}(k) = [C_1, \dots, C_i, \dots, C_M]^T, \quad (12)$$

in which

$$\begin{aligned}
 C_1 &= c^T(kM), \\
 C_i &= c^T(kM + i - 1) \prod_{j=2}^i A(kM + i - j), \quad i = 2, 3, \dots, M.
 \end{aligned}$$

The $M \times M$ matrix $\bar{D}(k)$ is defined by

$$\bar{D}(k) = [d_{ij}], \quad i, j = 1, 2, \dots, M, \quad (13)$$

in which

$$d_{ij} = 0, \quad \text{if } i < j;$$

$$d_{ij} = d(kM + i - 1), \quad \text{if } i = j;$$

$$d_{ij} = c^T(kM + i - 1)b(kM + j - 1), \quad \text{if } i = j + 1;$$

$$d_{ij} = c^T(kM + i - 1) \prod_{l=2}^{i-j} A(kM + i - l)b(kM + j - 1), \quad \text{if } i > j + 1;$$

$$\bar{y}(k) = [y(kM), y(kM + 1), \dots, y(kM + M - 1)]^T.$$

For LPTV discrete-time systems, $A(k)$, $b(k)$, $c^T(k)$, and $d(k)$ are L -periodic, i.e., $A(k + L) = A(k)$, $b(k + L) = b(k)$, $c^T(k + L) = c^T(k)$, and $d(k + L) = d(k)$. In case the pipelining level M is equal to the periodicity L of a LPTV system, we get, from Eqs. 8 and 9, simpler expressions for calculating matrices $\bar{A}(k) = \bar{A}$ and $\bar{B}(k) = \bar{B}$

$$\bar{A} = \prod_{i=1}^L A(L - i), \quad (14)$$

$$\bar{B} = [B_1, \dots, B_j, \dots, B_L], \quad (15)$$

in which

$$B_j = \prod_{i=1}^{L-j} A(L - i)b(j - 1), \quad j = 1, 2, \dots, L - 1,$$

$$B_L = b(L - 1).$$

In the case the periodicity is equal to the block size, for the LPTV discrete-time system, we obtain simpler expressions for calculating $\bar{C}(k) = \bar{C}$, and $\bar{D}(k) = \bar{D}$. Then, using Eqs. 12 and 13, we get

$$\bar{C} = [C_1, \dots, C_i, \dots, C_L]^T, \quad (16)$$

in which

$$C_1 = c^T(0),$$

$$C_i = c^T(i - 1) \prod_{j=2}^i A(i - j), \quad i = 2, 3, \dots, L,$$

and the $L \times L$ matrix

$$\bar{D} = [d_{ij}], \quad (17)$$

in which

$$\begin{aligned} d_{ij} &= 0, \quad \text{if } i < j; \\ d_{ij} &= d(i-1), \quad \text{if } i = j; \\ d_{ij} &= c^T(i-1)b(j-1), \quad \text{if } i = j + 1; \\ d_{ij} &= c^T(i-1) \prod_{l=2}^{i-j} A(i-l)b(j-1), \quad \text{if } i > j + 1. \end{aligned}$$

Eq. 7 with matrices (8) and (9) can be used for pipelining of the N th order LTV discrete-time system and N th order LPTV discrete-time system with an arbitrary pipelining level M . In expressions (12) and (13) M is the block size, which is not the same (or may be the same) as the periodicity L of the LPTV discrete-time system.

For LTI discrete-time systems, $A(k) = A$, $b(k) = b$, $c^T(k) = c^T$, and $d(k) = d$. Hence, matrices (14), (15), (16), and (17) are of the forms, respectively

$$\bar{A}(k) = \bar{A} = A^L, \quad (18)$$

$$\bar{B}(k) = \bar{B} = [A^{L-1}b, \dots, Ab, b], \quad (19)$$

$$\bar{C}(k) = \bar{C} = [c^T, c^T A, \dots, c^T A^{L-1}]^T, \quad (20)$$

$$\bar{D}(k) = \bar{D} = [d_{ij}], \quad (21)$$

in which

$$\begin{aligned} d_{ij} &= 0, \quad \text{if } i < j; \\ d_{ij} &= d, \quad \text{if } i = j; \\ d_{ij} &= c^T b, \quad \text{if } i = j + 1; \\ d_{ij} &= c^T A^{i-j-1} b, \quad \text{if } i > j + 1. \end{aligned}$$

3. Statistical characteristics of pipelined-block LTV discrete-time systems in state space. Consider a pipelined-block LTV discrete-time system in state space described by

$$\bar{x}(k+1) = \bar{A}(k)\bar{x}(k) + \bar{B}(k)\bar{u}(k) + \bar{v}(k), \quad (22a)$$

$$\bar{y}(k) = \bar{C}(k)\bar{x}(k) + \bar{D}(k)\bar{u}(k) + \bar{w}(k), \quad k = 0, 1, 2, \dots, \quad (22b)$$

where $\bar{A}(k)$, $\bar{B}(k)$, $\bar{C}(k)$, and $\bar{D}(k)$ are $N \times N$, $N \times M$, $M \times N$, and $M \times M$ matrices defined in (8), (9), (12), and (13), respectively; $\bar{x}(k+1) = x[(k+1)M]$, $\bar{x}(k) = x(kM)$, $\bar{u}(kM) = [u(kM), \dots, u(kM+M-1)]^T$, $\bar{y}(kM) = [y(kM), \dots, y(kM+M-1)]^T$; $\bar{v}(k)$ and $\bar{w}(k)$ are $N \times 1$ and $M \times 1$ zero-mean white noise vectors, respectively. For noise vectors, it holds $E[\bar{v}(n)\bar{v}^T(m)] = K_{\bar{v}}(n)\delta(n-m)$, $E[\bar{v}(k)] = 0$; $E[\bar{w}(n)\bar{w}^T(n)] = K_{\bar{w}}(n)\delta(n-m)$, $E[\bar{w}(k)] = 0$, where E is the expectation; $\delta(n-m) = 1$, if $n = m$ and $\delta(n-m) = 0$, if $n \neq m$.

3.1. State covariance matrix. The solution of state equation (22a) is given by

$$\begin{aligned} \bar{x}(n) = & \Phi(n, k_0)\bar{x}(k_0) + \sum_{k=k_0}^{n-1} \Phi(n, k+1)\bar{B}(k)\bar{u}(k) \\ & + \sum_{k=k_0}^{n-1} \Phi(n, k+1)\bar{v}(k), \quad n = 0, 1, 2, \dots, \end{aligned} \quad (23)$$

where

$$\begin{aligned} \Phi(n, k_0) = & \prod_{i=1}^{n-k_0} \bar{A}(n-i), \quad \Phi(n, k+1) = \prod_{i=1}^{n-k-1} \bar{A}(n-i), \\ \bar{x}(n) = & x(nM), \quad \bar{x}(k_0) = x(k_0M). \end{aligned}$$

The expected mean value of the state variable $\bar{x}(n)$ is as follows:

$$E[\bar{x}(n)] = \Phi(n, k_0)E[\bar{x}(k_0)] + \sum_{k=k_0}^{n-1} \Phi(n, k+1)\bar{B}(k)\bar{u}(k). \quad (24)$$

Subtracting (24) from (23), we get

$$\hat{\bar{x}}(n) = \Phi(n, k_0)\hat{\bar{x}}(k_0) + \sum_{k=k_0}^{n-1} \Phi(n, k+1)\bar{v}(k),$$

where

$$\hat{\bar{x}}(n) = \bar{x}(n) - E[\bar{x}(n)].$$

Assume, that initial values $E[\bar{x}(k_0)]$ and $R_{\bar{x}}(k_0) = E[\bar{x}(k_0)\bar{x}^T(k_0)]$ are known; the state $\bar{x}(k_0)$ and the noise vector are uncorrelated. In this case the expression of the state variables covariance matrix is given by

$$\begin{aligned} K_{\bar{x}}(n, m) &= E \left[\bar{x}(n) \bar{x}^T(m) \right] \\ &= E \left[\Phi(n, k_0) \bar{x}(k_0) \bar{x}^T(k_0) \Phi^T(m, k_0) \right. \\ &\quad + \Phi(n, k_0) \bar{x}(k_0) \left(\sum_{k=k_0}^{m-1} \bar{v}^T(k) \Phi^T(m, k+1) \right) \\ &\quad + \left(\sum_{k=k_0}^{n-1} \Phi(n, k+1) \bar{v}(k) \right) \bar{x}^T(k_0) \Phi^T(m, k_0) \\ &\quad \left. + \sum_{k=k_0}^{n-1} \sum_{l=k_0}^{m-1} \Phi(n, k+1) \bar{v}(k) \bar{v}^T(l) \Phi^T(m, l+1) \right]. \end{aligned}$$

Under the assumption that the vectors $\bar{x}(k_0)$ and $\bar{v}(m)$ are uncorrelated, the state covariance matrix can now be expressed as

$$\begin{aligned} K_{\bar{x}}(n, m) &= \Phi(n, k_0) R_{\bar{x}}(k_0) \Phi^T(m, k_0) \\ &\quad + \sum_{k=k_0}^{n-1} \sum_{l=k_0}^{m-1} \Phi(n, k+1) K_{\bar{v}}(k) \delta(k-l) \Phi^T(m, l+1). \quad (25) \end{aligned}$$

It follows from (25) that the state covariance matrix is given by

$$\begin{aligned} K_{\bar{x}}(n, m) &= \Phi(n, k_0) R_{\bar{x}}(k_0) \Phi^T(m, k_0) \\ &\quad + \sum_{k=k_0}^{m-1} \Phi(n, k+1) K_{\bar{v}}(k) \Phi^T(m, k+1), \quad \text{if } n > m, \quad (26a) \end{aligned}$$

$$\begin{aligned} K_{\bar{x}}(n, m) &= \Phi(n, k_0) R_{\bar{x}}(k_0) \Phi^T(m, k_0) \\ &\quad + \sum_{k=k_0}^{n-1} \Phi(n, k+1) K_{\bar{v}}(k) \Phi^T(m, k+1), \quad \text{if } n < m, \quad (26b) \end{aligned}$$

$$\begin{aligned} K_{\bar{x}}(n, n) &= \Phi(n, k_0) R_{\bar{x}}(k_0) \Phi^T(m, k_0) \\ &\quad + \sum_{k=k_0}^{n-1} \Phi(n, k+1) K_{\bar{v}}(k) \Phi^T(n, k+1), \quad \text{if } m = n. \quad (26c) \end{aligned}$$

For the pipelined LTI discrete-time system $\bar{A}(n) = \bar{A} = A^M$, $\bar{B}(n) = \bar{B} = [A^{M-1}b, \dots, Ab, b]$ and

$$\Phi(n, k_0) = \prod_{i=1}^{n-k_0} \bar{A}(n-i) = \bar{A}^{n-k_0} = P\Lambda^{n-k_0}S, \quad (27)$$

where Λ is an $N \times N$ diagonal matrix $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$ in which λ_i are eigenvalues of the matrix \bar{A} , P is an $N \times N$ matrix which is formed of the right row eigenvectors of the matrix \bar{A} , S is an $N \times N$ matrix which is formed of the left row eigenvalues of \bar{A} .

Assuming $k_0 = 0$ and substituting (27) into (26), one can show that

$$\begin{aligned} R_{\bar{x}}(n) &= P\Lambda^n S R_{\bar{x}}(0) S^T \Lambda^n P^T \\ &\quad + \sum_{k=0}^{n-1} P\Lambda^{n-k-1} S K_{\bar{v}} S^T \Lambda^{n-k-1} P^T, \end{aligned} \quad (28a)$$

$$\begin{aligned} K_{\bar{x}}(n, m) &= P\Lambda^n S R_{\bar{x}}(0) S^T \Lambda^m P^T \\ &\quad + \sum_{k=0}^{m-1} P\Lambda^{n-k-1} S K_{\bar{v}} S^T \Lambda^{n-k-1} P^T, \quad \text{if } n > m, \end{aligned} \quad (28b)$$

$$\begin{aligned} K_{\bar{x}}(n, m) &= P\Lambda^n S R_{\bar{x}}(0) S^T \Lambda^m P^T \\ &\quad + \sum_{k=0}^{n-1} P\Lambda^{n-k-1} S K_{\bar{v}} S^T \Lambda^{n-k-1} P^T, \quad \text{if } n < m. \end{aligned} \quad (28c)$$

Using expressions (3), we can write

$$\begin{aligned} \Phi(n, n) &= I_N, \quad \Phi(n+1, n) = \bar{A}(n), \quad \text{and} \\ \Phi(n+1, m) &= \bar{A}(n)\Phi(n, m). \end{aligned} \quad (29)$$

Then, from (26) and (29), we have recursive equations for calculating the state covariance matrix

$$K_{\bar{x}}(n+1, m) = \bar{A}(n)K_{\bar{x}}(n, m), \quad \text{if } n > m, \quad (30a)$$

$$K_{\bar{x}}(n, m+1) = K_{\bar{x}}(n, m)\bar{A}^T(m), \quad \text{if } n < m, \quad (30b)$$

$$R_{\bar{x}}(n, n) = \bar{A}(n-1)R_{\bar{x}}(n-1)\bar{A}^T(n-1) + K_{\bar{v}}(n-1), \quad (30c)$$

$$R_{\bar{x}}(0) = 0.$$

So the state covariance matrix is computed in two steps: 1) using (30c), we compute $R_{\bar{x}}(n, n)$; 2) keeping in mind, that $K_{\bar{x}}(n, n) = R_{\bar{x}}(n)$, we solve (30a) and (30b).

For stable pipelined LTI discrete-time system in a steady state (i.e., $n, m \rightarrow \infty$) define $n - m = l$. Then, from (30), we get

$$K_{\bar{x}}(l + 1) = \bar{A}K_{\bar{x}}(l), \quad \text{if } l > 0, \tag{31a}$$

$$K_{\bar{x}}(l - 1) = K_{\bar{x}}(l)\bar{A}^T, \quad \text{if } l < 0, \tag{31b}$$

$$R_{\bar{x}} = \bar{A}R_{\bar{x}}\bar{A}^T + K_{\bar{v}}, \quad K_{\bar{x}}(0) = R_{\bar{x}}. \tag{31c}$$

3.2. Output covariance matrix. Substituting (23) into (22b), we have

$$\begin{aligned} \bar{y}(n) = & \bar{C}(n)\Phi(n, k_0)\bar{x}(k_0) + \sum_{k=k_0}^{n-1} \bar{C}(n)\Phi(n, k + 1)\bar{B}(k)\bar{u}(k) \\ & + \sum_{k=k_0}^{n-1} \bar{C}(n)\Phi(n, k + 1)\bar{v}(k) + \bar{D}(n)\bar{u}(n) + \bar{w}(n), \tag{32} \\ & n = 0, 1, 2, \dots \end{aligned}$$

The expected mean value of the output $\bar{y}(n)$ is defined by

$$\begin{aligned} E[\bar{y}(n)] = & \bar{C}(n)\Phi(n, k_0)E[\bar{x}(k_0)] \\ & + \sum_{k=k_0}^{n-1} \bar{C}(n)\Phi(n, k + 1)\bar{B}(k)\bar{u}(k) + \bar{D}(n)\bar{u}(n). \tag{33} \end{aligned}$$

Subtracting (33) from (32), we get

$$\begin{aligned} \overset{\circ}{\bar{y}}(n) = & \bar{y}(n) - E[\bar{y}(n)] = \bar{C}(n)\Phi(n, k_0)\overset{\circ}{\bar{x}}(k_0) \\ & + \sum_{k=k_0}^{n-1} \bar{C}(n)\Phi(n, k + 1)\bar{v}(k) + \bar{w}(n), \tag{34} \end{aligned}$$

where

$$\overset{\circ}{\bar{x}}(k_0) = \bar{x}(k_0) - E[\bar{x}(k_0)].$$

The output covariance matrix is given by

$$\begin{aligned}
K_{\bar{y}}(n, m) &= E \left[\overset{\circ}{\bar{y}}(n) \overset{\circ}{\bar{y}}^T(m) \right] \\
&= E \left[\bar{C}(n) \Phi(n, k_0) \overset{\circ}{\bar{x}}(k_0) \overset{\circ}{\bar{x}}^T(k_0) \Phi^T(m, k_0) \bar{C}^T(m) \right. \\
&\quad + \bar{C}(n) \Phi(n, k_0) \overset{\circ}{\bar{x}}(k_0) \sum_{k=k_0}^{m-1} \bar{v}^T(k) \Phi^T(m, k+1) \bar{C}^T(m) \\
&\quad + \bar{C}(n) \Phi(n, k_0) \overset{\circ}{\bar{x}}(k_0) \bar{w}^T(m) \\
&\quad + \left(\sum_{k=k_0}^{n-1} \bar{C}(n) \Phi(n, k+1) \bar{v}(k) \right) \overset{\circ}{\bar{x}}^T(k_0) \Phi^T(m, k_0) \bar{C}^T(m) \\
&\quad + \sum_{k=k_0}^{n-1} \sum_{l=l_0}^{m-1} \bar{C}(n) \Phi(n, k+1) \bar{v}(k) \bar{v}^T(l) \Phi^T(m, l+1) \bar{C}^T(m) \\
&\quad + \left(\sum_{k=k_0}^{n-1} \bar{C}(n) \Phi(n, k+1) \bar{v}(k) \right) \bar{w}^T(m) \\
&\quad + \bar{w}(n) \overset{\circ}{\bar{x}}^T(k_0) \Phi^T(m, k_0) \bar{C}^T(m) \\
&\quad \left. + \bar{w}(n) \sum_{k=k_0}^{m-1} \bar{v}^T(k) \Phi^T(m, k+1) \bar{C}^T(m) + \bar{w}(n) \bar{w}^T(m) \right].
\end{aligned}$$

Under the assumption that the vectors $\bar{x}(k_0)$, $\bar{v}(n)$, and $\bar{w}(n)$ are uncorrelated, the output covariance matrix can now be expressed as follows

$$\begin{aligned}
K_{\bar{y}}(n, m) &= E \left[\bar{C}(n) \Phi(n, k_0) \overset{\circ}{\bar{x}}(k_0) \overset{\circ}{\bar{x}}^T(k_0) \Phi^T(m, k_0) \bar{C}^T(m) \right. \\
&\quad + \sum_{k=k_0}^{n-1} \sum_{l=l_0}^{m-1} \bar{C}(n) \Phi(n, k+1) \bar{v}(k) \bar{v}^T(l) \Phi^T(m, l+1) \bar{C}^T(m) \\
&\quad \left. + \bar{w}(n) \bar{w}^T(m) \right] \tag{35} \\
&= \bar{C}(n) \Phi(n, k_0) R_{\bar{x}}(k_0) \Phi^T(m, k_0) \bar{C}^T(m) \\
&\quad + \sum_{k=k_0}^{n-1} \bar{C}(n) \Phi(n, k+1) K_{\bar{v}}(k) \Phi^T(m, k+1) \bar{C}^T(m) + K_{\bar{w}}(n, m),
\end{aligned}$$

where

$$\Phi(n, k_0) = \prod_{i=1}^{n-k_0} \bar{A}(n-i), \quad \Phi(n, k+1) = \prod_{i=1}^{n-k-1} \bar{A}(n-i),$$

$$R_{\bar{x}}(k_0) = E \left[\bar{x}^{\circ}(k_0) \bar{x}^{\circ T}(k_0) \right], \quad K_{\bar{w}}(n, m) = E \left[\bar{w}(n) \bar{w}^T(m) \right].$$

4. Roundoff noise in pipelined-block LTV discrete-time systems in state space. In digital discrete-time systems implemented with fixed point arithmetic, after multiplication of two numbers, the roundoff operation is performed through the limited register length. For recursive discrete-time systems, this roundoff can cause roundoff limit cycles, even though the discrete-time system is stable.

In this section we shall statistically model the effects of the roundoff as stationary zero mean white noise. We shall derive general expressions for the autocovariance matrix of the roundoff noise in the vector output of pipelined-block state space realization.

We assume that the pipelined-block discrete-time system is stable, and the roundoff is performed only at the outputs of state variable summing nodes, and at the outputs of the summing nodes at the discrete-time system output. We model the effects of roundoff as stationary zero mean white noise sources.

In the additive roundoff noise model, we can apply superposition and compute contribution to output due to noise sources separately. Thus, we obtain the following equations as a model of the roundoff noise:

$$\bar{x}(k+1) = \bar{A}(k)\bar{x}(k) + \bar{v}(k), \quad (36a)$$

$$\bar{y}(k) = \bar{C}(k)\bar{x}(k) + \bar{w}(k), \quad k = 0, 1, 2, \dots, \quad (36b)$$

where $\bar{A}(k)$, $\bar{C}(k)$, $\bar{x}(k)$, $\bar{y}(k)$, $\bar{v}(k)$, and $\bar{w}(k)$ are the same as in (22).

The solution of dynamic equation (36a) is given by

$$\bar{x}(n) = \Phi(n, k_0)\bar{x}(k_0) + \sum_{k=k_0}^{n-1} \Phi(n, k+1)\bar{v}(k), \quad (37)$$

or

$$\bar{x}(n) = \Phi(n, k_0)\bar{x}(k_0) + \sum_{k=k_0}^{n-1} \Phi(n, n-k)\bar{v}(n-k-1), \quad n = 0, 1, 2, \dots, \quad (38)$$

where

$$\Phi(n, k_0) = \prod_{i=1}^{n-k_0} \bar{A}(n-i),$$

$$\Phi(n, n-k) = \prod_{i=1}^k \bar{A}(n-i).$$

For the LTV discrete-time system $\bar{A}(n)$ is computed using (8), and for LPTV discrete-time system $\bar{A}(n)$ is computed using (14).

From (36b) and (37), we have

$$\bar{y}(n) = \bar{C}(n)\Phi(n, k_0)\bar{x}(k_0) + \sum_{k=k_0}^{n-1} \bar{C}(n)\Phi(n, k+1)\bar{v}(k) + \bar{w}(n), \quad (39)$$

where for the pipelined-block LTV discrete-time system $\bar{C}(n)$ is computed using (12), and for LPTV discrete-time system $\bar{C}(n)$ is computed using (16).

Since the white noise inputs are zero mean, then the mean of the vector output $\bar{y}(n)$ is given by

$$E[\bar{y}(n)] = \bar{C}(n)\Phi(n, k_0)E[\bar{x}(k_0)].$$

By observing statistical independence of the white noise input and using (26) and (35), we obtain the autocovariance matrix $K_{\bar{y}}(n, m)$ of the roundoff noise $\bar{y}(n)$ of the pipelined-block LTV discrete-time system as follows:

$$\begin{aligned} K_{\bar{y}}(n, m) &= E[(\bar{y}(n) - E[\bar{y}(n)])(\bar{y}(m) - E[\bar{y}(m)])^T] \\ &= \bar{C}(n)\Phi(n, k_0)R_{\bar{x}}(k_0)\Phi^T(m, k_0)\bar{C}^T(m) \\ &\quad + \sum_{k=k_0}^{m-1} \bar{C}(n)\Phi(n, k+1)K_{\bar{v}}(k)\Phi^T(m, k+1)\bar{C}^T(m) \\ &\quad + K_{\bar{w}}(n, m), \quad \text{if } n > m, \\ K_{\bar{y}}(n, m) &= \bar{C}(n)\Phi(n, k_0)R_{\bar{x}}(k_0)\Phi^T(m, k_0)\bar{C}^T(m) \\ &\quad + \sum_{k=k_0}^{n-1} \bar{C}(n)\Phi(n, k+1)K_{\bar{v}}(k)\Phi^T(m, k+1)\bar{C}^T(m) \\ &\quad + K_{\bar{w}}(n, m), \quad \text{if } n < m, \end{aligned}$$

$$\begin{aligned}
 K_{\bar{y}}(n, m) = R_{\bar{y}}(n) &= \bar{C}(n)\Phi(n, k_0)R_{\bar{x}}(k_0)\Phi^T(m, k_0)\bar{C}^T(m) \\
 &+ \sum_{k=k_0}^{n-1} \bar{C}(n)\Phi(n, k+1)K_{\bar{v}}(k)\Phi^T(m, k+1)\bar{C}^T(m) \\
 &+ K_{\bar{w}}(n, m), \quad \text{if } n = m,
 \end{aligned}$$

$R_{\bar{x}}(k_0) = E[\overset{\circ}{\bar{x}}(k_0)\overset{\circ}{\bar{x}}^T(k_0)]$ is the covariance matrix of the initial state.

5. Roundoff noise in pipelined-block LTI discrete-time systems in state space. Substituting (38) into (36b), we get

$$\begin{aligned}
 \bar{y}(n) &= \bar{C}(n)\Phi(n, k_0)\bar{x}(k_0) \\
 &+ \sum_{k=k_0}^{n-1} \bar{C}(n)\Phi(n, n-k)\bar{v}(n-k-1) + \bar{w}(n), \quad n = 0, 1, 2, \dots (40)
 \end{aligned}$$

For the pipelined-block LTI discrete-time system $\bar{A}(n) = \bar{A}$, $\bar{C}(n) = \bar{C}$,

$$\Phi(n, k_0) = \prod_{i=1}^{n-k_0} \bar{A}(n-i) = \bar{A}^{n-k_0}, \quad (41)$$

and

$$\Phi(n, n-k) = \prod_{i=1}^k \bar{A}(n-i) = \bar{A}^k. \quad (42)$$

Substituting (41) and (42) into (40), we get roundoff noise $\bar{y}(n)$ of the pipelined-block LTI discrete-time system as follows:

$$\bar{y}(n) = \bar{C}\bar{A}^{n-k_0}\bar{x}(k_0) + \sum_{k=k_0}^{n-1} \bar{C}\bar{A}^k\bar{v}(n-k-1) + \bar{w}(n), \quad (43)$$

where $\bar{v}(n)$ and $\bar{w}(n)$ are noise vectors with covariance matrices $K_{\bar{v}} = \sigma^2 I_N$, and $K_{\bar{w}} = \sigma^2 I_M$, respectively. Since the white noise input is zero mean, the mean of the output $\bar{y}(n)$ is given by

$$E[\bar{y}(n)] = \bar{C}\bar{A}^{n-k_0}E[\bar{x}(k_0)].$$

If the discrete-time system is stable, then the steady state mean of the output converges to a zero vector $E[\bar{y}(\infty)] = 0$. By observing statistical independence

of the white noise input, using (43) and assuming $k_0 = 0$, we obtain the autocovariance matrix of the roundoff noise as follows:

$$\begin{aligned} K_{\bar{y}}(n, m) &= E \left[(\bar{y}(n) - E[\bar{y}(n)]) (\bar{y}(m) - E[\bar{y}(m)])^T \right] \\ &= \bar{C} \bar{A}^n R_{\bar{x}}(0) (\bar{A}^n)^T \bar{C}^T + \sigma^2 \left[\sum_{k=0}^{n-1} \bar{C} \bar{A}^k (\bar{A}^k)^T \bar{C}^T + I_M \right], \end{aligned} \quad (44)$$

where

$$R_{\bar{x}}(0) = E \left[(\bar{x}(0) - E[\bar{x}(0)]) (\bar{x}(0) - E[\bar{x}(0)])^T \right].$$

The steady state output covariance matrix is defined by

$$K_{\bar{y}}(n - m) = K_{\bar{y}}(n, m) \Big|_{n \rightarrow \infty}.$$

Since the discrete-time system is stable, in the steady state the first term in (44) converges to a zero matrix. Thus we obtain the *steady state* autocovariance matrix of the roundoff noise as follows:

$$K_{\bar{y}}(n - m) = \sigma^2 \left[\sum_{k=0}^{\infty} \bar{C} \bar{A}^k (\bar{A}^k)^T \bar{C}^T + I_M \right]. \quad (45)$$

The block output is combined by a parallel-in, serial-out register to form a scalar output. Then the autocovariance function K_y and the variance σ_y^2 of the roundoff noise at the j th output summing node are given by

$$\begin{aligned} K_y(nM + i, mM + j) &= K_{\bar{y}}(n, m)_{i+1, j+1}, \\ n, m &= 0, 1, 2, \dots; \quad i, j = 0, 1, \dots, M - 1, \end{aligned}$$

and

$$\sigma_y^2(mM + j) = K_{\bar{y}}(m, m)_{j+1, j+1},$$

respectively.

In the *steady state* the autocovariance function and the variance of the round-off noise at the j th output summing node are given by Barnes and Shinnaka (1980b):

$$K_y(nM + i, mM + j) = K_{\bar{y}}(n - m)_{i+1, j+1},$$

and

$$\sigma_y^2(mM + j) = K_{\bar{y}}(0)_{j+1, j+1}, \quad (46)$$

respectively.

So in the steady state, the autocovariance function of a roundoff noise in a pipelined-block implementation is periodic with period M .

Substitution of (45) into (46) yields

$$\sigma_y^2(mM + j) = \sigma^2 \left[\sum_{k=0}^{\infty} \bar{C} \bar{A}^k (\bar{A}^k)^T \bar{C}^T + I_M \right]_{j+1, j+1}, \quad (47)$$

$$j = 0, 1, \dots, M - 1.$$

For LTI discrete-time system, from (18), (20), and (47), we get the same result as in Parhi and Messerschmitt (1989c).

$$\sigma_y^2(mM + j) = \sigma^2 \left[c A^j \sum_{k=0}^{\infty} A^{kM} (A^{kM})^T (A^j)^T c^T + 1 \right], \quad (48)$$

$$j = 0, 1, \dots, M - 1.$$

Eq. 48 demonstrates that the steady state variance of the roundoff noise at the j th output in a pipelined-block implementation is periodic with period M .

We define the average steady state roundoff noise variance at the outputs in a pipelined-block implementation by

$$(\sigma_y^2)_{av} = \frac{1}{M} \sum_{j=0}^{M-1} \sigma_y^2(mM + j) |_{m \rightarrow \infty}. \quad (49)$$

Substituting (48) into (49), we have

$$(\sigma_y^2)_{av} = \frac{\sigma^2}{M} \left[\sum_{j=0}^{M-1} \sum_{k=0}^{\infty} c A^{kM+j} (A^{kM+j})^T c^T + 1 \right]. \quad (50)$$

For the state space realization of LTI discrete-time system, the roundoff noise variance at the scalar output is given by

$$\sigma_y^2(\infty) = \sigma^2 \left[\sum_{k=0}^{\infty} c A^k (A^k)^T c^T + 1 \right]. \quad (51)$$

However,

$$\sum_{j=0}^{M-1} \sum_{k=0}^{\infty} c A^{kM+j} (A^{kM+j})^T c^T = \sum_{k=0}^{\infty} c A^k (A^k)^T c^T + 1. \quad (52)$$

Then, comparing (50) and (51), it can be observed, that in a pipelined-block implementation the average roundoff noise variance at the outputs is reduced by the factor M , the dimension of the input and the output vectors.

6. Scaling in the pipelined LTV discrete-time systems in state space. In this section we shall introduce a scaling rule for the pipelined LTV discrete-time system in state space.

In digital discrete-time systems implemented with fixed point arithmetic, overflow can occur during addition. In the state space realization, it is common to place dynamic range constraints on the states variable to control overflow. This procedure is called scaling.

We scale a pipelined-block discrete-time system as follows: for stationary zero mean white noise vector input with the covariance matrix $\sigma_u^2 I$, we require that each state variable variance $\sigma_{\bar{x}_i}^2$ in steady state satisfies the inequality

$$\frac{\sigma_{\bar{x}_i}^2}{\sigma_u^2} \leq \gamma^2, \quad i = 1, 2, \dots, N, \quad (53)$$

where γ is a parameter that controls the overflow.

The variance $\sigma_{\bar{x}_i}^2$ of the i th state variable \bar{x}_i is the i th diagonal element of the covariance matrix of the state vector \bar{x} . Then we are concerned with the properties of the state covariance matrix $K_{\bar{x}}(n, m)$.

Let us consider the state equation with zero mean white noise block input $\bar{u}(k)$ of the pipelined LTV discrete-time system described by (7):

$$\bar{x}(k+1) = \bar{A}(k)\bar{x}(k) + \bar{B}(k)\bar{u}(k), \quad k = 0, 1, 2, \dots \quad (54)$$

We assume that the system is stable. The solution of (54) is given by

$$\bar{x}(n) = \Phi(n, k_0)\bar{x}(k_0) + \sum_{k=k_0}^{n-1} \Phi(n, n-k)\bar{B}(k)\bar{u}(n-k-1),$$

where

$$\Phi(n, k_0) = \prod_{i=1}^{n-k_0} \bar{A}(n-i),$$

$$\Phi(n, n-k) = \prod_{i=1}^k \bar{A}(n-i).$$

The mean of the state vector $\bar{x}(n)$ is given by

$$E[\bar{x}(n)] = \Phi(n, k_0)E[\bar{x}(k_0)].$$

The covariance matrix of the state vector $\bar{x}(n)$ is given as follows:

$$\begin{aligned} K_{\bar{x}}(n, m) &= E\left[(\bar{x}(n) - E[\bar{x}(n)])(\bar{x}(m) - E[\bar{x}(m)])^T\right] \\ &= E\left[\Phi(n, k_0)\overset{\circ}{\bar{x}}(k_0)\overset{\circ}{\bar{x}}^T(k_0)\Phi^T(m, k_0) \right. \\ &\quad + \Phi(n, k_0)\overset{\circ}{\bar{x}}(k_0)\sum_{k=k_0}^{m-1}\bar{u}^T(m-k-1)\bar{B}^T(k)\Phi^T(m, m-k) \\ &\quad + \sum_{k=k_0}^{n-1}\Phi(n, n-k)\bar{B}(k)\bar{u}(n-k-1)\overset{\circ}{\bar{x}}^T(k_0)\Phi^T(m, k_0) \quad (55) \\ &\quad + \sum_{k=k_0}^{n-1}\sum_{l=l_0}^{m-1}\Phi(n, n-k)\bar{B}(k)\bar{u}(n-k-1) \\ &\quad \left. \times \bar{u}^T(m-l-1)\bar{B}^T(l)\Phi^T(m, m-l)\right] \\ &= \Phi(n, k_0)K_{\bar{x}}(k_0)\Phi^T(m, k_0) \\ &\quad + \sigma_u^2\sum_{k=k_0}^{n-1}\Phi(n, n-k)\bar{B}(k)\bar{B}^T(k)\Phi^T(m, m-k), \end{aligned}$$

where

$$\overset{\circ}{\bar{x}}(k_0) = \bar{x}(k_0) - E[\bar{x}(k_0)].$$

Since the discrete-time system is stable, for the steady state the first term on the right side in (55) converges to a zero matrix. Thus we obtain the covariance matrix of the state vector \bar{x} in the steady state as follows:

$$K_{\bar{x}}(n, m)|_{n \rightarrow \infty} = \sigma_u^2\sum_{k=k_0}^{\infty}\Phi(n, n-k)\bar{B}(k)\bar{B}^T(k)\Phi^T(m, m-k).$$

Thus, for scaling the pipelined LTV or LPTV discrete-time system, we select $\bar{A}(k)$ and $\bar{B}(k)$ as follows:

$$\frac{\sigma_{\bar{x}_i}^2}{\sigma_u^2} = \left[\sum_{k=k_0}^{\infty} \Phi(n, n-k)\bar{B}(k)\bar{B}^T(k)\Phi^T(m, m-k) \right]_{ii} \leq \gamma^2, \quad (56)$$

$$i = 1, 2, \dots, N,$$

where $\sigma_{\bar{x}_i}^2$ is the steady state variance of the i th state variable.

For the pipelined LTI discrete-time systems $\bar{A}(n) = \bar{A}$, $\bar{B}(n) = \bar{B}$, $\Phi(n, n-k) = \bar{A}(n-1) \dots \bar{A}(n-k) = \bar{A}^k$. Assuming $k_0 = 0$, and using (56), we get

$$\frac{\sigma_{\bar{x}_i}^2}{\sigma_u^2} = \left[\sum_{k=0}^{\infty} \bar{A}^k \bar{B} \bar{B}^T (\bar{A}^T)^k \right]_{ii} \leq \gamma^2, \quad i = 1, 2, \dots, N. \quad (57)$$

Therefore for scaling the pipelined LTI discrete-time system, we select \bar{A} and \bar{B} from (57).

7. Concluding remarks. This paper proposed a unified approach to the modeling of pipelined-block LTV discrete-time systems. Using the general solution of an LTV discrete-time system, we can get any form of a pipelined-block LTV, LPTV, or LTI discrete-time system. Models of pipelined-block and scalar discrete-time systems in state space are of the same form. So, we derived the state covariance and output covariance matrices for pipelined-block models, using the ordinary way. Analysis of the roundoff noise error in pipelined state space digital systems shows, that the roundoff error strictly improves with an increase in number of pipeline stages. We assumed that the roundoff operation was performed at the output of the state variables and at the system outputs. The noise sources are assumed to be white stationary with zero mean and statistically independent of signals. The roundoff noise is nonstationary, and the maximal variance of the roundoff noise in a pipelined-block realization is never greater than the variance of the roundoff noise in the associated scalar state space realization. The effect of the pipelined-block structure is to reduce the internally generated roundoff noise. Pipelined-block realizations can result in reduced roundoff noise with the greatest reduction occurring in digital systems with poles near the unit circle.

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KONVEJERINIŲ-BLOKINIŲ TIESINIŲ KINTAMŲ PARAMETRŲ DISKRETINIŲ SISTEMŲ STATISTINĖS SAVYBĖS

Kazys KAZLAUSKAS

Nagrinėjamos konvejerinių-blokinių tiesinių kintamų parametrų diskretinių sistemų statistinės savybės. Konvejerinės-blokinės lygtys išvedamos naudojant kintamų parametrų diskretinių sistemų būsenų erdvės bendrąjį sprendinį. Analizuojamos konvejerinių-blokinių kintamų ir pastovių parametrų diskretinių sistemų būsenų ir išėjimo signalo kovariacinės matricos. Išvestos išraiškos apvalinimo triukšmams apskaičiuoti. Parodyta, kaip parenkamas konvejerinių kintamų parametrų diskretinių sistemų mastelis.